Appendix to
"Modeling Behavioral Responses to COVID-19"

Ben Craig, Thomas Phelan, Jan-Peter Siedlarek, and Jared Steinberg.

Federal Reserve Bank of Cleveland, *Economic Commentary*, 2021-05.

1 Mathematical model

An individual's preferences are represented by

$$U = \mathbf{E}\left[\int_0^{\tau_D} \rho_S e^{-(\rho_S+\rho_V)t} \ln a_t + e^{-(\rho_S+\rho_V)\tau_D} u_D\right] \tag{1}$$

where ρ_S, ρ_V and u_D are constants and τ_D is the (random) date of death. Here ρ_S represents the subjective discount factor of the individual and ρ_V represents the arrival rate of the vaccine. The parameter u_D captures the desire of the individual to avoid exposure to the virus and the attendant possibility of death. At any time the health status of each individual may be in one of four possible states, S, I, R or D, referred to as *Susceptible, Infected, Recovered* or *Dead*. The population is indexed by the unit interval and $\Gamma_{St}, \Gamma_{It} \subseteq [0,1]$ denote the sets of susceptible and infected individuals, respectively, at time $t \geq 0$. If the actions taken at date t are given by $(a_{it})_{i\in[0,1]}$ then a susceptible individual who adheres to activity a_t becomes infected at rate $\beta a_t \int_{\Gamma_I} a_{it} di$. Infected cases are *resolved* at a constant rate γ. A fraction $\delta(I_t)$ of resolved cases end in death and the remaining fraction end in recovery. Once recovered an individual is assumed to be immune, unable to contract the virus again or transmit it to others. We assume that when susceptible and infected individuals take actions a_1 and a_2 that population shares evolve according to

$$\begin{aligned} S' &= -\beta S I a_1 a_2 \\ I' &= \beta S I a_1 a_2 - \gamma I \\ R' &= \gamma(1-\delta(I))I \\ D' &= \gamma\delta(I)I \end{aligned}$$

where $S + I + R + D = 1$. We assume that the mortality function is given by

$$\delta(I) = \underline{\delta} + (\overline{\delta} - \underline{\delta}) \min\{(I/H)^\kappa, 1\} \tag{2}$$

for some positive constants $\underline{\delta}, \overline{\delta}, H$ and κ with $\overline{\delta} > \underline{\delta}$. The specification (2) is a simple way of capturing the fact that fatality rates are increasing with the strain on the healthcare system (i.e. when I is large). The parameters $\underline{\delta}$ and $\overline{\delta}$ are then the lower and upper bounds of the fatality rate.

1.1 Planner's problem

We first analyze the problem of a utilitarian social planner who may directly control the actions of susceptible agents, where the actions of the infected are fixed at a_2^*. We write the planning problem in terms of actions and the original state variables before simplifying. The flow payoff function is

$$F(S, I, R, D, a_1) = \rho_S[Du_D + S \ln a_1 + I \ln a_2^* + R \ln(1)]$$

where we used the fact that there is no loss in assuming recovered symptomatic individuals take the highest action. For simplicity we suppose that $a_2^* = 1$ and treat the arrival of the vaccine as if it were a cure, so that the flow payoff to the planner when the vaccine arrives is Du_D. The planner's value function then solves

$$\begin{aligned}\rho_S W(S, I, D) &= \rho_S Du_D + \rho_V(Du_D - W(S, I, D)) + \gamma\delta(I)IW_3(S, I, D) \\ &\quad + \max_{a_1 \in [\underline{a}_1, 1]} \rho_S S \ln a_1 - \beta S a_1 I W_1(S, I, D) + [\beta S a_1 - \gamma] I W_2(S, I, D).\end{aligned}$$

Direct substitution shows that we may remove one more state variable.

Lemma 1.1. *The value function is of the form $W(S, I, D) = Du_D - V(S, I)$ where V solves*

$$(\rho_S + \rho_V)V(S, I) = \min_{a_1 \in [\underline{a}_1, 1]} -\rho_S S \ln a_1 - \gamma\delta(I)Iu_D - \beta S I a_1 V_1(S, I) + [\beta S a_1 - \gamma] I V_2(S, I).$$

The function V has a natural interpretation as the "cost of the pandemic" in terms of utility, and is the payoff from a cost minimization problem with flow payoff

$$\tilde{C}(S, I, a_1) = -\rho_S S \ln a_1 - \gamma\delta(I)Iu_D \tag{3}$$

and state variable (S, I) evolving according to

$$(\dot{S}, \dot{I}) = (-\beta S I a_1, \beta S I a_1 - \gamma I). \tag{4}$$

1.2 Stationary Markov competitive equilibrium

We now consider the competitive equilibria. All individuals will maximize expected discounted utility, taking as given the actions of everyone else. The value functions for the dead and recovered individuals are constant at u_D and $u(1)$, respectively. Denote the average action of susceptible individuals when the aggregate state is (S, I) by $M(S, I)$, and note that the aggregate states again evolves according to

$$(\dot{S}, \dot{I}) = (-\beta S M(S, I) I, \beta S M(S, I) I - \gamma I). \tag{5}$$

Denote the value functions of susceptible and infected individuals by $U(S, I; 1)$ and $U(S, I; 2)$, respectively. The Hamilton-Jacobi-Bellman equations for the susceptible and infected agents are

$$\begin{aligned}(\rho_S + \rho_V)U(S, I; 1) &= \max_{a_1 \in [\underline{a}_1, 1]} \rho_S \ln a_1 + \beta a_1 I[U(S, I; 2) - U(S, I; 1)] \\ &\quad - \beta S M(S, I) I U_1(S, I; 1) + (\beta S M(S, I) - \gamma) I U_2(S, I; 1) \\ (\rho_S + \rho_V)U(S, I; 2) &= \gamma(\delta(I)[u_D - U(S, I; 2)] - (1 - \delta(I))U(S, I; 2)) \\ &\quad - \beta S M(S, I) I U_1(S, I; 2) + (\beta S M(S, I) - \gamma) I U_2(S, I; 2).\end{aligned}$$

Given an aggregate law of motion M there is an associated policy function $m(S, I; M)$ solving the problem of the susceptible individuals. Define an operator J on functions of the form $M : [0,1]^2 \to \mathbf{R}$, by

$$J(M)(S, I) = m(S, I; M) - M. \tag{6}$$

The equilibrium notion is then standard: all individuals solve their individual problems taking the aggregate law of motion as given, and the associated law of motion is consistent with individual behavior.

Definition 1.1. A stationary Markov competitive equilibrium consists of value functions $U(S, I; 1)$ and $U(S, I; 2)$ together with a policy function $m(S, I; M)$ for the infected, such that:

- The functions $U(S, I; 1), m(S, I; M)$ and $U(S, I; 2)$ solve the problems of the susceptible and infected individuals, respectively.
- The law of motion of the aggregate state is consistent with the policy function of the susceptible individuals, or $J(M) = 0$.

2 Numerical algorithm

This section outlines the numerical algorithm used to solve both the social planner's problem and the competitive equilibrium. Since the mass of infected agents is often an order of magnitude smaller than the mass of susceptible agents, we will use a non-uniform grid for I and a uniform grid for S. For fixed $N_S, N_I \geq 1$, define $\Sigma_S = \{0, 1/N_S, \ldots, 1 - 1/N_S, 1\}$ and $\Sigma_I = \exp(u) - s$, where for some $c > 0$, $s := c^2/(1-2c)$ and u is a uniform grid on $[\ln s, \ln(1+s)]$ with N_I points. Writing $\Sigma_I := \{I_0, I_1, \ldots, I_{N_I}\}$, define

$$\begin{aligned} \Delta_{Ii}^- &= I_i - I_{i-1} & i &= 1, \ldots, N_I \\ \Delta_{Ii}^+ &= I_{i+1} - I_i & i &= 0, \ldots, N_I - 1 \end{aligned}$$

and declare $\Delta_{I0}^- = \Delta_{IN_I}^+ = 0$ and $\Delta_S = 1/N_S$. We then write $\Sigma := \Sigma_S \times \Sigma_I$.

2.1 Planning problem

We now wish to solve the control problem defined by (3) and (4). We must therefore construct a locally consistent Markov chain for the law of motion (4). The local consistency requirements are given by

$$\begin{aligned} \mathbf{E}[\Delta X_S] &= -\Delta_t \beta S I a_1 + o(\Delta_t) \\ \mathbf{E}[\Delta X_I] &= \Delta_t (\beta S a_1 - \gamma) I + o(\Delta_t). \end{aligned} \tag{7}$$

For an arbitrary $(S, I) \in \Sigma$ there are three possible transitions, to $(S - \Delta_S, I), (S, I - \Delta_I^-)$ and $(S, I + \Delta_I^+)$, with associated probabilities p^{-S}, p^{-I} and p^{+I}. The local consistency requirements (7) are then

$$\begin{aligned} -\Delta_S p^{-S} &= -\Delta_t \beta S I a_1 + o(\Delta_t) \\ -\Delta_I^- p^{-I} + \Delta_I^+ p^{+I} &= \Delta_t (\beta S a_1 - \gamma) I + o(\Delta_t). \end{aligned}$$

Inspection reveals it will suffice to set

$$p^{-S} = \frac{\Delta_t}{\Delta_S}\beta S I a_1 \qquad\qquad p^{\pm I} = \frac{\Delta_t}{\Delta_I^{\pm}} \max\left\{\pm(\beta S a_1 - \gamma)I, 0\right\}. \tag{8}$$

At $I = 1$ we impose $a_1 \leq \tilde{a}_1(S) = \gamma/[\beta S]$. We then have the Bellman equation

$$V(S,I) = \min_{a_1 \in [\underline{a}_1, 1]} \; -\Delta_t \rho_S S \ln a_1 - \Delta_t \gamma \delta(I) I u_D + e^{-(\rho_S+\rho_V)\Delta_t} V(S,I)$$
$$+ e^{-(\rho_S+\rho_V)\Delta_t}\big(p^{-S}V(S-\Delta_S, I) + p^{+I}V(S, I+\Delta_I^+) + p^{-I}V(S, I-\Delta_I^-) - (p^{-S}+p^{+I}+p^{-I})V(S,I)\big).$$

Omitting terms independent of the controls, using (8), dividing by $-\Delta_t \rho_S S$ and sending $\Delta_t \to 0$ gives

$$\max_{a_1 \in [\underline{a}_1, 1]} \; \ln a_1 + \beta S a_1 D V^{BS} + (\beta S a_1 - \gamma) D\big((a_1 > \tilde{a}(S))[-V^{FI}] + [1 - (a_1 > \tilde{a}(S))][-V^{BI}]\big)$$

where $D := I/[\rho_S S]$. Subtracting $\gamma D V^{BS}$, the above maximization is equivalent to

$$\max_{a_1 \in [\underline{a}_1, 1]} \ln a_1 + (\beta S a_1 - \gamma)\big[(a_1 > \tilde{a}(S))[V^{BI} - V^{FI}] + V^{BS} - V^{BI}\big] D.$$

On each interval $[\underline{a}_1, \tilde{a}_1]$ and $[\tilde{a}_1, 1]$ this is of the form

$$G(a,b,c) := \max_{a_1 \in [a,b]} \; \ln a_1 + c a_1, \tag{9}$$

where $c := \big((a_1 > \tilde{a}(S))[V^{BI} - V^{FI}] + V^{BS} - V^{BI}\big) D \beta S$. If $c \geq 0$ then $a_1 = b$. Otherwise the objective is concave, with first-order condition $0 = v'(a_1^{FOC}) + c$. For any $0 < a < b$, the solution to (9) is

$$a_1(c) = b 1_{c \geq 0} + (1 - 1_{c \geq 0}) \max\left\{a, \min\left\{-1/c, b\right\}\right\}. \tag{10}$$

Finally, to avoid overflow in the numerical examples it is useful to divide all quantities by Δ_t and consider the limit of the above as $\Delta_t \to 0$. The linear system we wish to solve at each stage is $0 = b + TV$ where

$$b = -\rho_S S \ln a_1 - \gamma \delta(I) I u_D$$
$$TV = -(\rho_S + \rho_V + \overline{p}^{-S} + \overline{p}^{+I} + \overline{p}^{-I})V + \overline{p}^{-S}V(S - \Delta_S, I) + \overline{p}^{+I}V(S, I + \Delta_I^+) + \overline{p}^{-I}V(S, I - \Delta_I^-)$$

where for each probability we have $\overline{p} = p/\Delta_t$.

2.2 Competitive equilibrium

The state space and transition probabilities associated with the aggregate state (S, I) will coincide with those constructed in Section 2.1. For the symptomatic agents, the belief state vanishes identically and is therefore omitted, so we need only specify the transition probabilities for the aggregate state and health status. For an arbitrary $(S, I) \in \Sigma$ there are three possible transitions, to the adjacent points $(S - \Delta_S, I), (S, I - \Delta_I^-)$ and $(S, I + \Delta_I^+)$, with probabilities p^{-S}, p^{-I} and p^{+I} given by

$$\begin{aligned} p^{-S} &= \frac{\Delta_t}{\Delta_S}\beta S M(S,I) I \\ p^{\pm I} &= \frac{\Delta_t}{\Delta_I^{\pm}}\{\pm[\beta S M(S,I) - \gamma] I, 0\}. \end{aligned} \tag{11}$$

The expressions (11) give the probabilities of transitions for the aggregate state and are common to all agents. The probabilities with which an infected agent transitions to the recovered or dead states are defined to be $\Delta_t \gamma(1-\delta(I))$ and $\Delta_t \gamma \delta(I)$, respectively. Writing $\overline{p} = p/\Delta_t$ and sending $\Delta_t \to 0$ gives the Bellman equation for infected agents

$$\begin{aligned} 0 &= \gamma\delta(I)u_D + \overline{p}^{-S}U(S-\Delta_S, I; 2) + \overline{p}^{+I}U(S, I+\Delta_I^+; 2) + \overline{p}^{-I}U(S, I-\Delta_I^-; 2) \\ &\quad - \left(\rho_S + \rho_V + \overline{p}^{-S} + \overline{p}^{+I} + \overline{p}^{-I} + \gamma\right)U(S, I; 2) \end{aligned} \tag{12}$$

which is of the form $0 = b + TU$. Susceptible agents contract the virus with probability $\Delta_t \beta a_1 I$, so the Bellman equation for susceptible agents is

$$\begin{aligned} U(S, I; 1) = \max_{a_1 \in [\underline{a}_1, 1]} \; & \Delta_t \rho_S \ln a_1 + e^{-(\rho_S+\rho_V)\Delta_t}\Delta_t \rho\beta a_1 I[U(S, I; 2) - U(S, I; 1)] \\ & + e^{-(\rho_S+\rho_V)\Delta_t}\left(p^{-S}U(S-\Delta_S, I+\Delta_S; 1) + p^{-I}U(S, I-\Delta_I^-; 1) + p^{+I}U(S, I+\Delta_I^+; 1)\right) \\ & + e^{-(\rho_S+\rho_V)\Delta_t}\left(1 - p^{-S} - p^{-I} - p^{+I}\right)U(S, I; 1). \end{aligned} \tag{13}$$

Omitting terms independent of a_1, dividing by $r\Delta_t$ and sending $\Delta_t \to 0$, the maximization becomes

$$\max_{a_1 \in [\underline{a}_1, 1]} \; \ln a_1 + \rho_S^{-1}\beta a_1 I[U(S, I; 2) - U(S, I; 1)]$$

The optimal policy is then (10), where

$$c = \rho_S^{-1}\beta[U(S, I; 2) - U(S, I; 1)]I.$$

Sending $\Delta_t \to 0$, the linear system associated with (13) is $0 = b + TU$ where

$$\begin{aligned} b &= \rho_S \ln a_1 + \rho\beta a_1 I[U(S, I; 2) - U(S, I; 1)] \\ TU &= \overline{p}^{-S}U(S-\Delta_S, I+\Delta_S; 1) + \overline{p}^{-I}U(S, I-\Delta_I^-; 1) + \overline{p}^{+I}U(S, I+\Delta_I^+; 1) \\ &\quad - \left(\rho_S + \rho_V + \overline{p}^{-S} + \overline{p}^{-I} + \overline{p}^{+I}\right)U(S, I; 1). \end{aligned}$$

We then iterate upon the policy function a_1: we begin with an arbitrary guess a_1, solve the problem of the asymptomatic agent, replace a_1 with the implied policy function a_1', and repeat until convergence.

3 Choice of parameters

The full list of parameters in the model is: $\rho_S, \rho_V, \beta, \gamma, u_D, \underline{\delta}, \overline{\delta}, H$ and κ. The subjective discount factor $\rho_S = 0.05$ is standard and ρ_V is chosen to match an expected arrival of 1.5 years at the beginning of the pandemic, as in Farboodi et al. (2020). The parameters $\beta = 3/10$ and $\gamma = 1/7$ are also taken from Farboodi et al. (2020). The parameter u_D is chosen in a manner similar to that done in Hall et al. (2020) and Farboodi et al. (2020). Hall et al. (2020) estimate the value of a statistical life of a typical COVID-19 victim to be approximately \$3,915,000 and quote US income per capita to be roughly \$45,000. An individual would therefore be willing to pay \$39,150δ once (or $\$39,150\delta\rho$ daily if ρ is the daily discount rate), or a fraction

$$\frac{\$39,150\delta\rho}{\$45,000/365} \approx 317\delta\rho$$

of daily consumption to avoid a $\delta\%$ chance of death. We determine u_D by equating the utility across the following two scenarios: 1. consume fraction $1 - 317\delta\rho$ of output every day and face no possibility of death; and 2. consume all output every day but face $\delta\%$ chance of incurring disutility u_D. The utility from the first scenario is $\ln(1 - 317\delta\rho)$ and that of the second is $(1 - \delta/100)\ln(1) + (\delta/100)u_D$. We therefore have

$$u_D = \frac{100}{\delta}\ln(1 - 317\delta\rho) \approx \frac{100}{\delta}[-317\delta\rho] = -31,700\rho$$

if $\delta\rho$ is very small. Finally, the lower and upper bounds for fatality rates are $\underline{\delta} = 0.5\%$ and $\overline{\delta} = 1.0\%$, as a simple way of encompassing the value 0.062% used in Farboodi et al. (2020) and the upper estimate of 0.081% in Hall et al. (2020). The Python code is available upon request.

References

Farboodi, Maryam, Gregor Jarosch, and Robert Shimer (2020). "Internal and external effects of social distancing in a pandemic." Working paper 27059, National Bureau of Economic Research. doi:10.3386/w27059.

Hall, Robert E., Charles I. Jones, and Peter J. Klenow (2020). "Trading off consumption and covid-19 deaths." Working paper 27340, National Bureau of Economic Research. doi:10.3386/w27340.