working paper

14 37

The Effect of Safe Assets on Financial Fragility in a Bank-Run Model

Toni Ahnert and Mahmoud Elamin

FEDERAL RESERVE BANK OF CLEVELAND

Working Paper 14-37 December 2014

The Effect of Safe Assets on Financial Fragility in a Bank-Run Model

Toni Ahnert and Mahmoud Elamin

Risk-averse investors induce competitive intermediaries to hold safe assets, thereby lowering the probability of a run and reducing financial fragility. We revisit Goldstein and Pauzner (2005), who obtain a unique equilibrium in the banking model of Diamond and Dybvig (1983) by introducing risky investment and noisy private signals. We show that, in the optimal demand-deposit contract subject to sequential service, banks hold safe assets to insure investors against investment risk. Consequently, fewer investors withdraw prematurely, which reduces the probability of a bank run. Safe asset holdings increase investor welfare and may increase the bank's provision of liquidity.

Keywords: bank runs, demand deposits, global games, liquidity provision, safe assets.

JEL classications: D8, G21.

Suggested citation: Ahnert, Toni, and Mahmoud Elamin, 2014. "The Effect of Safe Assets on Financial Fragility in a Bank-Run Model," Federal Reserve Bank of Cleveland, working paper no. 14-37.

Toni Ahnert is at the Bank of Canada (tahnert@bankofcanada.ca.), and Mahmoud Elamin is at the Federal Reserve Bank of Cleveland (mahmoud.elamin@clev.frb.org). The authors thank Jason Allen, Nabil AlNajjar, Joe Haubrich, and Filippo Occhino for helpful comments. The views they express in this paper are theirs and should not be interpreted as reflecting the views of the Bank of Canada, the Federal Reserve Bank of Cleveland, or the Federal Reserve System.

1 Introduction

While runs experienced in the recent financial crisis involved intermediation activities beyond traditional banking, the financial arrangement and the associated risk of runs often resembled those of traditional banks.[1] Financial intermediaries invest in risky assets funded by short-term liabilities similar to demand deposits. These short-term liabilities allow investors to satisfy their liquidity needs, while participating in profitable long-term investments. The maturity mismatch of assets and liabilities expose intermediaries to the risk of runs. An investor's decision whether to roll over funding is determined by considerations about the risk of the intermediary's investment and the strategic behavior of other investors. Premature liquidation by some investors exerts a negative externality on those investors who roll over funding. Does the availability of a safe asset decrease financial fragility by mitigating runs?

In a seminal paper, Diamond and Dybvig (1983) analyze a setup with risk-less investment and risk-averse investors who face unobservable idiosyncratic liquidity risk. The first-best allocation provides liquidity risk-sharing among investors. A demand-deposit contract implements the first-best allocation by promising an interim return above the liquidation value of the investment. As a result, the demand-deposit contract is run-prone and two equilibria emerge. In the good equilibrium, only investors with liquidity needs withdraw prematurely. In the bad equilibrium, all investors withdraw, resulting in the bank's insolvency. However, the probability of a run remains outside the perimeter of the analysis.

In another seminal paper, Goldstein and Pauzner (2005) introduce investment risk and noisy private information in the Diamond/Dybvig setup. They derive a unique equilibrium in which the probability of a bank run depends on the promised interim return. Resolving the multiplicity of equilibria allows for a meaningful examination of welfare, whereby the costs of a run ex post are internalized ex ante. That is, the bank's promised interim return trades off a larger run probability with enhanced liquidity provision (greater liquidity risk-sharing).

[1] See Covitz et al. (2013) for a run on the Asset-Backed Commercial Paper market, Kacperczyk and Schnabl (2013) and Schmidt et al. (2014) for a run on money market mutual funds, and Gorton and Metrick (2012) for a run on the repo market. See Brunnermeier (2009) for a review.

We expand the portfolio of the bank by introducing investment in a safe asset. Building on the Goldstein/Pauzner setup, banks can now hold safe assets, such as cash, government debt or central bank reserves. This allows the bank to insure those risk-averse investors who roll over funding by reducing their exposure to investment risk.[2] Expanding the analysis of Goldstein and Pauzner (2005), we study the effect of safe assets holdings on financial stability. This allows us to examine the effect of both liquidity risk-sharing and insurance against investment risk on bank fragility.[3] More broadly, we aim to contribute to the ongoing discussion about the role of safe assets in society (Gorton et al. (2012)).[4]

We show that safe asset holdings decrease the probability of a bank run and enhance financial stability. Banks can decrease the probability of a run by either providing less liquidity to investors or holding safe assets. The demand-deposit contract we analyze consists of two parts. First, the bank promises a non-contingent interim return to investors who withdraw prematurely. Second, after all withdrawing investors have been served, the bank may also liquidate more investment. Holding the proceeds as a safe asset offers insurance against investment risk.

To analyze the withdrawal behavior of investors when the bank holds safe assets, we use the methods of one-sided strategic complementarity proposed by Goldstein and Pauzner (2005). When the bank insures patient investors against investment risk, fewer investors withdraw prematurely. Thus, the probability of a bank run is overstated in Goldstein and Pauzner (2005), and the ex-ante welfare of investors is understated. Figure 1 shows the unique thresholds with and without safe asset holdings as the interim return varies.

[2] Positive safe asset holdings always occurs if the marginal utility at zero is sufficiently high. The Inada conditions assumed in Diamond and Dybvig (1983) are an extreme assumption satisfying this requirement.

[3] In both Goldstein and Pauzner (2005) and our model, there is no role for holding safe assets at the initial date since it is dominated by risky investment. At the interim date, however, it is optimal to hold safe assets, through partial liquidation of investment.

[4] Gorton et al. (2012) argue that the safe asset share of total assets remained stable at about one third since 1952. This is striking because economic conditions and regimes varied extensively in this long period. Gorton et al. (2012) focus on the liabilities side of bank's balance sheet, and consider banks as safe asset producers. In contrast, we focus on the asset side of bank's balance sheet and provide a rationale for bank demand for safe assets produced outside of the banking sector. Banks demand safe assets produced outside the banking sector to decrease financial fragility and increase stability.

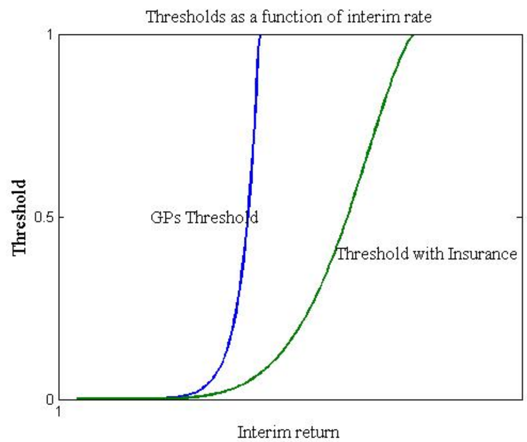


Figure 1: Withdrawal thresholds and the interim return with and without safe asset holdings.

Safe asset holdings also affect the optimal provision of liquidity. Revisiting the trade-off between liquidity provision and bank fragility, we derive the optimal interim-rate promised by a bank that holds safe assets. Numerical exercises and preliminary evaluation confirm our intuition that the ex-post insurance against investment risk allows the bank to provide more ex-ante insurance against liquidity risk. The rise in the optimal interim-rate with safe assets bridges part of the gap between the first-best interim-rate of Diamond and Dybvig (1983) and the one in Goldstein and Pauzner (2005). Figure 2 depicts the ex-ante expected utility with and without insurance against investment risk as the interim rate varies. It also shows that optimality with ex-post insurance against investment risk is achieved at a higher interim return than without insurance, implying greater ex-ante liquidity risk-sharing.

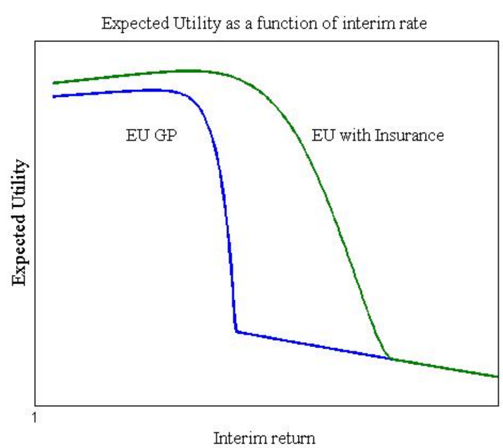


Figure 2: Expected utility and the interim return with and without safe asset holdings.

Besides showing the role of safe asset holdings in financial stability, we make some technical contributions to the literature. Most importantly, we show that the equilibrium threshold below which a run occurs is continuous in the noise. Continuity allows us to obtain a unique limit that was not otherwise guaranteed: uniqueness for every positive noise does not imply uniqueness with vanishing noise. Hence, we prove Goldstein and Pauzner (2005)'s conjecture about the existence of such a unique limit with vanishing noise. Second, we provide a condition for the lower dominance region to emerge naturally. Third, we provide a strictly tighter bound on the promised interim return, above which an investor wishes to withdraw prematurely, irrespective of the strategies of other investors. Fourth, we provide a more extensive treatment than the proofs in Goldstein and Pauzner (2005). Some technical issues do arise, it is shown that these do not affect the main results.[5]

This paper proceeds as follows. Section 2 describes the model. In section 3 we analyze a competitive bank's safe asset holdings. We study the impact of these holdings on the optimal withdrawal threshold in section 4. Section 4.1 contains an important technical contribution regarding the continuity of the withdrawal threshold. In section 4.2 we establish a link between the withdrawal threshold and the promised return, which gives rise to a trade-off between bank fragility and liquidity risk-sharing as in Goldstein and Pauzner (2005). Finally, in section 5 we study how the withdrawal threshold, investor welfare and the bank's provision of liquidity are affected by safe asset holdings.

[5]For example, the fact that the probability is strictly increasing does not guarantee that the net incentive integral is increasing in the fundamental. For small enough noise, this integral initially *decreases* as the fundamental increases. We show that there exists a level of the fundamental above which it increases, as in Goldstein and Pauzner (2005). Furthermore, we show that the utility differential *increases* in the number of withdrawals, when the fundamental is in the upper dominance region. This result arises since the promised interim return is less than the liquidation value. This fact contrasts with the case when the fundamental is below the upper dominance threshold. Consequently, we show that there is a discontinuity in the utility differential as the threshold approaches the upper dominance region. This complicates the one-sided strategic complementarities argument.

2 Environment

We revisit the global games model of banking proposed by Goldstein and Pauzner (2005). The economy extends over three dates $t \in \{0, 1, 2\}$ and there is no discounting. A single good is used for consumption and investment. There is a unit continuum of investors $i \in [0, 1]$, each endowed with one unit of the good at the initial date. As in Diamond and Dybvig (1983), each investor privately learns his individual preference $\omega_i \in \{0, 1\}$ at the interim date. An impatient investor ($\omega_i = 1$) values interim-date consumption only. By contrast, a patient investor ($\omega_i = 0$) is indifferent between interim-date and final-date consumption:

$$U_i(c_1, c_2) = \omega_i \, u(c_1) + (1 - \omega_i) \, u(c_1 + c_2) \tag{1}$$

where c_t is consumption at date t. The initial-date probability of facing idiosyncratic liquidity risk, $\Pr\{\omega_i = 1\} \equiv \lambda \in (0, 1)$, is identical and independent across investors and equals the economy-wide proportion of impatient investors at the interim date. The utility function $u(c)$ is twice continuously differentiable, strictly increasing, and strictly concave. The relative risk aversion, $\frac{-cu''(c)}{u'(c)}$, exceeds unity for $c \geq 1$.[6] We follow Goldstein and Pauzner (2005) in imposing $u(0) \equiv 0$, which is without loss of generality if the utility at zero is bounded.

A constant-return-to-scale investment technology is publicly available at the initial date. Departing from Diamond and Dybvig (1983), Goldstein and Pauzner (2005) introduce investment risk, whereby the gross return on investment is R in the good state and zero in the bad state. Let θ be the economy's fundamental, which can be interpreted as a macroeconomic indicator. The good state occurs with probability $p(\theta)$, an arbitrary continuously differentiable function that strictly increases in the fundamental, $p'(\theta) > 0$. The investment technology has a positive expected net present value, $\mathrm{E}_\theta[p(\theta)]R > 1$. Investment can be liquidated at par at the interim date. Storage is available at the interim date and yields a

[6] Diamond and Dybvig (1983) show that this assumption generates risk-sharing in the first-best allocation without investment risk. In the Goldstein and Pauzner (2005) setting with investment risk, Elamin (2013) provides a stronger condition for risk sharing to occur in the first-best allocation.

unit gross return at the final date. Holding liquidity at $t = 0$ is dominated by investment.

There is incomplete information about the fundamental. At the interim date it is drawn from a uniform distribution, $\theta \sim U[0,1]$, but not publicly observed. Each investor receives a noisy private signal at the interim date:

$$\theta_i \equiv \theta + \epsilon_i, \tag{2}$$

where the idiosyncratic noise is independent of the fundamental and identically and independently distributed across investors, $\epsilon_i \sim U[-\epsilon, \epsilon]$, for some $\epsilon > 0$.

At the initial date, a competitive bank offers a demand-deposit contract to maximize the expected utility of its investors.[7] Unlike in Goldstein and Pauzner (2005), where the contract only specified a promised return at the interim date, the demand-deposit contract we consider consists of two parts.

First, an investor who withdraws at the interim date is promised a fixed return $1 < r_1 < \frac{R}{1+\lambda(R-1)}$.[8] By pooling resources, the demand-deposit contract insures investors against the idiosyncratic liquidity risk as in Diamond and Dybvig (1983) and Goldstein and Pauzner (2005). Upon observing their private information, investors simultaneously decide whether to withdraw from the bank or not. Let $n \geq \lambda$ denote the proportion of investors who withdraw at the interim date. The bank can fulfill the promised interim-date return r_1 only if sufficiently few investors withdraw, $n \leq \frac{1}{r_1}$. When the bank is insolvent, $n > \frac{1}{r_1}$, an investor is paid r_1 only with probability $\frac{1}{nr_1}$ (sequential service constraint).[9]

[7] Since the autarky allocation is feasible, the contract offered by the bank yields at least the same expected utility to investors. Hence, investors always deposit their endowment with the bank at the initial date.

[8] The provided upper bound on r_1 is more restrictive than that in Goldstein and Pauzner (2005). In particular, $\frac{R}{1+\lambda(R-1)} < min\{\frac{1}{\lambda}, R\}$, where the right-hand side is the bound in Goldstein and Pauzner (2005). In more subgames than considered by Goldstein and Pauzner (2005) investors always withdraw from the bank, irrespective of the fundamental. This upper bound also applies to Goldstein and Pauzner (2005), where insurance against investment risk is absent. See also section 3.4.

[9] In Diamond and Dybvig (1983), the sequential service constraint prevents the interim-date payment r_1 from depending on the withdrawal volume n. Likewise, the promised interim-date payment in Goldstein and Pauzner (2005) is independent of withdrawals and any information about the realized fundamental θ. This is identical to what we consider for the first part of the contract.

Second, an investor who does not withdraw at the interim date receives an equal share of the bank's final-date assets. These assets comprise the investment return and the storage of additional interim-date liquidation (safe asset holdings). Although Goldstein and Pauzner (2005) introduce risky investment into the setup of Diamond and Dybvig (1983), they do not consider insurance against such investment risk. Our contract allows a solvent bank at the interim date to liquidate an additional amount of the investment to insure the remaining risk-averse investors against the risk of a low return. Investors who do not withdraw at the interim date must be patient. The bank's investment after withdrawals at the interim date is $1 - nr_1$ and it liquidates a per-capita amount $0 \leq y(n, r_1) \leq \frac{1-nr_1}{1-n}$ for insurance purpose.

Working backwards, we start by finding the bank's liquidation policy $y^*(n, r_1)$ at the end of the interim date for any given withdrawal volume n and promised interim return r_1. Next, we move to the withdrawal decision of investors at the beginning of the interim date. Using their private information, investors update their belief about the fundamental and the proportion of withdrawing investors. Each investor withdraws if and only if the private signal is sufficiently low, $\theta_i < \theta^*(r_1)$. Finally, we find the optimal interim-date return r_1^* promised by the bank at the initial date. Table 1 summarizes.

Initial date	**Interim date**	**Final date**
1. Banks offer demand-deposit contract.	1. Private information about investment.	1. Investment matures.
2. Investors deposit their endowment.	2. Investors may withdraw.	2. Remaining investors withdraw.
	3. Bank liquidates investment to hold safe assets.	3. Late Consumption.
	4. Early Consumption.	

Table 1: Timeline.

Similar to Goldstein and Pauzner (2005), we make assumptions to establish dominance regions in each withdrawal subgame. When investors are certain that the fundamental is in any of these regions, they act without regard to the strategies or information of other investors (Carlsson and van Damme (1993), Morris and Shin (2003)). That is, for high values of the fundamental, the liquidation value increases from 1 to R and the investment always succeeds. That is, there exists a $\overline{\theta} \in (0,1)$ such that $p(\theta) = 1$ for all $\theta > \overline{\theta}$ and the liquidation value is R.[10] Except for this upper dominance region, the probability of the good state strictly increases in the fundamental and the liquidation value is 1. We assume $\bar{\theta} < 1-2\epsilon$, so this bound can be arbitrarily close to 1 as $\epsilon \to 0$. Subsequently, we consider this limit and the upper dominance region shrinks to a point. The improvement in the liquidation value is crucial for establishing the upper dominance region $(\bar{\theta}, 1]$. If the fundamental is in the upper dominance region, withdrawing does not pose a negative externality on those who do not withdraw. Each unit liquidated at the interim date is worth R but only $r_1 < R$ is paid out.[11] The upper dominance region exists in all subgames by assumption.

To show that a lower dominance region exists, we assume that the lowest possible fundamental leads to certain default, $p(0) = 0$. This natural assumption is sufficient for the existence of a lower dominance region in all subgames.[12]

3 Insurance against investment risk

We start by analyzing the optimal insurance against the investment risk. If the bank is insolvent, $1 - nr_1 \leq 0$, each investor receives r_1 with probability $\frac{1}{nr_1}$, as implied by the sequential service constraint. No resources remain for non-withdrawing investors and insurance is absent, $y^*(n, r_1) = 0$. This result is unchanged from Goldstein and Pauzner (2005).

[10]Unlike the lower dominance region, this upper dominance region is assumed and is independent of r_1.

[11]If the fundamental is in the upper dominance region, the incentive of an investor to withdraw actually *decreases* in the proportion of withdrawing investors. Each investor who withdraws generates an extra amount of resources $R - r_1$, which can be distributed to non-withdrawing investors.

[12]We show that the lower dominance region has an endogenously generated non-zero bound which depends on the interim return r_1, as we explain in section 3.4.

In contrast, if the bank is solvent, the amount $1 - nr_1 > 0$ of investment is available for non-withdrawing patient investors. Departing from Goldstein and Pauzner (2005), the bank optimally chooses how much of this amount to liquidate, while keeping the remainder invested. Let $y(n, r_1)$ denote the additional liquidation per capita. Optimal insurance against investment risk maximizes the interim-date expected utility of non-withdrawing patient investors by shifting resources from the good state to the bad state. Hence, patient investors consume $R[\frac{1-nr_1}{1-n}] - (R-1)y$ in the good state and y in the bad state.

What information is available to the bank when determining the optimal insurance of the remaining patient investors against investment risk? We assume that the bank observes the realization of the fundamental at the interim date. This assumption might seem innocuous, given that we consider vanishing noise. It is innocuous in the following sense: the unique equilibrium under this assumption is still an equilibrium when the bank receives a private signal, and forms a posterior on θ, given the signal and the realized n.[13] Because the bank's information is more precise, we base the optimization problem on its information. Therefore, the bank solves the following optimization problem at the end of the interim date:

$$y^* \equiv \arg\max_y p(\theta)\, u\left(R[\frac{1-nr_1}{1-n}] - (R-1)y\right) + (1-p(\theta))\, u\,(y) \quad \text{s.t.} \quad 0 \leq y \leq \frac{1-nr_1}{1-n}. \quad (3)$$

All investment is liquidated, $y^* = \frac{1-nr_1}{1-n}$, if the realized fundamental is sufficiently low, $p(\theta)R \leq 1$. If the fundamental is high enough, $p(\theta)R > 1$, it is optimal to keep some of the resources invested, $y^* < \frac{1-nr_1}{1-n}$. Provided that marginal utility at zero is sufficiently high, which we assume throughout this paper, some insurance is always optimal, $y^* > 0$.[14] This interior solution is determined by the first order condition:

[13]Each patient investor bases his withdrawal decision at the interim date on his private information θ_i of the fundamental. This gives rise to the aggregate number of withdrawals n observed by the bank. The bank uses the observed n and its signal, to form a posterior about the fundamental θ. Therefore, the insurance would be based on a posterior, but for small ϵs that would be very close to what we have here.

[14]Elamin (2013) provides a more detailed analysis of optimal insurance when investors do not receive private information about investment risk.

$$\frac{u'(y^*)}{u'\left[\frac{1-nr_1}{1-n}R - y^*\left(R-1\right)\right]} = \frac{p(R-1)}{1-p} \tag{4}$$

We characterized the optimal insurance against investment risk in terms of n and θ. The number of early withdrawals n is an equilibrium object that depends on the realized fundamental θ through the investors' equilibrium strategies. Nevertheless, we now solve for the optimal expected utility in three auxiliary problems, where this dependence between n and θ is broken. These will be useful in establishing subsequent results on existence of lower dominance region and the one-sided strategic complementarity property. In section 3.1, we fix the fundamental at an arbitrary level $p(\theta)$, and vary n. In section 3.2, we fix both n and $p(\theta)$ and vary r_1. And in section 3.3, we fix n at an arbitrary level where the bank is solvent and vary the fundamental.

3.1 Interim-date expected utility and proportion of withdrawals

In this subsection, we fix the information of the bank at some level θ, so $p(\theta)$ is constant. We derive the optimal expected utility of a non-withdrawing patient investor as a function of n. As n increases, consumption decreases in both the good state and the bad state. The logic of this result is explained in Elamin (2013): an increase in n decreases the resources available for consumption by a patient investor. As a result, it will be optimal to decrease both consumption levels.

Lemma 1 *Fix $\theta < \overline{\theta}$. If the bank is solvent, $\lambda \leq n \leq \frac{1}{r_1}$, both the consumption in the good state and the bad state decrease in n. Therefore, the expected utility decreases in n.*

Proof: See Appendix A.1. □

3.2 Interim-date expected utility and interim-date return

We now fix $\lambda \leq n \leq \frac{1}{r_1}$ (solvent bank) and θ. We show that the optimal expected utility of a non-withdrawing investor decreases in r_1, the interim-date return that insures investors at the initial date against idiosyncratic liquidity risk. Just as in subsection 3.1, higher values of r_1 leave less resources for remaining patient investors. Therefore, the optimal expected utility decreases in r_1.

Lemma 2 *For a solvent bank, $\lambda \leq n \leq \frac{1}{r_1}$, fix both n and θ. The consumption level in both the good and the bad state decrease in r_1 and so does the optimal expected utility of non-withdrawing patient investor.*

Proof: See Appendix A.2. □

3.3 Interim-date expected utility and the fundamental

In this subsection, we fix $\lambda \leq n \leq \frac{1}{r_1}$. We show that the optimal expected utility is increasing and continuous in the fundamental θ if the fundamental takes an intermediate value. In the higher and lower dominance regions, no extra liquidation or full liquidation occurs, respectively. In both these cases, the optimal expected utility does not depend on the fundamental. Let p^{-1} denote the inverse of $p(\theta)$ for all $0 \leq \theta \leq \overline{\theta}$. Over this domain, $p(.)$ is strictly increasing and thus invertible.

Lemma 3 *If the bank is solvent, $\lambda \leq n \leq \frac{1}{r_1}$, fix n independent of the fundamental θ. If the fundamental takes an intermediate value, $p^{-1}\left(\frac{1}{R}\right) < \theta < \overline{\theta}$, the interim-date optimal expected utility of a non-withdrawing patient investor is continuous and strictly increasing in θ. Otherwise, the optimal expected utility is independent of the fundamental.*

Proof: See Appendix A.3. □

3.4 Lower Dominance Region

The upper dominance region exists and is independent of r_1 by assumption. By contrast, the lower dominance region naturally arises from the withdrawal subgame between investors and thus depends on r_1. Specifically, the range of the lower dominance region increases in the interim return r_1. Every r_1 in the domain $1 < r_1 < \frac{R}{1+\lambda(R-1)}$ defines a withdrawal subgame. The global games techniques we use require the existence of a nontrivial upper dominance region and lower dominance region in each subgame defined by r_1. In this subsection, we focus on the sufficiency of the assumption $p(0) = 0$ for the existence of a lower dominance regions across the entire domain of r_1, and on the necessity of the upper bound $\frac{R}{1-\lambda(R-1)}$.

If $r_1 \geq \frac{R}{1+\lambda(R-1)}$, the lower dominance region is the whole domain of θ. First, when $r_1 = \frac{R}{1-\lambda(R-1)}$, we have $u(\frac{1-\lambda r_1}{1-\lambda}R) = u(r_1)$. Thus, $\forall p(\theta) < 1$ and all $n \geq \lambda$, each patient investor has a dominant strategy to withdraw from the bank, so $n = 1$ holds independent of θ. Second, when $r_1 > \frac{R}{1-\lambda(R-1)}$, we even have $u(\frac{1-\lambda r_1}{1-\lambda}R) < u(r_1)$. Thus, $\forall p(\theta) \leq 1$ and all $n \geq \lambda$, each patient investor has a dominant strategy to withdraw, and again $n = 1$ independent of θ. Both cases yield $\underline{\theta} = 1$. Hence, if we assume an upper dominance region, there would not exist intermediate region, since these dominance regions collide at $\overline{\theta}$.

Next, we consider the interesting case of $r_1 < \frac{R}{1+\lambda(R-1)}$, so $u(\frac{1-\lambda r_1}{1-\lambda}R) > u(r_1)$. When $p(\theta) = 1$ and $n = \lambda$, each patient investor prefers to wait. Thus, we can show that a lower dominance region exists with some bound $\underline{\theta} < 1$. The interval $1 - \underline{\theta} > 0$ allows us to assume an upper dominance region on a portion of it, leaving space for an intermediate region.[15]

Lemma 4 *If $p(0) = 0$ then there exists a non-trivial lower dominance region in every subgame defined by $r_1 \in \left(1, \frac{R}{1+\lambda(R-1)}\right)$.*

Proof: See Appendix A.4. □

[15]The exactly same logic carries through in the setup Goldstein and Pauzner (2005). Hence, the bound on r_1 we provide should replace $min\{\frac{1}{\lambda}, R\}$ in their paper. Our bound is lower, $\frac{R}{1+\lambda(R-1)} < min\{\frac{1}{\lambda}, R\}$.

We next show that $\underline{\theta}(r_1)$ increases in r_1.

Lemma 5 *The bound that defines the lower dominance region, $\underline{\theta}(r_1)$, increases in r_1. It converges to* 1 *as r_1 goes to the upper bound, $\frac{R}{1+\lambda(R-1)}$.*

Proof: See Appendix A.5. □

4 Withdrawal behavior of investors

Having solved for the optimal insurance against investment risk, we now analyze the withdrawal behavior of investors. At the interim date, each feasible interim return r_1 defines a subgame in which investors simultaneously decide whether to withdraw.

An investor's decision is based on their type (patient or impatient) and their signal of the fundamental. An impatient investor always withdraws at the interim date. The incentive of a patient investor to withdraw depends on both the realized fundamental θ and the proportion of withdrawals n. The signal provides a patient investor with relevant information about the proportion of withdrawing patient investors, and the possible level of fundamental. Following the notation of Goldstein and Pauzner (2005), let $v(\theta, n)$ denote the utility differential between not withdrawing and withdrawing at the interim date. If the fundamental is in the upper dominance region, $\theta \geq \overline{\theta}$, the utility differential is $u(\frac{R-nr_1}{1-n}) - u(r_1)$ if $n < 1$ and $u(R) - u(r_1)$ if $n = 1$. If the fundamental is not in the upper dominance region, the utility differential is:[16]

$$v(\theta, n; r_1) = \begin{cases} (1-p(\theta))u(y^*) + p(\theta)u((\frac{1-nr_1}{1-n} - y^*)R + y^*) - u(r_1) & if \lambda \leq n \leq \frac{1}{r_1} \\ -\frac{1}{nr_1}u(r_1) & if \frac{1}{r_1} \leq n \leq 1 \end{cases} \tag{5}$$

[16] If insurance against investment risk was absent, $y^* = 0$, the utility differential collapses to the expression in Goldstein and Pauzner (2005), which is $p(\theta)u(\frac{1-nr_1}{1-n}R) - u(r_1)$ for a solvent bank.

To establish one-sided strategic complementarity as in Goldstein and Pauzner (2005), we show that the utility differential decreases in the proportion of early withdrawals when the bank is solvent ($\lambda \leq n < \frac{1}{r_1}$). This is already shown in Lemma 1. At $n = \frac{1}{r_1}$, the differential becomes $-u(r_1) < 0$. If the bank is insolvent $n \geq \frac{1}{r_1}$, then there is no change arising from insurance against investment risk. The utility differential increases in early withdrawals n but remains negative since $-\frac{u(r_1)}{nr_1} < 0$ even for $n = 1$, as in Goldstein and Pauzner (2005). Figure 4 shows how the utility differential is affected by insurance against investment risk.

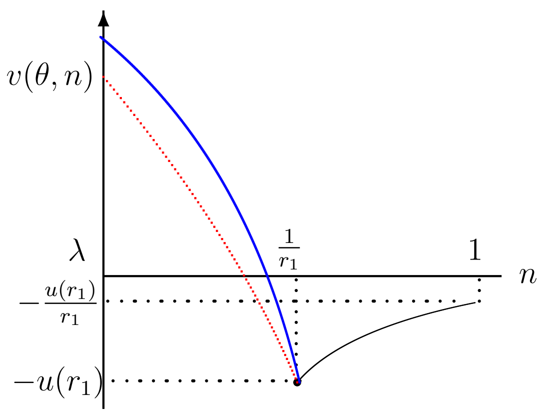


Figure 3: The utility differential with insurance against investment risk (blue, solid) and without (red, dotted). For a solvent bank, $n \in [\lambda, \frac{1}{r_1}]$, insurance against investment risk increases the interim-date expected utility of a patient investor, provided that $\theta < \overline{\theta}$. In contrast, for an insolvent bank, $n \geq \frac{1}{r_1}$, there is no role for such insurance.

Proposition 1, the analogue of Theorem 1 in Goldstein and Pauzner (2005), states that the withdrawal subgame determined by an interim return r_1 has a unique equilibrium, when noise is presents. To prevent possible confusion, we highlight the distinction between an equilibrium in the subgame, and an equilibrium of the overall game. An equilibrium of the overall game comprises the optimal r_1 chosen by the bank at the initial date and, in every subgame determined by every possible r_1, the optimal withdrawal behavior of investors at the interim date, and optimal insurance in each subgame given the realized proportions of early withdrawals and the realized fundamental. Having clarified that, we will abuse the concepts from now on and refer to an equilibrium in the subgame as equilibrium.

Proposition 1 ***Unique equilibrium in each withdrawal subgame.*** *Given subsequent optimal insurance against investment risk, in every subgame determined by* $r_1 \in \left(1, \frac{R}{1+\lambda(R-1)}\right)$*, there exists an upper bound on private noise* $\bar{\epsilon}(r_1) > 0$ *such that a unique Bayesian equilibrium exists for every* $0 < \epsilon < \bar{\epsilon}$*. This equilibrium is characterized by a threshold* $\theta_\epsilon(r_1)$*, below which a bank run occurs,* $\theta < \theta_\epsilon(r_1)$*.*[17]

Proof: See Appendix A.6. □

The proof provided here simplifies, explains and extends the proof in Goldstein and Pauzner (2005). We identify some issues and caveats that arise, and shows that these do not affect the main result. The proof works in three steps. First, we show there is a unique *symmetric* threshold candidate for an equilibrium. Therefore, there exists only one θ that all patient investors can use as a threshold, where each investor is indifferent between running and not when receiving this threshold as a signal. That is, the net incentive from withdrawing, the integral $\Delta^{r_1}(\theta', n(., \theta'))$, is zero at exactly one θ'.[18] Second, we show that the candidate threshold of step 1 is actually an equilibrium. Assume that all investors except investor i play threshold strategies at the θ' computed in step 1, $i's$ unique best response is to use a threshold strategy at that θ'. Third, we show that any equilibrium has to be the threshold equilibrium identified.

4.1 Continuity of withdrawal threshold in noise

Uniqueness of the threshold $\theta^*(\epsilon)$ at every positive $\epsilon > 0$ does not imply convergence to a unique limit as private noise vanishes. Indeed, it may converge to two or many limits. In this subsection, we prove that it converges to a unique limit. The same result holds in Goldstein and Pauzner (2005), and the proofs of that are very similar to those in our environment.

[17] The threshold $\theta_\epsilon(r_1)$ depends on both ϵ and r_1, while the upper bound on private noise $\bar{\epsilon}(r_1)$ depends on the interim return r_1. As r_1 approaches 1, $\bar{\epsilon}(r_1)$ approaches zero.

[18] This rules out the existence of two symmetric threshold equilibria. It does not rule out that some mass of investors may use one threshold and another mass uses another threshold.

Note that $\theta^*(\epsilon)$ is defined for every $\epsilon > 0$ by Proposition 1. We show that it is continuous for every $\epsilon > 0$. Next, we show that, as $\epsilon \to 0$, $\theta^*(\epsilon)$ converges to a unique limit that we define as $\theta^*(0)$.

Proposition 2 ***Continuity.*** *The withdrawal threshold $\theta^*(\epsilon)$ is continuous in ϵ for every $\epsilon > 0$.*

Proof: See Appendix A.7. □

We show next that there is a unique limit which $\theta^*(\epsilon)$ converges to with every sequence going to zero. This limit will be defined as $\theta^*(0)$.

Proposition 3 *There exists a unique limit of $\theta^*(\epsilon_n)$ for every sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$.*

Proof: See Appendix A.8. □

We have already shown that $\theta^*(\epsilon)$ converges when $\epsilon \to 0$. By dominated convergence theorem, the following explicit expression of $\theta^*(0)$ obtains and we call it θ^* from now on.

Proposition 4 *As private noise vanishes, the unique limit of the withdrawal threshold θ^* is defined by:*

$$p(\theta^*) = \frac{\frac{u(r_1)}{r_1}\left[1 - \lambda r_1 + ln(r_1)\right] - \int_{\lambda}^{\frac{1}{r_1}} u(y^*)dn}{\int_{\lambda}^{\frac{1}{r_1}} u(\left[\frac{1-nr_1}{1-n} - y^*\right] R + y^*)dn - \int_{\lambda}^{\frac{1}{r_1}} u(y^*)dn}. \tag{6}$$

Proof: See Appendix A.9. □

4.2 Liquidity risk sharing and bank fragility

As in Goldstein and Pauzner (2005), increased liquidity risk sharing has a destabilizing effect. A higher interim return r_1 allows for more risk sharing, but comes at the cost of an increased probability of a bank run. We show that the threshold increases in the interim return r_1.

Goldstein and Pauzner (2005)'s proof of this monotonicity theorem relies on operating on the indifference equation at strictly positive noise, and then showing that the theorem holds when ϵ is small. Therefore, they prove this result for small levels of private noise. When the threshold is continuous with respect to noise, as we showed in the previous section, operating as in Goldstein and Pauzner (2005) is equivalent to working directly with the limit indifference condition (which we do here). We expand on this equivalence in Appendix B.

Proposition 5 ***Liquidity risk sharing and bank fragility.*** *The withdrawal threshold $\theta^*(r_1)$ increases in the interim return r_1.*

Proof: See Appendix A.10. □

5 Comparison with Goldstein and Pauzner (2005)

In this section, we describe the effect of insurance against investment risk on the results of Goldstein and Pauzner (2005). In section 5.1, we show that such insurance lowers the probability of a run, and therefore that the costs of liquidity provision are overestimated in Goldstein and Pauzner (2005). Risk-averse investors have less incentive to withdraw at the interim date when insured against investment risk.

In section 5.2, we describe the effect of this insurance on ex-ante expected utility. Insurance affects three contingencies that comprise the ex-ante expected utility: expected utility when the bank fails ($\frac{u(r_1)}{r_1}$), expected utility when the bank does not fail, but the investor is impatient ($\lambda u(r_1)$), and when the investor is patient. Because insurance lowers the run probability, less weight is put on the first contingency than the second, yielding a lower expected utility in the combination of these two contingencies with insurance.[19]

[19]Note that $\frac{u(r_1)}{r_1} > \lambda u(r_1)$.

$\theta^* \frac{u(r_1)}{r_1} + (1-\theta^*)\lambda u(r_1) < \theta^*_{GP} \frac{u(r_1)}{r_1} + (1-\theta^*_{GP})\lambda u(r_1)$. The third contingency, yields higher expected utility because of the combined effect of the decrease in the run probability, and the increase in expected utility from insurance. The increase in this third contingency dominates the decrease in the convex combination of the first two, and we get that the ex-ante expected utility of an investor increases in every subgame.

5.1 Insurance lowers the withdrawal threshold in each subgame

We start by identifying the subgames in which the bank provides so much liquidity (very high r_1) that a bank run always occurs ($\theta^* = 1$).[20] First, there is no guarantee that the withdrawal threshold θ^* computed in Proposition 4 is below 1 for all r_1. Indeed, we showed in Lemma 5 that $\underline{\theta}(r_1)$ converges to 1 as r_1 goes to the upper bound, $\frac{R}{1+\lambda(R-1)}$. Second, when r_1 is close to 1, the threshold satisfies $\theta^* < 1$. By Proposition 5, $\theta^*(r_1)$ is increasing in r_1. At $r_1 = \frac{R}{1+\lambda(R-1)}$, we already showed that $\underline{\theta} = 1$, therefore $\theta^* = 1$. Therefore, by the intermediate value theorem, we have that there exists an $\underline{r_1}$ such that $\theta^*(r_1) = 1 \; \forall r_1 \geq \underline{r_1}$.

The next proposition shows that in every subgame defined by $r_1 < \underline{r_1}$, the threshold we compute with insurance against investment risk θ^* is lower than θ^*_{GP}, the threshold in Goldstein and Pauzner (2005). Both thresholds are unity in subgames with $r_1 > \underline{r_1}$. Note that θ^*_{GP} hits 1 before θ^* does. Therefore, some subgames would have $\theta^*_{GP} = 1$, while $\theta^* < 1$. This proposition shows that insurance against investment risk, or holding safe assets, decreases the probability of a bank run, and therefore decreases financial fragility.

Proposition 6 ***Lower withdrawal threshold than Goldstein and Pauzner (2005).*** *Consider subgames in which the withdrawal threshold with insurance against investment risk is below one, $\theta^* < 1$ for all $r_1 < \underline{r_1}$. Then, the withdrawal threshold with insurance is lower*

[20] It is important to note that the ex-ante optimal interim return r_1^* should be different than these.

than the withdrawal threshold without insurance:

$$\theta^* < \theta^*_{GP}. \tag{7}$$

Proof: See Appendix A.11. □

For convenience, we include the formulas for these thresholds here:

$$p(\theta^*) = \frac{\frac{u(r_1)}{r_1}(1 - \lambda r_1 + ln(r_1)) - \int_\lambda^{\frac{1}{r_1}} u(y^*)dn}{\int_\lambda^{\frac{1}{r_1}} u((\frac{1-nr_1}{1-n} - y^*)R + y^*)dn - \int_\lambda^{\frac{1}{r_1}} u(y^*)dn}, \quad p(\theta^*_{GP}) = \frac{\frac{u(r_1)}{r_1}(1 - \lambda r_1 + ln(r_1))}{\int_{n=\lambda}^{\frac{1}{r_1}} u(\frac{1-nr_1}{1-n}R)dn}.$$

The intuition for this result is clear. Fix a subgame given by r_1 and consider each proportion of early withdrawals n. The total amount of resources available to patient investors is the same, irrespective of insurance against investment risk. However, the given amount of resources are used more efficiently from the perspective of a patient investor, when insurance is present. That is, the expected utility with insurance is strictly higher than the expected insurance without such insurance (GP). Note that the expected utility of withdrawing is fixed at $\frac{u(r_1)}{r_1}(1-\lambda r_1+ln(r_1))$ in both cases, but the expected utility of a patient investor who does not withdraw decreases in the threshold θ^*. Therefore, the marginal patient investor who is indifferent between withdrawing and waiting must have received a lower private signal under insurance than without insurance.

5.2 Insurance Raises the Ex-ante Expected Utility

The optimal insurance against investment risk and the optimal threshold equilibrium in each subgame determine an ex-ante expected utility for any given interim rate (r_1). In this subsection, we write the expected utility as a function of r_1, after the upper dominance region shrinks to a point ($\overline{\theta} \to 1$), and private noise vanishes ($\epsilon \to 0$). In each subgame determined by the interim rate r_1, the ex-ante expected utility with insurance against investment risk is higher than without insurance.

We start with the case of no insurance. The ex-ante expected utility in Goldstein and Pauzner (2005) for any given r_1 is:

$$\begin{aligned}\lim_{\bar{\theta}\to 1,\epsilon\to 0} EU_{GP}(r_1) &= \int_0^{\theta^*_{GP}(r_1)} \frac{1}{r_1}u(r_1)d\theta + \int_{\theta^*_{GP}(r_1)}^1 \lambda u(r_1) + (1-\lambda)p(\theta)u(\frac{1-\lambda r_1}{1-\lambda}R)d\theta \\ &= \frac{u(r_1)}{r_1}\theta^*_{GP} + \lambda(1-\theta^*_{GP})u(r_1) + (1-\lambda)u(\frac{1-\lambda r_1}{1-\lambda}R)\int_{\theta^*_{GP}}^1 p(\theta)d\theta \quad (8)\end{aligned}$$

With insurance against investment risk, the ex-ante expected utility for a given r_1 is:

$$\lim_{\bar{\theta}\to 1,\epsilon\to 0} EU(r_1) = \frac{u(r_1)}{r_1}\theta^* + \lambda(1-\theta^*)u(r_1) + (1-\lambda)\int_{\theta^*}^1 p(\theta)u(\frac{1-\lambda r_1}{1-\lambda}R-(R-1)y^*) + (1-p(\theta))u(y^*)d\theta, \quad (9)$$

where $y^* = y^*(r_1, \lambda, p(\theta))$ and therefore y^* depends on the realized θ whenever $\theta \geq \theta^*$. This complicates the integrated expression, because unlike in Equation 8, the utility itself depends on the realized fundamental through the insurance.

The next proposition shows that insurance against investment risk raises the initial-date expected utility of investors for each interim rate r_1.

Proposition 7 *At any fixed interim rate* $\forall\ r_1 < \underline{r_1}$, *the expected utility with insurance against investment risk is greater than that without such insurance:*

$$EU(r_1) > EU_{GP}(r_1). \quad (10)$$

Proof: See Appendix A.12. $\square$

6 Conclusion

We study the optimal demand-deposit contract subject to sequential service in the bank-run model of Goldstein and Pauzner (2005). We show that competitive banks hold safe assets in order to insure risk-averse investors against investment risk. As a result, fewer investors withdraw prematurely, which reduces the probability of a bank run and increases investor welfare. Furthermore, safe asset holdings may also allow banks to provide more liquidity.

A Appendix: Proofs

A.1 Proof of Lemma 1

Proof: This is an application of Lemma 3 in Elamin (2013).

Case 1: We start by considering the corner solutions. If $p(\theta)R \leq 1$, then full liquidation occurs, so $y = \frac{1-nr_1}{1-n}$ and $y' = \frac{1-n'r_1}{1-n'}$. Thus $n' > n$ implies $y' < y$. Since the respective expected utility in case of full liquidation equals $u(y)$ and $u(y')$, it decreases in n.

Case 2: If $p(\theta)R > 1$ but $\theta < \overline{\theta}$, we get an interior solution. Let $n' > n$, $y' > 0$ and $y > 0$ be the optimal insurance that solve the respective optimization problems in 3. The first order condition of the problems give:

$$\frac{u'(y)}{u'\left[\frac{1-nr_1}{1-n}R - y\,(R-1)\right]} = \frac{p(R-1)}{1-p} = \frac{u'(y')}{u'\left[\frac{1-n'r_1}{1-n'}R - y'\,(R-1)\right]} \tag{11}$$

First, assume $y' = y$. Thus $\frac{1-nr_1}{1-n} = \frac{1-n'r_1}{1-n'}$ by strict concavity, resulting in $n = n'$. Contradiction.

Second, assume $y' > y$. Then, by strict concavity, we have $u'(y') < u'(y)$. Thus, for equation 11 to hold, we require:$u'\left[\frac{1-n'r_1}{1-n'}R - y'\,(R-1)\right] < u'\left[\frac{1-nr_1}{1-n}R - y\,(R-1)\right]$. Therefore, again by strict concavity, $\frac{1-n'r_1}{1-n'}R - y'\,(R-1) > \frac{1-nr_1}{1-n}R - y\,(R-1)$ and $(y-y')\frac{(R-1)}{R} > \frac{1-nr_1}{1-n} - \frac{1-n'r_1}{1-n'}$. Since $y - y' < 0$, $\frac{1-n'r_1}{1-n'} > \frac{1-nr_1}{1-n}$, which implies $n' < n$. Contradiction. This shows that $y' < y$ and the consumption level in the bad state decreases in n.

$y' < y$ implies $u'(y') > u'(y)$. Equation 11 implies that: $u'\left[\frac{1-n'r_1}{1-n'}R - y'\,(R-1)\right] > u'\left[\frac{1-nr_1}{1-n}R - y\,(R-1)\right]$. Therefore, $\frac{1-n'r_1}{1-n'}R - y'\,(R-1) < \frac{1-nr_1}{1-n}R - y\,(R-1)$ and the consumption level in the good state also decreases in n. Since the consumption in both states decrease in n, so does the expected utility of a non-withdrawing investor at the interim date.

□

A.2 Proof of Lemma 2

Proof: This is another application of Lemma 3 in Elamin (2013). Cases 1 and 2 deal with corner solutions.

Case 1: If $\theta \geq \overline{\theta}$, then $p(\theta) = 1$ and the investment is riskless. All resources remain invested and there is no need for insurance, $y^* = 0$. The expected utility at the interim date is $u(\frac{R-nr_1}{1-n})$, which decreases in r_1.

Case 2: If $p(\theta)R \leq 1$, the expected return on investment is below the liquidation value, so full liquidation occurs. The expected utility at the interim date becomes $u(\frac{1-nr_1}{1-n})$, which also decreases in r_1.

Case 3: We focus next on an interior solution, which occurs if $p(\theta)R > 1$ but $\theta < \overline{\theta}$. Let $r_1' > r_1$, and let y' and y solve the respective maximization problems. The first-order conditions of these problems give:

$$\frac{u'(y)}{u'\left[\frac{1-nr_1}{1-n}R - y\,(R-1)\right]} = \frac{p(R-1)}{1-p} = \frac{u'(y')}{u'\left[\frac{1-nr_1'}{1-n}R - y'\,(R-1)\right]} \tag{12}$$

First, assume $y' = y$. This implies $\frac{1-nr_1}{1-n} = \frac{1-nr_1'}{1-n}$ by strict inequality and thus $r_1 = r_1'$. Contradiction.

Second, assume $y' > y$. Then, by strict concavity: $u'(y') < u'(y)$. Steps very similar to Lemma 1 lead to the required contradiction.

Therefore, the consumption levels in both states decrease in r_1, and so does the interim-date expected utility of the non-withdrawing patient investor. $\square$

A.3 Proof of Lemma 3

Proof: **Case 1:** Corner solutions. If $p(\theta)R \leq 1$, full liquidation occurs and the optimal interim-date expected utility becomes $u(\frac{1-nr_1}{1-n})$. In this case, the expected utility does not depend on the fundamental. Likewise, if $\theta \geq \overline{\theta}$, there is no need for insurance, $y^* = 0$. The expected utility at the interim date is $u(\frac{R-nr_1}{1-n})$, which is independent of the fundamental.

Case 2: We now focus on the (more relevant) range of fundamentals that yield an interior solution, $p(\theta)R > 1$ but $\theta < \overline{\theta}$. The continuity of y^* in θ follows from continuity of marginal utility, $u'(.)$, and the continuity of $p(.)$. Thus, the continuity of the optimal expected utility in θ follows.

To prove that the optimal expected utility increases in the fundamental, let $p' > p$ as a short hand for $\theta' > \theta$. Moreover, let the optimal insurance level against investment risk be y' and y, corresponding to p' and p. By optimality of y' and strict concavity of utility function:

$$p'u\left[\frac{1-nr_1}{1-n}R - y'(R-1)\right] + (1-p')u(y') > p'u\left[\frac{1-nr_1}{1-n}R - y(R-1)\right] + (1-p')u(y).$$

We also have $y < \frac{1-nr_1}{1-n}$ from the interior solution of this case (see also the constraint of Problem 3). Therefore, $u\left[\frac{1-nr_1}{1-n}R - y\left(R-1\right)\right] > u\left(y\right)$. Since $p' > p$, we have:

$$p'u\left[\frac{1-nr_1}{1-n}R - y(R-1)\right] + (1-p')u(y) > pu\left[\frac{1-nr_1}{1-n}R - y(R-1)\right] + (1-p)u(y).$$

Combining both of these inequalities proves strict monotonicity. $\square$

A.4 Proof of Lemma 4

Proof: Consider a fixed $r_1 \in (1, \frac{R}{1+\lambda(R-1)})$. If the bank is insolvent, withdrawing from the bank at the interim date yields $\frac{u(r_1)}{nr_1}$ in expected utility terms, while waiting yields $u(0) = 0$. Therefore, a patient investor always prefers to withdraw. When the bank is solvent, to use the intermediate value theorem, we show that the expected utility differential is negative at $\theta = 0$ and positive on the upper dominance region. Fix $n = \lambda$ for now. At $\theta = 0$, the investment fails for sure by assumption of $p(0) = 0$. Therefore, all investment is liquidated, $y^* = \frac{1-\lambda r_1}{1-\lambda}$. Thus, a non-withdrawing patient investor's expected utility at the interim date is: $u(\frac{1-\lambda r_1}{1-\lambda})$, which is strictly below $u(r_1)$ because $r_1 > 1$, the lower bound on r_1. When $\theta \geq \overline{\theta}$, the investment is guaranteed to succeed, $p(\theta) = 1$. Optimal insurance against investment risk is zero, $y^* = 0$, and all resources remain invested. The interim-date expected utility becomes $u(\frac{1-\lambda r_1}{1-\lambda}R)$, which is strictly above $u(r_1)$, because $r_1 < \frac{R}{1+\lambda(R-1)}$, the upper bound on r_1.

For each r_1, by the continuity of expected utility in θ at $n = \lambda$ (Lemma 3) and the intermediate value theorem, there exists a cut-off $\underline{\theta}(r_1)$ defined by:

$$p(\underline{\theta}(r_1))\, u \left[\frac{1-\lambda r_1}{1-\lambda} R - y^* \left(R-1\right)\right] + (1 - p(\underline{\theta}(r_1)))\, u\left(y^*\right) = u(r_1). \tag{13}$$

Since the expected utility increases in θ when $n = \lambda$ by Lemma 3, we have for any $\theta < \underline{\theta}(r_1)$ that $p(\theta)u\left[\frac{1-\lambda r_1}{1-\lambda}R - y^*(R-1)\right] + (1-p(\theta))u(y^*) < u(r_1)$.

Because the optimal expected utility at the interim date decreases in n at every $\theta < \underline{\theta}$, Lemma 1 shows that for all $n > \lambda$, the patient investor still has a dominant strategy to withdraw at the interim date. This establishes a lower dominance region for every r_1 in the domain and concludes our proof. □

A.5 Proof of Lemma 5

Proof: Let $r_1' > r_1$, y' and y solve the respective maximization problems with the associated bound $\underline{\theta}'$ and $\underline{\theta}$. We have that: $p(\underline{\theta})\, u\left[\frac{1-\lambda r_1}{1-\lambda}R - y\,(R-1)\right] + (1-p(\underline{\theta}))\, u\,(y) = u\,(r_1)$.

When r_1 increases to r_1', the expected utility evaluated at $p(\underline{\theta})$ decreases by Lemma 2. But to raise the expected utility to $u(r_1') > u(r_1)$, Lemma 3 shows that we need to increase $\underline{\theta}$. This establishes strict monotonicity of $\underline{\theta}(r_1)$ in r_1. The discussion at the beginning of the section shows that $\underline{\theta} = 1$ when $r_1 = \frac{R}{1+\lambda(R-1)}$. This concludes our short proof. □

A.6 Proof of Proposition 1

A threshold strategy for investor i is a mapping from possible interim-date returns r_1 to a threshold signal: $\theta_i^{'} : \Re^+ \rightarrow [0,1]$ s.t. i runs when his signal is below the threshold $(\theta_i < \theta_i^{'})$ and does not run when it is above $(\theta_i \geq \theta_i^{'})$. We assume that the strategy is doubly measurable in i and r_1.

The first step assumes we are looking for a **symmetric** threshold equilibrium. By showing a potential candidate at a unique $\theta^{'}$, we rule out the possibility of two symmetric threshold equilibria $\theta_1^{'}(r_1)$ and $\theta_2^{'}(r_1)$. Assume all investors except i use the same threshold $\theta^{'}(r_1)$. By the law of large numbers, $i's$ belief $n(\theta, \theta^{'}(r_1))$ about the proportion of investors who run at the true fundamental θ is degenerate[21] and defined by:

$$n(\theta, \theta^{'}(r_1)) = \begin{cases} 1 & if \theta \leq \theta^{'}(r_1) - \epsilon \\ \lambda + (1-\lambda)(\frac{1}{2} + \frac{\theta^{'}(r_1) - \theta}{2\epsilon}) & if \theta^{'}(r_1) - \epsilon \leq \theta \leq \theta^{'}(r_1) + \epsilon \\ \lambda & if \theta \geq \theta^{'}(r_1) + \epsilon \end{cases}$$

i's utility differential at the interim date becomes:

[21] This assumption is morally correct, but technically wrong. We follow the literature in assuming it. The analysis of continuum of random variables have well known and widely discussed problems, which we abstract from here, and refer the reader to the literature discussing it. See Judd (1985)and Al-Najjar (1995) and the references therein, for identification of the problem and a possible solution.

$$v(\theta,n;r_1)=\begin{cases}(1-p(\theta))u(y^*)+p(\theta)u((\frac{1-nr_1}{1-n}-y^*)R+y^*)-u(r_1) & if \lambda\leq n\leq \frac{1}{r_1}\\ -\frac{1}{nr_1}u(r_1) & if \frac{1}{r_1}\leq n\leq 1\end{cases}$$

Combining these two definitions, let $v(\theta)=v(\theta,n(\theta,\theta^{'}))$. This is i's utility difference at the true fundamental θ, when all investors beside i use a threshold strategy at $\theta^{'}$. i does not observe the true fundamental θ however, only a signal θ_i. His posterior of θ at signal θ_i is uniform over the interval $[\theta_i-\epsilon,\theta_i+\epsilon]$. Therefore, i's incentive at signal θ_i is: $\Delta^{r_1}(\theta_i,n(.,\theta^{'}))=\frac{1}{2\epsilon}\int_{\theta_i-\epsilon}^{\theta_i+\epsilon}v(\theta)d\theta$. We will show that i's integral of net incentives when he receives the threshold $\theta^{'}$ as a signal: $\Delta^{r_1}(\theta^{'},n(.,\theta^{'}))$ intersects zero at exactly one point $\theta^{'}=\theta^*$, is negative below it and positive above it, and therefore provide a unique potential threshold that could serve as a symmetric equilibrium.[22]

To analyze the integral, we first plot the function $v(\theta)$ and focus on when the threshold $\theta^{'}$ is far from the upper dominance region ($\theta^{'}+\epsilon\leq\overline{\theta}$). $v(\theta)$ achieves its minimum value at fundamental $\hat{\theta}\in(\theta^{'}-\epsilon,\theta^{'}+\epsilon)$, where the proportion of investors running is equal to $\frac{1}{r_1}$.[23] On the interval $[\theta^{'}-\epsilon,\hat{\theta}]$, $v(\theta)=-\frac{u(r_1)}{nr_1}$. As θ increases from $\theta^{'}-\epsilon$ to $\hat{\theta}$, n decreases from 1 to $\frac{1}{r_1}$ and $v(\theta)$ decreases to its minimum value $-u(r_1)$. On the interval $[\hat{\theta},\theta^{'}+\epsilon]$, $v(\theta)=(1-p(\theta))u(y^*)+p(\theta)u((\frac{1-nr_1}{1-n}-y^*)R+y^*)-u(r_1)$. As θ increases from $\hat{\theta}$ to $\theta^{'}+\epsilon$, n decreases from $\frac{1}{r_1}$ to λ and by Lemma 3, $v(\theta)$ increases in θ. At the upper dominance region, $v(\theta)$ is discontinuous because the liquidation value discontinuously increases from 1 to R at $\overline{\theta}$. Because of continuity of $p()$ and because $p(\overline{\theta})=1$, $v(\theta)$ approaches $u(\frac{\mathbf{1}-\lambda r_1}{1-\lambda}R)-u(r_1)$ as θ approaches $\overline{\theta}$. But $v(\theta)=u(\frac{\mathbf{R}-\lambda r_1}{1-\lambda}R)-u(r_1)\ \forall\theta\geq\overline{\theta}$. The following picture codes what has been discussed here.[24]

We now analyze what happens as the threshold $\theta^{'}$ increases, but is still below the upper dominance region. Evaluating $n(\hat{\theta},\theta^{'})=\frac{1}{r_1}$ leads to $\theta^{'}-\hat{\theta}=(\frac{1}{r_1}-\frac{1+\lambda}{2})\frac{2\epsilon}{1-\lambda}$. Therefore, the

[22] This only establishes a possible candidate. To show the candidate is an equilibrium, we have to show monotonicity of $\Delta^{r_1}(\theta_i,n(.,\theta^{'}))$ as a function of the signal θ_i.

[23] For $\lambda<n(\theta,\theta^{'}(r_1))<1$, $n(.,\theta^{'}(r_1))$ has an inverse function since it is linearly decreasing over that segment. Therefore, let $\hat{\theta}=\theta:n(\theta,\theta^{'}(r_1))=\frac{1}{r_1}$. By the intermediate value theorem $\hat{\theta}$ exists, and $\theta^{'}-\epsilon<\hat{\theta}<\theta^{'}+\epsilon$.

[24] Although by assumption the upper dominance region is greater than 2ϵ, for expositional purposes, it is drawn as smaller than the interval $[\theta^{'}-\epsilon,\theta^{'}+\epsilon]$.

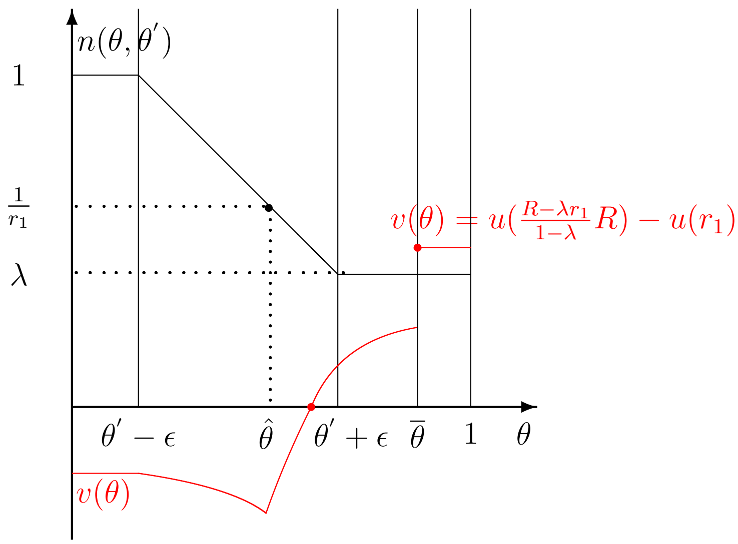


Figure 4: The utility differential when the threshold is away from the upper dominance region ($\theta' + \epsilon \leq \overline{\theta}$).

interval $[\theta' - \epsilon, \hat{\theta}]$ stays constant in length, does not shrink or expand, as the threshold θ' changes. On the interval $[\theta' - \epsilon, \hat{\theta}]$, $v(\theta) = -\frac{u(r_1)}{nr_1}$, which is independent of θ'. Therefore, the function $v(\theta)$ and the integral $\int_{\theta'-\epsilon}^{\hat{\theta}} v(\theta)d\theta$ are constant when the threshold θ' changes. The change happens on the interval $[\hat{\theta}, \theta' + \epsilon]$. As the threshold θ' is raised, the function $v(\theta)$ is evaluated at a translation to the right of the interval $[\hat{\theta}, \theta' + \epsilon]$, therefore again by Lemma 3, the function $v(\theta)$ increases. This information is coded in the following picture, which plots $v(\theta)$ at two different thresholds. The function $v(\theta)$ on interval $[\hat{\theta}, \theta' + \epsilon]$, is drawn in blue at the higher threshold. It is clear that $\int_{\hat{\theta}}^{\theta'+\epsilon} v(\theta)d\theta$ increases in the threshold θ', when this threshold is sufficiently far from the upper dominance region.

We now focus on the case when the threshold is closer to the upper dominance region ($\theta' + \epsilon > \overline{\theta}$). An investor, sure that the fundamental θ is in the upper dominance region, has a dominant strategy not to run. Investors with signals $\theta_i \geq \overline{\theta} + \epsilon$, do not run. Therefore, $\theta' \leq \overline{\theta} + \epsilon$. If not, then when true fundamental is at θ', investors with signals $\overline{\theta} + \epsilon \leq \theta_i \leq \theta'$

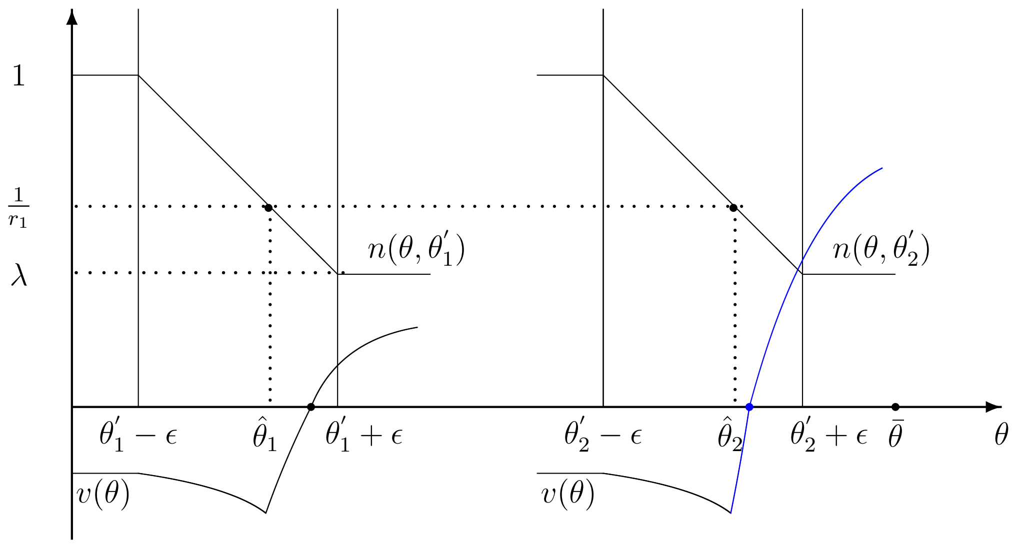


Figure 5: The utility differential with two thresholds ($\hat{\theta}_2 > \hat{\theta}_1$). Differences highlighted in blue in the graph with the higher threshold.

are expected to run, when they are sure they are in the upper dominance region. Therefore, irrespective of where the threshold θ' is, $v(\theta)$ is constant at $u(\frac{R-\lambda r_1}{1-\lambda}R) - u(r_1)$ for $\theta \geq \overline{\theta} + \epsilon$.

On the interval $[\theta' - \epsilon, \overline{\theta})$, the graph of $v(\theta)$ is similar to when the threshold was far from the upper dominance region. A minor difference from before concerns $\hat{\theta}$. If $\hat{\theta} < \overline{\theta}$, then $v(\theta)$ decreases first, achieves its minimum at $\hat{\theta}$, then increases and exhibits a point of discontinuity at $\overline{\theta}$. When $\hat{\theta} \geq \overline{\theta}$ however, $v(\theta)$ decreases and approaches an infimum value at $\overline{\theta}$ but never achieves it. It still exhibits the point of discontinuity at $\overline{\theta}$. At $\theta \geq \overline{\theta}$, $v(\theta)$ has a peculiar feature. When the threshold was far from the upper dominance region, $v(\theta)$ was constant at $u(\frac{R-\lambda r_1}{1-\lambda}R) - u(r_1)$ for $\theta \geq \overline{\theta}$. But now, for $\theta \in [\overline{\theta}, \theta' + \epsilon]$, $v(\theta) = u(\frac{R-nr_1}{1-n}R) - u(r_1) > 0$.[25] Unlike in regions below the upper dominance region, $u(\frac{R-nr_1}{1-n}R) - u(r_1)$ is actually **increasing** in n. We experience a reversal in the incentives, more investors running is good news. Since there is no default in this region, an increase in θ mainly means an increase in the proportion

[25] This expression is positive, since $\frac{R-nr_1}{1-n}$ is increasing in n ($r_1 < R$), $R > 1$, and $\frac{R-\lambda r_1}{1-\lambda} > r_1$.

of running investors (n). Therefore, on the domain $\theta \in [\overline{\theta}, \theta' + \epsilon]$, $v(\theta)$ decreases in θ but remains positive, until it becomes constant at $u(\frac{R-\lambda r_1}{1-\lambda}R) - u(r_1)$, when θ is above $\theta' + \epsilon$. It is useful to keep in mind that $u(\frac{R-\lambda r_1}{1-\lambda}R) - u(r_1)$ is still greater than the greatest expected utility that could be achieved on the domain below $\overline{\theta}$. This will be useful when we show the integral still increases later on. To minimize the clutter on the graph, Figure 6 plots $v(\theta)$ assuming that $\theta' = \overline{\theta}$, and $\hat{\theta} > \overline{\theta}$. It is easy to see how the graph would be different when $\hat{\theta} < \overline{\theta}$. We note here that this analysis is crucial for understanding what happens in some of the subgames when r_1 is high enough. We will prove later that the threshold $\theta^*(r_1)$ is increasing in r_1, we know it is definitely 1 when it is close to its upper bound $\frac{R}{1-\lambda(R-1)}$. Therefore, the analysis here is crucial for understanding the resulting behavior in some of the subgames even when noise vanishes and the upper dominance region shrinks to a point.

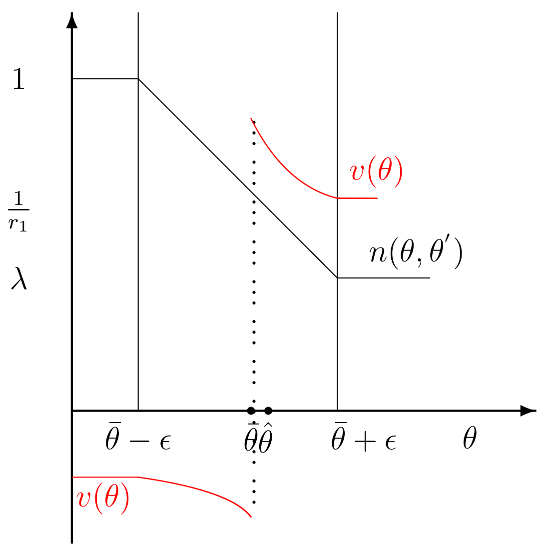


Figure 6: Utility differential when the threshold $\theta' = \overline{\theta}$.

What happens to the graph as the threshold θ' increases? When $\theta' + \epsilon = \overline{\theta}$, the figure would be very close to the one we drew in Figure 4. As θ' increases, the resulting figure would resemble Figure 4 on the domain below $\overline{\theta}$, and Figure 6 on the domain above $\overline{\theta}$. As θ' increases, the interval $[\theta' - \epsilon, \overline{\theta}]$ shrinks. The integral increases because higher values of

$v(\theta)$, values when θ is above $\overline{\theta}$, are included in the integral instead of lower values, values when $\theta < \overline{\theta}$. At the point when $\theta^{'} - \epsilon = \overline{\theta}$, the negative portion of $v(\theta)$ disappears from the integral. The lower domain of integration starts to slide off the decreasing part of the graph above $\overline{\theta}$. As $\theta^{'}$ increases, we are dropping off higher values of $v(\theta)$ (the part that declines in θ) and adding lower values (the constant portion over $\theta^{'} + \epsilon$). At this point, the integral decreases. However, by then all values of $v(\theta)$ are positive and the integral stays positive. The crucial aspect here is that the decreasing portion of the integral does not cross zero, since $u(\frac{R-\lambda r_1}{1-\lambda}R) - u(r_1) > 0$.

Having understood the $v(\theta)$ function and how it changes with the threshold, we are ready to tackle how $\Delta^{r_1}(\theta^{'}, n(., \theta^{'}))$ behaves. To show existence and uniqueness of a potential threshold, we will first show that $\Delta^{r_1}(\theta^{'}, n(., \theta^{'}))$ is continuous in $\theta^{'}$. It starts negative when $\theta^{'}$ is close to 0, increases until it is positive, and then decreases while remaining positive. The intermediate value theorem then proves our claim.

Lemma 6 $\Delta^{r_1}(\theta^{'}, n(., \theta^{'}))$ *is cont in* $\theta^{'}$.

Proof: The integrand is bounded making the continuity of the integral straightforward. Because the discontinuity in $v(\theta)$ is a simple discontinuity, it has no effect on the continuity of the integral. □

Lemma 7 $\exists\, \epsilon$ *s.t.* $\Delta^{r_1}(\theta^{'}, n(., \theta^{'}))$ *increases in* $\theta^{'}$ *then decreases. It is positive when it starts decreasing and remains positive after the decrease.*

Proof: The graphical description of the $v(\theta)$ function and how it changes with the threshold constitutes most of the proof. We only provide here a lower bound on the decreasing portion of the integral and show that it is bounded away from zero. Once the integral starts to decline, this means that as the threshold $\theta^{'}$ is increasing, portions of $v(\theta) = u(\frac{R-nr_1}{1-n}R) - u(r_1)$ are replaced by $v(\theta) = u(\frac{R-\lambda r_1}{1-\lambda}R) - u(r_1)$. Remember that $v(\theta) = u(\frac{R-nr_1}{1-n}R) - u(r_1)$ is increasing with n and therefore decreasing in θ. Therefore, the decreasing portion of the integral is bounded below by $2\epsilon(u(\frac{R-\lambda r_1}{1-\lambda}R) - u(r_1)) > 0$. $\square$

That the integral is positive on the higher part of the upper dominance region when ϵ is small enough is clear by the positive lower bound on the decreasing portion we showed in the proof before. Similarly in the lower dominance region, fix $r_1 > 1$, we have that $\underline{\theta}(r_1) > 0$. Therefore, $\exists\ \bar{\epsilon}(r_1) > 0$ small enough, such that $\underline{\theta}(r_1) - 3\bar{\epsilon}(r_1) > 0$. When $\theta^{'} < \underline{\theta}(r_1) - \epsilon(r_1)$ $v(\theta) < 0\ \forall\theta \in [\theta^{'} - \epsilon, \theta^{'} + \epsilon)$, therefore $\Delta^{r_1}(\theta^{'}, n(., \theta^{'})) < 0$. Note that this still holds for any $\epsilon \leq \bar{\epsilon}(r_1)$. We have established by the intermediate value theorem, the unique state θ^* where $\Delta^{r_1}(\theta^*, n(., \theta^*)) = 0$.

In step 2, we look at how $\Delta^{r_1}(\theta_i, n(., \theta^*))$ varies as θ_i the signal that i receives increases. We know it is zero at $\theta_i = \theta^*$. We show it is negative for all $\theta < \theta^*$, and positive for all $\theta > \theta^*$, thereby showing a threshold best response by investor i at θ^*, when all investors use a threshold strategy at θ^*. We also show continuity and use intermediate value theorem to wrap up step 2 of the proof.

Lemma 8 *Function $\Delta^{r_1}(\theta_i, n(., \theta^*(r_1)))$ is continuous in θ_i.*

Proof: $v(\theta)$ is bounded and only has simple discontinuities, therefore the integral is continuous. $\square$

Lemma 9 *$\exists\ \epsilon$ s.t. the function $\Delta^{r_1}(\theta_i, n(., \theta^*(r_1)))$ starts negative and decreasing in θ_i and then bottoms out and increases till it is positive, and remains positive after that.*

Proof: The integral $\Delta^{r_1}(\theta_i, n(., \theta^*(r_1)))$ starts constant at $-2\epsilon\frac{u(r_1)}{r_1}$ when $\theta_i < \theta^{'} - 2\epsilon$. It decreases on the downward sliding chunk of $v(\theta)$. A bit of thinking shows that it bottoms out at the θ that achieves the following infimum: $inf\{\theta_i : v(\theta_i - \epsilon) < v(\theta_i + \epsilon)\}$.[26] After that point,

[26] if this is not achieved because $\theta^{'}$ is close to the upper dominance region, the same logic carries through.

as θ_i increases, the integral substitutes values of $v(\theta+\epsilon)$ which are higher than the $v(\theta_i-\epsilon)$ that get dropped out of the integral. Therefore, the integral increases. Straightforwardly, the integral is positive when $\theta_i > \bar{\theta}+\epsilon$. If θ' is close to the upper dominance region, there is a decreasing part of the integral once it is positive, but it remains positive after the decrease.

□

We now focus on step 3. We consider the finitely many thresholds case first, and then a special case of the proof for the general case. We refer the reader to Goldstein and Pauzner (2005) for an expansion on the general case. Rank the finite thresholds, pick the highest two and apply the following procedure. Call these two thresholds: (θ_1^* and θ_2^*). Now note that at each θ in the interval (θ_1^*, θ_2^*), some investors are supposed to run at that θ and some are not. This can only happen when $\Delta^{r_1}(\theta, n(.)) = 0$. Therefore, the integral is constant at zero for fundamentals in $[\theta_1^*, \theta_2^*]$. Now consider the point $\theta_2^*+\epsilon$, $\exists$ an interval $[\theta_2^*+\epsilon-\eta, \theta_2^*+\epsilon]$ over which $v(\theta, n) > 0$. If not, how can the integral be positive? We will show that the integral can not be zero at both θ and $\theta-\eta$. Note that because of threshold strategies, $n(\theta) \leq n(\theta-\eta)$. As we move from θ to $\theta-\eta$, we drop off these positive values from the integral and add values of $v(\theta, n)$ that has possibly lower n, and definitely lower θs. If the values we are adding are negative, which definitely encompasses the situation when $n > \frac{1}{r_1}$, then since the values we dropped are positive the integral has got to decline. If the values we are adding are positive, then we are definitely in the portion of $\lambda \leq n \leq \frac{1}{r_1}$ and on this portion $v(\theta, n)$ increases in θ, so again we are dropping higher and adding lower values to the integral, and the integral declines. In both cases, the integral can not be constant at zero. This wraps up the finite threshold case.

For the general case, we assume all the measurability restrictions on the strategies that would get us a measurable $n(.)$ function and an integrable $v(.)$ function. Now the integral of net incentives at signal θ_i has the usual definition: $\Delta^{r_1}(\theta_i, n(.))) = \frac{1}{2\epsilon}\int_{\theta_i-\epsilon}^{\theta_i+\epsilon} v(\theta)d\theta$. Moreover, because $v(\theta)$ is bounded, the integral is continuous.

Let $\theta_B = sup\{\theta_i : \Delta^{r_1}(\theta_i, n(.))) \leq 0\}$. By continuity, $\Delta^{r_1}(\theta_B, n(.))) = 0$. If we are not in a threshold equilibrium, there are signals below θ_B at which $\Delta^{r_1}(\theta_i, n(.))) \geq 0$. Let$\theta_A$ be their supremum. $\theta_A = sup\{\theta_i < \theta_B : \Delta^{r_1}(\theta_i, n(.))) \geq 0\}$. By continuity, $\Delta^{r_1}(\theta_A, n(.))) = 0$. We focus here only on the case where the distance between θ_A and θ_B is greater than 2ϵ. Let the range $(\theta_B - \epsilon, \theta_B + \epsilon)$ be called d_B and similarly $(\theta_A - \epsilon, \theta_A + \epsilon)$ be called d_A. Because of the threshold nature of θ_B and θ_A, as θ declines from θ_B to θ_A, $n(.)$ can not decrease. So both θ and $n(.)$ decrease, and therefore $v(\theta)$ can only decrease. Assuming $\Delta^{r_1}(\theta_B, n(.))) = 0$, then $\Delta^{r_1}(\theta_A, n(.))) \neq 0$. Figure 7 illustrates the points discussed. This contradiction concludes the proof.

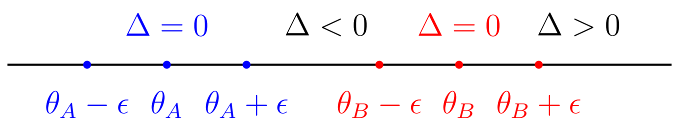


Figure 7: θ_A and θ_B are the thresholds at which $\Delta = 0$. Distance betwen them is more than 2ϵ.

A.7 Proof of Proposition 2

The first lemma establishes that the optimal insurance does not depend directly on the noise ϵ, but only indirectly through n and θ.

Lemma 10 *For a given n, y^* depends on ϵ through θ.*

Proof: First assume $\epsilon_1 \neq \epsilon_2$. At the same n and θ, both problems have the same first order condition 4. Therefore by strict concavity, $y^*(n, \theta, \epsilon_1) = y^*(n, \theta, \epsilon_2)$. And y^* does not depend directly on ϵ.

Now we show the dependence through $\theta(\epsilon)$. When $\theta(\epsilon_1) \neq \theta(\epsilon_2)$ and at the same n, the first order condition 4, shows that:

$$\frac{u'(y^*(\epsilon_1))}{u'\left[\frac{1-nr_1}{1-n}R - y^*(\epsilon_1)(R-1)\right]} = \frac{p(\theta(\epsilon_1))(R-1)}{1-p(\theta(\epsilon_1))}.$$

Therefore by strict concavity, $y^*(n,\theta(\epsilon_1)) \neq y^*(n,\theta(\epsilon_2))$. $\square$

The next lemma determines the relationship between $\theta^*(\epsilon_1)$ and $\theta^*(\epsilon_2)$ for different ϵs.

Lemma 11 $\forall\ \epsilon_1 \neq \epsilon_2 > 0$, *the following equation holds:*

$$\begin{aligned}\int_\lambda^{\frac{1}{r_1}} p[\theta^*(\epsilon_1)+\epsilon_1(1-2\tfrac{n-\lambda}{1-\lambda})]u((\tfrac{1-nr_1}{1-n}-y^*)R+y^*)+p[\theta^*(\epsilon_1)+\epsilon_1(1-2\tfrac{n-\lambda}{1-\lambda})]\big]u(y^*)dn = \\ \int_\lambda^{\frac{1}{r_1}} p[\theta^*(\epsilon_2)+\epsilon_2(1-2\tfrac{n-\lambda}{1-\lambda})]u((\tfrac{1-nr_1}{1-n}-y^*)R+y^*)+p[\theta^*(\epsilon_2)+\epsilon_2(1-2\tfrac{n-\lambda}{1-\lambda})]u(y^*)dn\end{aligned} \tag{14}$$

Proof: At any ϵ, an investor with signal $\theta^*(\epsilon)$ is indifferent between running and not running. The following definition will be useful. Define $\hat{\theta}$ as the θ at which $n(\theta,\theta^*(r_1)) = \frac{1}{r_1}$. We know that $\hat{\theta} \in (\theta^*-\epsilon, \theta^*+\epsilon)$.

Expected utility of not running when the investor receives the threshold as a signal: $\frac{1}{2\epsilon}\int_{\hat{\theta}}^{\theta^*+\epsilon} p(\theta)u((\frac{1-nr_1}{1-n}-y^*)R+y^*)+(1-p(\theta))u(y^*)d\theta$
Expected utility of running: $\int_{\theta^*-\epsilon}^{\hat{\theta}} \frac{u(r_1)}{nr_1}\frac{1}{2\epsilon}d\theta + \int_{\hat{\theta}}^{\theta^*+\epsilon} u(r_1)\frac{1}{2\epsilon}d\theta$

So we have that:

$$\frac{1}{2\epsilon}\int_{\hat{\theta}}^{\theta^*+\epsilon} p(\theta)u((\frac{1-nr_1}{1-n}-y^*)R+y^*)+(1-p(\theta))u(y^*)d\theta = \int_{\theta^*-\epsilon}^{\hat{\theta}} \frac{u(r_1)}{nr_1}\frac{1}{2\epsilon}d\theta+\int_{\hat{\theta}}^{\theta^*+\epsilon} u(r_1)\frac{1}{2\epsilon}d\theta \tag{15}$$

We use the fact that $n = \lambda+(1-\lambda)(\frac{1}{2}+\frac{\theta^*-\theta}{2\epsilon})$ to change the integration variables: $d\theta = -\frac{2\epsilon}{1-\lambda}dn$, also using Lemma 10, we get:
$\int_\lambda^{\frac{1}{r_1}} p(\theta^*+\epsilon(1-2\frac{n-\lambda}{1-\lambda}))u((\frac{1-nr_1}{1-n}-y^*)R+y^*)+(1-p(\theta^*+\epsilon(1-2\frac{n-\lambda}{1-\lambda})))u(y^*)dn = \int_{\frac{1}{r_1}}^{1} \frac{u(r_1)}{r_1 n}dn + \int_\lambda^{\frac{1}{r_1}} u(r_1)dn$

Therefore we have the following equation:

$$\begin{aligned}\int_{\lambda}^{\frac{1}{r_1}} p(\theta^* + \epsilon(1-2\tfrac{n-\lambda}{1-\lambda}))u((\tfrac{1-nr_1}{1-n} - y^*)R + y^*) + (1-p(\theta^* + \epsilon(1-2\tfrac{n-\lambda}{1-\lambda})))u(y^*)dn \\ = \int_{\lambda}^{\frac{1}{r_1}} u(r_1)dn + \int_{\frac{1}{r_1}}^{1} \frac{u(r_1)}{r_1 n}dn\end{aligned} \tag{16}$$

Note that the right hand side of the equation is independent of ϵ. Plugging in ϵ_1 and ϵ_2, transitivity gives us Equation 14. □

Contrapositive. Assume $\exists x > 0$ at which $\theta^*(\epsilon)$ is not continuous. Then $\exists\{\epsilon_n\}$ a sequence with $\epsilon_n > 0$ s.t. $\epsilon_n \to x$ but $\theta^*(\epsilon_n) \nrightarrow \theta^*(x)$.

$\theta^*(\epsilon_n) \in [0,1]$ so it has a convergent subsequence with limit $\theta \neq \theta^*(x)$. Let $|\theta - \theta^*(x)| = \eta$. There exists $\bar{n}$ s.t. $\forall n > \bar{n}$, $|\epsilon_n - x| < \delta$ but $|\theta^*(\epsilon_n) - \theta^*(x)| > \frac{\eta}{2}$.

Assume $\theta > \theta^*(x)$ (similar argument holds for $\theta < \theta^*(x)$), then $\theta(\epsilon_n) - \theta^*(x) > 0$ $\forall n > \bar{n}$. But then by choosing ϵ_n close enough to x, $\exists\ \bar{n}$ (we abuse notation here) s.t. $\forall n > \bar{n}$, $\theta(\epsilon_n) - \theta(x) - (1-2\frac{n-\lambda}{1-\lambda})(\epsilon_n - x) > \frac{\eta}{4}$ for all run proportions $\lambda \leq n \leq \frac{1}{r_1}$. Note that the n inside the equation denotes run proportion while the n in ϵ_n is for counting.

By using Lemma 3, we show that the expected utility at $\theta(\epsilon_n) - (1-2\frac{n-\lambda}{1-\lambda})\epsilon_n$ is greater than at $\theta(x) - (1-2\frac{n-\lambda}{1-\lambda})x$ for all n. But then the integral of the expected utility across n, still preserve the same sign which contradicts Lemma 11.

In other words,

$$\begin{aligned}\int_{\lambda}^{\frac{1}{r_1}} p[\theta^*(\epsilon_n) + \epsilon_n(1-2\tfrac{n-\lambda}{1-\lambda})]u((\tfrac{1-nr_1}{1-n} - y^*)R + y^*) + p[\theta^*(\epsilon_n) + \epsilon_n(1-2\tfrac{n-\lambda}{1-\lambda})]\big]u(y^*)dn - \\ \int_{\lambda}^{\frac{1}{r_1}} p[\theta^*(x) + x(1-2\tfrac{n-\lambda}{1-\lambda})]u((\tfrac{1-nr_1}{1-n} - y^*)R + y^*) + p[\theta^*(x) + x(1-2\tfrac{n-\lambda}{1-\lambda})]u(y^*)dn > 0\end{aligned} \tag{17}$$

The contradiction with Lemma 11 completes the proof.

A.8 Proof of Proposition 3

$\theta^*(\epsilon_n)$ has a convergent subsequence, for any sequence $\{\epsilon_n\}$ going to zero. We will show there can not be two subsequential limits and that proves the proposition.

The idea of the proof follows the proof of Proposition 2. We sketch it here. Contrapositive, assume there are two subsequential limits. Then there are two sequences converging to those two different limits. One limit is greater than the other. But then we can find two elements along the two sequences, call them x and ϵ_n, close enough to each other (since they are close to zero) s.t. Inequality 17 holds.

A.9 Proof of Proposition 4

We first note that the proof of Lemma 3 contains a discussion on why y^* is continuous in p, which we will use here.

Let $g(n) = p(\theta^*)u\left[\frac{1-nr_1}{1-n}R - y^*(R-1)\right] + (1-p(\theta^*))u(y^*)$. Let $g^\epsilon(n) = p(\theta^*(\epsilon)+\epsilon[1-2\frac{n-\lambda}{1-\lambda}])u\left[\frac{1-nr_1}{1-n}R - y^*(R-1)\right] + (1-p(\theta^*(\epsilon)+\epsilon[1-2\frac{n-\lambda}{1-\lambda}]))u(y^*)$. We direct the reader to Lemma 10 to point out that y^* depends only indirectly on ϵ through $\theta = \theta^*(\epsilon)+\epsilon[1-2\frac{n-\lambda}{1-\lambda}]$.

By continuity of $p(.)$, $u(.)$, and y^* in θ, $g^\epsilon \to g$ point-wise. Now, focus on Equation 15. Taking limits:

$$lim_{\epsilon\to 0}\int_\lambda^{\frac{1}{r_1}} p(\theta^*+\epsilon(1-2\frac{n-\lambda}{1-\lambda}))u((\frac{1-nr_1}{1-n}-y^*)R+y^*) + (1-p(\theta^*+\epsilon(1-2\frac{n-\lambda}{1-\lambda})))u(y^*)dn$$
$$= \int_\lambda^{\frac{1}{r_1}} u(r_1)dn + \int_{\frac{1}{r_1}}^1 \frac{u(r_1)}{r_1 n}dn \tag{18}$$

We use the dominated convergence theorem to get:

$$\int_\lambda^{\frac{1}{r_1}} lim_{\epsilon\to 0}p(\theta^*+\epsilon(1-2\frac{n-\lambda}{1-\lambda}))u((\frac{1-nr_1}{1-n}-y^*)R+y^*) + lim_{\epsilon\to 0}(1-p(\theta^*+\epsilon(1-2\frac{n-\lambda}{1-\lambda})))u(y^*)dn$$
$$= \int_\lambda^{\frac{1}{r_1}} u(r_1)dn + \int_{\frac{1}{r_1}}^1 \frac{u(r_1)}{r_1 n}dn$$

This becomes:

$$\int_{\lambda}^{\frac{1}{r_1}} p(\theta^*)u((\tfrac{1-nr_1}{1-n} - y^*)R + y^*) + (1-p(\theta^*))u(y^*)dn = \int_{\lambda}^{\frac{1}{r_1}} u(r_1)dn + \int_{\frac{1}{r_1}}^{1} \frac{u(r_1)}{r_1 n}dn$$

y^* in the last equation is $y^* = y^*(p(\theta^*), n)$. Simplifying we get:

$$p(\theta^*)\int_{\lambda}^{\frac{1}{r_1}} u((\tfrac{1-nr_1}{1-n} - y^*)R + y^*)dn + (1-p(\theta^*))\int_{\lambda}^{\frac{1}{r_1}} u(y^*)dn = \int_{\lambda}^{\frac{1}{r_1}} u(r_1)dn + \int_{\frac{1}{r_1}}^{1} \frac{u(r_1)}{r_1 n}dn$$

$$p(\theta^*) = \int_{\lambda}^{\frac{1}{r_1}} u(r_1)dn + \int_{\frac{1}{r_1}}^{1} \frac{u(r_1)}{r_1 n}dn - \int_{\lambda}^{\frac{1}{r_1}} u(y^*)dn \Big/ \Big(\int_{\lambda}^{\frac{1}{r_1}} u((\tfrac{1-nr_1}{1-n} - y^*)R + y^*)dn - \int_{\lambda}^{\frac{1}{r_1}} u(y^*)dn \Big)$$

$$p(\theta^*) = \frac{\int_{\lambda}^{\frac{1}{r_1}} u(r_1)dn + \int_{\frac{1}{r_1}}^{1} \frac{u(r_1)}{r_1 n}dn - \int_{\lambda}^{\frac{1}{r_1}} u(y^*)dn}{\int_{\lambda}^{\frac{1}{r_1}} u((\frac{1-nr_1}{1-n} - y^*)R + y^*)dn - \int_{\lambda}^{\frac{1}{r_1}} u(y^*)dn}$$

This proves the claim.

A.10 Proof of Proposition 5

$$r_1 \int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*)u(\tfrac{1-nr_1}{1-n}R - (R-1)y^*(r_1)) + (1-p(\theta^*))u(y^*(r_1))dn = u(r_1)(1 - \lambda r_1 + ln(r_1)).$$

Taking derivatives:

$$\begin{aligned}
&\int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*)u(\tfrac{1-nr_1}{1-n}R - (R-1)y^*(r_1)) + (1-p(\theta^*))u(y^*(r_1))dn \\
&+ r_1 \frac{\partial}{\partial r_1}\int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*)u(\tfrac{1-nr_1}{1-n}R - (R-1)y^*(r_1)) + (1-p(\theta^*))u(y^*(r_1))dn \\
&\qquad = u'(r_1)[1 - \lambda r_1 + ln(r_1)] + (u(r_1)/r_1)(1 - \lambda r_1)
\end{aligned}$$

which becomes:

$$\begin{aligned}
&\int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*)u(\tfrac{1-nr_1}{1-n}R - (R-1)y^*(r_1)) + (1-p(\theta^*))u(y^*(r_1))dn \\
&+ r_1 \int_{n=\lambda}^{\frac{1}{r_1}} p'(\theta^*)\frac{\partial\theta^*}{\partial r_1}u(\tfrac{1-nr_1}{1-n}R - (R-1)y^*(r_1)) - p'(\theta^*)\frac{\partial\theta^*}{\partial r_1}u(y^*(r_1))dn \\
&+ r_1 \int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*)\frac{\partial}{\partial r_1}u(\tfrac{1-nr_1}{1-n}R - (R-1)y^*(r_1)) + (1-p(\theta^*))\frac{\partial}{\partial r_1}u(y^*(r_1))dn \\
&\qquad = u'(r_1)[1 - \lambda r_1 + ln(r_1)] + (u(r_1)/r_1)(1 - \lambda r_1)
\end{aligned}$$

Therefore:

$$\begin{aligned}\frac{\partial\theta^*}{\partial r_1}p'(\theta^*)r_1\int_{\lambda}^{\frac{1}{r_1}}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*)-u(y^*)dn=\\-r_1\int_{n=\lambda}^{\frac{1}{r_1}}p(\theta^*)\tfrac{\partial}{\partial r_1}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial}{\partial r_1}u(y^*(r_1))dn\\-\int_{n=\lambda}^{\frac{1}{r_1}}p(\theta^*)u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))u(y^*(r_1))dn\\+u'(r_1)[1-\lambda r_1+ln(r_1)]+(u(r_1)/r_1)(1-\lambda r_1)\end{aligned}\tag{19}$$

In other words, the lhs is equal to the negative of r_1 multiplied by the expected derivative minus the expected utility and the last two positive terms. The difference between us and Goldstein and Pauzner (2005)'s equation is that our expected utility and expected derivative differs from theirs.

The last two terms are positive. $p(.)$ is increasing, and the difference $u(\frac{1-nr_1}{1-n}R-(R-1)y^*)-u(y^*)>0$ in an interior y^*, therefore we have that $\frac{\partial\theta^*}{\partial r_1}>0$ if the negative of the expected derivative minus the expected utility is positive. When:

$$\begin{aligned}&-r_1\int_{n=\lambda}^{\frac{1}{r_1}}p(\theta^*)\tfrac{\partial}{\partial r_1}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial}{\partial r_1}u(y^*(r_1))dn\\&-\int_{n=\lambda}^{\frac{1}{r_1}}p(\theta^*)u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))u(y^*(r_1))dn>0\end{aligned}$$

We now focus on the expected derivative wrt r_1 term, we write it as a derivative wrt n to use integration by parts:

$\frac{\partial}{\partial r_1}u(\frac{1-nr_1}{1-n}R-(R-1)y^*(r_1))=u'(\frac{1-nr_1}{1-n}R-(R-1)y^*(r_1))(-\frac{nR}{1-n}-(R-1)\frac{\partial y^*}{\partial r_1})$

$\frac{\partial}{\partial r_1}u(y^*)=u'(y^*)\frac{\partial y^*}{\partial r_1}$

Therefore,

$$\begin{aligned}p(\theta^*)\tfrac{\partial}{\partial r_1}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial}{\partial r_1}u(y^*(r_1))=\\-p(\theta^*)u'((\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1)))(\tfrac{nR}{1-n})\\-p(\theta^*)(R-1)\tfrac{\partial y^*}{\partial r_1}u'(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial y^*}{\partial r_1}u'(y^*)\end{aligned}$$

Collecting terms we get:

$$\begin{aligned}p(\theta^*)\tfrac{\partial}{\partial r_1}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial}{\partial r_1}u(y^*(r_1))=\\-p(\theta^*)u'((\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1)))\tfrac{nR}{1-n})\\-\tfrac{\partial y^*}{\partial r_1}[p(\theta^*)(R-1)u'(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))-(1-p(\theta^*))u'(y^*)]\end{aligned}$$

Using the foc on y^*, makes the last term null and we get:

$$p(\theta^*)\tfrac{\partial}{\partial r_1}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial}{\partial r_1}u(y^*(r_1))=$$
$$-u'((\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1)))p(\theta^*)(\tfrac{nR}{1-n})$$

Similarly,

$$p(\theta^*)\tfrac{\partial}{\partial n}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial}{\partial n}u(y^*(r_1))=$$
$$-p(\theta^*)u'((\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1)))(\tfrac{R(r_1-1)}{(1-n)^2})$$
$$-p(\theta^*)(R-1)\tfrac{\partial y^*}{\partial n}u'(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial y^*}{\partial n}u'(y^*)$$

Using the foc on y^* makes the last two terms null again and we get:

$$p(\theta^*)\tfrac{\partial}{\partial n}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial}{\partial n}u(y^*(r_1))=$$
$$-p(\theta^*)u'((\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1)))(\tfrac{R(r_1-1)}{(1-n)^2})$$

Therefore,

$$\frac{p(\theta^*)\tfrac{\partial}{\partial r_1}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial}{\partial r_1}u(y^*(r_1))}{p(\theta^*)\tfrac{\partial}{\partial n}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial}{\partial n}u(y^*(r_1))}=\frac{n(1-n)}{r_1-1} \tag{20}$$

Using Equation 20, rewrite the second term of Equation 19 as:

$$\begin{gathered}-r_1\textstyle\int_{n=\lambda}^{\frac{1}{r_1}}p(\theta^*)\tfrac{\partial}{\partial r_1}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))\tfrac{\partial}{\partial r_1}u(y^*(r_1))dn=\\ -\tfrac{r_1}{r_1-1}\textstyle\int_{n=\lambda}^{\frac{1}{r_1}}p(\theta^*)n(1-n)\tfrac{\partial}{\partial n}u(\tfrac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))n(1-n)\tfrac{\partial}{\partial r_1}u(y^*(r_1))dn\end{gathered} \tag{21}$$

Now integrate by parts the two terms (we only do the harder term the other one follows identically), let $u=n(1-n)$ and $dv=\frac{\partial u}{\partial n}(\frac{1-nr_1}{1-n}R-(R-1)y^*)$, then $du=(1-2n)dn$ and $v=u(\frac{1-nr_1}{1-n}R-(R-1)y^*)$. We get:

$-p(\theta^*)\frac{r_1}{r_1-1}[-\lambda(1-\lambda)u(\frac{1-\lambda r_1}{1-\lambda}R-(R-1)y^*(\lambda))-\int_{n=\lambda}^{\frac{1}{r_1}}(1-2n)u(\frac{1-r_1n}{1-n}R-(R-1)y^*)dn]$

and

$-(1-p(\theta^*))\frac{r_1}{r_1-1}[-\lambda(1-\lambda)u(y^*(\lambda))-\int_{n=\lambda}^{\frac{1}{r_1}}(1-2n)u(y^*)dn]$

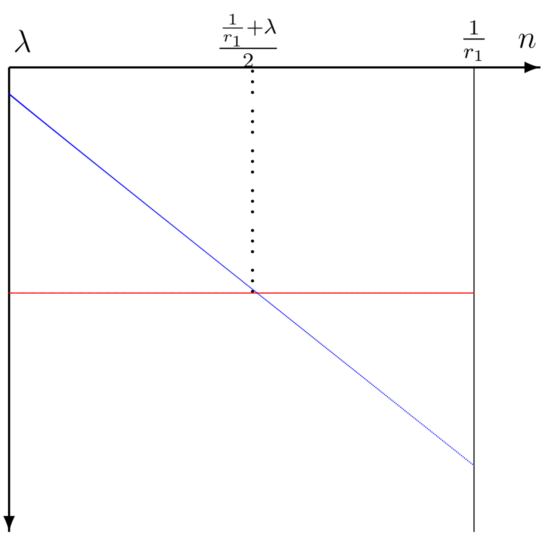
We get that:

$$-r_1 \int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*) \tfrac{\partial}{\partial r_1} u(\tfrac{1-nr_1}{1-n} R - (R-1)y^*(r_1)) + (1-p(\theta^*)) \tfrac{\partial}{\partial r_1} u(y^*(r_1)) dn =$$
$$-p(\theta^*) \tfrac{r_1}{r_1-1} [-\lambda(1-\lambda) u(\tfrac{1-\lambda r_1}{1-\lambda} R - (R-1)y^*(\lambda)) - \int_{n=\lambda}^{\frac{1}{r_1}} (1-2n) u(\tfrac{1-r_1 n}{1-n} R - (R-1)y^*) dn]$$
$$-(1-p(\theta^*)) \tfrac{r_1}{r_1-1} [-\lambda(1-\lambda) u(y^*(\lambda)) - \int_{n=\lambda}^{\frac{1}{r_1}} (1-2n) u(y^*) dn] \tag{22}$$

Therefore going back to the two terms of Equation 19 gives,

$$p(\theta^*) \tfrac{r_1}{r_1-1} \lambda(1-\lambda) u(\tfrac{1-\lambda r_1}{1-\lambda} R - (R-1)y^*(\lambda)) + \tfrac{p(\theta^*)}{r_1-1} \int_{n=\lambda}^{\frac{1}{r_1}} (1-2nr_1) u(\tfrac{1-nr_1}{1-n} R - (R-1)y^*) dn$$
$$(1-p(\theta^*)) \tfrac{r_1}{r_1-1} \lambda(1-\lambda) u(y^*(\lambda)) + \tfrac{1-p(\theta^*)}{r_1-1} \int_{n=\lambda}^{\frac{1}{r_1}} (1-2nr_1) u(y^*) dn \tag{23}$$

We need to show Expression 23 is positive and we are done.



A picture helps with the next step. The picture could be drawn in two ways, and in both cases we get the desired result. Note that if $1-2\lambda r_1 < 0$ then $1-2nr_1$ never hits zero on the domain. If it is positive it does hit zero. In both cases $1-2nr_1$ evaluated at the midpoint of the domain segment $(\frac{\frac{1}{r_1}+\lambda}{2})$ is negative. We will assume it does not hit zero, the other case follows similarly. Note that there are two triangles: one to the left and one to the right of the dotted line. These two triangles are congruent. Since $u(.)$ is decreasing, substituting $\frac{\frac{1}{r_1}+\lambda}{2}$ instead of n inside the integral just shifts weights on different $u(.)$ values

inside the integral. Now note that the blue line is a plot of the line $1-2nr_1$, while the red line is the constant function at $1-2r_1\frac{\frac{1}{r_1}+\lambda}{2}$. By Lemma 3.1, we have that $\frac{1-nr_1}{1-n}R-(R-1)y^*$ and y^* are both decreasing in n. Because $u(.)$ is decreasing, this exchange shifts weight from higher valued $u()$s to lower valued ones. Therefore, we have the following:
$\frac{p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}(1-2nr_1)u(\frac{1-nr_1}{1-n}R-(R-1)y^*)dn >$
$\frac{p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}(1-2(\frac{\frac{1}{r_1}+\lambda}{2})r_1)u(\frac{1-nr_1}{1-n}R-(R-1)y^*)dn = -\frac{\lambda r_1 p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}u(\frac{1-nr_1}{1-n}R-(R-1)y^*)dn$
And also:
$\frac{p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}(1-2nr_1)u(y^*)dn > \frac{p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}(1-2(\frac{\frac{1}{r_1}+\lambda}{2})r_1)u(y^*)dn = -\frac{\lambda r_1 p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}u(y^*)dn$

Moreover, since by Lemma 3.1, $\frac{1-nr_1}{1-n}R-(R-1)y^*$ and y^* are both decreasing in n and $u(.)$ is decreasing,
$(\frac{1}{r_1}-\lambda)u(\frac{1-\lambda r_1}{1-\lambda}R-(R-1)y^*(\lambda)) > \int_{n=\lambda}^{\frac{1}{r_1}}u(\frac{1-nr_1}{1-n}R-(R-1)y^*)dn$. We get that:
$-\frac{\lambda r_1 p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}u(\frac{1-nr_1}{1-n}R-(R-1)y^*)dn > -p(\theta^*)\frac{r_1}{r_1-1}\lambda(\frac{1}{r_1}-\lambda)u(\frac{1-\lambda r_1}{1-\lambda}R-(R-1)y^*)$

Similarly,
$(\frac{1}{r_1}-\lambda)u(y^*(\lambda)) > \int_{n=\lambda}^{\frac{1}{r_1}}u(y^*)dn$. We get that:
$-\frac{\lambda r_1 p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}u(y^*)dn > -p(\theta^*)\frac{r_1}{r_1-1}\lambda(\frac{1}{r_1}-\lambda)u(y^*(\lambda))$.

But then looking at the sum on the top again we see that,

$$p(\theta^*)\frac{r_1}{r_1-1}\lambda(1-\lambda)u(\frac{1-\lambda r_1}{1-\lambda}R-(R-1)y^*(\lambda)) + \frac{p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}(1-2nr_1)u(\frac{1-nr_1}{1-n}R-(R-1)y^*)dn > p(\theta^*)\frac{r_1}{r_1-1}\lambda(1-\lambda)u(\frac{1-\lambda r_1}{1-\lambda}R-(R-1)y^*(\lambda)) - p(\theta^*)\frac{r_1}{r_1-1}\lambda(\frac{1}{r_1}-\lambda)u(\frac{1-\lambda r_1}{1-\lambda}R-(R-1)y^*(\lambda)) \tag{24}$$

$$p(\theta^*)\frac{r_1}{r_1-1}\lambda(1-\lambda)u(\frac{1-\lambda r_1}{1-\lambda}R-(R-1)y^*(\lambda)) + \frac{p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}(1-2nr_1)u(\frac{1-nr_1}{1-n}R-(R-1)y^*)dn > p(\theta^*)\lambda u(\frac{1-\lambda r_1}{1-\lambda}R-(R-1)y^*(\lambda)) > 0 \tag{25}$$

Similarly,

$$p(\theta^*)\frac{r_1}{r_1-1}\lambda(1-\lambda)u(y^*(\lambda)) + \frac{p(\theta^*)}{r_1-1}\int_{n=\lambda}^{\frac{1}{r_1}}(1-2nr_1)u(y^*)dn > \\ p(\theta^*)\lambda u(y^*(\lambda)) > 0 \tag{26}$$

This completes our proof.

A.11 Proof of Proposition 6

$\forall\ 1 < r_1 < \underline{r_1}$, the equation that determines the threshold with insurance is:
$\int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*)u(\frac{1-nr_1}{1-n}R-(R-1)y^*(r_1)) + (1-p(\theta^*))u(y^*(r_1))dn = \frac{u(r_1)}{r_1}(1-\lambda r_1 + ln(r_1))$.

The equation that determines the threshold without insurance, θ^*_{GP} is:
$\int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*_{GP})u(\frac{1-nr_1}{1-n}R)dn = \frac{u(r_1)}{r_1}(1-\lambda r_1 + ln(r_1))$

Assume for a contradiction that $\theta^* \geq \theta^*_{GP}$. Fix a proportion of early runners $n : \lambda \leq n \leq \frac{1}{r_1}$. n determines the resources left for the remaining patient investors. Under $p(\theta^*)$ and without insurance, these resources are all invested in the project, netting: $p(\theta^*)u(\frac{1-nr_1}{1-n}R)$.
With insurance, these resources are used to net:
$p(\theta^*)u(\frac{1-nr_1}{1-n}R-(R-1)y^*(r_1)) + (1-p(\theta^*))u(y^*(r_1))$.
Since investors are risk averse and marginal utility is high enough at zero, we get $y^*(r_1) > 0$ and so:
$p(\theta^*)u(\frac{1-nr_1}{1-n}R-(R-1)y^*(r_1)) + (1-p(\theta^*))u(y^*(r_1)) > p(\theta^*)u(\frac{1-nr_1}{1-n}R)$

$\theta^* \geq \theta^*_{GP}$ implies that $p(\theta^*) \geq p(\theta^*_{GP})$, and therefore,
$\int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*)u(\frac{1-nr_1}{1-n}R)dn \geq \int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*_{GP})u(\frac{1-nr_1}{1-n}R)dn$.

But then, $\int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*)u(\frac{1-nr_1}{1-n}R-(R-1)y^*(r_1))+(1-p(\theta^*))u(y^*(r_1))dn > \int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*)u(\frac{1-nr_1}{1-n}R)dn \geq \int_{n=\lambda}^{\frac{1}{r_1}} p(\theta^*_{GP})u(\frac{1-nr_1}{1-n}R)dn = \frac{u(r_1)}{r_1}(1-\lambda r_1 + ln(r_1))$ contradiction.

Therefore, $\theta^* < \theta^*_{GP}$. This completes the proof.

A.12 Proof of Proposition 7

Fix any $\theta > \theta^*$. Remember that $\theta^* < \theta^*_{GP}$. We know that when no patient player runs, a player with a signal above the equilibrium threshold strictly prefers to wait than run. At $n = \lambda$, $v(\theta, \lambda) > 0$, therefore $(1 - p(\theta))u(y^*) + p(\theta)u((\frac{1-nr_1}{1-n} - y^*)R + y^*) > u(r_1) > \frac{u(r_1)}{r_1}$. The second inequality follows because $r_1 > 1$. But then by taking a convex combination of the two terms strictly greater than $\frac{u(r_1)}{r_1}$, the result is still greater than $\frac{u(r_1)}{r_1}$:
$(1 - \lambda)[(1 - p(\theta))u(y^*) + p(\theta)u((\frac{1-nr_1}{1-n} - y^*)R + y^*)] + \lambda u(r_1) > \frac{u(r_1)}{r_1}$. but then:
$\int_{\theta^*}^{\theta^*_{GP}} (1 - \lambda)[(1 - p(\theta))u(y^*) + p(\theta)u((\frac{1-nr_1}{1-n} - y^*)R + y^*)] + \lambda u(r_1) > \frac{u(r_1)}{r_1}(\theta^*_{GP} - \theta^*)$.
Therefore,
$\frac{u(r_1)}{r_1}\theta^* + \int_{\theta^*}^{\theta^*_{GP}} \lambda u(r_1) + (1 - \lambda)[(1 - p(\theta))u(y^*) + p(\theta)u((\frac{1-nr_1}{1-n} - y^*)R + y^*)]d\theta > \frac{u(r_1)}{r_1}\theta^*_{GP}$.
Now since: $\int_{\theta^*_{GP}(r_1)}^{1} \lambda u(r_1) + (1-\lambda)[(1-p(\theta))u(y^*) + p(\theta)u((\frac{1-nr_1}{1-n} - y^*)R + y^*)]d\theta > \int_{\theta^*_{GP}(r_1)}^{1} \lambda u(r_1) + (1 - \lambda)p(\theta)u(\frac{1-\lambda r_1}{1-\lambda}R)d\theta$.

Adding the inequality to the one before, we get:
$\frac{u(r_1)}{r_1}\theta^* + \lambda(1-\theta^*)u(r_1) + (1-\lambda)\int_{\theta^*}^{1} p(\theta)u(\frac{1-\lambda r_1}{1-\lambda}R - (R-1)y^*) + (1-p(\theta))u(y^*)d\theta > \frac{u(r_1)}{r_1}\theta^*_{GP} + \lambda(1 - \theta^*_{GP})u(r_1) + (1 - \lambda)u(\frac{1-\lambda r_1}{1-\lambda}R)\int_{\theta^*_{GP}}^{1} p(\theta)d\theta$. This completes the proof.

B Appendix: Limit of derivatives of f

$$\hat{f}(\theta^*, r_1) = r_1 \int_{\lambda}^{\frac{1}{r_1}} p(\theta^* + \epsilon(1 - 2\frac{n-\lambda}{1-\lambda}))u(\frac{1-nr_1}{1-n}R)dn - u(r_1)[1 - \lambda r_1 + ln(r_1)] = 0 \quad (27)$$

First we look into $\frac{\partial \hat{f}}{\partial \theta^*}$.
By Leibniz rule,
$\frac{\partial \hat{f}}{\partial \theta^*} = r_1 \int_{\lambda}^{\frac{1}{r_1}} p'(\theta^* + \epsilon(1 - 2\frac{n-\lambda}{1-\lambda}))u(\frac{1-nr_1}{1-n}R)dn$

Because of continuity of $\theta^*(r_1, \epsilon)$ in ϵ, p is continuously differentiable and because of the dominated convergence theorem, we get that

$lim_{\epsilon\to 0}\frac{\partial\hat{f}}{\partial\theta^*} = r_1\int_\lambda^{\frac{1}{r_1}} p'(\theta^*)u(\frac{1-nr_1}{1-n}R)dn = r_1 p'(\theta^*)\int_\lambda^{\frac{1}{r_1}} u(\frac{1-nr_1}{1-n}R)dn$

Next we look into $\frac{\partial\hat{f}}{\partial r_1}$. By proof of theorem 2 section above we get:

$$\begin{gathered}\frac{\partial\hat{f}}{\partial r_1} = -\int_\lambda^{\frac{1}{r_1}}(\frac{1-2nr_1}{r_1-1})p(\theta^*+\epsilon)u(\frac{1-nr_1}{1-n}R)dn - \frac{r_1}{r_1-1}\lambda(1-\lambda)p(\theta^*+\epsilon)u(\frac{1-\lambda r_1}{1-\lambda}R)\\ +\frac{2\epsilon}{1-\lambda}\frac{r_1}{r_1-1}\int_\lambda^{\frac{1}{r_1}} p'(\theta)n(1-n)u(\frac{1-nr_1}{1-n}R)dn - \frac{2\epsilon}{1-\lambda}\int_\lambda^{\frac{1}{r_1}}(\frac{2nr_1-1}{r_1-1})\int_\lambda^n p'(\theta)dnu(\frac{1-nr_1}{1-n}R)dn\\ -u'(r_1)[1-\lambda r_1+ln(r_1)] - \frac{u(r_1)}{r_1}(1-\lambda r_1)\end{gathered} \tag{28}$$

$$\begin{gathered}lim_{\epsilon\to 0}\frac{\partial\hat{f}}{\partial r_1} = -p(\theta^*)\int_\lambda^{\frac{1}{r_1}}(\frac{1-2nr_1}{r_1-1})u(\frac{1-nr_1}{1-n}R)dn - \frac{r_1}{r_1-1}\lambda(1-\lambda)p(\theta^*)u(\frac{1-\lambda r_1}{1-\lambda}R)\\ -u'(r_1)[1-\lambda r_1+ln(r_1)] - \frac{u(r_1)}{r_1}(1-\lambda r_1)\end{gathered} \tag{29}$$

Let $g(\theta^*, r_1) = lim_{\epsilon\to 0}\hat{f}(\theta^*, r_1)$:

$$g(\theta^*, r_1) = r_1\int_\lambda^{\frac{1}{r_1}} p(\theta^*)u(\frac{1-nr_1}{1-n}R)dn - u(r_1)[1-\lambda r_1+ln(r_1)] = 0 \tag{30}$$

$\frac{\partial g}{\partial\theta^*} = r_1 p'(\theta^*)\int_\lambda^{\frac{1}{r_1}} u(\frac{1-nr_1}{1-n}R)dn$

$\frac{\partial g}{\partial r_1} = \int_\lambda^{\frac{1}{r_1}} p(\theta^*)u(\frac{1-nr_1}{1-n}R)dn + r_1\int_\lambda^{\frac{1}{r_1}} p(\theta^*)\frac{\partial u}{\partial r_1}(\frac{1-nr_1}{1-n}R)dn - u'(r_1)[1-\lambda r_1+ln(r_1)] - \frac{u(r_1)}{r_1}(1-\lambda r_1)$

We focus on first two terms, and do a change of variable as in the proof of GP to get: $p(\theta^*)\int_\lambda^{\frac{1}{r_1}} u(\frac{1-nr_1}{1-n}R)dn + p(\theta^*)\frac{r_1}{r_1-1}\int_\lambda^{\frac{1}{r_1}} n(1-n)\frac{\partial u}{\partial n}(\frac{1-nr_1}{1-n}R)dn$

Integrating by parts, let $u = n(1-n)$ and $dv = \frac{\partial u}{\partial n}(\frac{1-nr_1}{1-n}R)$, then $du = (1-2n)dn$ and $v = u(\frac{1-nr_1}{1-n}R)$. We get:
$p(\theta^*)\int_\lambda^{\frac{1}{r_1}} u(\frac{1-nr_1}{1-n}R)dn + p(\theta^*)\frac{r_1}{r_1-1}\big[-\lambda(1-\lambda)u(\frac{1-\lambda r_1}{1-\lambda}R) - \int_\lambda^{\frac{1}{r_1}}(1-2n)u(\frac{1-nr_1}{1-n}R)dn\big]$
$-p(\theta^*)\frac{r_1}{r_1-1}\lambda(1-\lambda)u(\frac{1-\lambda r_1}{1-\lambda}R) - p(\theta^*)\int_\lambda^{\frac{1}{r_1}}\frac{1-2nr_1}{r_1-1}u(\frac{1-nr_1}{1-n}R)dn$

Collecting terms we see that:

$$\begin{aligned}\frac{\partial g}{\partial r_1} = -p(\theta^*)\tfrac{r_1}{r_1-1}\lambda(1-\lambda)u(\tfrac{1-\lambda r_1}{1-\lambda}R) - p(\theta^*)\int_{\lambda}^{\frac{1}{r_1}} \tfrac{1-2nr_1}{r_1-1}u(\tfrac{1-nr_1}{1-n}R)dn \\ -u'(r_1)[1-\lambda r_1 + ln(r_1)] - \tfrac{u(r_1)}{r_1}(1-\lambda r_1)\end{aligned} \tag{31}$$

We note here that $lim_{\epsilon \to 0}\frac{\partial \hat{f}}{\partial r_1} = \frac{\partial g}{\partial r_1}$ and $lim_{\epsilon \to 0}\frac{\partial \hat{f}}{\partial \theta^*} = \frac{\partial g}{\partial \theta^*}$.

This is an application of the following theorem from real analysis:

Theorem 1 *Let [a,b] be an interval, and* $\forall\ n \geq 1$, *let* $f_n : [a,b] \to \Re$ *be a differentiable function whose derivative* $f_n' : [a,b] \to \Re$ *is continuous. Suppose that the derivatives* f_n' *converge uniformly to a function* $g : [a,b] \to \Re$ *and that there exists a point* x_0 *such that* $lim_{n\to\infty} f_n(x_0)$ *exists. Then the functions* f_n *converge uniformly to a differentiable function* f, *and the derivative of* f *equals* g.

References

Al-Najjar, Nabil, "Decomposition And Characterization of Risk with a Continuum of Random Variables," *Econometrica*, 1995.

Brunnermeier, Markus K., "Deciphering the Liquidity and Credit Crunch 2007-2008," *Journal of Economic Perspectives*, 2009, *23* (1), 77–100.

Carlsson, Hans and Eric van Damme, "Global Games and Equilibrium Selection," *Econometrica*, 1993, *61* (5), 989–1018.

Covitz, Daniel, Nellie Liang, and Gustavo A Suarez, "The Evolution of a Financial Crisis: Collapse of the Asset-Backed Commercial Paper Market," *The Journal of Finance*, 2013, *68* (3), 815–848.

Diamond, D.W. and P.H. Dybvig, "Bank Runs, Deposit Insurance and Liquidity," *Journal of Political Economy*, 1983, *91*, 401–419.

Elamin, Mahmoud, "On the Non-Optimality of a Diamond-Dybvig Contract in the Goldstein-Pauzner Environment," *FRB of Cleveland Working Paper No. 13-06. Available at SSRN: http://ssrn.com/abstract=2258757*, April 2013.

Goldstein, Itay and Ady Pauzner, "Demand Deposit Contracts and The Probability of Bank Runs," *Journal of Finance*, 2005, *60* (3), 1293–1327.

Gorton, Gary and Andrew Metrick, "Securitized Banking and the Run on Repo," *Journal of Financial Economics*, 2012.

_ , Stefan Lewellen, and Andrew Metrick, "The Safe-Asset Share," *American Economic Review: Papers and Proceedings*, 2012.

Judd, Kenneth, "The Law of Large Numbers with a Continuum of IID Random Variables," *Journal of Economic Theory*, 1985.

Kacperczyk, Marcin and Philipp Schnabl, "How Safe are Money Market Funds?," *The Quarterly Journal of Economics*, 2013, *128* (3), 1073–1122.

Morris, Stephen and Hyun Song Shin, "Global Games: Theory and Applications," 2003.

Schmidt, Lawrence, Allan Timmermann, and Russ Wermers, "Runs on Money Market Mutual Funds," *Mimeo*, 2014.