working paper

15 12

Forecasts from Reduced-form Models under the Zero-Lower-Bound Constraint

Mehmet Pasaogullari

FEDERAL RESERVE BANK OF CLEVELAND

Working Paper 15-12 July 2015

Forecasts from Reduced-form Models under the Zero-Lower-Bound Constraint

Mehmet Pasaogullari

In this paper, I consider forecasting from a reduced-form VAR under the zero lower bound (ZLB) for the short-term nominal interest rate. The ZLB constraint expands the number of states exponentially, making the exact computation of forecast moments infeasible. I develop a method that a) computes the exact moments for the first $n + 1$ periods when n previous periods are tracked and b) approximates moments for the periods beyond $n + 1$ period using techniques for truncated normal distributions and approximations a la Kim (1994). In its simplest form, the algorithm tracks only the previous forecast period. The approximations become more accurate as additional previous periods are tracked at the cost of longer computational time, although when the method is tracking two or three previous periods, it is competitive with Monte Carlo simulation in terms computational time. I show that the algorithm produces satisfactory results for VAR systems with moderate to high persistence even when only one previous period is tracked. For very persistent VAR systems, however, tracking more periods is needed in order to obtain reliable approximations. I also show that the method is suitable for affine term-structure modeling, where the underlying state vector includes the short-term interest rate as in Taylor rules with inertia.

JEL Codes: E42, E43, E47, C53.

Keywords: monetary policy, forecasting from VARs, zero lower bound, normal mixtures.

Suggested citation: Pasaogullari, Mehmet, 2015. "Forecasts from Reduced-form Models under the Zero-Lower-Bound Constraint," Federal Reserve Bank of Cleveland, working paper, no. 15-12.

Mehmet Pasaogullari is at the Federal Reserve Bank of Cleveland (mehmet.pasaogullari@clev.frb.org). He thanks seminar participants at the Federal Reserve Bank of Cleveland and Cleveland State University.

1 Introduction

Facing the global financial crisis and the associated recession, many central banks around the world cut their short-term interest rates and many of these rates among the developed countries hit the zero lower bound (ZLB). The ZLB puts a nonlinearity in the otherwise linearized models that central banks and academics use to describe the evolution of the economy. Such linearized models are also used widely for analyzing the term structure models in affine models. There have been many attempts to deal with ZLB and the associated nonlinearity.

The first paper to deal with the ZLB and optimal monetary policy is Krugman (1998). That paper and the influential Eggertson and Woodford (2003) consider monetary policy when there is always a chance to be stuck at the ZLB. However, once out of the ZLB the lower bound can never be hit again. Although these papers provide important insights for monetary policy, recent papers study more realistic cases. For example, Fernandez-Villaverde et al. (2012) consider a medium-scale DSGE model and uses global projection methods to deal with the ZLB.

There are two other strands of literature that also study the ZLB problem. In the forecasting literature, there are papers such as Clark and McCracken (2014), which assess the predictive ability of conditional forecasts. However, these papers only consider forecasts for the case in which interest rate is zero; they do not explicitly consider that the case in which the short rate is zero because of an explicit lower bound. In the term-structure literature, there has been a wide interest in analyzing the effect of the ZLB for the yield curve. Black (1995) first proposes that the short-term rate can be thought of as an option since the arbitrage opportunity with cash creates the ZLB. Filipovic et al. (2014) and Andreasen and Meldrun (2014) use models where the interest rates cannot take values lower than a threshold via parameter restrictions. There are also papers analyzing what is called "shadow interest rate models". In those models, it is the shadow rate that is the driver of the short rate (along with other latent or observable variables) and thus the term structure. This shadow rate can take any value. As long as it is above a threshold, the actual/observed interest rate is equal to this shadow rate. However, if it is less than the threshold, the observed interest rate takes the threshold value. Under such a system, Kim and Singleton (2012) uses simulations to come up with conditional expectations. Krippner (2012), Wu and Xia (2014), and Priebsch (2013) instead compute analytical expressions to compute those expectations.

However, economic relationships such as the investment–interest rate or asset pricing relationships are about the actual interest rate, not an artificial shadow rate. Accordingly, in this paper I use a setting where the actual/observed interest rate is endogenous. In other words, it is the actual (observed) short-term interest rate that directly affects other variables, and it is the actual (observed) short interest rate that is directly affected by other variables and shocks. In such a forecasting exercise, one has to pay attention to all of the previous interest rate forecasts, not only the forecast in the previous period in order to compute the exact moments of the future values of the variables. However, this causes the number of state variables to grow exponentially as the forecast horizon increases. The contribution of this paper is therefore twofold: First, it computes the exact moments of the variables for the first $n+1$ periods when n previous periods are tracked. However, this does come at the cost of tracking 2^n different states of binding and no-binding ZLB constraint. The second contribution of the paper is to find an approximation method to deal with the exponentially growing number of states.

In its simplest form, the proposed method tracks the ZLB only in the previous period and uses an approximation originally proposed by Kim (1994) for regime-switching state space models. I show that the method works pretty well for a moderate degree of persistence. However, the method's accuracy worsens as the persistence of the VAR system increases. I also show that we can improve the approximation by keeping track of more periods than just the previous one. As we track more periods, the algorithm gets numerically more complicated and more costly in terms of computational time but results show that the method where we track two or three periods can compete with the Monte Carlo simulations in terms of computational time and accuracy.

In the next section I set up the forecasting problem: I show the nonlinearity induced by the lower bound and the way this setting differs from the setting of the shadow rate models. In Section III, I develop the algorithm of this paper. I first show how to compute the exact moments for the first $n+1$ periods. I then present the approximation method that keeps the number of states constant. Section IV goes over some numerical examples showing the performance of the algorithm in terms of the accuracy of the approximation and computational speed. The method is quite suitable for VAR(1) models, which is also commonly used in the affine term structure models. In Section V, I show the results for such a numerical example. On the other hand, by increasing the number of periods tracked we can at the same time have a better ability to make the method work for VAR systems with more lags. In Section VI, I present the results for a VAR model with 2 lags. Section VII provides further possible avenues to increase computational accuracy and concludes.

2 Forecasting from the VAR under a lower bound

In this section, I present the forecasting problem and introduce how the existence of the zero-lower bound introduces a nonlinearity and brings about complexity in the forecasting and simulations for an otherwise ordinary reduced-form VAR. Throughout the presentation, I work with a VAR of order one. Note that in finance term-structure models (such as affine term structure models), the law of motion for the state is typically a VAR of order one. Thus, the method developed in this paper suits those models well. VAR models for economic forecasting, however, typically have more than one lag. As will be clear in the next section, the introduction of more lags will make the forecasting/simulation problem more complex. Later in the paper, I present the case with higher order VARs and suggest some ways to reduce the complexity.

Let X_t be an $n_x \times 1$ column vector of endogenous variables including the nominal interest rate, i_t. Without loss of generality we can order the variables such that i_t is the first variable. Let y_t denote other endogenous variables. If there is no bound on any of the endogenous variables, the endogenous variables will follow a VAR(1):

$$X_{t+1} = \mu + \Phi X_t + \Sigma \varepsilon_{t+1}.$$

Here, μ is the constant, Φ is a VAR (1) matrix of coefficients, Σ is the volatility matrix, and ε_t is the vector of errors (multi-) normally distributed with mean $\mathbf{0}$ and variance term I_{n_x}; $\varepsilon_t \sim N(\mathbf{0}, I_{n_x})$. Let's decompose the VAR system in a way that is helpful in the exposition of this paper's method. First, let e_i denote the row vector that picks out i_t from X_t, i.e. $e_i = [1, 0, .., 0]_{1 \times n_x}$ and $i_t = e_i X_t$. Similarly, let e_y denote the $(n_x - 1) \times n_x$ matrix that picks out other endogenous variables, y_t, from X_t, i.e. $e_y = \begin{bmatrix} \mathbf{0}_{n_x - 1 \times 1} & I_{n_x - 1} \end{bmatrix}$ and $y_t = e_y X_t$. The introduction of a lower bound on the nominal interest rate, $\bar{\imath}$, makes the law of motion for the endogenous variables as follows:

$$\begin{array}{|l|}
\hline
\tilde{X}_t = X_t, \\
\hline
\tilde{X}_{t+1} = \mu + \Phi \tilde{X}_t + \Sigma \varepsilon_{t+1}, \\
\hline
X_{t+1} = \left\{ \begin{array}{|l|l|} \hline \tilde{X}_{t+1} & \text{if } e_i \tilde{X}_{t+1} \geq \bar{\imath} \\ \hline \begin{array}{l} i_{t+1} = \bar{\imath} \\ y_{t+1} = e_y \tilde{X}_{t+1} \end{array} & \text{if } e_i \tilde{X}_{t+1} < \bar{\imath} \\ \hline \end{array} \right. . \\
\hline
\end{array}$$

Note that if the lower bound does not bind, $X_{t+1} = \mu + \Phi X_t + \Sigma \varepsilon_{t+1}$. In addition, the value of y_{t+1} does not differ whether the lower bound binds or not. However, since the next period's value for the other variables depends on the nominal rate in the current period, the value of those other variables in period $t+2$ will depend on whether the $t+1$ lower bound binds or not.

The way I am modeling the ZLB and its effects on other endogenous variables seems to be similar to the use of a shadow interest rate, as modeled in Krippner (2012), Wu and Xia (2014), and Priebsch (2013) but it has quite different implications . In these models, there is a shadow rate, s_t, that is affected by the state vector, X_t directly. X_t follows a VAR(1), and s_t may take any value as dictated by the reduced-form model, including values lower than $\bar{\imath}$. On the other hand, the nominal interest rate, i_t is equal to s_t if s_t is greater than $\bar{\imath}$ and equal to $\bar{\imath}$ otherwise:

$$i_t = \max\left(\bar{\imath}, s_t\right).$$

There are a number of differences between that modeling choice and mine. The endogenous relationship is between the shadow rate and the other variables in those models, not with the nominal interest rate and the other variables. Importantly, there is no feedback loop from the actual interest rate to the other endogenous variables or to the shadow rate. This seems to be in contrast with a) how the economy works or b) the monetary models that analyze the effects of the ZLB on the economy and optimal policy. As for the first point, for example, the level of investment in the economy is a function of the nominal interest rate. Similarly, the aggregate demand relationship (i.e., the relationship between the level of output gap, the short rate and the inflation rate) and asset pricing relationships are all about the nominal interest rate, not an artificial variable like the shadow rate. Second, imposing the ZLB constraint via shadow rate modeling does not lead to any of the concerns that Krugman (1998), and Eggertson and Woodford (2003)have raised. If the nominal rate is only an indicator and does not directly affect other variables, there would be no need to worry about whether there is a lower bound on it. As a final note, the ZLB is a recent phenomenon in the US and most other developed economies, so we don't really know much about the relationship between a shadow rate and other variables. For these reasons, I model the ZLB differently than in the term-structure papers cited above.

It can be argued that a longer duration of the nominal interest rate at zero (i.e., the persistence of a binding ZLB constraint) can easily be satisfied in a shadow rate model. If a large enough shock moves the shadow rate far away from the lower bound and if the process is persistent enough, no further large shocks are needed in order to be stuck at the ZLB. In contrast, since the VAR model is mean-reverting and the main driving variable for the interest rate is its own lag[1], which is at least zero, the tendency to stay at the ZLB is relatively smaller in my model. However, the approach chosen should relate to the kind of force one thinks causes a binding ZLB. If a persistent shock (like a discount rate shock in macro models) is thought to be responsible, not a big one-time shock, it can

[1]Either because of persistence or because of a monetary policy rule with a lagged interest rate.

easily be modeled within the framework I introduce. Or if one thinks that other variables, such as a big and persistent negative output gap, causes the ZLB constraint to bind (via a Taylor-type policy rule), that, too, can easily be modeled within my framework. In contrast, the main mechanism that shadow rate papers propose is that the nominal short rate is zero today because it was zero yesterday because of an underlying shadow rate at a very low level .

Let's continue with the decomposition of the variables. First, note that we can decompose the error term into $\varepsilon_{1,t}$ and $\tilde{\varepsilon}_t$ as $\varepsilon_t = \begin{bmatrix} \varepsilon_{1,t+1} & \tilde{\varepsilon}'_t \end{bmatrix}'$. Next we define following the submatrices for the decomposition:

$$X_t = \begin{bmatrix} i_t \\ y_t \end{bmatrix}, \varepsilon_t = \begin{bmatrix} \varepsilon_{1,t} \\ \tilde{\varepsilon}_t \end{bmatrix}, \mu = \begin{bmatrix} \mu_i \\ \mu_y \end{bmatrix},$$
$$\Phi = \begin{bmatrix} \Phi_i \\ \Phi_y \end{bmatrix}, \Phi = \begin{bmatrix} \lambda_i & \lambda_y \end{bmatrix}, \Sigma = \begin{bmatrix} \Sigma_i \\ \Sigma_y \end{bmatrix}.$$

With this decomposition at hand we can write the system as

$$y_{t+1} = e_y \left(\mu + \Phi X_t + \Sigma \varepsilon_{t+1}\right) = \mu_y + \Phi_y X_t + \Sigma_y \varepsilon_{t+1}.$$
$$i_{t+1} = \begin{cases} \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1} & \text{if } \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} \\ i_{t+1} = \bar{\imath} & \text{if } \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1} < \bar{\imath} \end{cases},$$
$$X_{t+1} = \begin{bmatrix} i_{t+1} \\ y_{t+1} \end{bmatrix}.$$

In the next section, we use these decompositions to come up with a solution for the computation of forecast moments under a lower bound.

3 Simulations and forecasts under a lower bound

In this section, I explain how we can simulate the VAR system under a lower bound period by period. Then, I explain how we can compute the exact moments of the system for the first 2 periods when we track 1 previous period. I go over the main steps and leave the details to Appendix A. Although I do not go over the general case of tracking n previous periods, Appendix B provides the constraints for tracking different number of previous periods.

3.1 The ZLB constraint at $t+1$

3.1.1 Simulations

Consider the case of the ZLB at $t+1$. We have:

$$\begin{aligned}&\text{If } \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} \Longrightarrow i_t = \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1}, \text{ and}\\&\text{If } \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1} < \bar{\imath} \Longrightarrow i_t = \bar{\imath}.\end{aligned}$$

We can equivalently write these two cases as:

$$\begin{aligned}&\text{If } \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t \Longrightarrow i_t = \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1}, \text{ and}\\&\text{If } \Sigma_i \varepsilon_{t+1} < \bar{\imath} - \mu_i - \Phi_i X_t \Longrightarrow i_t = \bar{\imath}.\end{aligned}$$

Accordingly, we have two sets of ε_{t+1} on the realization of which the interest rate takes either the minimum value ($\bar{\imath}$) or the value dictated by the VAR dynamics. These sets are:

$$\begin{aligned}\mathcal{F}^1_{t+1} &= \left\{ \varepsilon_{t+1} \middle| \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t \right\},\\\mathcal{F}^2_{t+1} &= \left\{ \varepsilon_{t+1} \middle| \Sigma_i \varepsilon_{t+1} \leq \bar{\imath} - \mu_i - \Phi_i X_t \right\} = R^{n_x} \backslash \mathcal{F}^1_{t+1} = \left(\mathcal{F}^1_{t+1}\right)^c.\end{aligned}$$

Whether the constraint binds or not, the value of other endogenous variables at time $t+1$ is given by the VAR law of motion:

$$y_{t+1} = \mu_y + \Phi_y X_t + \Sigma_y \varepsilon_{t+1}.$$

We can summarize the simulation for $t+1$ as follows: Given ε_{t+1}, y_{t+1} takes the value given by VAR. If the lower bound does not bind, that's the case for the short rate; if not, the short rate is $\bar{\imath}$.

3.1.2 Computing the moments

The conditional expectation of X_{t+1} at time t can be decomposed into the expectations coming from two mutually exclusive sets of ε_{t+1}:

$$\begin{aligned}E_t X_{t+1} = &\Pr\left(\varepsilon_{t+1} \in \mathcal{F}^1_{t+1}\right) \times E_t\left[X_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}^1_{t+1}\right]\\&+ \Pr\left(\varepsilon_{t+1} \in \mathcal{F}^2_{t+1}\right) \times E_t\left[X_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}^2_{t+1}\right].\end{aligned}$$

Since the value of y_t does not depend on whether the lower bound binds or not the one-period expectation is equal to that of a model without a lower bound. Formally,

$$\begin{aligned}E_t\left[y_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}^1_{t+1}\right] &= \mu_y + \Phi_y X_t + E_t\left[\Sigma_y \varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}^1_{t+1}\right],\\E_t\left[y_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}^2_{t+1}\right] &= \mu_y + \Phi_y X_t + E_t\left[\Sigma_y \varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}^2_{t+1}\right],\\E_t y_{t+1} &= \mu_y + \Phi_y X_t.\end{aligned}$$

The variance of y_{t+1} is

$$var_t y_{t+1} = \Sigma_y \Sigma_y'.$$

Thus, y_{t+1} is normally distributed with $N\left(\mu_y + \Phi_Y X_t, \Sigma_y \Sigma_y'\right)$. On the other hand, since

$$\begin{aligned}
E_t\left[i_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right] &= \mu_i + \Phi_i X_t + E_t\left[\Sigma_i \varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right], \\
E_t\left[i_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^2\right] &= \bar{\imath},
\end{aligned}$$

the conditional expectation of i_{t+1} at t is:

$$\begin{aligned}
E_t i_{t+1} = {} & \Pr\left(\varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right) \times \left\{\Phi_i X_t + E_t\left[\Sigma_i \varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right]\right\} \\
& + \bar{\imath}\left[1 - \Pr\left(\varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right)\right].
\end{aligned}$$

Thus, we have to compute moments such as:

$$\begin{aligned}
& \Pr\left(\varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right), E_t\left[\varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right], var_t\left[\varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right], \\
& \Pr\left(\varepsilon_{t+1} \in \mathcal{F}_{t+1}^2\right), E_t\left[\varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^2\right], var_t\left[\varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^2\right].
\end{aligned}$$

Remember that the set $\mathcal{F}_{t+1}^1$ is defined by a single linear constraint, $\Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t$; thus, it is the combination of shocks, not the value of a particular shock, that determines whether the lower-bound constraint binds or not. This constraint leads to a truncated normal distribution, as the lowest value of the short rate is $\bar{\imath}$. In order to find this truncated distribution, I follow Tallis (1965), who proposes an exact solution to compute moments under linear constraints for multivariate normal settings. His method uses a transformation of the original shock process into another shock process, which turns the single constraint consisting of a combination of shocks into one that consists of only one shock. The derivation of his method for a single constraint is given in Appendix A.

With some abuse of notation, let $E_t Z_{t+k}^1$ denote $E_t\left[Z_{t+k} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right]$ and $E_t Z_{t+k}^2$ denote $E_t\left[Z_{t+k} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^2\right]$. Given the expectation and the variance of the shocks conditional on whether the lower bound binds or not, the conditional and the unconditional expectation and the variance of the short-rate is straightforward:

$$\begin{aligned}
E_t i_{t+1}^1 &= \mu_i + \Phi_i X_t + E_{1,t}\left[\Sigma_i \varepsilon_{t+1}\right], \\
E_t i_{t+1}^2 &= \bar{\imath}, \\
var_t i_{t+1}^1 &= var_{1,t}\left[\Sigma_i \varepsilon_{t+1}\right] = \Sigma_i var_t\left[\varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right] \Sigma_i', \\
var_t i_{t+1}^2 &= 0.
\end{aligned}$$

The unconditional (time t-conditional) expectation of i_{t+1} is then

$$E_t i_{t+1} = p_1 E_{1,t} i_{t+1} + (1 - p_1) E_{2,t} i_{t+1},$$

where $p_1 = \Pr\left(\varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right)$. Appendix A shows the moments for y_t as well.

3.2 The ZLB constraint at $t+2$

3.2.1 Simulation

Now consider the simulation of X_{t+2}.

$$
\begin{aligned}
y_{t+2} &= \mu_y + \Phi_y X_{t+1} + \Sigma_y \varepsilon_{t+2}, \\
i_{t+2} &= \left\{ \begin{array}{|l|l|} \hline \mu_i + \Phi_i X_{t+1} + \Sigma_i \varepsilon_{t+2} & \text{if } \mu_i + \Phi_i X_{t+1} + \Sigma_i \varepsilon_{t+2} \geq \bar{\imath} \\ \hline i_{t+1} = \bar{\imath} & \text{if } \mu_i + \Phi_i X_{t+1} + \Sigma_i \varepsilon_{t+2} < \bar{\imath} \\ \hline \end{array} \right. .
\end{aligned}
$$

The constraint $\mu_i + \Phi_i X_{t+1} + \Sigma_i \varepsilon_{t+2} \geq \bar{\imath}$ defines a set $\mathcal{F}^1_{t+2}$ and its complement $\mathcal{F}^2_{t+2}$

$$
\begin{aligned}
\mathcal{F}^1_{t+2} &= \{ \varepsilon_{t+2} | \, \Sigma_i \varepsilon_{t+2} \geq \bar{\imath} - \mu_i - \Phi_i X_{t+1} \}, \\
\mathcal{F}^2_{t+2} &= \{ \varepsilon_{t+2} | \, \Sigma_i \varepsilon_{t+2} < \bar{\imath} - \mu_i - \Phi_i X_{t+1} \} = R^{n_x} \backslash \mathcal{F}^1_{t+2} = \left(\mathcal{F}^1_{t+2} \right)^c .
\end{aligned}
$$

However, remember that the value of X_{t+1} depends on whether ε_{t+1} belongs to $\mathcal{F}^1_{t+1}$ or its complement, $\mathcal{F}^2_{t+1}$, since

$$
\begin{aligned}
\Phi_y X_{t+1} &= \Phi_y \left(\lambda_i i_{t+1} + \lambda_y y_{t+1} \right), \\
\Phi_i X_{t+1} &= \Phi_i \left(\lambda_i i_{t+1} + \lambda_y y_{t+1} \right),
\end{aligned}
$$

and $i_{t+1} = \bar{\imath}$ if $\varepsilon_{t+1} \in \mathcal{F}^2_{t+1}$ and $i_{t+1} = \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1}$. Thus, $\mathcal{F}^1_{t+2}$ and $\mathcal{F}^2_{t+2}$ actually depend on $\mathcal{F}^1_{t+1}$ and $\mathcal{F}^2_{t+1}$. Thus, for the $t+2$ simulations we have four cases depending on whether $\varepsilon_{t+1} \in \mathcal{F}^1_{t+1}$ or $\varepsilon_{t+1} \in \mathcal{F}^2_{t+1}$. We denote these cases by using one superscript (i) for $t+1$ variables and two superscripts in the variables (i, j), where i and j take 1 for the case in which the lower bound does not bind or 2 for the case in which the lower bound binds. The superscript i is for the period $t+1$, and j is for the period $t+2$. Thus, X^1_{t+1} is the (simulated) value of the variable at period $t+1$ when the lower bound does not bind, and $X^{1,1}_{t+2}$ is the (simulated) value of the variable at period $t+2$ where the lower bound does not bind in both periods. Similarly, $X^{2,2}_{t+2}$ means the (simulated) value of the variable where the lower bound binds in both periods. Our four cases are as follows:

1. Case 1:

$$
\begin{aligned}
y^{1,1}_{t+2} &= e_y \left(\mu + \Phi X^1_{t+1} + \Sigma \varepsilon_{t+2} \right), \\
i^{1,1}_{t+2} &= e_i \left(\mu + \Phi X^1_{t+1} + \Sigma \varepsilon_{t+2} \right).
\end{aligned}
$$

2. Case 2:

$$
\begin{aligned}
y^{1,2}_{t+2} &= e_y \left(\mu + \Phi X^1_{t+1} + \Sigma \varepsilon_{t+2} \right), \\
i^{1,2}_{t+k+1} &= \bar{\imath}.
\end{aligned}
$$

3. Case 3:

$$y_{t+2}^{2,1} = e_y \left(\mu + \Phi X_{t+1}^2 + \Sigma \varepsilon_{t+2} \right),$$
$$i_{t+2}^{2,1} = e_i \left(\mu + \Phi X_{t+1}^2 + \Sigma \varepsilon_{t+2} \right).$$

4. Case 4:

$$y_{t+2}^{2,2} = e_y \left(\mu + \Phi X_{t+1}^2 + \Sigma \varepsilon_{t+2} \right),$$
$$i_{t+2}^{2,2} = \bar{\imath}.$$

3.2.2 Computing the moments

Since there are four different cases depending on whether $\varepsilon_{t+1} \in \mathcal{F}_{t+1}^1$ or not and $\varepsilon_{t+2} \in \mathcal{F}_{t+2}^1$ or not, we have conditional moments corresponding to those four cases. In this section I go over the main steps to compute the conditional moments for these four cases. I then aggregate them to find the conditional moment with less information, going all the way to time t conditional moments. I present the method to compute only one particular conditional probability, that of the event of a nonbinding ZLB constraint both in the first and the second periods. Following our convention of using the superscripts above, let p_{t+2}^{11} denote this probability; i.e. $p_{t+2}^{11} = \Pr\left(i_{t+2} \geq \bar{\imath} \middle| i_{t+1} > \bar{\imath} \right)$.

Remember that $\Pr\left(i_{t+2} \geq \bar{\imath}, i_{t+1} > \bar{\imath} \right)$ is the probability that $\Pr\left(\varepsilon_{t+2} \in \mathcal{F}_{t+2}^1, \varepsilon_{t+1} \in \mathcal{F}_{t+1}^1 \right)$. Thus, there are two linear restrictions for this event.

$$\Pr\left(\Sigma_i \varepsilon_{t+2} \geq \bar{\imath} - \mu_i - \Phi_i X_{t+1}, \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t \right),$$
$$\Pr\left(\left(\Sigma_i \varepsilon_{t+2} \geq \bar{\imath} - \mu_i - \Phi_i X_{t+1} \right) \& \left(\Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t \right) \right).$$

We need to find the joint event of a no-binding restriction in both periods. To do that write the time $t+2$ restriction in terms of period t variables along with time $t+1$ and $t+2$ shocks. For example, for the case of $i_{t+2}, i_{t+1} > \bar{\imath}$, we have

$$\begin{aligned}
e_i \Sigma \varepsilon_{t+2} &\geq \bar{\imath} - e_i \left(\mu + \Phi X_{t+1} \right) \\
e_i \Sigma \varepsilon_{t+2} &\geq \bar{\imath} - e_i \left[\mu + \Phi \left(\mu + \Phi X_t + \Sigma \varepsilon_{t+1} \right) \right] \\
e_i \left(\Phi \Sigma \varepsilon_{t+1} + \Sigma \varepsilon_{t+2} \right) &\geq \bar{\imath} - e_i \left[\mu + \Phi \mu + \Phi^2 X_t + \Phi \Sigma \varepsilon_{t+1} \right] \\
e_i C_2' \eta &\geq \bar{\imath} - e_i a_2,
\end{aligned}$$

with η, a_2 and C_2' defined as

$$\begin{aligned}
\eta &= \left[\varepsilon_{t+1}', \varepsilon_{t+2}' \right]' \\
a_2 &\equiv \mu + \Phi \mu + \Phi^2 X_t + \Phi \Sigma \varepsilon_{t+1} \\
C_2' &\equiv \begin{bmatrix} \Phi \Sigma & \Sigma \end{bmatrix}.
\end{aligned}$$

Hence our two restrictions are

$$e_i C_2' \eta \geq \bar{\imath} - e_i a_2$$
$$\Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t.$$

Notice that these two restrictions are a linear combination of $2 \times n_x$ and n_x shock variables, respectively. Just as we reduce the dimensionality of the relevant errors to one for the restriction at time $t+1$, using the method of Tallis (1965) for the multiple constraint case, we reduce the dimensionality the problem to two in this case. Appendix A goes over the derivation in detail. With this transformation we can find the exact values of the conditional moments, say the expected value of the short rate at $t+2$ given the lower bound not binding in periods $t+1$ and $t+2$. From these conditional moments we can find other moments such as the expected value of the short rate at $t+2$ given the lower bound not binding in $t+2$ or the unconditional (time t-conditional) expected value of the short rate at $t+2$. Notice that these are exact as well. In the next section we consider periods $t+3$ and beyond.

3.3 The ZLB constraint at $t+3$ and beyond

3.3.1 Simulation

For simulations at period $t+3$ and beyond, we can continue running the law of motion *period by period*, given the realization of shock processes for different simulations, and check whether the lower bound binds or not. If it does not bind, we can use that particular draw and continue building the path further using those values. If the lower bound binds, however, we set the short rate to the lower bound while keeping the values of the other endogenous variables the same and then continue building the path further using these revised values.

3.3.2 Computing the moments

Our analysis of $t+1$ shows that we have two cases for this period, depending on whether the lower bound is binding or not at this period. There are four different $t+2$ cases, depending on whether the constraint binds in either of the two periods. Similarly, there are eight different cases in period $t+3$, depending on whether the lower bound binds in any of the $t+1, t+2$ or $t+3$ periods. By the same token, the number of cases will continue expanding at an exponential rate (it doubles every period), so that it becomes impossible to manage even after a few periods.

This problem of an increasing number of states is the same as Kim (1994) faces in his state-space regime-switching model. He proposed to solve this problem by collapsing the distributions for the different states so that the number of cases going forward does not expand. I follow this idea of collapsing the number of states using his method. To be concrete suppose at period $t+1$ we have the distribution for the endogenous variables in the nonbinding state (X^1_{t+1} in the notation of the previous section) and the binding state (X^2_{t+1} in the notation of the previous section). If we track n previous periods, for period $t+s+n$, we can use the distribution of X^1_{t+s} for states that has the nonbinding state at period $t+s$. These include the distributions of X^{1jk}_{t+s+n}, etc. For example, when 1 previous period is tracked, we use the distribution of X^1_{t+s} to find the distributions of X^{11}_{t+s+1} and X^{12}_{t+s+1}. Consider the first one" X^{11}_{t+s+1} is the distribution of X_{t+s+1} that has a nonbinding state in both $t+s$ and $t+s+1$. Since X^1_{t+s} is random, I characterize it with the following:

$$X^1_{t+s} = E_t\left[X^1_{t+s}\right] + \Gamma^1\omega_{t+s},$$

where Γ is the Cholesky decomposition of $var_t\left[X^1_{t+s}\right]$. Thus, to find the distribution of X^{11}_{t+s+1}, we will find the set of ω_{t+s} and ε_{t+s+1} that satisfies

$$e_i X^1_{t+s} \geq \bar{\imath}; e_i\left(\mu + \Phi X^1_{t+s+1} + \Sigma\varepsilon_{t+s+1}\right) \geq \bar{\imath}.$$

Similarly, the distribution of X^{12}_{t+s+1} is associated with the set of ω_{t+s} and ε_{t+s+1} that satisfies

$$e_i X^1_{t+s} \geq \bar{\imath}; e_i\left(\mu + \Phi X^1_{t+s+1} + \Sigma\varepsilon_{t+s+1}\right) \leq \bar{\imath}.$$

Just as at time $t+2$, we have 2 restrictions, and transform the constraints on ε_{t+1} and ε_{t+2} using the method of Tallis (1965), following the same logic we transform the constraints on ω_{t+s} and ε_{t+s+1} here. One note for the cases that start with a binding state at period $t+s$ is in order. Since the interest rate at period $t+s$ is always zero for those cases, thus, nonrandom; we do not check whether it is less than the lower bound leading to one less restriction compared to the case of a nonbinding starting value.

For n-previous period tracking at period $t+s+n$ we have 2^{n+1} different cases. For example, for one-period tracking we have the four cases of $X^{11}_{t+s+1}, X^{12}_{t+s+1}, X^{21}_{t+s+1}, X^{22}_{t+s+1}$. However, for period $t+s+2$, we are going to use X^1_{t+s+1} and X^2_{t+s+1}. Hence, I use Kim's idea to collapse the mixture distribution of $X^{1,1}_{t+s+1}$ and. $X^{2,1}_{t+s+1}$ into a normal distribution of X^1_{t+s+1}. Appendix D computes the moments of the mixture of normals. This appendix gives the appropriate formulas for the new distributions such as:

$$\begin{aligned} p^1_{t+k+1} &= p^1_{t+k} p^{11}_{t+k+1} + p^2_{t+k} p^{21}_{t+k+1} \\ E_t X^1_{t+k+1} \equiv E_t \left[X_{t+k+1} \right| i_{t+k+1} \geq \bar{\imath}] &= \frac{p^1_{t+k} p^{11}_{t+k+1}}{p^1_{t+k} p^{11}_{t+k+1} + p^2_{t+k} p^{21}_{t+k+1}} E_t X^{11}_{t+k+1} \\ &+ \frac{p^2_{t+k} p^{21}_{t+k+1}}{p^1_{t+k} p^{11}_{t+k+1} + p^2_{t+k} p^{21}_{t+k+1}} E_t X^{12}_{t+k+1} \end{aligned}$$

The first formula shows that the probability of the lower bound constraint not binding at time $t+k+1$, p^1_{t+k+1} is equal to the sum of the two probabilities: The first term is the probability of the lower bound constraint not binding at time $t+k+1$ given it does not bind at time $t+k$, p^{11}_{t+k+1} times the probability of the lower bound constraint not binding at time $t+k$. The second term is the probability of the lower bound constraint not binding at time $t+k+1$ conditional on a binding constraint at time $t+k$ times the probability of lower bound constraint binding at time $t+k$. Similarly, the expected value of a variable when the lower bound is not binding at time $t+k+1$ is the weighted average of its expected value when the constraint does not bind in the previous period and its expected value when the constraint binds in the previous period with the weights computed by the Bayes formula. The collapse of the mixture of distributions induces an inevitable approximation error. In the next section, numerical examples will show under what conditions these approximations work best. Appendix B shows how we can find the restrictions induced by different binding and nonbinding states for the case of tracking more periods.

4 Numerical Examples

In this section I go over some numerical examples to show the performance of the method. For the first three numerical examples, I consider a three-variable VAR with one lag. These three variables are the interest rate (i_t), the output gap (x_t), and the inflation rate (π_t). Since the numerical examples serve the purpose of presenting the method and the assessment of the approximation, I did not estimate a VAR using the historical data. A VAR with historical data probably understates the importance of the ZLB because there is only one case of the ZLB constraint binding in the postwar US data, typically the longest span of data considered for empirical macro studies. Rather, I come up with arbitrary but plausible coefficients to enable larger chances of long periods of binding ZLB. In the VARs employed for the analysis of monetary policy shocks, the interest rate is ordered after the inflation and output gap so that the shocks to the interest rate affect the inflation and output gap with a lag, whereas the reverse is not the case. I follow this ordering. As the exposition above put the short rate as the first variable, I follow that as well. Now, let $X_t = \begin{bmatrix} i_t & x_t & \pi_t \end{bmatrix}'$, and the VAR coefficients and the steady-state and the initial value for the first numerical example are:

$$\Phi = \begin{bmatrix} 0.8 & -0.1 & 0.2 \\ 0.05 & 0.7 & 0.1 \\ -0.2 & 0.1 & 0.7 \end{bmatrix}, \Sigma = \begin{bmatrix} 1.5 & 0.3 & 0.2 \\ 0 & 0.8 & 0 \\ 0 & 0.1 & 1 \end{bmatrix},$$

$$\bar{X} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, X_0 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}.$$

I set the interest rate lower bound at zero. In Figure 1, I depict the probability of a binding lower bound and the conditional and unconditional expectations of the three endogenous variables. I compute the expectations using two different methods. The first is the Monte Carlo simulation using 10^6 different shock realizations for each period, and the other is the method described in this paper. I use different numbers of tracking periods to show a) how we can increase the number of periods and can get the exact moments and b) how we can improve the approximation accuracy for the rest of the periods. In particular, I track up to four previous periods. The way I use this paper's method relies on the Kim-style approximation after the second period for tracking one previous period, after the third period for tracking two previous periods and so on. As such, the results of the first two periods are exactly the same for the simulation method and my method. Similarly, I get the third period moment exactly for the cases of tracking two and more previous periods. The results from the following periods show that even the simplest case produces approximations that are reasonable, and by tracking more periods we reduce the approximation error quite a lot. Table I shows the average approximation error at different periods when tracking different numbers of previous periods and a comparison of the computational time. For example, by tracking two previous periods, we will have a 5 basis point difference for the conditional expectation of the interest rate at the nonbinding state and a 10 basis points of difference for the unconditional expectation of inflation rates between the Monte Carlo method and the proposed analytical method tracking at periods 20 and 40, respectively. We can further reduce the approximation errors by tracking three and four previous periods. These come at a cost of complexity and computational time as Table 1 shows. In particular, while two-period tracking takes about 2 % of the computational time[2] that Monte Carlo exercise with a 10^6 draws

[2] All computational time comparisons are done by running Monte Carlo simulation and cases of tracking different numbers of periods10 times and averaging out the computational times of these 10 trials.

takes, the same figures for the three and four-period tracking are 35% and 237%. Thus tracking two or three periods is a serious contender in terms of approximation error and computing time vis-a-vis Monte Carlo simulation.

In Figure 2 and Table 2, I use the same VAR coefficients but start with a different initial value, which is equal to the nonstochastic steady-state value. The coefficients, the steady-state and initial value are given along with Figure 2. The figure shows that the approximation is also reasonable for one-period tracking and much better for the cases of tracking more previous periods. Next, I look at a system that is more persistent. I keep the VAR coefficients of the first numerical example, except the VAR(1) term, where it is now:

$$\Phi = \begin{bmatrix} 0.9 & -0.1 & 0.2 \\ 0.05 & 0.9 & 0.1 \\ -0.2 & 0.1 & 0.8 \end{bmatrix}.$$

The maximum of the absolute value of the eigenvalues of the first two VAR systems is 0.76, whereas that of the last model is 0.96. As Figure 3 and Table 3 show, there is a deterioration in the approximation in the case of tracking one previous period. The approximations become better as we increase the number of previous periods tracked. For example,. for the two-period tracking, the deviation of the unconditional value of the inflation rate at period 5 is only 4 basis points, whereas at period 40 is higher at 19 basis points. However, the same figures for the three-period tracking are 3 and 11 basis points, respectively. Hence, a highly persistent VAR system requires more periods tracked in order to produce a good approximation.

In Figure 4 and Table 4, I present the results for a larger model. We add three more variables but keep the maximum of the absolute eigenvalue at 0.76. In general, the approximation from the larger model seems a little worse for the case of one-period tracking and quite good for the case of tracking two or more periods.

5 An affine term-structure model under the ZLB constraint

We can use the algorithm described in this paper to compute the bond prices in an affine term structure, where the underlying law of motion, the VAR(1) model contains the short-rate. I still use the three-variable VAR(1) model that has been used so far in the paper: VAR(1) with the short-rate, the output gap, and the inflation rate. Notice that such a model can be used where the short-rate is governed not by latent factors but by a Taylor-rule with inertia. The state vector follows the same VAR with one lag. The short-rate is an affine function of the state

$$i_t = \delta_0 + \delta_1' X_t,$$

with

$$\delta_0 = 0 \text{ and } \delta_1' = e_i'.$$

The nominal stochastic discount factor used for pricing nominal bonds is:

$$m_{t+1} = exp\left(-i_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\right),$$

where the market price of risk is also an affine function of the state.

$$\lambda_t = \lambda_0 + \lambda_1 X_t,$$

I assume that the market price of risk depends on the present variables; thus the first column of λ_1 is a vector of 0's[3]. We can still use the Q-measure for pricing the bonds as the ZLB is about the short-rate[4]. The VAR system under the Q-measure has the parameters:

$$\tilde{\mu} = \mu - \Sigma\lambda_0,$$
$$\tilde{\Phi} = \Phi - \Sigma\lambda_1.$$

The n-period bond price can be computed as:

$$P_t^n = E_t^Q\left[\exp\left(-\sum_{i=0}^{n-1} i_{t+i}\right)\right].$$

Finally the (log-) yield for maturity n is:

$$y_t^n = \frac{-\log\left(P_t^n\right)}{n}.$$

If there is no ZLB constraint, we can use the usual bond-pricing recursion relationships, expressing the n-period (log) bond prices as an affine function of the state:

$$\log\left(P_t^n\right) = \alpha_n + \beta_n' X_t$$

[3]Notice that this assumption does not affect the method's ability to compute the expected values of the endogenous variables or that of the bond yields.

[4]One can argue that other nominal rates should also have a lower bound. However, no-arbitrage assumption takes care of that constraint for other bond yields. In the numerical examples, we shall see that if the ZLB constraint for the short-rate is satisfied, so is the ZLB constraint for the longer rates.

Appendix E.1 shows that the bond-price coefficients are:

$$\alpha_1 = 0, \beta n_1' = -e_i,$$
$$\alpha_n = \alpha_{n-1} + \beta_{n-1}'\tilde{\mu} + \frac{1}{2}\beta_{n-1}'\Sigma\Sigma'\beta_{n-1},$$
$$\beta_n' = -e_i + \beta_{n-1}'\tilde{\Phi}.$$

With the ZLB constraint, however, we can either use the simulation method or the method described in this paper. We approximate the distribution of $b_{t+n-1} = \sum_{i=0}^{n-1} i_{t+i}$, which is a mixture of normals and truncated normals, with a normal approximation where we use the expectation and the variance derived from the paper's algorithm. In other words, I approximate the n-period yield with

$$\begin{aligned} y_t^n &\cong \frac{-1}{n}\log\left\{\exp\left[\left(-\sum_{i=0}^{n-1}E_t i_{t+i} + \frac{1}{2}var_t i_{t+i}\right)\right]\right\} \\ &= \frac{1}{n}\left[E_t b_{t+n-1} - \frac{1}{2}var_t\left(b_{t+n-1}\right)\right] \end{aligned}$$

with the moments coming from the method's algorithm. In order to find the moments of b_{t+n-1}, I append it to the vector of endogenous variables, $Z_{t+k}^i = \left[X_{t+k}^{i\prime}, b_{t+k}^i\right]'$, for each nonbinding and binding state at every period $t+k$. The expectation of b_{t+k} is fairly straightforward

$$\mathfrak{b}_{t+k} \equiv E_t b_{t+k} = E_t b_{t+k-1} + E_t i_{t+k}.$$

Appendix E.2 goes over the derivation of the variance of b_{t+k}. Figure 5 and Table 5 show the results of a numerical example for the bond-yield computation with a Monte Carlo simulation and the paper's method. For the numerical example of Figure 4, I use the VAR parameters of the first numerical example (that of Figure 1). Given the VAR parameters, I estimate the market price of risk parameters in order to fit an upwardly sloping yield curve at the nonstochastic steady state under the assumption of no ZLB constraint. The yield curve I try to match has the following nonstochastic steady state values:

Yields (percent)	y_t^1	y_t^4	y_t^8	y_t^{20}	y_t^{40}
Objective	3.00	3.25	3.50	4.25	5.25
Fitted	3.00	3.41	3.69	3.87	3.91

Although such a simple term-structure model does not generate a large enough nominal slope, the 10-year-1quarter slope is still about 91 basis points, which is reasonable given the simplicity of the model. Given the VAR parameters and the market price of risk, I transform the law of motion from the P-measure to the Q-measure and compute the time t-yields, where X_0 is equal to that of the first numerical example. In Figure 5, I compare the results of the Monte-Carlo simulation with the different cases calculated with the method in which I allow tracking 1, 2, and 3 previous periods. The results show that although the approximation for one-period tracking is not quite satisfactory, those for the cases of tracking 2 and 3 periods approximate the yield curve pretty well. The difference between the Monte Carlo simulation (with 10^6 simulations) is 2.84 and 1.60 basis points at a two-year maturity,-6.84 and -3.89 basis points at a 5-year maturity and -7.53 and -4.11 basis points at a 10-year maturity for the cases of tracking two or three periods, respectively. The approximation error increases over the maturity horizon but is still comparable to the usual one-standard deviation of the measurement error typically found in the estimated for affine term structure models with a Taylor-rule[5] for the case of tracking 2 previous periods and lower for the case of tracking 3 previous periods.

6 Models with more than one lag

Although the law of motion in many affine term-structure models is a VAR with one lag, macro forecasting generally requires VARs with more lags. In Figure 6 and Table 6, I show the results of the method when it is applied to a VAR with two lags. I still keep the maximum of the absolute of the eigenvalues of the system at a moderate level, 0.78. In principle, one has to keep track of the previous period not only for the first lag of the interest rate but also for the second lag. I decompose X^i_{t+s} and X^j_{t+s-1}

$$
\begin{aligned}
X^i_{t+s} &= E_t X^i_{t+s} + \Gamma^i_s \omega_{t+s} \\
X^j_{t+s-1} &= E_t X^j_{t+s} + \Gamma^j_{s-1} \omega_{t+s}
\end{aligned}
$$

and compute the constraints induced by the ZLB accordingly. Appendix F goes over the derivation of these constraints for the case of VARs with more than one lag. The approximation is still reasonable but a little bit worse than that of a one-lag model as expected. Although the nonbinding case still performs well, the binding case performs worse. However, since over large periods hitting the probability of the ZLB becomes lower (for example, it is 7 percent at 20 periods for this example), the deviation of the method with Monte Carlo simulation is -0.13 percentage point for the unconditional expectation of the interest rate, and -0.03 percentage point for the output gap, and -0.05 percentage point for the inflation rate at 20 periods.

[5] For example, Ang and Piazzesi report 18 basis points for the measurement error's standard deviation for a 1-quarter yield and 6 basis points for that of a 5-year yield.

7 Conclusion

In this paper, I develop a new analytical method to obtain forecasts from a reduced-form VAR model under the ZLB constraint. The introduction of the ZLB in my setting introduces a nonlinearity similar to the settings of Wu and Xia (2014), Priebsch (2013). However, unlike a shadow-rate model, the ZLB constraint in my setting prevents computing the exact moments of forecasts for any forecast horizon because the variables in all of the previous periods need to be tracked. I first show in a forecast exercise of a VAR under the ZLB constraint how we can compute the exact moments for the first $n + 1$ periods when we track n previous periods. Then, for the periods beyond $n + 1$, I developed an approximation similar to the one employed by Kim (1994), which he derives for a different setting, that of regime-switching. The results show that my algorithm, even for the simplest case of tracking one-period, works pretty well when there is a moderate degree of persistence and involves much less computational cost than a Monte Carlo simulation, even when the initial point for the variables are quite different than the nonstochastic steady state value. I also show that one can produce better approximations by tracking more previous periods, though it does require more computational time. In particular, I show that two- and three-period tracking is a viable alternative to Monte Carlo simulation in terms of computational time, and produces good results for persistent cases, larger systems, and systems with more than one lag. Going over three periods creates better approximations but it requires more computational time than a Monte Carlo simulation with a large number of draws.

Since most of the affine term-structure models use a VAR(1) as the law of motion for the state variable, I also present a way to compute the yields in an affine terms structure model where the state also contains the short-rate itself, which is subject to the ZLB constraint. The results show that the method achieves results where the deviation from the exact moments are lower than one-standard deviation of usual measurement error in estimated affine term-structure models where short-rate is explained by a Taylor-type policy rule. Hence, the method proves useful in computing bond prices for affine term-structure models in which the lagged interest rate is among the state variables.

I would like to end with two suggestions that may improve the method in terms of accuracy and computing time. The first relies on the mean-reverting property for the VAR systems. By tracking more periods in the beginning -where there is a higher difference between the expected value and the ergodic (conditional) expectation- we can minimize the approximation error where it is more important. After this initial number of periods, we can always go back to tracking a lower number of previous periods. The second way is exploiting the ergodic moments coming from the VAR model with the ZLB. One can use the ergodic moments conditional on the cases of binding and nonbinding interest rate constraint. One can then incorporate this information and make the expectation of the endogenous variables as a convex combination of the moment coming from the method and the conditional ergodic moment, where the weight for the ergodic moment increases over time.

8 References

References

[1] Andreasen, M. M. and Meldrun, A. (2014) "Dynamic term structure models: The best way to enforce the zero lower bound," *CREATES Research Paper* 2014-47.

[2] Ang, A. Dong, S. and Piazzesi, M. (2007) "No-Arbitrage Taylor Rules." *unpublished manuscript.*

[3] Black, F. (1995), "Interest Rates as Options," *Journal of Finance*, 50(5):1371-1376.

[4] Clark, T. and McCracken, M. W. (2014) "Evaluating Conditional Forecasts from Vector Autoregressions," *Federal Reserve Bank of Cleveland Working Paper* No:14-13.

[5] Eggertson, G.B. and M. Woodford (2003) "The Zero Bound on Interest Rates and Optimal Monetary Policy," *Brookings Papers on Economic Activity* 34, 139-235.

[6] Fernandez-Villaverde, J., Gray, G., Guerron-Quintana, P. and Rubio-Ramirez, J.F. (2012) "Nonlinear Adventures at the Zero Lower Bound," *Federal Reserve Bank of Philadelphia Working Paper* No:12-10.

[7] Filipovic, D., Larsson, M. and Trolle, A. (2014) "Linear-Rational Term Structure Models," *Swiss Finance Institute Research Paper* No. 14-15.

[8] Kim, C. J. (1994), "Dynamic Linear Models with Markov-Switching," *Journal of Econometrics* 60, 1-22.

[9] Kim, D., and Singleton, K. (2012), "Term Structure Models and the Zero Bound: An Empirical Investigation of Japanese Yields," *Journal of Econometrics*, 170(1):32 49.

[10] Krippner, L. (2012), "Modifying Gaussian Term Structure Models When Interest Rates are Near the Zero Lower Bound," *Reserve Bank of New Zealand Discussion Paper* 2012/02.

[11] Krugman, P. (1998) "Krugman, Paul (1998), "It's Baaack! Japan's Slump and the return of the Liquidity Trap," *Brookings Papers on Economic Activity* 1998:2, 137-187.

[12] Priebsch, M. A. (2013) "Computing Arbitrage-Free Yields in Multi-Factor Gaussian Shadow-Rate Term Structure Models," *Finance and Economic Discussion Series* No:2013-63 Board of Governors Federal Reserve Board.

[13] Tallis, G.M. (1961) "The Moment Generating Function of the Truncated Multi-normal Distribution," *Journal of the Royal Statistical Society. Series B (Methodological)*, 23(1), 223-229.

[14] Tallis, G.M. (1961) "Plane Truncation in Normal Populations," *Journal of the Royal Statistical Society. Series B (Methodological)*, 27(2), .301-307.

[15] Wu, J. C. and Xia, F. D. (2014) "Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound" *Chicago Booth Research Paper* No. 13-77.

A Appendix A

A.1 The ZLB constraint at $t+1$

In this appendix I show the method of Tallis (1965), which provides an exact solution to compute moments under linear constraints for multivariate normal settings. Consider the linear constraint on ε_{t+1}

$$\begin{aligned}&\text{If } \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} \Longrightarrow i_t = \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1}, \text{ and}\\&\text{If } \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1} < \bar{\imath} \Longrightarrow i_t = \bar{\imath}.\end{aligned}$$

We can equivalently write these two cases as:

$$\begin{aligned}&\text{If } \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t \Longrightarrow i_t = \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1}, \text{ and}\\&\text{If } \Sigma_i \varepsilon_{t+1} < \bar{\imath} - \mu_i - \Phi_i X_t \Longrightarrow i_t = \bar{\imath}.\end{aligned}$$

Accordingly, we have two sets of ε_{t+1}, on the realization of which the interest rate takes either the minimum value ($\bar{\imath}$) or the value dictated by the VAR dynamics. These sets are:

$$\begin{aligned}\mathcal{F}^1_{t+1} &= \{\varepsilon_{t+1} | \, \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t\},\\\mathcal{F}^2_{t+1} &= \{\varepsilon_{t+1} | \, \Sigma_i \varepsilon_{t+1} \leq \bar{\imath} - \mu_i - \Phi_i X_t\} = R^{n_\varepsilon} \backslash \mathcal{F}^1_{t+1} = \left(\mathcal{F}^1_{t+1}\right)^c.\end{aligned}$$

Whether the constraint binds or not, the value of other endogenous variables at time $t+1$ is given by the VAR law of motion:

$$y_{t+1} = \mu_y + \Phi_y X_t + \Sigma_y \varepsilon_{t+1}.$$

First, consider the probability:

$$\Pr\left(\varepsilon_{t+1} \in \mathcal{F}^1_{t+1}\right) = \int_{\mathcal{F}^1_{t+1}} \psi\left(\varepsilon_{t+1}\right) d\varepsilon_{t+1} = \underset{\varepsilon_i' s \text{ such that } \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \Phi_i X_t}{\int_{\varepsilon_1} \cdots \int_{\varepsilon_{n_\varepsilon}}} \psi\left(\varepsilon_{t+1}\right) d\varepsilon_{1,t+1} \ldots d\varepsilon_{n_\varepsilon,t+1}$$

Here $\psi\left(\varepsilon_{t+1}\right)$ is the multinormal pdf with a mean $\mathbf{0}$ and I_n. Tallis's idea was to transform the set $\mathcal{F}^1_{t+1}$ so that the constraint would be on a single transformed shock, $\eta_{1,t+1}$, instead of a set of shocks. The transformation we are going to use is the orthogonal transformation with $\varepsilon_{t+1} = H\eta_{t+1}$, where H has $\dfrac{\Sigma_i'}{\|\Sigma_i\|}$ as its first column and the remaining columns being orthogonal to Σ_i and orthonormal amongst themselves. Then,

$$\begin{aligned}\mathcal{F}^1_{t+1}\left(\varepsilon_{t+1}\right) &\equiv \{\varepsilon_{t+1}; \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t\}\\\mathcal{F}^1_{t+1}\left(\eta_{1,t+1}\right) &= \left\{\eta_{1,t+1}; \eta_{1,t+1} \geq c_1 \equiv \frac{\bar{\imath} - \mu_i - \Phi_i X_t}{\|\Sigma_i\|}\right\}\\\mathcal{F}^1_{t+1}\left(\varepsilon_{t+1}\right) &= \mathcal{F}^1_{t+1}\left(\eta_{1,t+1}\right).\end{aligned}$$

Tallis showed that if $\varepsilon_{t+1} \sim N(0, I)$,

$$\Pr\left(\varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right) = 1 - \Psi(c_1),$$
$$E_t\left[\varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^1\right] = \frac{\psi(c_1)}{1 - \Psi(c_1)} \frac{\Sigma_i'}{\|\Sigma_i\|},$$

where $\Psi(.)$ is the cumulative distribution of a standard normal at c_1 and $\psi(c_1)$ is its density at the same point. Similarly, we can find the corresponding moments for the domain $\mathcal{F}_{t+1}^1$ by multiplying both sides by -1:

$$\begin{aligned}\mathcal{F}_1^c(\varepsilon_{t+1}) &\equiv \{\varepsilon_{t+1}; \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t\} \\ &= \{\varepsilon_{t+1}; -\Sigma_i \varepsilon_{t+1} \leq \bar{\imath} - \mu_i - \Phi_i X_t\}\end{aligned}$$

so that

$$\Pr\left(\varepsilon_{t+1} \in \mathcal{F}_{t+1}^2\right) = 1 - \Psi(-c_1) = \Psi(c_1)$$
$$E_t\left[\varepsilon_{t+1} \middle| \varepsilon_{t+1} \in \mathcal{F}_{t+1}^2\right] = \frac{\psi(-c_1)}{1 - \Psi(-c_1)} \frac{-\Sigma_i'}{\|\Sigma_i\|} = -\frac{\psi(c_1)}{\Psi(c_1)} \frac{\Sigma_i'}{\|\Sigma_i\|}$$

I will not show the variance computation for this case directly. However, when discussing the multirestriction case, we transform the multi-restriction case to the case of moment computation for the truncated multinormal distribution (Tallis, 1961) for which I provide the computation for both the expectation and the variance.

A.2 The ZLB constraint at $t + 2$

A.2.1 Simulation

Now consider the simulation of the variable at period $t + 2$:

$$y_{t+2} = \mu_y + \Phi_y X_{t+1} + \Sigma_y \varepsilon_{t+2}$$
$$i_{t+2} = \begin{cases} \mu_i + \Phi_i X_{t+1} + \Sigma_i \varepsilon_{t+2} & \text{if } \mu_i + \Phi_i X_{t+1} + \Sigma_i \varepsilon_{t+2} \geq \bar{\imath} \\ i_{t+1} = \bar{\imath} & \text{if } \mu_i + \Phi_i X_{t+1} + \Sigma_i \varepsilon_{t+2} < \bar{\imath} \end{cases}$$

The constraint $\mu_i + \Phi_i X_{t+1} + \Sigma_i \varepsilon_{t+2} \geq \bar{\imath}$ defines a set $\mathcal{F}_{t+2}^1$ and its complement $\mathcal{F}_{t+2}^1$

$$\begin{aligned}\mathcal{F}_{t+2}^1 &= \{\varepsilon_{t+2} | \Sigma_i \varepsilon_{t+2} \geq \bar{\imath} - \mu_i - \Phi_i X_{t+1}\} \\ \mathcal{F}_{t+2}^2 &= \{\varepsilon_{t+2} | \Sigma_i \varepsilon_{t+2} < \bar{\imath} - \mu_i - \Phi_i X_{t+1}\} = R^{n_\varepsilon} \backslash \mathcal{F}_{t+2}^1 = \left(\mathcal{F}_{t+2}^1\right)^c\end{aligned}$$

However, remember that the value of X_{t+1} depends on whether ε_{t+1} belong to $\mathcal{F}_{t+1}^1$ or its complement, $\mathcal{F}_{t+1}^2$ since

$$\Phi_y X_{t+1} = \Phi_{yi} i_{t+1} + \Phi_{yy} y_{t+1}$$
$$\Phi_i X_{t+1} = \Phi_{ii} i_{t+1} + \Phi_{iy} y_{t+1}$$

and $i_{t+1} = \bar{\imath}$ if $\varepsilon_{t+1} \in \mathcal{F}^2_{t+1}$ and $i_{t+1} = \mu_i + \Phi_i X_t + \Sigma_i \varepsilon_{t+1}$. Thus, $\mathcal{F}^1_{t+2}$ and $\mathcal{F}^2_{t+2}$ actually depend on $\mathcal{F}^1_{t+1}$ and $\mathcal{F}^2_{t+1}$. Thus, for the $t+2$ simulations we have four cases depending on whether $\varepsilon_{t+1} \in \mathcal{F}^1_{t+1}$ or $\varepsilon_{t+1} \in \mathcal{F}^2_{t+1}$. We denote these cases by using one superscript (i) for $t+1$ variables and two superscripts in the variables (i, j), where i and j take 1 for the case in which the lower bound does not bind or 2 for the case in which the lower bound binds. The superscript i is for the period $t+1$, and j is for the period $t+2$. Thus, X^1_{t+1} is the (simulated) value of the variable at period $t+1$ when the lower bound does not bind, and $X^{1,1}_{t+2}$ is the (simulated) value of the variable at period $t+2$ where the lower bound does not bind in both periods. Similarly, $X^{2,2}_{t+2}$ means the (simulated) value of the variable where the lower bound binds in both periods. Our four cases are as follows:

1. Case 1:$X^{1,1}_{t+2}$

$$\begin{aligned} y^{1,1}_{t+2} &= \mu_y + \Phi_y X^1_{t+1} + \Sigma_y \varepsilon_{t+2} \\ i^{1,1}_{t+2} &= \mu_i + \Phi_i X^1_{t+1} + \Sigma_i \varepsilon_{t+2} \end{aligned}$$

2. Case 2:$X^{1,2}_{t+2}$

$$\begin{aligned} y^{1,2}_{t+2} &= \mu_y + \Phi_y X^1_{t+1} + \Sigma_y \varepsilon_{t+2} \\ i^{1,2}_{t+k+1} &= \bar{\imath} \end{aligned}$$

3. Case 3:$X^{2,1}_{t+2}$

$$\begin{aligned} y^{2,1}_{t+2} &= \mu_y + \Phi_y X^2_{t+1} + \Sigma_y \varepsilon_{t+2} \\ &= \mu_y + \Phi_{yi} \bar{\imath} + \Phi_{yy} y^2_{t+1} + \Sigma_y \varepsilon_{t+2} \\ i^{2,1}_{t+2} &= \mu_i + \Phi_i X^2_{t+1} + \Sigma_i \varepsilon_{t+2} \\ &= \mu_i + \Phi_{ii} \bar{\imath} + \Phi_{iy} y^2_{t+1} + \Sigma_i \varepsilon_{t+2} \end{aligned}$$

4. Case 4: $X^{2,2}_{t+2}$

$$\begin{aligned} y^{2,2}_{t+2} &= \mu_y + \Phi_y X^2_{t+1} + \Sigma_y \varepsilon_{t+2} \\ &= \mu_y + \Phi_{yi} \bar{\imath} + \Phi_{yy} y^2_{t+1} + \Sigma_y \varepsilon_{t+2} \\ i^{2,2}_{t+2} &= \bar{\imath} \end{aligned}$$

A.2.2 Computing the moments

Since there are four different cases depending on whether $\varepsilon_{t+1} \in \mathcal{F}^1_{t+1}$ or not and $\varepsilon_{t+2} \in \mathcal{F}^1_{t+2}$ or not, we have conditional moments corresponding to those four cases. In this section I first review the method to compute the moments for those conditional moments.

I then aggregate them to find the conditional moment with less information going all the way to time t conditional moments. I present the method to compute only one particular conditional probability, that of the event that lower bound does not bind in the second period given that it does not bind in the first period. Following our convention of using the superscripts above, let p^{11}_{t+2} denote this probability. i.e. $p^{11}_{t+2} = \Pr\left(i_{t+2} \geq \bar{\imath} \middle| i_{t+1} > \bar{\imath}\right)$.

Remember that $\Pr\left(i_{t+2} \geq \bar{\imath}, i_{t+1} > \bar{\imath}\right)$. is the probability that $\Pr\left(\varepsilon_{t+2} \in \mathcal{F}^1_{t+2}, \varepsilon_{t+1} \in \mathcal{F}^1_{t+1}\right)$. Thus, there are two linear restrictions for this event.

$$\Pr\left(\Sigma_i \varepsilon_{t+2} \geq \bar{\imath} - \mu_i - \Phi_i X_{t+1}, \Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t\right)$$
$$\Pr\left(\left(\Sigma_i \varepsilon_{t+2} \geq \bar{\imath} - \mu_i - \Phi_i X_{t+1}\right) \& \left(\Sigma_i \varepsilon_{t+1} \geq \bar{\imath} - \mu_i - \Phi_i X_t\right)\right)$$

We need to find the joint event of no binding restriction in both periods. To do that let's write the time $t+2$ restriction in terms of period t variables along with time $t+1$ and $t+2$ shocks.

$$\begin{aligned}
e_i \Sigma \varepsilon_{t+2} &\geq \bar{\imath} - e_i\left(\mu + \Phi X_{t+1}\right) \\
e_i \Sigma \varepsilon_{t+2} &\geq \bar{\imath} - e_i\left[\mu + \Phi\left(\mu + \Phi X_t + \Sigma \varepsilon_{t+1}\right)\right] \\
e_i\left(\Phi \Sigma \varepsilon_{t+1} + \Sigma \varepsilon_{t+2}\right) &\geq \bar{\imath} - e_i\left[\mu + \Phi \mu + \Phi^2 X_t + \Phi \Sigma \varepsilon_{t+1}\right] \\
e_i C_2' \eta &\geq \bar{\imath} - e_i a_2
\end{aligned}$$

with η, a_2 and C_2' defined as

$$\begin{aligned}
\eta &= \left[\varepsilon'_{t+1}, \varepsilon'_{t+2}\right]' \\
a_2 &\equiv \mu + \Phi \mu + \Phi^2 X_t + \Phi \Sigma \varepsilon_{t+1} \\
C_2' &\equiv \left[\begin{array}{cc} \Phi \Sigma & \Sigma \end{array}\right].
\end{aligned}$$

Hence our two restrictions are

$$\begin{aligned}
e_i C_2' \eta &\geq \bar{\imath} - e_i a_2 \\
\Sigma_i \varepsilon_{t+1} &\geq \bar{\imath} - \mu_i - \Phi_i X_t.
\end{aligned}$$

Notice that these two restrictions are a linear combination of $2 \times n_x$ and n_x shock variables, respectively. Just as we reduce the dimensionality of the relevant errors to one for the restriction at time $t+1$, using the method of Tallis (1965) for the multiple constraint case, we reduce the dimensionality the problem to two in this case. Define

$$\begin{aligned}
\upsilon &= \left[\begin{array}{cc} \varepsilon'_{t+1} & \varepsilon'_{t+2} \end{array}\right]' \\
p &= \left[\begin{array}{cc} c_1 & c_2 \end{array}\right]' \\
C &= \left[\begin{array}{cc} \Sigma_i & \mathbf{0} \\ \Sigma_i & C_2' \end{array}\right]
\end{aligned}$$

so that

$$Cv \geq p$$

Let $B = \begin{bmatrix} C' & H' \end{bmatrix}'$,where the columns of H are again orthogonal to C and orthonormal to each other. Finally define

$$\tilde{\eta} = Bv$$

by which I transform the linear restriction of many variables into a plane truncation for which Tallis (1961) provides the methods to compute the moments. I present the relevant resultsof Tallis (1961) in Appendix C. Then, the 2 linear constraints of a combination of $2 \times n_x$ normal variables is transformed to a truncated normal distribution of $2 \times n_x$ variables where only two of them is truncated below.

A.3 The ZLB constraint at $t + 3$ and beyond

First we get the Cholesky decomposition of X_{t+2} for both cases:

$$X^1_{t+2} = \mu^1_{X,2} + \Gamma^1 \omega_{t+2}$$
$$X^2_{t+2} = \mu^2_{X,2} + \Gamma^2 \omega_{t+2}$$

In period $t + 3$,we have four possibilities: Let's denote non-binding state as nb and binding state b from here onwards and continue to write the value of X^{ij}_{t+s} where $i, j = 1$ for non-binding and $i, j = 2$ for binding cases. It can be (nb, nb), (nb, b), (b, nb) and (b, b) where the first symbol shows the state for the period $t + 2$ and the next shows for $t + 3$. For example the interest rate for the (nb, nb) and (b, nb) cases are:

$$\begin{aligned} X^{11}_{t+3} &= \mu + \Phi X^1_{t+2} + \Sigma \varepsilon_{t+3} \\ &= \mu + \Phi \mu^1_{X,2} + \Phi \Gamma^1 \omega_{t+2} + \Sigma \varepsilon_{t+3} \end{aligned}$$

$$X^{21}_{t+3} = \mu + \Phi \mu^2_{X,2} + \Phi \Gamma^2 \omega_{t+2} + \Sigma \varepsilon_{t+3}$$

For all of the four cases, I then solve for the set of $(\omega_{t+1}, \varepsilon_{t+2})$ that satisfies constraints induced by the ZLB. Notice that for the b state in period $t + 2$, $i^2_{t+2} = 0$ thus it is not random. Hence, for states starting with b we have only one constraint, i.e. that of period $t + 3$ and for the states that start with nb state we have two constraints those of $t + 2$ and $t + 3$. For example, the contraints for the (nb, nb) are:

$$\mu_i + \Phi_i \left(\mu^{1,nb}_{X,1} + \Gamma^{1,nb}_{X,1} \omega^{nb}_{t+1} \right) + \Sigma_i \varepsilon_{t+2} \geq \bar{\imath}$$
$$e_i \left(\mu^{1,nb}_{X,1} + \Gamma^{1,nb}_{X,1} \omega^{nb}_{t+1} \right) \geq \bar{\imath}$$

We can compute the moments accordingly. For the cases of $nb-b$ and $b-b$ we impose $i_{t+2}^{i2}=\bar{\imath}$ and thus have no variance. Since we are tracking one-period we collapse the X_{t+3}^{11} and X_{t+3}^{21} into X_{t+3}^{1} and X_{t+3}^{12} and X_{t+3}^{22} into X_{t+3}^{2}. We do that using the formulas provided in Appendix C.

B Appendix B

In this appendix, I explain how we can find the constraints for the cases of tracking different number of previous periods. I will first go over the first $n+1$ periods when we track n previous periods and then move to the periods after $n+1$.

B.1 The first n periods

Suppose at any branch at period k we have the following constraints for the case in which period k is nb

$$e_i C_k' \eta \geq \bar{\imath} - e_i a_k,$$

with

$$\eta = [\varepsilon_1', \varepsilon_2', ..., \varepsilon_k']'$$

I compute C_{k+1}' and a_{k+1} for period k is nb and when period k is b. In the following I present the case where period $k+1$ is nb. The case where period $k+1$ is b is the same except when the last row of both a_k and C_k' is multiplied by -1.

B.1.1 Period k is nb:

If period k is nb, we have

$$X_{k+1} = \mu + \Phi X_k + \Sigma\varepsilon_{k+1}.$$

Suppose I am computing the constraints for the case in which period $k+1$ is nb. Then, the constraint for period $k+1$ is

$$e_i\left(\mu + \Phi X_k + \Sigma\varepsilon_{k+1}\right) \geq \bar{\imath}.$$

Note that the period k constraint for $e_i X_k > \bar{\imath}$ is

$$e_i C_k' \eta \geq \bar{\imath} - e_i a_k \Leftrightarrow e_i X_k \geq \bar{\imath},$$

making

$$X_k = a_k + C_k'\eta.$$

Thus,

$$\begin{aligned} e_i\left(\mu+\Phi X_k+\Sigma\varepsilon_{k+1}\right) &\geq \bar{\imath} \\ e_i\left(\mu+\Phi\left(a_k+C_k'\eta\right)+\Sigma\varepsilon_{k+1}\right) &\geq \bar{\imath} \\ e_i\left(\Phi C_k'\eta+\Sigma\varepsilon_{k+1}\right) &\geq \bar{\imath}-e_i\left(\mu+\Phi a_k\right). \end{aligned}$$

Thus,

$$\begin{aligned} a_{k+1} &= \mu+\Phi a_k \\ C_{k+1}' &= \left[\begin{array}{cc} \Phi C_k' & \Sigma \end{array}\right] \end{aligned}$$

Obviously, for the case period in which $k+1$ is b, we have

$$e_i C_k'\eta \geq -\bar{\imath}-e_i a_k,$$

with

$$\begin{aligned} a_{k+1} &= -\left(\mu+\Phi a_k\right), \\ C_{k+1}' &= -\left[\begin{array}{cc} \Phi C_k' & \Sigma \end{array}\right]. \end{aligned}$$

B.1.2 Period k is b:

If period k is nb, we have

$$X_{k+1}=\mu+\lambda_i\bar{\imath}+\lambda_y y_k+\Sigma\varepsilon_{k+1}.$$

Then, the constraint for period $k+1$ is

$$e_i\left(\mu+\lambda_i\bar{\imath}+\lambda_y y_k+\Sigma\varepsilon_{k+1}\right) \geq \bar{\imath}.$$

Using

$$\begin{aligned} y_k &= e_y X_k \\ &= e_y\left(a_k+C_k'\eta\right), \end{aligned}$$

We have

$$\begin{aligned} e_i\left(\mu+\lambda_i\bar{\imath}+\lambda_y e_y\left(a_k+C_k'\eta\right)+\Sigma\varepsilon_{k+1}\right) &\geq \bar{\imath} \\ e_i\left(\lambda_y e_y C_k'\eta+\Sigma\varepsilon_{k+1}\right) &\geq \bar{\imath}-e_i\left(\mu+\lambda_i\bar{\imath}+\lambda_y e_y a_k\right). \end{aligned}$$

Thus,

$$a_k = \mu + \lambda_i \bar{\imath} + \lambda_y e_y a_k,$$
$$C'_{k+1} = \left[\begin{array}{cc} \lambda_y e_y C'_k & \Sigma \end{array} \right].$$

Obviously, for the case period $k+1$ is b, we have

$$e_i C'_k \eta \geq -\bar{\imath} - e_i a_k,$$

with

$$a_k = -\left(\mu + \lambda_i \bar{\imath} + \lambda_y e_y a_k\right),$$
$$C'_{k+1} = -\left[\begin{array}{cc} \lambda_y e_y C'_k & \Sigma \end{array} \right].$$

B.2 For periods after $n+1$

Suppose we are using X_{t+s} as our initial period. If this initial period state is nb, we first decompose it as follows:

$$X_{t+s} = m_{t+s} + \Gamma \varepsilon_{t+s}.$$

Note that if we track n previous periods we will have the following $(n+1)$ constraints for periods:

$$e_i X_{t+s} \lesseqgtr \bar{\imath},$$
$$e_i X_{t+s+1} \lesseqgtr \bar{\imath},$$
$$...$$
$$e_i X_{t+s+k} \lesseqgtr \bar{\imath}.$$

What are they? Assume we want to check whether all of them are greater than $\bar{\imath}$, i.e., we are checking whether it is $(nb - nb - ... - nb)$ Then, we have

$$X_{t+s} = m_{t+s} + \Gamma \varepsilon_{t+s},$$
$$X_{t+s+1} = \mu + \Phi X_{t+s} + \Sigma \varepsilon_{t+s+1},$$
$$...$$
$$X_{t+s+k} = \mu + \Phi X_{t+s+k-1} + \Sigma \varepsilon_{t+s+k}.$$

Then, the constraints are:

$$e_i \left(m_{t+s} + \Gamma \varepsilon_{t+s}\right) > \bar{\imath},$$
$$e_i \left(\mu + \Phi X_{t+s} + \Sigma \varepsilon_{t+s+1}\right) > \bar{\imath},$$
$$...$$
$$e_i \left(\mu + \Phi X_{t+s+k-1} + \Sigma \varepsilon_{t+s+k}\right) > \bar{\imath}.$$

The first constraint can be written in the $e_i C_k' \eta \geq \bar{\imath} - e_i a_k$ fashion with

$$a_s = m_{t+s},$$
$$C_s' = \Gamma.$$

Similarly, the second constraint can be written

$$\begin{aligned} & e_i \left(\mu + \Phi X_{t+s} + \Sigma \varepsilon_{t+s+1}\right) \\ & = e_i \left(\mu + \Phi \left(m_{t+s} + \Gamma \varepsilon_{t+s}\right) + \Sigma \varepsilon_{t+s+1}\right) > \bar{\imath} \end{aligned}$$

$$a_{s+1} = \mu + \Phi m_{t+s}$$
$$C_{s+1}' = \left[\begin{array}{cc} \Phi\Gamma & \Sigma \end{array} \right]$$

Thus, with the exception of the first constraint, the recursion is same for the case in which X_{t+s} is nb.

If X_{t+s} is b,by definition we have $i_{t+s} = \bar{\imath}$, and we have

$$X_{t+s} = m_{t+s} + \Gamma \varepsilon_{t+s}$$

with a Γ that has its $(1, 1)$ entry set to zero. Thus we don't need have the constraint for the period $t + s$. For the period $t + s + 1$, we have

$$X_{t+s+1} = \mu + \lambda_i \bar{\imath} + \lambda_y y_{t+s} + \Sigma \varepsilon_{t+s+1},$$

and the constraint is

$$\begin{aligned} & e_i \left(\mu + \lambda_i \bar{\imath} + \lambda_y y_{t+s} + \Sigma \varepsilon_{t+s+1}\right) \\ & = e_i \left(\mu + \lambda_i \bar{\imath} + \lambda_y e_y X_{t+s} + \Sigma \varepsilon_{t+s+1}\right) \\ & = e_i \left(\mu + \lambda_i \bar{\imath} + \lambda_y e_y \left(m_{t+s} + \Gamma \varepsilon_{t+s}\right) + \Sigma \varepsilon_{t+s+1}\right) \\ & > \bar{\imath}. \end{aligned}$$

Thus

$$a_{s+1} = \mu + \lambda_i \bar{\imath} + \lambda_y e_y m_{t+s},$$
$$C_{s+1}' = \left[\begin{array}{cc} \lambda_y e_y \Gamma & \Sigma \end{array} \right].$$

Suppose we are computing the case for $(b - nb - nb)$, for period $t + s + 2$ we have

$$X_{t+s+2} = \mu + \Phi X_{t+s+1} + \Sigma \varepsilon_{t+s+2}.$$

Hence, the constraint will be

$$a_{s+2} = \mu + \Phi a_{s+1},$$
$$C'_{s+2} = \left[\begin{array}{cc} \Phi C'_{s+1} & \Sigma \end{array} \right].$$

To sum up, if X_{t+s} is b we have k constraints with the first constraint being

$$a_{s+1} = \mu + \lambda_i \bar{\imath} + \lambda_y e_y m_{t+s},$$
$$C'_{s+1} = \left[\begin{array}{cc} \lambda_y e_y \Gamma & \Sigma \end{array} \right].$$

and the rest evolving as above.

C Appendix C

In this appendix, I provide the results of Tallis (1961) for the computation of the first and second moment for the truncated multinormal distribution. Let X have the multivariate normal distribution with $N(\mathbf{0}, R)$ with the correlation matrix R, and let X_s be truncated at a_s so that $X = [X_1, X_2, ..., X_n]'$ and $X_1 > a_1$, $X_2 > a_2, ..., X_n > a_n$.

$$\alpha = \Pr(X_1 > a_1, X_2 > a_2, ... > X_n > a_n)$$

Then

$$\alpha E(X_i) = \sum_{q=1}^{n} \rho_{iq} \phi(a_q) \Phi_{n-1}(A_{qs} : R_q)$$
$$\alpha E(X_i X_j) = \alpha \rho_{ij} + \sum_{q=1}^{n} \rho_{qi} \rho_{qj} a_q \phi(a_q) \Phi_{n-1}(A_{qs} : R_q)$$
$$+ \sum_{q=1}^{n} \left\{ \rho_{qi} \left(\sum_{r \neq q} \phi(a_q, a_r; \rho_{qr}) \right) \Phi_{n-2}(A^q_{rs} : R_{qr}) (\rho_{rj} - \rho_{qr}\rho_{qj}) \right\},$$

where ρ_{ij} is the correlation coefficient between X_i and X_j; ϕ_n is the normal probability distribution function, and Φ is the normal cumulative distribution function for dimension n; R_q and R_{qr} are the first and second order partial correlation coefficients with

$$A_{qs} = \frac{a_s - \rho_{sq} a_q}{\sqrt{1 - \rho_{sq}^2}},$$
$$A^q_{rs} = \frac{a_s - \beta_{sq.r} a_q - \beta_{sr.q} a_r}{\sqrt{\left(1 - \rho_{sq}^2\right)\left(1 - \rho_{sr.q}^2\right)}},$$

and $s \neq q$ in Φ_{n-1} and $s \neq q \neq r$ in Φ_{n-2}.$\beta_{sq.r}$ and $\beta_{sr.q}$ are the partial regression coefficients of X_s on X_q and X_r, respectively, and $\rho_{sr.q}$ is the partial correlation coefficient between X_s and X_r for fixed X_q.

D Appendix D

In this appendix, I derive the computation of the moments for mixed normals. Suppose I have S mixed normals with $N(\mu_i, \Sigma_i)$ each with probabilities π_i. Note that

$$var(X) = EX^2 - (EX)^2.$$

Thus,

$$EX_i^2 = \Sigma_i + \mu_i\mu_i',$$
$$\mu_X = \sum_{i=1}^{S} \pi_i\mu_i,$$

$$\begin{aligned} var(X) &= EX^2 - (EX)^2 \\ &= EX^2 - \mu_X\mu_X', \end{aligned}$$

$$\begin{aligned} EX^2 &= \sum_{i=1}^{S} \pi_i EX_i^2 \\ &= \sum_{i=1}^{S} \pi_i (\Sigma_i + \mu_i\mu_i'). \end{aligned}$$

Thus,

$$var(X) = \left[\sum_{i=1}^{S} \pi_i (\Sigma_i + \mu_i\mu_i')\right] - \mu_X\mu_X'.$$

What about the covariance between X and Y, where both X and Y are linear functions of Z a mixed normal? Suppose

$$X = \alpha + AZ$$
$$Y = \beta + BZ$$

where S mixed normals with $N(\mu_i, \Sigma_i)$. Then,

$$\begin{aligned} cov(X, Y) &= Acov(Z, Z)B' \\ &= Avar(Z)B' \end{aligned}$$

E Appendix E

This appendix provides a detailed computation for Section 5. First, I present a simple term structure model and derive the yields for different maturities and then I explain how we can find the moments of the sum of the short-rates, $b_{t+n-1} = \sum_{i=0}^{n-1} i_{t+i}$ under the ZLB constraint using the method of the paper.

E.1 A simple term-structure model with lagged interest rate, output gap and inflation

The state vector follows a VAR with one lag.

$$X_t = \mu + \Phi X_{t-1} + \Sigma\varepsilon_t$$

The short-rate is an affine function of the state

$$i_t = \delta_0 + \delta_1' X_t$$

The nominal stochastic discount factor iswith

$$m_{t+1} = exp\left(-i_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\right)$$

where the market price of risk is also an affine function of the state.

$$\lambda_t = \lambda_0 + \lambda_1 X_t$$

The state vector consists of the short-rate, output gap and inflation: $X_t = \begin{bmatrix} i_t & g_t & \pi_t \end{bmatrix}'$
I assume the market price of the risk takes the below functional form:

$$\lambda_0 = \begin{bmatrix} \lambda_{0,1} \\ \lambda_{0,2} \\ \lambda_{0,3} \end{bmatrix}, \lambda_1 = \begin{bmatrix} 0 & \lambda_{1,12} & \lambda_{1,13} \\ 0 & \lambda_{1,22} & \lambda_{1,23} \\ 0 & \lambda_{1,32} & \lambda_{1,33} \end{bmatrix}$$

Given the parameters μ, Φ and Σ, the $\mathbf{Q}-$measure is defined by

$$\tilde{\mu} = \mu - \Sigma\lambda_0$$
$$\tilde{\Phi} = \Phi - \Sigma\lambda_1$$

Once I have that

$$P_t^n = E_t^Q\left[\exp\left(-\sum_{i=0}^{n-1} i_{t+i}\right)\right]$$

E.1.1 Bond-prices under Q-measure

1-period bond

$$
\begin{aligned}
P_t^1 &= E_t^Q\left[m_{t+1}\right] \\
&= E_t^Q\left[\exp\left(-i_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\right)\right] \\
&= \exp\left(-i_t - \frac{1}{2}\lambda_t'\lambda_t\right) \times E_t^Q\left[\exp\left(\lambda_t'\varepsilon_{t+1}\right)\right] \\
&= \exp\left(-i_t - \frac{1}{2}\lambda_t'\lambda_t\right) \times \exp\left(\frac{1}{2}\lambda_t'\lambda_t\right) \\
&= \exp\left(-i_t\right)
\end{aligned}
$$

Since

$$i_t = e_i X_t$$

Thus

$$\log\left(P_t^1\right) = a_1 + b_1' X_t$$

with

$$
\begin{aligned}
\alpha_1 &= 0 \\
\beta_1' &= -e_i
\end{aligned}
$$

n-period bond

$$P_t^n = E_t^Q\left[\exp\left(-\sum_{i=0}^{n-1} i_{t+i}\right)\right]$$

$$
\begin{aligned}
P_t^n &= E_t^Q\left[\exp\left(-\sum_{i=0}^{n-1} i_{t+i}\right)\right] \\
&= E_t^Q\left[E_{t+1}^Q\left[\exp\left(-\sum_{i=0}^{n-1} i_{t+i}\right)\right]\right] \\
&= E_t^Q\left[E_{t+1}^Q\left[\exp\left(-i_t - \sum_{i=0}^{n-1} i_{t+i}\right)\right]\right] \\
&= E_t^Q\left[E_{t+1}^Q\left[\exp\left(-\sum_{i=1}^{n-1} i_{t+i}\right)\right] \times \exp\left(-i_t\right)\right] \\
&= E_t^Q\left[E_{t+1}^Q\left[\exp\left(-\sum_{i=0}^{n-2} i_{t+1+i}\right)\right] \times \exp\left(-i_t\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
P_t^n &= E_t^Q \left[P_{t+1}^{n-1} \times \exp\left(-i_t\right)\right] \\
&= E_t^Q \left[\exp\left(\alpha_{n-1} + \beta_{n-1}' X_{t+1}\right) \times \exp\left(-i_t\right)\right] \\
&= \exp\left(-i_t + \alpha_{n-1} + \beta_{n-1}'\tilde{\mu} + \beta_{n-1}'\tilde{\Phi} X_t\right) \times E_t^Q \left[\exp\left(\beta_{n-1}'\Sigma\varepsilon_{t+1}\right)\right] \\
&= \exp\left(-i_t + \alpha_{n-1} + \beta_{n-1}'\tilde{\mu} + \beta_{n-1}'\tilde{\Phi} X_t + \frac{1}{2}\beta_{n-1}'\Sigma\Sigma'\beta_{n-1}\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\alpha_n &= \alpha_{n-1} + \beta_{n-1}'\tilde{\mu} + \frac{1}{2}\beta_{n-1}'\Sigma\Sigma'\beta_{n-1} \\
\beta_n' &= -e_i + \beta_{n-1}'\tilde{\Phi}
\end{aligned}
$$

E.1.2 Yields

$$
\begin{aligned}
y_t^n &= \frac{-\log\left(P_t^n\right)}{n} \\
&= \frac{-1}{n}\left[\alpha_n + \beta_n X_t\right]
\end{aligned}
$$

E.2 Finding the moments of $b_{t+n-1} = \sum_{s=0}^{n-1} i_{t+s}$ under the ZLB constraint

In the affine term structure model of the paper, the yields are given by

$$
\exp\left(-ny_t^n\right) = E_t^Q \left[\exp\left(-\sum_{s=0}^{n-1} i_{t+s}\right)\right]
$$

I first define the cumulative interest rate variable,

$$
\begin{aligned}
b_t &= i_t \\
b_{t+k} &= i_{t+k} + b_{t+k-1}
\end{aligned}
$$

and approximate the yields from the unconditional (time $t-$conditional) mean and the variance of b_t, i.e:

$$
\exp\left(-ny_t^n\right) \cong \exp\left\{-E_t^Q b_{t+n-1} + \frac{1}{2} var_t^Q\left[b_{t+n-1}\right]\right\}
$$

The expectation term is computed directly from the algorithm:

$$
\begin{aligned}
b_t &= i_t \\
E_t^Q b_{t+k} &= E_t^Q i_{t+k} + E_t^Q b_{t+k-1}
\end{aligned}
$$

I denote the expectation of b_{t+s} with $\mathfrak{b}_{t+s}$, i.e. $\mathfrak{b}_{t+s} = E_t b_{t+s}$ (this is also the case for the different cases).

$$\mathfrak{b}^{ijk}_{t+s} = E_t \left[b^{ijk}_{t+s} \right]$$

Next, I explain how the variance term is computed.

E.2.1 The variance of b_{t+s}

Below, first I explain how to compute the expectation and the variance of b_{t+1} and then move to the other periods.For period 1 where $i_t > \bar{\imath}\,(nb)$, I have

$$\begin{aligned} X^1_{t+1} &= \mu + \Phi X_t + \Sigma \varepsilon_{t+1} \text{with C1 holding} \\ b^1_{t+1} &= i^1_{t+1} + b_0 \end{aligned}$$

Thus,

$$b^1_{t+1} = \text{constant} + e_i \Sigma \varepsilon_{t+1}$$

$$\begin{aligned} \mathfrak{b}^1_{t+1} &= i_t + E_t i^1_{t+1} \\ var_t \left(b^1_{t+1} \right) &= var_t \left(i^1_{t+1} \right) \\ cov_t \left(b^1_{t+1}, X^1_{t+1} \right) &= cov_t \left(i^1_{t+1}, X^1_{t+1} \right) \end{aligned}$$

Since everything is correlated because of the constraints, these can be computed with

$$\begin{aligned} \mathcal{A} &= [\Sigma] \\ \mathcal{B} &= [\Sigma] \\ var_t \left(b^1_{t+1} \right) &= \mathcal{A} var_t \left(\varepsilon_{t+1} \right) \mathcal{A}' \\ cov_t \left(b^1_{t+1}, X^1_{t+1} \right) &= \mathcal{A} var_t \left(\varepsilon_{t+1} \right) \mathcal{B}' \end{aligned}$$

Hence, in order to compute the variance of b_{t+s}, we need to compute $\mathcal{A}$ and $\mathcal{B}$ terms. For any period and any state notice that

$$\begin{aligned} var_t \left(b^{ijk}_{t+s} \right) &= \mathcal{A} var_t \left(\vartheta_{t+s} \right) \mathcal{A}' \\ cov_t \left(b^{ijk}_{t+s}, X^{ijk}_{t+s} \right) &= \mathcal{A} var_t \left(\vartheta_{t+s} \right) \mathcal{B}' \end{aligned}$$

, where $\vartheta_{t+s} = \left[\varepsilon'_{t+1}, \varepsilon'_{t+2}, .., \varepsilon'_{t+s} \right]'$ for $s \leq n+1$ and $\vartheta_{t+s} = \left[\omega'_{t+s-n}, \varepsilon'_{t+s-n+1}, .., \varepsilon'_{t+s} \right]$ for $s > n+1$. Notice that the $\mathcal{B}$ term is computed by the paper's algorithm; for period $t+s$, $\mathcal{B} = C_s$. Thus, I need to compute current $\mathcal{A}'s$ from past $\mathcal{A}$'s and past and current $\mathcal{B}'s$ In the next section, I explain how to compute these terms for the first $n+1$ periods and for the periods beyond $n+1$.

E.2.2 Computing $\mathcal{A}$

First $n+1$ periods: Suppose we know the term $\mathcal{A}^{ij}$ and we would like to compute $\mathcal{A}^{ij1}$ and $\mathcal{A}^{ij2}$. Note that $\mathcal{A}^{ij1}$ is the coefficient for the $i-j-nb$ state so that

$$\begin{aligned} b_{t+3}^{ij1} &= b_{t+2}^{ij} + i_{t+3}^{ij1} \\ &= \mathfrak{b}_{t+2}^{ij} + E_t\left[i_{t+3}^{ij1}\right] + e_i\mathcal{A}^{ij}\tilde{\varepsilon}_{t+2} + e_i\mathcal{B}^{ij1}\varepsilon_{t+3} \end{aligned}$$

Thus

$$\mathcal{A}^{ij1} = \begin{bmatrix} \mathcal{A}^{ij} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{ij1}$$

Similarly, for $i-j-2$, we have

$$b_{t+3}^{ij2} = b_{t+2}^{ij2} + i_{t+3}^{ij2} = \mathfrak{b}_{t+2}^{ij} + e_i\mathcal{A}^{ij}\tilde{\varepsilon}_{t+2}$$

so that

$$\mathcal{A}^{ij2} = \begin{bmatrix} \mathcal{A}^{ij} & \mathbf{0} \end{bmatrix}$$

Using the same logic for $i-1$ state we have,

$$\begin{aligned} b_{t+3}^{i1} &= b_{t+2}^{i} + i_{t+2}^{i1} \\ &= \mathfrak{b}_{t+1}^{i} + E_t\left[i_{t+2}^{i1}\right] + e_i\mathcal{A}^{i}\varepsilon_{t+1} + e_i\mathcal{B}^{i1}\varepsilon_{t+2} \end{aligned}$$

so that

$$\mathcal{A}^{i1} = \begin{bmatrix} \mathcal{A}^{i} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{i1}$$

and

$$\mathcal{A}^{i2} = \begin{bmatrix} \mathcal{A}^{i} & \mathbf{0} \end{bmatrix}$$

We can continue doing these if we track more periods like i-j-k-1 and i-j-k-2, like

$$\begin{aligned} \mathcal{A}^{ijk1} &= \begin{bmatrix} \mathcal{A}^{ijk} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{ijk1} \\ \mathcal{A}^{ijk2} &= \begin{bmatrix} \mathcal{A}^{ijk} & \mathbf{0} \end{bmatrix} \end{aligned}$$

The initial points (i.e. the $\mathcal{A}$ terms for the first period) are:

$$\begin{aligned} \mathcal{A}^{1} &= \Sigma \\ \mathcal{A}^{2} &= \mathbf{0} \end{aligned}$$

Periods after $n+1$ **:** In the first part of the Appendix E.2.2, I show how to compute expectation and the variance of b_{t+k} and the covariance of b_{t+k} with other variables for the first $n+1$ periods. Given these, I form the vector $Z^i_{t+k} = \left[X^{i\prime}_{t+k}, b^i_{t+k}\right]'$ and find the distribution of Z^1_{t+k} and Z^2_{t+k}. For periods after $n+1$, I use Cholesky decomposition to characterize the (approximate) distribution of Z^1_{t+k} and Z^2_{t+k}. For example, for two-period tracking we use the distribution of Z^1_{t+k-2} and Z^2_{t+k-2}:

$$\begin{aligned} Z^1_{t+k-2} &= E_t\left[Z^1_{t+k-2}\right] + \tilde{\Gamma}^1 \xi_{t+k-2}, \\ Z^2_{t+k-2} &= E_t\left[Z^2_{t+k-2}\right] + \tilde{\Gamma}^2 \xi_{t+k-2}. \end{aligned}$$

Note that this ordering results in the same Γ for $var_t X_{t+k-2}$. For periods $t+k-1$ and $t+k$, I do not check whether b_{t+k-2} and b_{t+k-1} are greater than zero as opposed to i_{t+k-2} and i_{t+k-1} so that in essence I only use the first n_x entries of ξ_{t+k-2} :

$$\omega_{t+k-2} = \xi_{t+k-2}\left(1 : n_x\right).$$

The additional part that I used in this decomposition relative to that of X_{t+k-2} is the part of the last row of $\tilde{\Gamma}^i, \Gamma^i_b$ row vector. I decompose $\tilde{\Gamma}$ and ξ_{t+k-2} as follows:

$$\begin{aligned} E_t\left[Z^i_{t+k-2}\right] = m^i_{t+k-2} &= \begin{bmatrix} \mu^i_{t+k-2} \\ \mathfrak{b}^i_{t+k-2} \end{bmatrix}, \\ \xi_{t+k-2} &= \left[\omega'_{t+k-2}, \eta_{t+k-2}\right]', \\ \tilde{\Gamma}^i &= \begin{bmatrix} \Gamma^i & \mathbf{0} \\ \Gamma^i_b & \Gamma^i_{bb} \end{bmatrix} \text{ so that} \\ \tilde{\Gamma}^i \xi_{t+k-2} &= \begin{bmatrix} \Gamma^i \omega_{t+k-2} \\ \Gamma^i_b \omega_{t+k-2} + \Gamma^i_{bb} \eta_{t+k-2} \end{bmatrix}. \end{aligned}$$

For one-period tracking, we have

$$Z^i_{t+k-1} = E_t\left[Z^i_{t+k-1}\right] + \tilde{\Gamma}^i \xi_{t+k-1},$$

with

$$b^i_{t+k-1} = \mathfrak{b}^i_{t+k-1} + \Gamma^i_b \omega_{t+k-1} + \Gamma^i_{bb} \eta_{t+k-1}.$$

Then for $t+k$,

$$\begin{aligned} b^{i1}_{t+k} &= b^i_{t+k-1} + i_{t+k} \\ &= \mathfrak{b}^i_{t+k-1} + E_t\left[i_{t+k}\right] + \Gamma^i_b \omega_{t+k-1} + \Gamma^i_{bb} \eta_{t+k-1} + e_i \mathcal{B}^{i1}. \end{aligned}$$

Thus,

$$\mathcal{A}^{i1} = \begin{bmatrix} \mathcal{A}^i & \mathbf{0} \end{bmatrix} + \mathcal{B}^{i1},$$

and

$$\mathcal{A}^{i2} = \left[\begin{array}{cc} \mathcal{A}^{i} & \mathbf{0} \end{array} \right].$$

For two-period tracking, we use

$$Z^{i}_{t+k-2} = E_t \left[Z^{i}_{t+k-2} \right] + \tilde{\Gamma}^{i} \xi_{t+k-2}$$

with

$$b^{i}_{t+k-2} = \mathfrak{b}^{i}_{t+k-2} + \Gamma^{i}_{b} \omega_{t+k-2} + \Gamma^{i}_{bb} \eta_{t+k-2}.$$

Then for $t+k-1$,

$$\begin{aligned} b^{i1}_{t+k-2} &= b^{i}_{t+k-2} + i_{t+k-1} \\ &= \mathfrak{b}^{i}_{t+k-2} + E_t \left[i_{t+k-1} \right] + \Gamma^{i}_{b} \omega_{t+k-2} + \Gamma^{i}_{bb} \eta_{t+k-2} + e_i \mathcal{B}^{i1}. \end{aligned}$$

Thus,

$$\mathcal{A}^{i1} = \left[\begin{array}{cc} \mathcal{A}^{i} & \mathbf{0} \end{array} \right] + \mathcal{B}^{i1},$$

and

$$\mathcal{A}^{i2} = \left[\begin{array}{cc} \mathcal{A}^{i} & \mathbf{0} \end{array} \right].$$

Denote the variance of the vector $\varrho = \left[\omega'_{t+k-2}, \varepsilon'_{t+k-1} \right]'$ with

$$var_t \left(\varrho \right) = \Omega.$$

Then,

$$\begin{aligned} var_t \left(b^{i1}_{t+k-2} \right) &= \Gamma^{i}_{bb} \Gamma^{i\prime}_{bb} + \mathcal{A}^{i1} \Omega \mathcal{A}^{i1\prime}, \\ cov_t \left(b^{i1}_{t+k-2}, X^{i1}_{t+k-2} \right) &= \mathcal{A}^{i1} \Omega \mathcal{A}^{i1\prime}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{A}^{ij1} &= \left[\begin{array}{cc} \mathcal{A}^{ij} & \mathcal{B}^{ij1} \end{array} \right], \\ \mathcal{A}^{ij2} &= \left[\begin{array}{cc} \mathcal{A}^{ij} & \mathbf{0} \end{array} \right]. \end{aligned}$$

Thus for our eight cases, we have

$$\begin{aligned} \mathcal{A}^{i11} &= \left[\begin{array}{ccc} \mathcal{A}^{i} & \mathbf{0} & \mathbf{0} \end{array} \right] + \left[\begin{array}{cc} \mathcal{B}^{i1} & \mathbf{0} \end{array} \right] + \mathcal{B}^{i11}, \\ \mathcal{A}^{i12} &= \left[\begin{array}{ccc} \mathcal{A}^{i} & \mathbf{0} & \mathbf{0} \end{array} \right] + \left[\begin{array}{cc} \mathcal{B}^{i1} & \mathbf{0} \end{array} \right], \\ \mathcal{A}^{i21} &= \left[\begin{array}{ccc} \mathcal{A}^{i} & \mathbf{0} & \mathbf{0} \end{array} \right] + \mathcal{B}^{i21}, \\ \mathcal{A}^{i22} &= \left[\begin{array}{ccc} \mathcal{A}^{i} & \mathbf{0} & \mathbf{0} \end{array} \right]. \end{aligned}$$

Since we do not directly follow $\mathcal{B}^{11}$ and $\mathcal{B}^{21}$ in the algorithm, we compute them here. $\mathcal{B}^{11}$ is the coefficient for X^{11},

$$X^{11}_{t+k-1} = \mu + \Phi X^{11}_{t+k-2} + \Sigma\varepsilon_{t+k-1},$$

so that

$$\mathcal{B}^{11} = \begin{bmatrix} \Phi\mathcal{B}^1 & \Sigma \end{bmatrix}.$$

On the other hand, things get a little interesting for $\mathcal{B}^{21}$, which is the coefficient for X^{21}, which is

$$X^{21}_{t+k-1} = \mu + \lambda_i \bar{\imath} + \lambda_y e_y \Phi X^2_{t+k-2} + \Sigma\varepsilon_{t+k-1},$$

so that

$$\mathcal{B}^{21} = \begin{bmatrix} \lambda_y e_y \mathcal{B}^2 & \Sigma \end{bmatrix}.$$

We can continue in this fashion and derive the $\mathcal{A}$'s for different cases. In the next section I summarize how to compute $\mathcal{A}$'s for the cases of one, two and three previous period tracking.

E.2.3 Coefficients for term structure computation ($\mathcal{A}$)

n periods tracking

First $n+1$ periods:

$$\begin{aligned}
\mathcal{A}^1 &= [\Sigma] \\
\mathcal{A}^2 &= [\mathbf{0}] \\
\mathcal{A}^{i1} &= \begin{bmatrix} \mathcal{A}^i & \mathbf{0} \end{bmatrix} + \mathcal{B}^{i1} \\
\mathcal{A}^{i2} &= \begin{bmatrix} \mathcal{A}^i & \mathbf{0} \end{bmatrix} \\
\mathcal{A}^{ij1} &= \begin{bmatrix} \mathcal{A}^{ij} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{ij1} \\
\mathcal{A}^{ij2} &= \begin{bmatrix} \mathcal{A}^{ij} & \mathbf{0} \end{bmatrix} \\
\mathcal{A}^{ijk1} &= \begin{bmatrix} \mathcal{A}^{ijk} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{ijk1} \\
\mathcal{A}^{ijk2} &= \begin{bmatrix} \mathcal{A}^{ijk} & \mathbf{0} \end{bmatrix}.
\end{aligned}$$

After $n+1$ periods:

$n = 1$

$$\mathcal{A}^i = \mathcal{B}^i$$

$$\begin{aligned}
\mathcal{A}^{i1} &= \left[\begin{array}{cc} \mathcal{A}^i & \mathbf{0} \end{array}\right] + \mathcal{B}^{i1} \\
\mathcal{A}^{i2} &= \left[\begin{array}{cc} \mathcal{A}^i & \mathbf{0} \end{array}\right].
\end{aligned}$$

$n = 2$

$$\mathcal{A}^i = \mathcal{B}^i$$

$$\begin{aligned}
\mathcal{A}^{i11} &= \left[\begin{array}{ccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{cc} \mathcal{B}^{i1} & \mathbf{0} \end{array}\right] + \mathcal{B}^{i11} \\
\mathcal{A}^{i12} &= \left[\begin{array}{ccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{cc} \mathcal{B}^{i1} & \mathbf{0} \end{array}\right] \\
\mathcal{A}^{i21} &= \left[\begin{array}{ccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} \end{array}\right] + \mathcal{B}^{i21} \\
\mathcal{A}^{i22} &= \left[\begin{array}{ccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} \end{array}\right],
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{B}^{11} &= \left[\begin{array}{cc} \mathcal{B}^1 & \Sigma \end{array}\right] \\
\mathcal{B}^{21} &= \left[\begin{array}{cc} \lambda_y e_y \mathcal{B}^2 & \Sigma \end{array}\right].
\end{aligned}$$

$n = 3$

$$\mathcal{A}^i = \mathcal{B}^i$$

$$\begin{aligned}
\mathcal{A}^{i111} &= \left[\begin{array}{cccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{ccc} \mathcal{B}^{i1} & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{cc} \mathcal{B}^{i11} & \mathbf{0} \end{array}\right] + \mathcal{B}^{i111} \\
\mathcal{A}^{i112} &= \left[\begin{array}{cccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{ccc} \mathcal{B}^{i1} & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{cc} \mathcal{B}^{i11} & \mathbf{0} \end{array}\right] \\
\mathcal{A}^{i121} &= \left[\begin{array}{cccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{ccc} \mathcal{B}^{i1} & \mathbf{0} & \mathbf{0} \end{array}\right] + \mathcal{B}^{i121} \\
\mathcal{A}^{i122} &= \left[\begin{array}{cccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{ccc} \mathcal{B}^{i1} & \mathbf{0} & \mathbf{0} \end{array}\right] \\
\mathcal{A}^{i211} &= \left[\begin{array}{cccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{cc} \mathcal{B}^{i21} & \mathbf{0} \end{array}\right] + \mathcal{B}^{i211} \\
\mathcal{A}^{i212} &= \left[\begin{array}{cccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] + \left[\begin{array}{cc} \mathcal{B}^{i21} & \mathbf{0} \end{array}\right] \\
\mathcal{A}^{i221} &= \left[\begin{array}{cccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] + \mathcal{B}^{i221} \\
\mathcal{A}^{i222} &= \left[\begin{array}{cccc} \mathcal{A}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]
\end{aligned}$$

with

$$
\begin{aligned}
\mathcal{A}^{1111} &= \begin{bmatrix} \mathcal{A}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{11} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{111} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{1111} \\
\mathcal{A}^{1112} &= \begin{bmatrix} \mathcal{A}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{11} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{111} & \mathbf{0} \end{bmatrix} \\
\mathcal{A}^{1121} &= \begin{bmatrix} \mathcal{A}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{11} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{1121} \\
\mathcal{A}^{1122} &= \begin{bmatrix} \mathcal{A}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{11} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\
\mathcal{A}^{1211} &= \begin{bmatrix} \mathcal{A}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{121} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{1211} \\
\mathcal{A}^{1212} &= \begin{bmatrix} \mathcal{A}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{121} & \mathbf{0} \end{bmatrix} \\
\mathcal{A}^{1221} &= \begin{bmatrix} \mathcal{A}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{1221} \\
\mathcal{A}^{1222} &= \begin{bmatrix} \mathcal{A}^1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}^{2111} &= \begin{bmatrix} \mathcal{A}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{21} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{211} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{2111} \\
\mathcal{A}^{2112} &= \begin{bmatrix} \mathcal{A}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{21} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{211} & \mathbf{0} \end{bmatrix} \\
\mathcal{A}^{2121} &= \begin{bmatrix} \mathcal{A}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{21} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{2121} \\
\mathcal{A}^{2122} &= \begin{bmatrix} \mathcal{A}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{21} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\
\mathcal{A}^{2211} &= \begin{bmatrix} \mathcal{A}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{221} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{2211} \\
\mathcal{A}^{2212} &= \begin{bmatrix} \mathcal{A}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathcal{B}^{221} & \mathbf{0} \end{bmatrix} \\
\mathcal{A}^{2221} &= \begin{bmatrix} \mathcal{A}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \mathcal{B}^{2221} \\
\mathcal{A}^{2222} &= \begin{bmatrix} \mathcal{A}^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}^{11} &= \begin{bmatrix} \Phi\mathcal{B}^1 & \Sigma \end{bmatrix} \\
\mathcal{B}^{21} &= \begin{bmatrix} \lambda_y e_y \mathcal{B}^2 & \Sigma \end{bmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{B}^{i11} &= \begin{bmatrix} \Phi\mathcal{B}^{i1} & \Sigma \end{bmatrix} \\
\mathcal{B}^{i21} &= \begin{bmatrix} \lambda_y e_y \mathcal{B}^{i2} & \Sigma \end{bmatrix}.
\end{aligned}
$$

F Appendix F

In this appendix, I explain how we can find the constraints for the different cases when we have lags. For the illustraton of the method, I present the case for a system with two lags, where I follow the previous two periods, which is the case in the example presented in the paper. I first explain the constraints for the first three periods and then explain the constraint for periods beyond three.

F.1 Systems with 2 lags

Suppose

$$X_t = \mu + \Phi_1 X_{t-1} + \Phi_2 X_{t-2} + \Sigma \varepsilon_t.$$

F.1.1 Period 1

$$X_1 = \mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma \varepsilon_{t+1}$$

We have two cases. For nb we have

$$\begin{aligned} e_i X_1 &> \bar{\imath}, \\ e_i \left(\Sigma \varepsilon_{t+1}\right) &> \bar{\imath} - e_i \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right), \end{aligned}$$

so

$$\begin{aligned} a_1 &= \mu + \Phi_1 X_0 + \Phi_2 X_{-1}, \\ C_1' &= \Sigma. \end{aligned}$$

and for b we have

$$\begin{aligned} a_1 &= -\left[\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right], \\ C_1' &= -\Sigma. \end{aligned}$$

F.1.2 Period 2

$$X_2 = \mu + \Phi_1 X_1 + \Phi_2 X_0 + \Sigma \varepsilon_{t+2}$$

Previous period nb**:** If previous period is nb we have

$$\begin{aligned} X_2 &= \mu + \Phi_1 X_1 + \Phi_2 X_0 + \Sigma\varepsilon_{t+2} \\ &= \mu + \Phi_1 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Phi_2 X_0 + \Sigma\varepsilon_{t+2}. \end{aligned}$$

For current period nb we have

$$\begin{aligned} a_2 &= \mu + \Phi_1 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right) + \Phi_2 X_0, \\ C_1' &= \left[\begin{array}{cc} \Phi_1\Sigma & \Sigma \end{array} \right]. \end{aligned}$$

For current period b we have

$$\begin{aligned} a_2 &= -\left[\mu + \Phi_1 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right) + \Phi_2 X_0\right], \\ C_1' &= -\left[\begin{array}{cc} \Phi_1\Sigma & \Sigma \end{array} \right]. \end{aligned}$$

Previous period b**:**

$$\begin{aligned} X_2 &= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} y_1 + \Phi_2 X_0 + \Sigma\varepsilon_{t+2} \\ &= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Phi_2 X_0 + \Sigma\varepsilon_{t+2}. \end{aligned}$$

For current period nb we have

$$\begin{aligned} a_2 &= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \mu + \lambda_{1,y} e_y \Phi_1 X_0 + \lambda_{1,y} e_y \Phi_2 X_{-1} + \Phi_2 X_0, \\ C_2' &= \left[\begin{array}{cc} \lambda_{1,y} e_y \Sigma & \Sigma \end{array} \right]. \end{aligned}$$

For current period b we have

$$\begin{aligned} a_2 &= -\left[\mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \mu + \lambda_{1,y} e_y \Phi_1 X_0 + \lambda_{1,y} e_y \Phi_2 X_{-1} + \Phi_2 X_0\right], \\ C_2' &= -\left[\begin{array}{cc} \lambda_{1,y} e_y \Sigma & \Sigma \end{array} \right]. \end{aligned}$$

F.1.3 Period 3:

$$X_3 = \mu + \Phi_1 X_2 + \Phi_2 X_1 + \Sigma\varepsilon_{t+3}$$

Decompose $\Phi_1 = \left[\begin{array}{cc} \lambda_{1,i} & \lambda_{1,y} \end{array} \right]$ such that $\Phi_1 X_{t-1} = \lambda_{1,i} i_{t-1} + \lambda_{1,y} y_{t-1}$ and $\Phi_2 = \left[\begin{array}{cc} \lambda_{2,i} & \lambda_{2,y} \end{array} \right]$.

Previous two periods: *nb*-*nb* If previous two periods are *nb*-*nb* we have

$$
\begin{aligned}
X_3 &= \mu + \Phi_1 X_2 + \Phi_2 X_1 + \Sigma\varepsilon_{t+3} \\
&= \mu + \Phi_1 \left(\mu + \Phi_1 X_1 + \Phi_2 X_0 + \Sigma\varepsilon_{t+2}\right) + \Phi_2 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Sigma\varepsilon_{t+3}.
\end{aligned}
$$

For current period *nb* we have

$$\bar{e}_i X_1 > \bar{\imath},$$

so that

$$
\begin{aligned}
a_3 &= \mu + \Phi_1 \left(\mu + \Phi_1 X_1 + \Phi_2 X_0\right) + \Phi_2 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right), \\
C_3' &= \left[\begin{array}{ccc} \Phi_2\Sigma & \Phi_1\Sigma & \Sigma \end{array}\right].
\end{aligned}
$$

For current period b, we have

$$
\begin{aligned}
a_3 &= -\left[\mu + \Phi_1 \left(\mu + \Phi_1 X_1 + \Phi_2 X_0\right) + \Phi_2 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right)\right], \\
C_3' &= -\left[\begin{array}{ccc} \Phi_2\Sigma & \Phi_1\Sigma & \Sigma \end{array}\right].
\end{aligned}
$$

Previous two periods: *b*-*nb* If previous period is *nb* but two periods ago is b we have

$$
\begin{aligned}
X_3 &= \mu + \Phi_1 X_2 + \lambda_{2,i}\bar{\imath} + \lambda_{2,y} y_1 + \Sigma\varepsilon_{t+3} \\
&= \mu + \Phi_1 \left(\mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Phi_2 X_0 + \Sigma\varepsilon_{t+2}\right) \\
&+ \lambda_{2,i}\bar{\imath} + \lambda_{2,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Sigma\varepsilon_{t+3}.
\end{aligned}
$$

If current period is *nb* we have

$$
\begin{aligned}
a_3 &= \left[\begin{array}{c} \mu + \\ \Phi_1 \left\{\left(\begin{array}{c} \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) \\ + \Phi_2 X_0 + \Sigma\varepsilon_{t+2} \end{array}\right)\right\} \\ + \lambda_{2,i}\bar{\imath} + \lambda_{2,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Sigma\varepsilon_{t+3} \end{array}\right], \\
C_3'\tilde{\varepsilon} &= \Phi_1 \left(\lambda_{1,y} e_y \Sigma\varepsilon_{t+1} + \Sigma\varepsilon_{t+2}\right) + \lambda_{2,y} e_y \Sigma\varepsilon_{t+1} + \Sigma\varepsilon_{t+3}, \\
C_3' &= \left[\begin{array}{ccc} \Phi_1 \lambda_{1,y} e_y \Sigma + \lambda_{2,y} e_y \Sigma & \Phi_1\Sigma & \Sigma \end{array}\right].
\end{aligned}
$$

If current period is b we have

$$
\begin{aligned}
a_3 &= -\left[\begin{array}{c} \mu + \\ \Phi_1 \left\{\left(\begin{array}{c} \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) \\ + \Phi_2 X_0 + \Sigma\varepsilon_{t+2} \end{array}\right)\right\} \\ + \lambda_{2,i}\bar{\imath} + \lambda_{2,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Sigma\varepsilon_{t+3} \end{array}\right], \\
C_3' &= -\left[\begin{array}{ccc} \Phi_1 \lambda_{1,y} e_y \Sigma + \lambda_{2,y} e_y \Sigma & \Phi_1\Sigma & \Sigma \end{array}\right].
\end{aligned}
$$

Previous two periods:nb**-**b If previous period is b but two periods ago is nb we have

$$\begin{aligned} X_3 &= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} y_2 + \Phi_2 X_1 + \Sigma\varepsilon_{t+3} \\ &= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu + \Phi_1 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Phi_2 X_0 + \Sigma\varepsilon_{t+2}\right) \\ &+ \Phi_2 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Sigma\varepsilon_{t+3}. \end{aligned}$$

If current period is nb we have

$$\begin{aligned} a_3 &= \left\{ \begin{array}{c} \mu + \lambda_{1,i}\bar{\imath} \\ +\lambda_{1,y} e_y \left[\mu + \Phi_1 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right) + \Phi_2 X_0\right] \\ +\Phi_2 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right) \end{array} \right\}, \\ C_3'\tilde{\varepsilon} &= \lambda_{1,y} e_y \left(\Phi_1 \Sigma\varepsilon_{t+1} + \Sigma\varepsilon_{t+2}\right) + \Phi_2 \Sigma\varepsilon_{t+1} + \Sigma\varepsilon_{t+3}, \\ C_3' &= \left[\begin{array}{ccc} \lambda_{1,y} e_y \Phi_1 \Sigma + \Phi_2 \Sigma & \lambda_{1,y} e_y \Sigma & \Sigma \end{array}\right]. \end{aligned}$$

If current period is b we have

$$\begin{aligned} a_3 &= -\left\{ \begin{array}{c} \mu + \lambda_{1,i}\bar{\imath} \\ +\lambda_{1,y} e_y \left[\mu + \Phi_1 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right) + \Phi_2 X_0\right] \\ +\Phi_2 \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right) \end{array} \right\}, \\ C_3' &= -\left[\begin{array}{ccc} \lambda_{1,y} e_y \Phi_1 \Sigma + \Phi_2 \Sigma & \lambda_{1,y} e_y \Sigma & \Sigma \end{array}\right]. \end{aligned}$$

Previous two periods:b**-**b

$$\begin{aligned} X_3 &= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} y_2 + \lambda_{2,i}\bar{\imath} + \lambda_{2,y} y_1 + \Sigma\varepsilon_{t+3} \\ &= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{2,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Phi_2 X_0 + \Sigma\varepsilon_{t+2}\right) \\ &+ \lambda_{2,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1} + \Sigma\varepsilon_{t+1}\right) + \Sigma\varepsilon_{t+3}. \end{aligned}$$

If current period is nb we have

$$\begin{aligned} a_3 &= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{2,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right) + \Phi_2 X_0\right) \\ &+ \lambda_{2,y} e_y \left(\mu + \Phi_1 X_0 + \Phi_2 X_{-1}\right), \\ C_3'\tilde{\varepsilon} &= \lambda_{1,y} e_y \left(\lambda_{1,y} e_y \Sigma\varepsilon_{t+1} + \Sigma\varepsilon_{t+2}\right) + \lambda_{2,y} e_y \Sigma\varepsilon_{t+1} + \Sigma\varepsilon_{t+3}, \\ C_3' &= \left[\begin{array}{ccc} \left(\lambda_{1,y} e_y \lambda_{1,y} e_y + \lambda_{2,y} e_y\right)\Sigma & \lambda_{1,y} e_y \Sigma & \Sigma \end{array}\right]. \end{aligned}$$

F.1.4 Periods after 3

nb-*nb* We have to consider two cases differently: If it is *nb*-*nb* two periods before, we have

$$\begin{aligned} X_{k-2} &= \mu^X_{k-2} + \Gamma_{k-2}\omega_{k-2}, \\ X_{k-1} &= \mu^X_{k-1} + \Gamma_{k-1}\omega_{k-1}. \end{aligned}$$

Then,

$$\begin{aligned} X_k &= \mu + \Phi_1 X_{k-1} + \Phi_2 X_{k-2} + \Sigma\varepsilon_k \\ &= \mu + \Phi_1\left(\mu^X_{k-1} + \Gamma_{k-1}\omega_{k-1}\right) + \Phi_2\left(\mu^X_{k-2} + \Gamma_{k-2}\omega_{k-2}\right) + \Sigma\varepsilon_k. \end{aligned}$$

If current is *nb*

$$\begin{aligned} a_k &= \mu + \Phi_1\mu^X_{k-1} + \Phi_2\mu^X_{k-2}, \\ C'_k &= \begin{bmatrix} \Phi_2\Gamma_{k-2} & \Phi_1\Gamma_{k-1} & \Sigma \end{bmatrix}. \end{aligned}$$

If current is *b*

$$\begin{aligned} a_k &= -\left[\mu + \Phi_1\mu^X_{k-1} + \Phi_2\mu^X_{k-2}\right], \\ C'_k &= -\begin{bmatrix} \Phi_2\Gamma_{k-2} & \Phi_1\Gamma_{k-1} & \Sigma \end{bmatrix}. \end{aligned}$$

The other periods are:

$$\begin{aligned} a_{k-1} &= \mu^X_{k-1}, \\ C'_{k-1} &= \Gamma_{k-1}, \end{aligned}$$

and

$$\begin{aligned} a_{k-2} &= \mu^X_{k-2}, \\ C'_{k-2} &= \Gamma_{k-2}. \end{aligned}$$

b-*nb*

$$\begin{aligned} X_{k-2} &= \mu^X_{k-2} + \Gamma_{k-2}\omega_{k-2}, \\ \mu^X_{k-2} &= \begin{bmatrix} \bar{\imath} \\ \mu^y_{k-2} \end{bmatrix}, \Gamma_{k-2} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \Gamma_{k-2,yy} \end{bmatrix}, \\ X_{k-1} &= \mu^X_{k-1} + \Gamma_{k-1}\omega_{k-1}. \end{aligned}$$

Then

$$\begin{aligned}X_k &= \mu + \Phi_1 X_{k-1} + \Phi_2 X_{k-2} + \Sigma\varepsilon_k \\ &= \mu + \Phi_1 \left(\mu^X_{k-1} + \Gamma_{k-1}\omega_{k-1}\right) + \lambda_{2,i}\bar{\imath} + \lambda_{2,y} e_y \left(\mu^y_{k-2} + \Gamma_{k-2,yy}\omega_{k-2}\right) + \Sigma\varepsilon_k.\end{aligned}$$

If current is nb

$$\begin{aligned}a_k &= \left[\mu + \Phi_1 \mu^X_{k-1} + \lambda_{2,i}\bar{\imath} + \lambda_{2,y} e_y \mu^y_{k-2}\right], \\ C'_k &= \left[\begin{array}{ccc} \lambda_{2,y} e_y \Gamma_{k-2,yy} & \Phi_1 \Gamma_{k-1} & \Sigma \end{array}\right].\end{aligned}$$

If current period is b

$$\begin{aligned}a_k &= -\left[\mu + \Phi_1 \mu^X_{k-1} + \lambda_{2,i}\bar{\imath} + \lambda_{2,y} e_y \mu^y_{k-2}\right], \\ C'_k &= -\left[\begin{array}{ccc} \lambda_{2,y} e_y \Gamma_{k-2,yy} & \Phi_1 \Gamma_{k-1} & \Sigma \end{array}\right].\end{aligned}$$

Period $k-1$ is

$$\begin{aligned}a_{k-1} &= \mu^X_{k-1}, \\ C'_{k-1} &= \Gamma_{k-1}.\end{aligned}$$

and we do not have any restriction for the period $k-2$'s b.

nb-*b*

$$\begin{aligned}X_{k-2} &= \mu^X_{k-2} + \Gamma_{k-2}\omega_{k-2}, \\ X_{k-1} &= \mu^X_{k-1} + \Gamma_{k-1}\omega_{k-1}, \\ \mu^X_{k-1} &= \left[\begin{array}{c} \bar{\imath} \\ \mu^y_{k-1} \end{array}\right], \Gamma_2 = \left[\begin{array}{cc} 0 & \mathbf{0} \\ \mathbf{0} & \Gamma_{k-1,yy} \end{array}\right].\end{aligned}$$

Then

$$\begin{aligned}X_k &= \mu + \Phi_1 X_{k-1} + \Phi_2 X_{k-2} + \Sigma\varepsilon_k \\ &= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu^y_{k-1} + \Gamma_{k-1,yy}\omega_{k-1}\right) + \Phi_2 \left(\mu^X_{k-2} + \Gamma_{k-2}\omega_{k-2}\right) + \Sigma\varepsilon_k,\end{aligned}$$

If current is nb

$$\begin{aligned}a_k &= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \mu^y_{k-1} + \Phi_2 \mu^X_{k-2}, \\ C'_k &= \left[\begin{array}{ccc} \Phi_2 \Gamma_{k-2} & \lambda_{1,y} e_y \Gamma_{k-1,yy} & \Sigma \end{array}\right].\end{aligned}$$

If current is b

$$a_k = -\left[\mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \mu^y_{k-1} + \Phi_2 \mu^X_{k-2}\right],$$
$$C'_k = -\left[\begin{array}{ccc} \Phi_2 \Gamma_{k-2} & \lambda_{1,y} e_y \Gamma_{k-1,yy} & \Sigma \end{array}\right].$$

We do not have any restriction for the period $k-1$'s b and period $k-2$ parameters are

$$a_{k-2} = \mu^X_{k-2},$$
$$C'_{k-2} = \Gamma_{k-2}.$$

b-*b*

$$X_{k-2} = \mu^X_{k-2} + \Gamma_{k-2}\omega_{k-2},$$
$$\mu^X_{k-2} = \left[\begin{array}{c} \bar{\imath} \\ \mu^y_{k-2} \end{array}\right], \Gamma_2 = \left[\begin{array}{cc} 0 & \mathbf{0} \\ \mathbf{0} & \Gamma_{k-2,yy} \end{array}\right],$$
$$X_{k-1} = \mu^X_{k-1} + \Gamma_{k-1}\omega_{k-1},$$
$$\mu^X_{k-1} = \left[\begin{array}{c} \bar{\imath} \\ \mu^y_{k-1} \end{array}\right], \Gamma_2 = \left[\begin{array}{cc} 0 & \mathbf{0} \\ \mathbf{0} & \Gamma_{k-1,yy} \end{array}\right].$$

Then,

$$X_k = \mu + \Phi_1 X_{k-1} + \Phi_2 X_{k-2} + \Sigma\varepsilon_k$$
$$= \mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \left(\mu^y_{k-1} + \Gamma_{k-1,yy}\omega_{k-1}\right) + \lambda_{2,i}\bar{\imath} + \lambda_{2,y} e_y \left(\mu^y_{k-2} + \Gamma_{k-2,yy}\omega_{k-2}\right) + \Sigma\varepsilon_k.$$

If current is nb

$$a_k = \left[\mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y} e_y \mu^y_{k-1} + \lambda_{2,i}\bar{\imath} + \lambda_{2,y} e_y \mu^y_{k-2}\right],$$
$$C'_k = \left[\begin{array}{ccc} \lambda_{2,y} e_y \Gamma_{k-2,yy} & \lambda_{1,y} e_y \Gamma_{k-1,yy} & \Sigma \end{array}\right].$$

If current is b

$$a_k = -\left[\mu + \lambda_{1,i}\bar{\imath} + \lambda_{1,y}\mu^y_{k-1} + \lambda_{2,i}\bar{\imath} + \lambda_{2,y}\mu^y_{k-2}\right],$$
$$C'_k = -\left[\begin{array}{ccc} \lambda_{2,y}\Gamma_{k-2,yy} & \lambda_{1,y}\Gamma_{k-1,yy} & \Sigma \end{array}\right].$$

and we do not have any restrictions for the previous two periods.

Panel 1A: Probability of hitting the ZLB (percentage point)

Period/Method Number of Periods Tracked	1	2	3	4
1	0.02	0.02	0.02	0.02
2	0.01	0.01	0.01	0.01
5	-2.17	-0.41	-0.04	0.01
20	-1.80	-0.39	-0.03	0.07
40	-1.80	-0.39	-0.03	0.07

Panel 1B: Expected Values of Variables in Different States (percentage point)

	i_t				x_t				π_t			
	Non-binding				Binding				Unconditonal			
Period/Method Number of Periods Tracked	1	2	3	4	1	2	3	4	1	2	3	4
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.12	0.04	0.01	0.00	0.01	0.00	0.00	0.00	-0.07	-0.04	-0.02	0.00
20	0.14	0.05	0.01	0.00	0.00	-0.01	-0.01	-0.01	-0.21	-0.10	-0.06	-0.04
40	0.14	0.05	0.01	0.00	0.00	0.00	0.00	0.00	-0.21	-0.10	-0.06	-0.03

Panel 1C: Computing Time Relative to a Monte Carlo with 10^6 draws (percent)

1	2	3	4
0.40	2.23	34.75	237.02

TABLE 1: Difference between the method and Monte Carlo simulation for numerical exercise 1

Panel A shows the difference between the method and Monte Carlo simulation for the probability of hitting the ZLB . Panel B shows the difference between the method and Monte Carlo simulation for the expected values of endogenous variables at different states. Panel C shows the computing time of each method relative to that of a Monte Carlo with 10^6 draws.

Panel 2A: Probability of hitting the ZLB (percentage point)

Period/Method Number of Periods Tracked	1	2	3	4
1	0.02	0.02	0.02	0.02
2	0.01	0.01	0.01	0.01
5	-1.74	-0.36	-0.02	0.04
20	-1.79	-0.39	-0.03	0.07
40	-1.80	-0.39	-0.03	0.06

Panel 2B: Expected Values of Variables in Different States (percentage point)

	i_t				x_t				π_t			
	Non-binding				Binding				Unconditonal			
Period/Method Number of Periods Tracked	1	2	3	4	1	2	3	4	1	2	3	4
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.14	0.04	0.02	0.00	0.01	0.00	0.00	0.00	-0.09	-0.03	-0.02	0.00
20	0.14	0.05	0.01	0.00	0.00	-0.01	-0.01	-0.01	-0.21	-0.10	-0.06	-0.04
40	0.14	0.05	0.01	0.00	0.00	0.00	0.00	0.00	-0.21	-0.10	-0.06	-0.03

Panel 2C: Computing Time Relative to A Monte Carlo with 10^6 draws (percent)

1	2	3	4
0.38	2.21	41.58	227.37

TABLE 2: Difference between the method and Monte Carlo simulation for numerical exercise 2
Panel A shows the difference between the method and Monte Carlo simulation for the probability of hitting the ZLB . Panel B shows the difference between the method and Monte Carlo simulation for the expected values of endogenous variables at different states. Panel C shows the computing time of each method relative to that of a Monte Carlo with 10^6 draws.

Panel 3A: Probability of hitting the ZLB (percentage point)

Period/Method Number of Periods Tracked	1	2	3	4
1	0.03	0.03	0.03	0.03
2	-0.01	-0.01	-0.01	-0.01
5	-3.08	-0.59	-0.06	-0.01
20	-2.75	-0.53	0.08	0.26
40	-2.76	-0.58	0.04	0.21

Panel 3B: Expected Values of Variables in Different States (percentage point)

	i_t				x_t				π_t			
	Non-binding				Binding				Unconditonal			
Period/Method Number of Periods Tracked	1	2	3	4	1	2	3	4	1	2	3	4
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.14	0.06	0.02	0.00	0.02	0.00	0.00	0.00	-0.08	-0.04	-0.03	0.00
20	0.18	0.05	0.00	-0.01	0.09	0.03	0.01	0.00	-0.38	-0.20	-0.12	-0.07
40	0.19	0.06	0.01	-0.01	0.12	0.04	0.01	0.00	-0.35	-0.19	-0.11	-0.07

Panel 3C: Computing Time Relative to A Monte Carlo with 10^6 draws (percent)

1	2	3	4
0.51	2.51	42.48	283.70

TABLE 3: Difference between the method and Monte Carlo simulation for numerical exercise 3

Panel A shows the difference between the method and Monte Carlo simulation for the probability of hitting the ZLB . Panel B shows the difference between the method and Monte Carlo simulation for the expected values of endogenous variables at different states. Panel C shows the computing time of each method relative to that of a Monte Carlo with 10^6 draws.

Panel 4A: Probability of hitting the ZLB (percentage point)

Period/Method Number of Periods Tracked	1	2	3	4
1	-0.04	-0.04	-0.04	-0.04
2	-0.03	-0.03	-0.03	-0.03
5	-2.19	-0.56	-0.13	-0.01
20	-2.47	-0.64	-0.10	-0.07
40	-2.47	-0.64	-0.10	-0.07

Panel 4B: Expected Values of Variables in Different States (percentage point)

	i_t				x_t				π_t			
	Non-binding				Binding				Unconditonal			
Period/Method Number of Periods Tracked	1	2	3	4	1	2	3	4	1	2	3	4
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	0.16	0.06	0.03	0.00	0.06	0.02	0.01	0.01	-0.08	-0.03	-0.03	0.00
20	0.21	0.08	0.03	0.01	0.05	0.01	0.00	0.00	-0.29	-0.15	-0.09	-0.06
40	0.21	0.08	0.03	0.01	0.05	0.02	0.01	0.00	-0.29	-0.15	-0.09	-0.06

Panel 4C: Computing Time Relative to A Monte Carlo with 10^6 draws (percent)

1	2	3	4
0.40	2.57	38.43	179.38

TABLE 4: Difference between the method and Monte Carlo simulation for numerical exercise 4

Panel A shows the difference between the method and Monte Carlo simulation for the probability of hitting the ZLB . Panel B shows the difference between the method and Monte Carlo simulation for the expected values of endogenous variables at different states. Panel C shows the computing time of each method relative to that of a Monte Carlo with 10^6 draws.

Panel 5A: VAR parameters under Q-measure

$$\mu = \begin{bmatrix} 0.72 \\ -0.26 \\ 0.92 \end{bmatrix} \Phi = \begin{bmatrix} 0.80 & -0.11 & 0.20 \\ 0.05 & 0.67 & 0.10 \\ -0.20 & 0.10 & 0.69 \end{bmatrix}$$

Panel 5B: Probability of hitting the ZLB under the Q-measure (percentage point)

Period/Method Number of Periods Tracked	1	2	3
1	0.19	0.19	0.19
2	-0.18	-0.18	-0.18
5	1.16	0.22	-0.03
20	0.92	0.30	0.12
40	0.87	0.25	0.04

Panel 5C: Yield Differences (percentage point)

	i_t		
Maturity/Method Number of Periods Tracked	1	2	3
1	0.00	0.00	0.00
4	-1.76	0.25	0.25
8	-10.70	-2.84	-1.60
20	-16.58	-6.84	-3.89
40	-17.76	-7.53	-4.11

Panel 5D: Computing Time Relative to A Monte Carlo with 10^6 draws (percent)

1	2	3
0.57	2.95	36.30

TABLE 5: Difference between the method and Monte Carlo simulation for numerical exercise 5
Panel A shows the VAR parameters under $Q-$measure. Panel B shows the difference between the method and Monte Carlo simulation for the probability of hitting the ZLB under Q-measure. Panel C shows the difference between the method and Monte Carlo simulation for the yields of different maturities. Panel D shows the computing time of each method relative to that of a Monte Carlo with 10^6 draws.

Panel 6A: Probability of hitting the ZLB (percentage point)

1	2	5	20	40
0.00	0.01	2.03	2.28	2.31

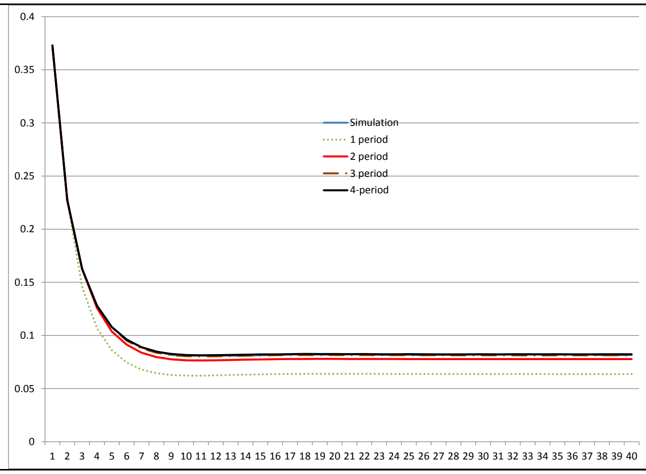
Panel 6B: Expected Values of Variables in Different States (percentage point)

	Non-binding			Binding			Unconditonal		
Period/Method Number of Periods Tracked	i_t	x_t	π_t	i_t	x_t	π_t	i_t	x_t	π_t
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	-0.01	-0.01	-0.01	0.00	0.00	0.00
5	0.08	0.02	0.05	0.00	-0.04	0.16	-0.13	-0.03	-0.05
20	-0.06	-0.02	0.14	0.00	-0.06	0.45	-0.13	-0.03	0.16
40	-0.06	-0.02	0.14	0.00	-0.07	0.45	-0.14	-0.04	0.16

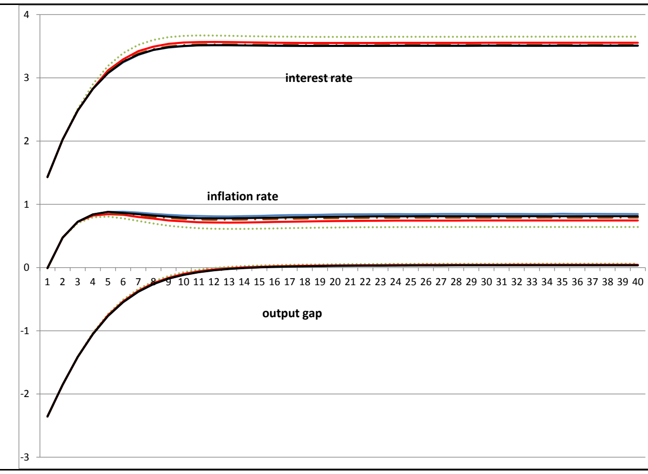
Panel 6C: Computing Time Relative to A Monte Carlo with 10^6 draws (percent)

2
1.81

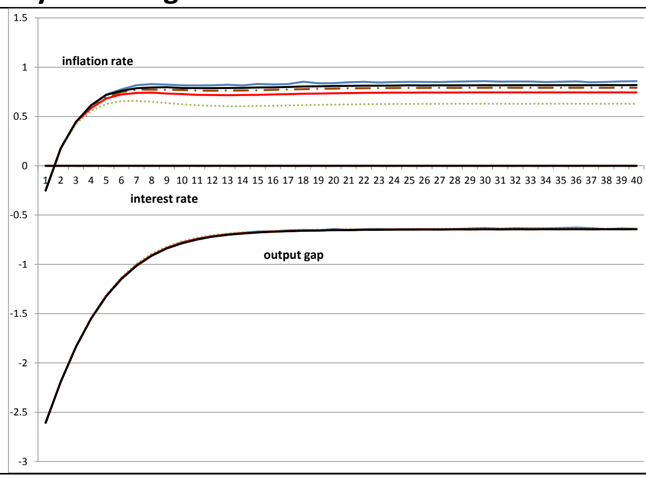
TABLE 6: Difference between the method and Monte Carlo simulation for numerical exercise 6
Panel A shows the difference between the method and Monte Carlo simulation for the probability of hitting the ZLB . Panel B shows the difference between the method and Monte Carlo simulation for the expected values of endogenous variables at different states. Panel C shows the computing time of each method relative to that of a Monte Carlo with 10^6 draws.



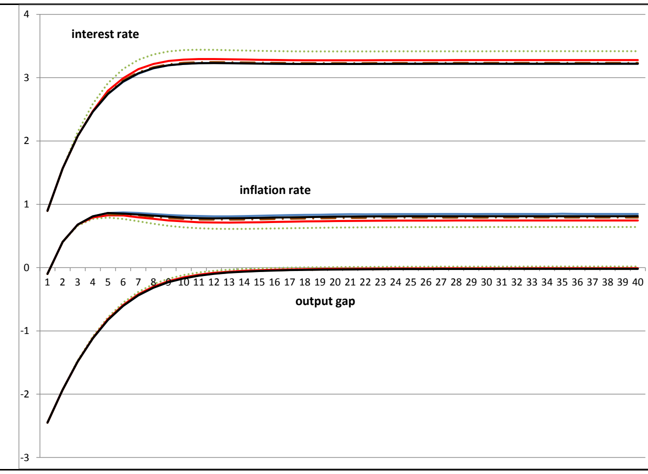
Probability of hitting the ZLB



Expected values conditional on ZLB does not bind

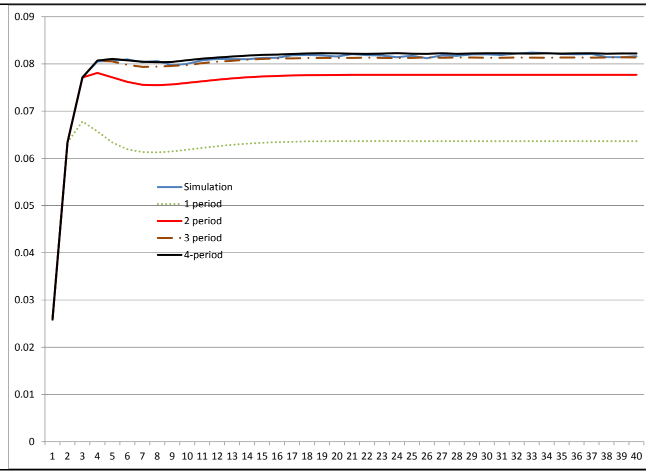


Expected values conditional on ZLB binds

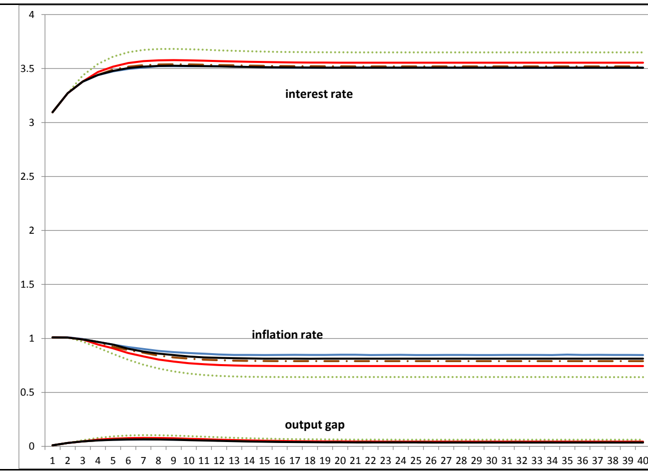


Unconditional (time-0 conditional) expected values

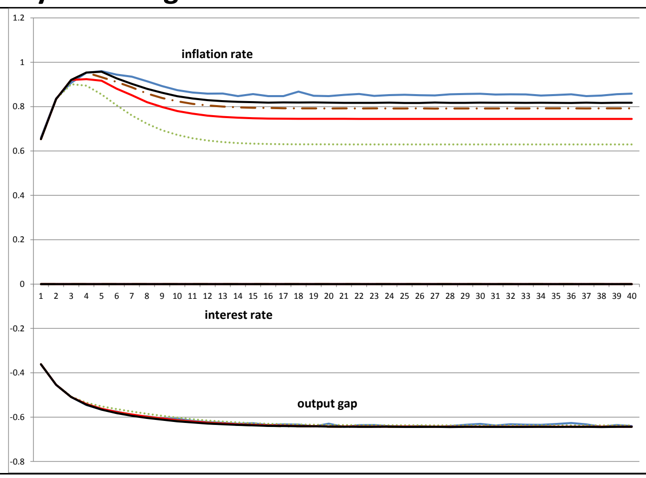
Figure 1: The results from numerical exercise 1: The upper left chart shows the probability of hitting the ZLB. The upper right chart shows the mean of the endogenous variables conditional on the ZLB constraint does not bind. The lower left chart shows the mean of the endogenous variables conditional on the ZLB constraint binds. The lower right chart shows the unconditional (time 0 conditional) mean of the endogenous variables.



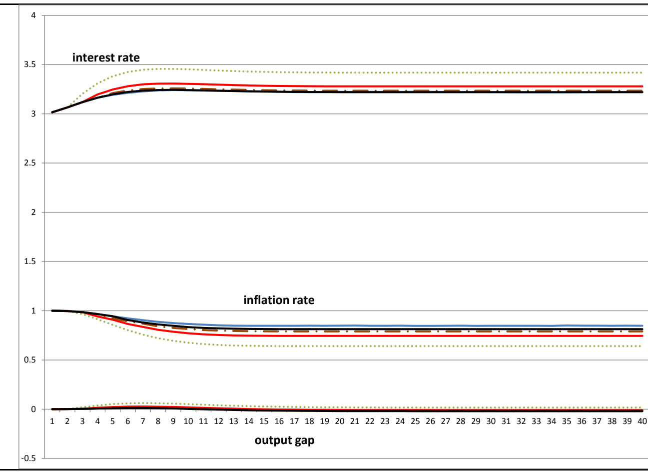
Probability of hitting the ZLB



Expected values conditional on ZLB does not bind

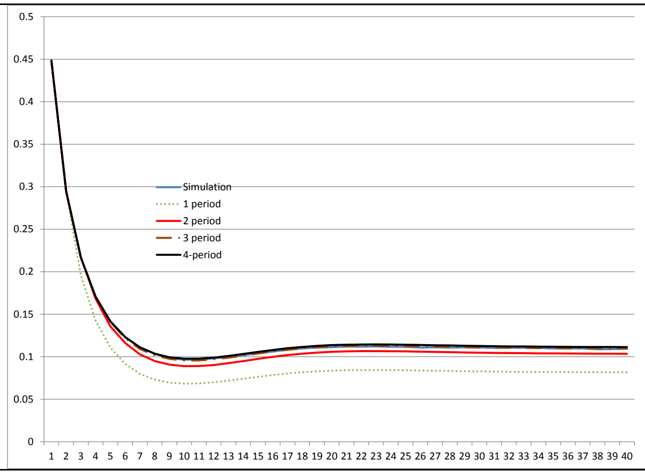


Expected values conditional on ZLB binds

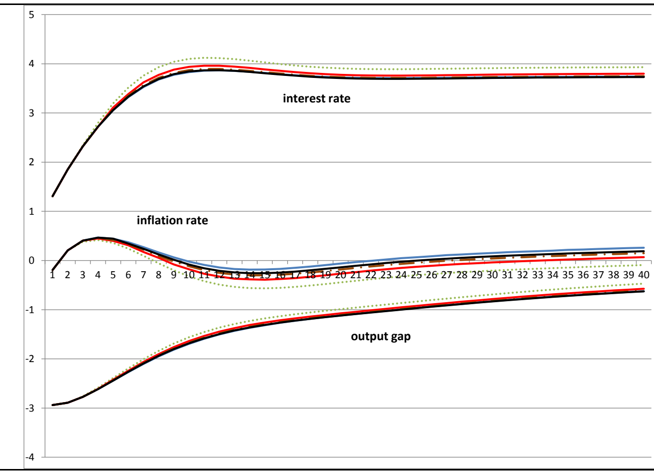


Unconditional (time-0 conditional) expected values

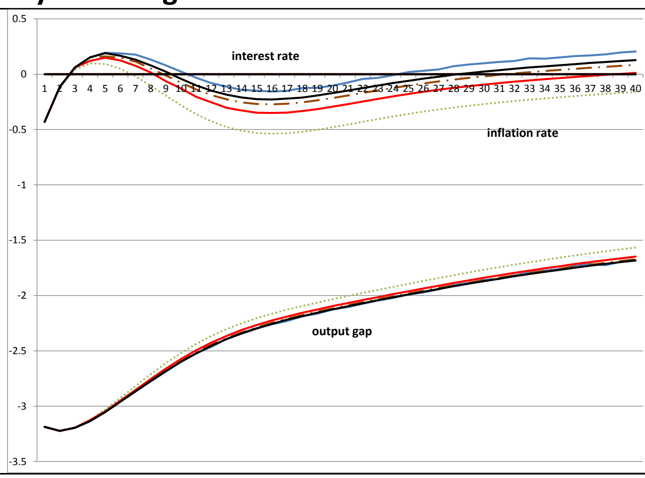
Figure 2: The results from numerical exercise 2: The upper left chart shows the probability of hitting the ZLB. The upper right chart shows the mean of the endogenous variables conditional on the ZLB constraint does not bind. The lower left chart shows the mean of the endogenous variables conditional on the ZLB constraint binds. The lower right chart shows the unconditional (time 0 conditional) mean of the endogenous variables.



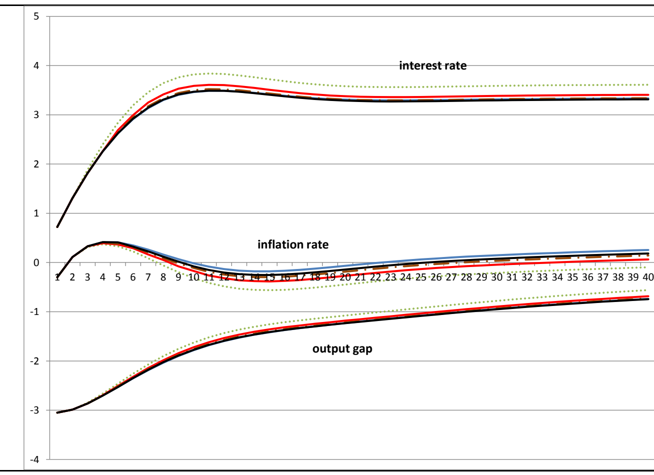
Probability of hitting the ZLB



Expected values conditional on ZLB does not bind



Expected values conditional on ZLB binds



Unconditional (time-0 conditional) expected values

Figure 3: The results from numerical exercise 3: The upper left chart shows the probability of hitting the ZLB. The upper right chart shows the mean of the endogenous variables conditional on the ZLB constraint does not bind. The lower left chart shows the mean of the endogenous variables conditional on the ZLB constraint binds. The lower right chart shows the unconditional (time 0 conditional) mean of the endogenous variables.

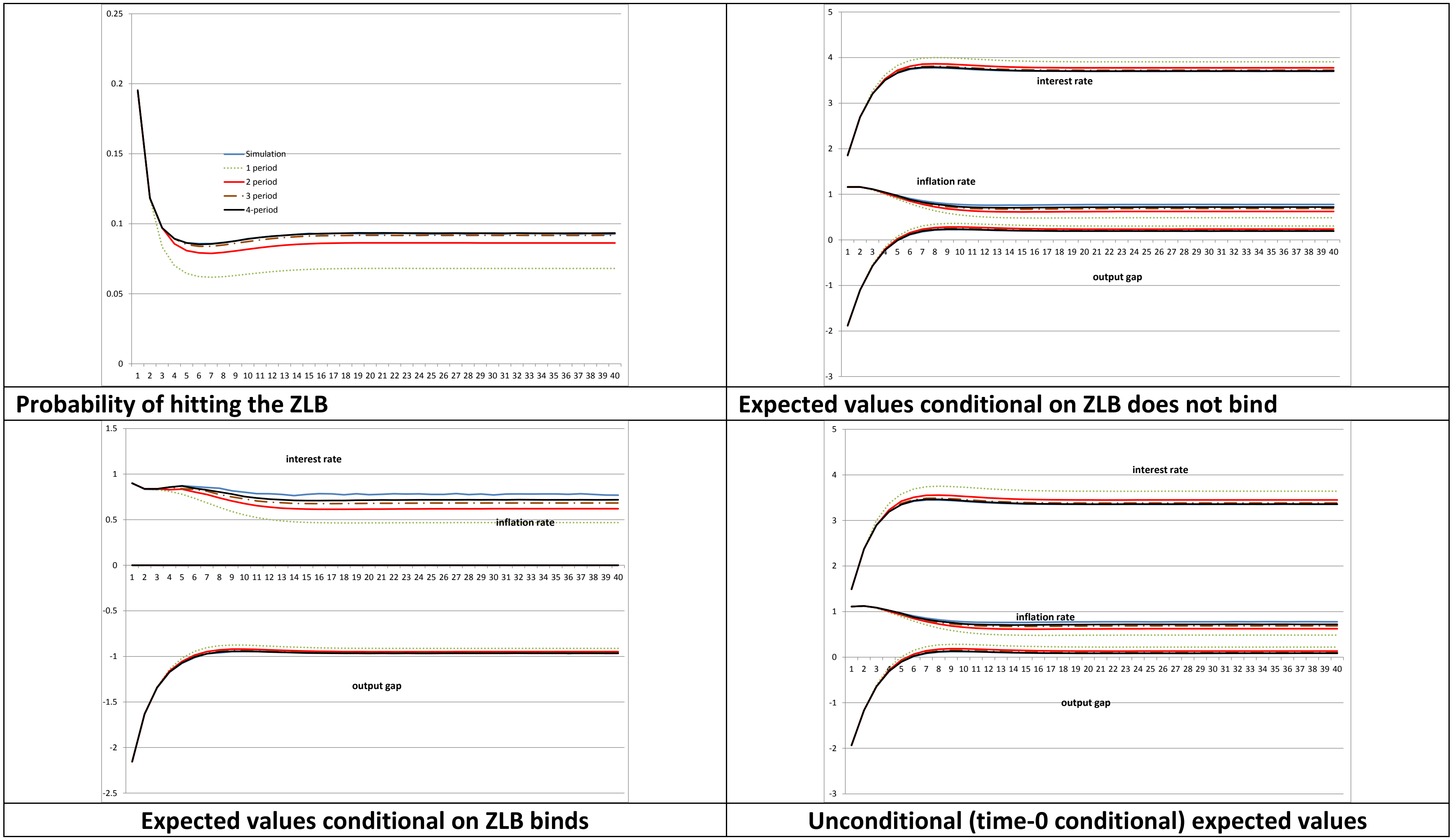
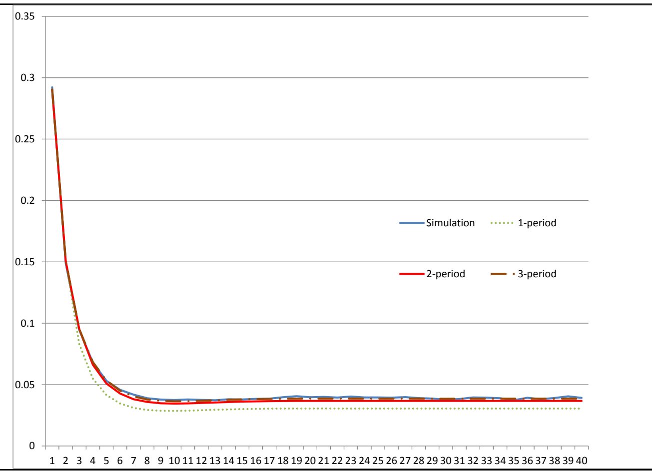
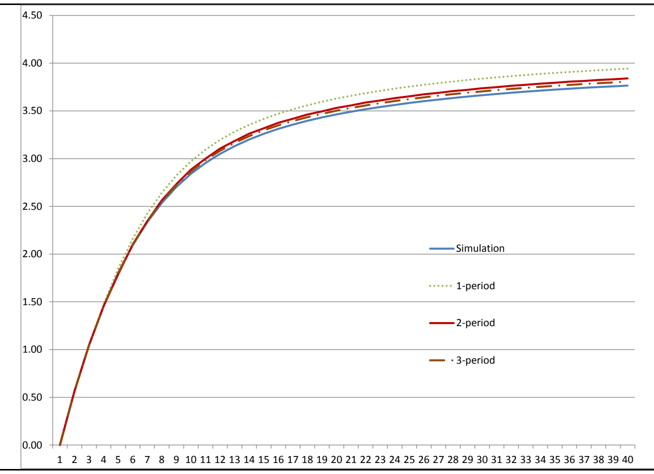


Figure 4: The results from numerical exercise 4: The upper left chart shows the probability of hitting the ZLB. The upper right chart shows the mean of the endogenous variables conditional on the ZLB constraint does not bind. The lower left chart shows the mean of the endogenous variables conditional on the ZLB constraint binds. The lower right chart shows the unconditional (time 0 conditional) mean of the endogenous variables.



Probability of hitting the ZLB



Yield curve

Figure 5: The results from numerical exercise 5: The upper left chart shows the probability of hitting the ZLB under the Q-measure. The upper right chart shows the term structure of nominal interest rates .

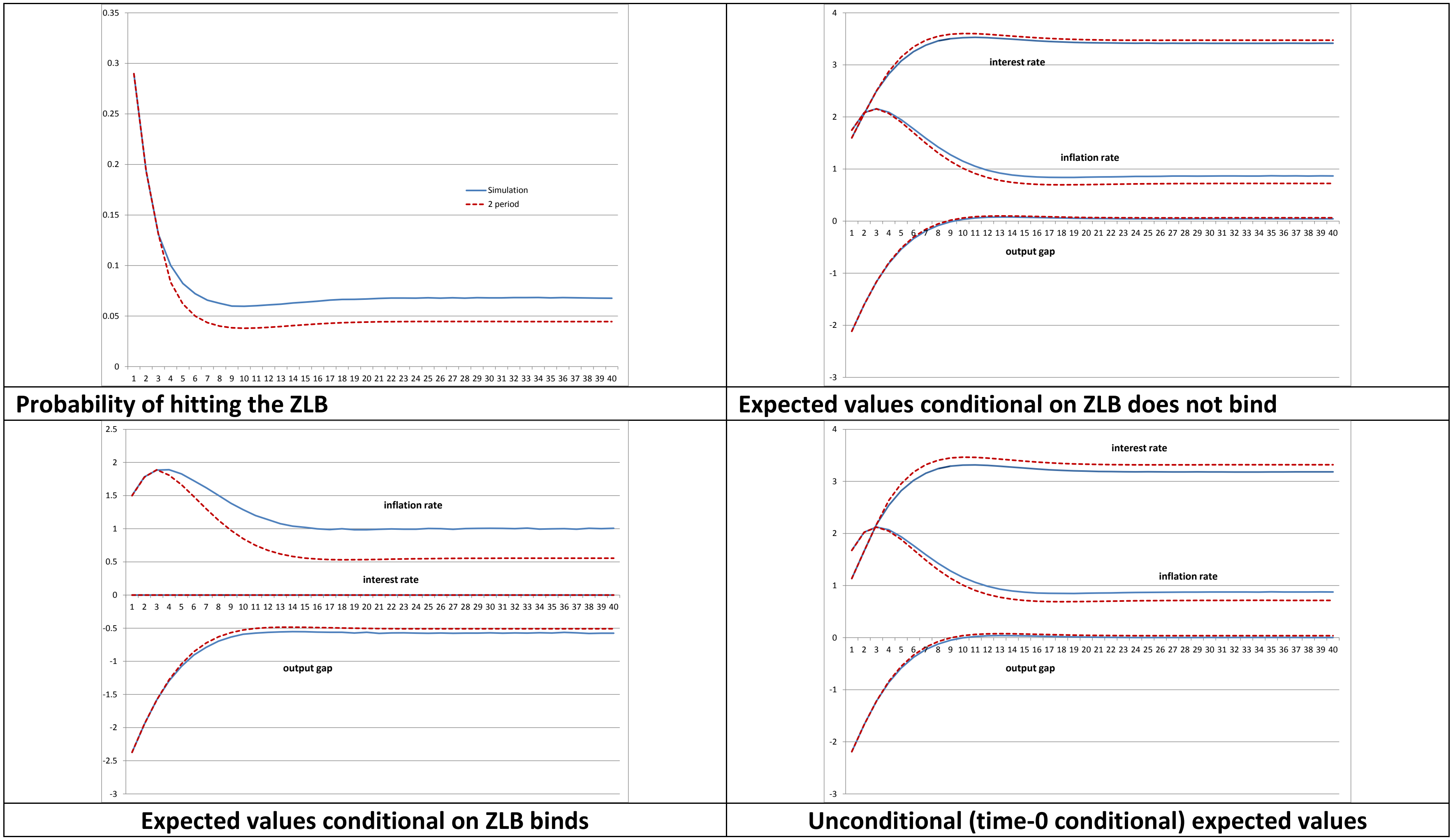


Figure 6: The results from numerical exercise 6: The upper left chart shows the probability of hitting the ZLB. The upper right chart shows the mean of the endogenous variables conditional on the ZLB constraint does not bind. The lower left chart shows the mean of the endogenous variables conditional on the ZLB constraint binds. The lower right chart shows the unconditional (time 0 conditional) mean of the endogenous variables.