Working Paper 8505

DYNAMICS OF FIXPRICE MODELS

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The author wishes to thank professors Truman Bewley, James Tobin, and John Geanakoplos of Yale University for a wide range of insights, inspirations, and corrections.

September 1985

Federal Reserve Bank of Cleveland

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Abstract

This paper examines the dynamics of a class of disequilibrium models developed in an earlier paper (Working Paper 8504) and uses both graphics and analysis to show that non-Walrasian equilibria can be steady states for disequilibrium models. In particular, it is shown that Keynesian (general excess supply) steady states are the most likely outcome in the model.

I. Introduction

This paper studies the time-paths of both prices and stock commodities in general equilibrium non-stochastic macromodels. Our objective is to show how parametric price constraints (short-run fixed prices) explain the stylized facts of a disequilibrium world. Our main result is that non-Walrasian equilibria can be stationary states of these models.

We have discussed objections to the fixprice methodology elsewhere and concluded that this approach is no more controversial than the assumption of instantaneous market clearing in all markets at all times. There are, however, some further general comments about modeling the dynamics of a disequilibrium economy that should be mentioned at the outset.

Our focus on dynamics stems from the strong case made by Fisher (1984) that before comparative statics can be used with confidence, the stability of equilibria must be shown. Further, the speed of adjustment must be rapid enough to allow close approximation by instantaneous adjustment. This observation is especially important to fixprice dynamics, since being out of equilibrium for extended periods of time greatly complicates the path of the economy between steady states. So although a great amount has been written about the comparative statics of fixprice models, a prerequisite for this work is dynamic studies of stability and adjustment speeds. This paper examines only stability issues.

The dynamics of fixprice models, for the most part, have very recent roots. But Patinkin (1965) should be mentioned in passing. He was the first to mention and attempt to study the effects of "spillovers" from rationing on one market to demand on another market. The canonical example is the Keynesian case, in which the inability of the laborers to sell a desired level of their services (thus lowering their income under fixed wages) leads them to demand less of the goods manufactured in the economy. To maintain our focus, we ignore a large literature studying these issues on a more fundamental level (e.g., Veendorp [1975]) and limit study to the dynamics of our specific models.

Our general dynamic framework is a sequence of temporary equilibria (Grandmont 1982). We imagine a discrete sequence of trading dates where goods and labor are traded for money. The distinguishing feature of fixprice models is that <u>at each date</u> prices are exogenously fixed and trades must clear by non-Walrasian methods. Price movements take place between periods. Although this approach seems like the only sensible framework for most of the fixprice literature, its use is not made explicit by all authors [e.g., van den Heuvel (1983), Muellbauer and Portes [1978], and Honkapohja [1979], among others).

In any dynamic model (even nonstochastic), expectations and information are key factors. For simplicity, though, we sidestep these issues with the simple assumption that all agents have complete information and rational expectations so that expectations do not need to be distinguished from outcomes in our certainty model. Note that this in no way bars our model from yielding Keynesian outcomes.

In the dynamic specification of the model, we must consider the paths of both prices and stock variables from temporary equilibrium to temporary equilibrium. Price movements are by far the more controversial. Like most work to date, we do not specify how prices actually change by the specific acts of agents in markets. We merely adopt the conventional "law of supply and demand": prices rise for goods in excess demand and fall for goods in excess supply. It must be emphasized that this does not reintroduce the auctioneer. The model economy studied does not mysteriously find an equilibrium price vector; we merely assume market forces work in the usual direction.

In fixprice models, there are a number of complications beyond this common arbitrariness. First, the law of supply and demand does not clearly apply to disequilbrium economies where, under our definitions, all goods may be in excess supply (or demand). If we are interested in relative price movements, how do we specify which excess is greater? And should this alone influence which relative price rises? Second, the very definition of excess demand in disequilibrium models is not clear; there will be a number of possibilities. No consensus exists on the correct measure

Once these details have been cleared up, and we add in the stock adjustment equations, we will find that we have not one, but many, sets of differential equations that may dictate the path of the economy. Each type of equilibria (i.e., each distinct constraint structure) will have its own set of differential equations. (Each dynamic system is called a regime.) This wouldn't be a problem if the economy could not move along a dynamic path from one regime to another, but in practice there is nothing to prevent this except direct assumption to the contrary, which we find too restrictive. Some models lack even continuity as the economy moves from one regime to another. Even assuming continuity, convergence is not easy to show. Standard methods do not apply, and when we can revise them to suit our economy, we still need extraordinary assumptions to establish stability.

Because matters become so messy in dynamic studies, we will first study the dynamic behavior of our simpler static models to gain some insights before trying to extend our results to the most general model.

Of particular interest will be what Hansen (1970) labeled "quasi-equilibria." These are dynamic paths where real variables are fixed (in equilibrium) but nominal variables move in proportion. We will find that although fully stationary points are impossible to locate except at the Walrasian outcome, interesting non-Walrasian quasi-equilibria exist.

II. The Static Model

The basic atemporal model consists of one aggregate household, one aggregate firm, and a government sector. The firm sells the good to the household and buys labor services from the household. Firms maximize profits; households maximize utility. The government finances its purchases by taxing all profits of the firms and finances deficits, if necessary, by printing money (or destroying money if it runs a surplus).

Notation

L : units of labor transacted,
Y : units of good transacted,
W : nominal wage,
p : nominal price of good,
w : real wage; $w = W/p$,
x : exogenous parameter vector; in this model $x=(p,W)$,
$\bar{M}$: end of period money holdings,
$\underline{M}$: beginning of period money holdings,
$\bar{m}$: real money holdings,
g : real government spending,
$\bar{i}$: end of period inventory holdings,
$\underline{i}$: beginning of period inventory holdings,

$U:C \quad R_+$: utility function of household.
We assume that this utility function has all the usual properties:

(1) -twice differentiable,
-quasi-concave,
-partial derivatives have signs $U_l < 0$; $U_Y > 0$; $U_m > 0$.

(2) $F(L)$: production function of firm,
-twice differentiable,
$-F' > 0$,
$-F'' < 0$.

Intertemporal adjustments are dictated by the following equations:

(3) $\bar{M} = \underline{M} + WL - pY$,

$\bar{i} = \underline{i} + F(L) - Y$.

Government expenditures are financed in two ways. First, all profits of the firms are taxed so that we need not worry about the firms holding money. Any resulting deficit or surplus is financed by the creation or destruction of money in trade for the good. This deficit must be accepted by the household as money savings. Analytically this says:

(4) $\Delta M = pg - r = WL - pY$.

Government demand is never rationed.

To model the firm's desire for inventories, we add a "valuation of stocks" function (van den Heuvel 1983) to their objective function. We label this function $v(i)$ or equivalently $v(x)$. v maps R_+ into R_+. We assume:

(5) $-v' > 0$,
$-v'' < 0$,
$-v$ is twice differentiable.

We then define the firms objective function as the sum of profits and valuation of inventories:

(6) $R(x) = r(x) + v(x)$.

Then our maximization problems are:

(7) Households: MAX $U(L,Y,M)$ s.t. $\bar{M} = \underline{M} + Wl - pY \geq 0$,

Firms: MAX $R(L,Y,i)$ s.t. $\bar{i} = \underline{i} + F(l) - y \geq 0$.

This economy fits the Arrow-Debreu framework (Debreu 1959), and Walrasian equilibria exist in this economy. To simplify matters in the dynamic analysis below, we desire the <u>uniqueness</u> of (Walrasian) equilibrium in our model. So we assume gross substitutablity for all goods. The content of this assumption for our model is discussed in <u>Working Paper 8503</u>; we find that it is not very restrictive.

We call the Walrasian quantity decisions of the agents (at a given, usually disequilibrium, parameter vector) <u>notional</u> quantities (Clower 1965). Notationally, these are marked with an asterisk superscript. Households are referenced by an h superscript; firms are denoted by an f. So, for example, we denote notional labor supply by L^{h*} or good demand by Y^{h*}.

Equilibrium theorists posit that the Walrasian price vector is somehow attained so that notional desires lead to balanced trade. Instead of assuming that this very special Walrasian price vector is found, the fixprice approach imagines that the price vector is truly parametric <u>at a given trading date</u> and will almost never be Walrasian. More structure must then be imposed to determine actual transactions. The most basic requirement imposed in fixprice models is <u>voluntary trade</u>: no agent is ever forced to trade (supply or demand) more of a good than he desires. But since markets will not, in general, clear in disequilibrium, agents will perceive quantity constraints in formulating demand. Quantity-constrained demands are called effective or Benassy demands. They are defined by:

(8) Households:

$$L^{h+} = \text{MAX } U(L,\overline{Y},x) \quad \text{subject to } \underline{m} + wL - Y \geq 0,$$

$$Y^{h+} = \text{MAX } U(L,Y,\overline{x}) \quad \text{subject to } \underline{m} + wL - Y \geq 0,$$

Firms:

$$L^{\cdot +} = \text{MAX } R(L,\overline{Y},x) \quad \text{subject to } \underline{i}+F(L)-Y \geq 0,$$

$$Y^{\cdot +} = \text{MAX } R(\overline{L},Y,\overline{x}) \quad \text{subject to } \underline{i}+F(L)-Y \geq 0,$$

where $\overline{L}$ and $\overline{Y}$ are perceived constraints on the other market when effective demands are formed on a given market.

These demands define a voluntary trade set that will, in general, have a large intersection. So more restrictions are necessary to determine transactions. We assume that only one side of a market can be rationed--the agent with the smaller effective demand will always have this demand fulfilled. Transactions are then determined by the intersection of two minimal effective demand curves. To insure uniqueness of disequilibrium we assume the monotonicity of demands and some restrictions on the first derivatives.

Fixprice equilibria are called by convention non-Walrasian. They are classified in aggregated macroeconomic models like ours, according to which sectors are rationed in which markets. In the following table we summarize the potential outcomes of the model and provide names for each.

Table I

Goods Market	Labor Market	Equilibrium Type
excess supply	excess supply	Keynesian (KE)
excess supply	excess demand	Classical (CE)
excess demand	excess supply	Underemployment (UE)
excess demand	excess demand	Inflationary (IE)
balanced	balanced	Walrasian (WE)

We will be most interested in KE and IE since it is not at all clear what direction real prices (the real wage) should change to alleviate the non-Walrasian structure of effective demands. The law of supply and demand fails to give a ready answer, and we may find stationary real price paths away from the WE.

We derive (assume) the signs of the derivatives of the notional and effective demands of the agents with respect to the parameters.

(9) $\partial L^{h*}/\partial m,\ \partial L^{h+}/\partial m < 0,$

$\partial L^{h*}/\partial w,\ \partial L^{h+}/\partial w > 0,$

$\partial Y^{h*}/\partial m,\ \partial Y^{h+}/\partial m > 0,$

$\partial Y^{h*}/\partial w,\ \partial Y^{h+}/\partial w > 0,$

$$\partial L^{f*}/\partial w = 1/(\partial^2 f/\partial L \partial L) < 0; \quad \partial L^{f+}/\partial w < 0,$$

$$\partial L^{f*}/\partial i = 0; \quad \partial L^{f+}/\partial i < 0,$$

$$\partial Y^{f*}/\partial w = w/(\partial^2 f/\partial L \partial L) \quad 0; \quad \partial Y^{f+}/\partial w = 0$$

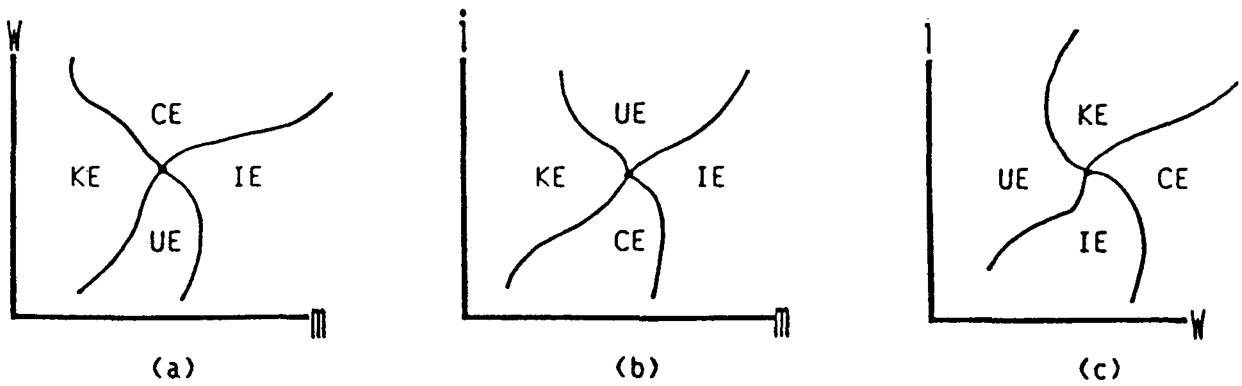
$$\partial Y^{f*}/\partial i = 1; \quad \partial Y^{f+}/\partial i = 0$$

In dynamic studies, we are interested in the convergence of the parameters (or state variables) to steady-states--dynamic equilibrium of money, inventory stocks, and prices. Thus we make use of graphs due to Malinvaud (1977), which show the range of parameter values for which each type of equilibria occurs (WE, KE, IE, CE, or UE). Under our assumptions we know that each set of parameter values implies a unique equilibrium.

The vector of state variables is $x = (w,m,i)$. We show the positions of the equilibria in all three 2-member subsets of the parameter vector [(m,w), (m,i), (i,w)].

To find these regions, we examine which constraints are binding <u>at the boundaries</u> between two states, and use the implicit function theorem to solve for the derivative of one of the state variables in terms of the other. In most cases the sign of the slope of the border is determinate under this procedure; we make clear graphically the cases where this is not true. Using the fact that all four such lines must meet at the Walrasian equilibrium and that we know which states are adjacent to which others (by comparing constraint structures) we are able to place the four regions in each parameter subspace.

Doing this for each boundary in each parameter subspace we derive the following diagrams:



<u>Figure 1</u> **Divisions of parameter spaces by equilibrium type**

Although this complete model is more satisfying than earlier models (Malinvaud [1977], Bohm [1978], Honkapohja [1979]) that lack a stock variable for the firm, the third state variable (inventories) greatly complicates dynamic analysis. Thus, our preliminary dynamic investigations will be conducted on simpler models lacking one of the stock variables. We graphically summarize the inventoryless economy to capture the essential differences when one stock variable is omitted.

Without inventories, the sole criterion in the firm's profit maximization problem is efficient production Its two effective demand curves (i.e. the Senassy demands L^{f+} and Y^{f+}) collapse into the production function in the trade space (L,Y). Then it makes no sense to say that the firm is constrained in both markets, and UE disappears. We still have WE, KE, IE, AND CE. In the two-dimensional state space (w,m) we can informally derive this graph by collapsing the UE region out of the diagram in (w,m) space derived above for the general model (see figure 1a).

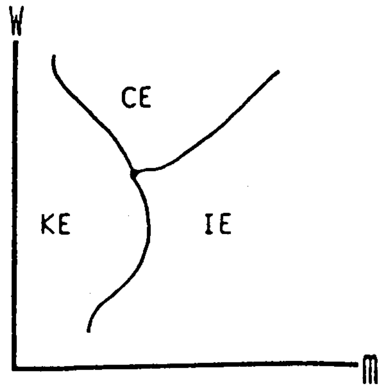


Figure 2 Equilibrium locations in the parameter space for the inventoryless model

For the inventoryless model, this single graph summarizes the entire system. A similar graph in (w,i) space, lacking CE, describes the model without money.

III. Dynamics

General Discussion

There are two distinct dynamics in the model. Money and inventory movements comprise stock dynamics, while price movements are market force dynamics. We first examine stocks.

The household retains money. In the one-period model above, the accounting identity for real money holdings at the end of a period was defined in terms of initial holdings plus the net of transactions:

$$(10)\quad \bar{M} = \underline{M} + (WL-pY) \geq 0.$$

We define the savings function as the increment to the wealth holdings of the household:

(11) $S(x) = WL(x)-pY(x)$.

Then our money stock identity in real terms is

(12) $\bar{M} = \underline{M} + S(x)$,

Then the discrete version of money dynamics is:

(13) $\Delta M = \bar{M} - \underline{M} = S(x)$,

To avoid the messiness of discrete systems, we approximate all difference equations with continuous analogs. Here we have:

(14) $\dot{M} = dM/dt = S(x)$.

Adding government bonds and allowing for a more realistic division of fiscal and monetary policy would complicate the model without changing the essentials of this story. For a steady-state, the behavior of the government in issuing or retiring debt in all forms must coincide with savings behavior of households. On the other hand, if households are allowed to hold other assets (inventories or newly introduced forms of wealth), then our simple accounting identities break down, and the model might yield different results.

The firm carries inventories across trading dates. The one-period model's inventory equation is:

(15) $\bar{i} = \underline{i} + f(L) - Y$.

For notational simplicity we define the inventory accumulation function:

(16) $I(x) = f(L) - Y$.

Then our inventory adjustment difference equation is:

(17) $\Delta i = \bar{i}(x) - \underline{i} = I(x),$

and its continuous counterpart is:

(18) $\dot{i} = I(x).$

Now we can completely describe the activity of the government. The profits of the firm (to be taxed 100 percent) are given by:

(19) $r(x) = pY - WL + pF(0),$

where F(0) is government demand. We can rewrite (19) as:

(20) $g = pF(0) = r(x) + s(x).$

This identity says that government expenditures are financed by profits and savings. A steady state in money and inventory stocks requires that government spending mesh with the aggregate behavior of the private sector.

As long as the firm cannot convert profits into other stores of wealth, the introduction of other assets will not change the results of the model. However, if the firm can hide profits by converting them into a different (non-taxed) form before the tax collector arrives, then our accounting identities again would become invalid, and we would have to model the dispensation of retained profits.

Price dynamics are much less stcaightforward than the almost accounting form of stock dynamics. There is little agreement on how price dynamics should be derived from the primitive elements of a general equilibrium system. Even in the simple Arrow-Debreu model, price adjustment by the tatonnement is entirely ad hoc. Although Arrow (1959) clearly outlined the difficulties involved, progress in this area has been slow.

Recently, some fresh efforts have been made to formulate more rigorous price dynamics based on the explicit behavior of maximizing agents. This requires the abandonment of all artificial constructs such as the auctioneer. Very detailed descriptions of individual actions (beyond choice criteria) must be given.

Fisher (1984) has constructed models where agents realize that markets do not find Walrasian equilibria and, based on such realizations, agents may change prices themselves. The framework reduces analytically to a Hahn Process with a Lyapunov function in target utilities. Agents initially believe they can transact all they desire at prevailing prices, and thus have a target (notional) utility in each tr'ading period. But disequilibrium is allowed, and these target transactions may not obtain. Then assuming (as we do) that only one side of a market can be constrained, agents realize that if they have excess target demand/supply on a market, so do other agents; thus, market pressures are going to move prices to all agents' detriment. They may then change these prices themselves to try to unload excess supplies or purchase unmet demand. But none of the "suprises" in unrealized target transactions can be beneficial. Target utility is always falling, and can be shown to converge under weak conditions.

Although Fisher's model is appealing as a more solid foundation for price adjustments than the usual law of supply and demand, our model is much richer than Fisher's in other ways (he does not model stocks and doesn't distinguish among different types of equilibrium). Superimposing Fisher's price dynamics on this class of disequilibrium models produces an analytically difficult set of equations.

Shapley and Shubik (1977) have introduced another appealing model of price formation derived from explicit asssumptions on the nature of market interactions. The economy is modeled as a noncooperative game with a commodity money. Agents send <u>quantity</u> signals to the market that subsequently determine prices in terms of the money commodity. The model is elegant, ingenious, and much less contrived than auctioneer worlds. Obviously a Walrasian outcome will not necessarily be reached; disequilibrium states are allowed. But this model determines prices endogenously <u>within</u> <u>each period</u>, and there is no production. Further, **it** is much more detailed than our models in specifying market interactions. For these reasons **it** is inapplicable to our price dynamics.

For lack of a superior alternative, we follow the rest of the literature, and use the standard law of supply and demand to model the adjustment of prices. Prices rise in the face of excess demand and fall when there is excess supply. Thus, in our model we have for the rates of change of nominal prices:

$$(21)\qquad \dot{p}/p = h_1(Z^Y),$$
$$\dot{W}/W = h_2(Z^l),$$

where Z^Y, Z^l are some measure of excess demand and h_1 and h_2 are sign-preserving functions. To simplify the study of dynamics we restrict h_1 and h_2 to linear functions in demands and supplies. We define D and S (as some measure of) demand and supply (the agent in each case is obvious). Then we have:

$$(22)\qquad \dot{p}/p = h_{11}(D^Y)-h_{12}(S^Y),$$
$$\dot{W}/W = h_{21}(D^l)-h_{22}(S^l).$$

These equations may be thought of as the linear approximation of more general price dynamics. The weights $h_{11},\ldots, h_{22}$ can be interpreted as speeds of adjustment for prices in reaction to the different demands.

In the canonical Arrow-Debreu model there is only one possible measure of excess demand (up to the functional form the unique demands and supplies take) The stricture on disequilibrium transactions eliminates further complications. But in this model, excess demand could conceivably involve notional demands, the larger of effective demand, and transacted quantities (the lessor of effective demands). Define the notation:

N^j : notional demand/supply for good **j**,
E^j : effective demand/supply for good **j**,
j : transacted quantity of good **j**,
Z^j : excess demand.

Then there is a number of potential definitions of excess demand in disequilibrium:

(23) $$Z^j = N^j - E^j,$$
$$Z^j = N^j - j,$$
$$Z^j = E^j - \mathbf{j}.$$

There is no formal method for selecting any of these. We have not modeled the market with enough detail to determine precisely which demands are communicated to the market. The LSD is not a specific description of the <u>mechanics</u> of price movements. We interpret notional quantities as merely wishful thinking that is never communicated to the market. Effective quantities are the forces that are felt by the economy, and thus drive price dynamics via the LSD. Further, since the lesser of the two effective demands determines transactions, our definition of excess demands involves transactions as well. Of course, the choice of the specific functional form of the definitions of excess demand (difference, ratio, . . .) remains arbitrary. For simplicity, we define excess demand in terms of differences (linearly):

(24) $$Z^Y = y^{h+} - Y^{f+},$$
$$Z^l = L^{f+} - L^{h+}.$$

Then our price dynamics equations are:

$$(25) \qquad \dot{p}/p = h_1(Z^Y) = h_{11}(Y^{h+}) - h_{12}(Y^{f+}),$$

$$\dot{W}/W = h_2(Z^l) = h_{21}(L^{f+}) - h_{22}(L^{h+}).$$

Thus the direction of price movements depends on what type of equilibrium prevails; the results are given by table 2:

Table 2: Price Movements Across Equilibria

	$\dot{p}$	$\dot{W}$
WE	0	0
KE	–	–
IE	+	+
UE	–	+
CE	+	–

The lack of strict stationarity at any point except the WE has led many theorists to unfairly reject the LSD in fixprice models. Even as respected a theorist as J. M. Grandmont disavows our approach because ". . . the stationary states of the resulting dynamic system cannot display unemployment." (Grandmont 1982, p. 916) Yet it is clear that the real wage may be stationary in KE or IE (both involving "unemployment" relative to the WE). Grandmont might mean that in such a case a fixed money supply (or a stockless model) would not permit a stationary state outside of WE. But with money dynamics in the model, we can have (as we show) a quasi-equilibria where the real wage and the real money supply are both stationary. Thus, although this objection is rigorously correct when "stationary states" is interpreted in terms of nominal variables, it entirely misses the point that real parameter values may be steady in this model at points in the KE or IE regions.

The indeterminateness of real price movements in the KE and IE regions makes our analysis more qualitative; the stationary locus of the real wage lies in the KE and IE regions, but we know little about its shape. We can detect general tendencies but cannot find closed-form solutions. The stationary wage locus must go through the WE point, and this provides some structure.

We now discuss some of the difficulties in solving these differential equations. The salient difficulty is that the specific functional forms of these generalized representations depend on which equilibria we are in (for example S(x) takes on a different form in the KE region than in the IE region since L and Y have different functional forms). As the economy evolves, the equilibrium type may switch, and a new dynamic system will then govern movements. All conventional techniques for solving systems of differential equations must be modified or abandoned. It is difficult to pinpoint the steady states of the model, yet perhaps this complexity is unavoidable in modeling disequilibrium.

Second, none of our assumptions on the uniqueness of fixprice transactions in a given period insures that there will be a unique stationary point to the dynamic system for our disequilibrium model. We have assumed that the Walrasian dynamic (tatonnement) analog of our economy has a unique equilibrium. This follows almost automatically from the assumption of gross substitutes and the equivalence of a dynamic tatonnment model with an atemporal one. However, since our dynamics cannot be tied to atemportal price movements (where prices are fixed), uniqueness does not carry over. We may have a denumerable, uncountable, or even generic set of stationary states to our dynamic model.

Non-uniqueness means that since the system will move toward different rest points based on its initial location, and the strength (not just direction) of various forces becomes relevant. This is much more difficult than the unique case and robs us of the purely qualitative Lyapunov function. We could assume uniqueness in one of the regions and apply a Lyapunov function to the set of differential equations guiding behavior at that point to test for stability. But even if this gave a positive result, we could not be sure that the system never evolved into another region somewhere during its evolution toward the unique rest point. Thus, the multiple regimes prevent even strong assumptions from admitting the use of the usual Lyapunov Theorem.

Dynamics of disequilibrium models like ours are usually analyzed qualitatively because of the switching regimes problem. Phase diagrams in the state varible space will be one tool in stability analysis. We will also examine linearized versions of our system at posited equilibrium points and test for stability of these approximations to the true dynamic path. Following our main body of dynamic results, we will show how Lyapunov's Second Theorem can be modified (Eckalbar 1980) to analyze models like ours, but our example will show the strong assumptions necessary to reach meaningful results via this route. Finally we will examine the application of Fillipov methods to the model (Ito and Honkapohja 1983), but here too the point is that more technical methods fail to improve on the conclusions of simpler qualitative techniques.

Almost all dynamic analyses of disequilibrium models focus on the special case of firms that carry no stock variables. The reason is historical; this version was formulated and understood much earlier than the more general model. Further, the complications of the general model encourage examination of simpler cases. For this reason, we will examine simpler models. We analyze the model first without inventories, then we will add inventories but remove money. This will give us some basic insights into the dynamics of each stock variable apart from more general complications. It might be hoped that we could directly gain solutions to the genera1 system by combining these two subsystems that together comprise the entire economy. Unfortunately savings and inventories are not independent in the determination of each period's transaction. Inventory decisions affect labor costs, which affect both consumer behavior and government finance (through profits). Savings decisions, similarly, influence the firm and the government. Thus, our system is too intertwined to admit solution by examining each stock variable separately. But these subsystem investigations can point to where we should and should not look for solutions to the general system.

There is an immediate implication of this procedure that reinforces a point that we have discussed above. In the model without inventories UE disappeared since with only the production function dictating (profit maximizing) behavior, the firm cannot be doubly constrained. We will see in the model without money that CE <u>disappears</u> since now the household lacks a stock variable, and so maximizes utility subject only to efficient consumption. Notice that in either case KE and IE exist; they are robust to different stock specifications of disequilibrium models. We have also observed that KE and IE (and WE) are the only regions where the real wage may be stationary. Combining these two results, we will focus most of our attention on the KE and IE regions of the state space in our search for steady states. We cannot completely ignore the CE and UE regions, since the economy may move <u>through</u> these regions, and this may affect the ultimate stability of the economy. Our reasons for ignoring the WE (unlike many other authors) have been discussed previously.

IV. Model without Inventories

We have already sketched the division of the parameter space between different equilibrium types for the inventoryless model in figure 2. In our dynamic systems, we will supress the variable p and consider only the movements of the real wage $w = W/p$ and the real money supply $m = M/p$. So we must modify our price and money dynamics so that they are in real terms. Taking logs and differentiating $w=W/p$ we have:

(26) $$\frac{1}{w}\cdot\frac{dw}{dt} = \frac{1}{W}\frac{dW}{dt} - \frac{1}{p}\frac{dp}{dt}.$$

Then our real wage differential equation, using (21), is:

(27) $$\dot{w} = w[h_2(Z^l) - h_1(Z^Y)].$$

With the linear LSD (25), we have:

(28) $$\dot{w} = w[h_{21}(L^{f+}) - h_{22}(L^{h+}) - h_{11}(Y^{h+}) + h_{12}(Y^{f+})].$$

A similar derivation on (14) and (21) yields our equation for the dynamics of the real money stock:

(29) $$\dot{m} = g - r(x) + mh_1[Z^Y(x)]$$

With our linear LSD, this reads:

(30) $$\dot{m} = g - r(x) + m[h_{11}(Y^{h+}) - h_{12}(Y^{f+})].$$

Since the LSD prevents the CE region from ever containing a steady state, we wish to simplify our first studies of this model by prohibiting the real wage from rising above the Walrasian level w*. By the monotonicity of the KE/CE and the CE/IE boundaries, this restriction on the wage prevents the system from ever entering the CE region. We are thus limiting ourselves to KE, IE and WE outcomes.

We now derive the the stationary locus for the KE and IE regions in (m,w) space. We have in KE that $L=L^{F+}$ and $Y=Y^{h+}$, while in IE $L=L^{h+}$ and $Y=Y^{\cdot+}$. Then using the savings expression for money dynamics from (14) we have the following derivatives for money stocks:

(31)

$$\left.\frac{\partial S}{\partial m}\right|_{KE} = \frac{\partial Y^{h+}}{\partial m} \cdot \frac{F' - w}{F' - \partial Y^{h+}/\partial L}$$

$$\left.\frac{\partial S}{\partial w}\right|_{KE} = \frac{L(F' - \partial Y^{h+}/\partial L) - \partial Y^{h+}/\partial w(F' - w)}{F' - \partial Y^{h+}/\partial L}$$

$$\left.\frac{\partial S}{\partial m}\right|_{IE} = \frac{-\partial Y^{h+}/\partial m(F' - w)}{1 - (F')\partial Y^{h+}/\partial Y}$$

$$\left.\frac{\partial S}{\partial w}\right|_{IE} = \frac{Y^{f+}[1 - (F')\partial Y^{f+}/\partial Y] - \partial Y^{f+}/\partial w(F' - w).}{1 - (F')\partial Y^{f+}/\partial Y.}$$

In the KE region, then:

(32)

$$\left.\frac{dm}{dw}\right|_{s=0} = \frac{L(F' - \partial Y^{h+}/\partial L) - \partial Y^{h+}/\partial w(F' - w),}{\partial Y^{h+}/\partial m(F' - w),}$$

and so the locus slopes upwards except for very low wage levels. In the IE region we have:

(33)

$$\left.\frac{dm}{dw}\right|_{s=0} = \frac{L^{h+}(1 - (F')\partial L^{h+}/\partial Y) - \partial L^{h+}/\partial w(F' - w),}{-\partial L^{h+}/\partial m(F' - w),}$$

and so the $\dot{m}=0$ locus slopes downward here except for low wage levels.

Finally, we have nothing in our assumptions to show that this locus will be continuous across different regimes. But without continuity we are lost, so we must assume it. Continuity is analytically a minimal assumption. We will not assume differentiability at the boundary since it is not a superfluous issue and doing so would eliminate the switching regimes problem; the different systems would, under differentiability, link up to form a continuously differentiable model that would be amenable to normal methods of solving differential equations.

Roughly, then, we have the following picture in the parameter space:

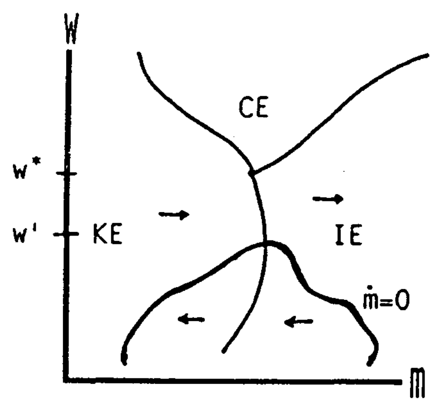


Figure 3 Stationary money locus in inventoryless model

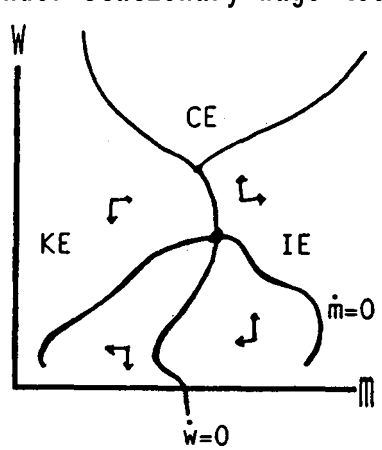
First, note that our restriction on the wage level leads to positive savings at the WE. This stems from the higher wage at WE (hence low profits) that forces deficit finance and allows households to accumulate wealth. Moreover, any wage above w' (the maximum of the m=0 locus with respect to w) leads to divergent inflationary outcomes.

Drawing in the phase arrows as dictated by our equations, we also see that stationary savings states in the KE area will be stable while those in the IE region are always unstable rest points. This is an indication that Keynesian states may be prevalent in the model.

Now we enrich this model by adding price adjustments (in w, the real wage) to the dynamics. To summarize what little we can be sure of with respect to price dynamics under our indefinite assumptions about them! We know that the w=0 locus must go through the unique WE point, and that the remainder of this set lies in the <u>union of</u> the KE and IE regions. Beyond, this nothing is definite.

Suprisingly no one has made a strong case for a very plausible possibility: the entire KE/IE border may be stable in the real wage. This would follow under the assumption that excess demands are <u>continuous</u> across regimes (though not necessarily differentiable) since both goods are in excess supply in KE but in excess demand in IE. In this case, we can easily see that the intersection of the m=0 and the w=0 loci gives a saddlepoint equilibrium on the KE/IE border:

<u>Figure 4</u> Saddlepoint equilibrium on KE/IE border under stationary wage locus I



This result certainly fits the stylized facts of recent economic performance well, with movements from Keynesian recessions to inflatiornary booms. And states of IE generate unemployment as well as price pressures and portray what has been dubbed stagflation.

The only reference in the literature to this issue (Honkapohja 1979) points out that along this border, the marginal product of labor (mpl) exceeds the real wage and that this places upward pressure on the wage. Yet this marginal condition for equilibrium holds only in WE, and it is not clear that mpl > w induces the firm to hire more labor in successive periods involving non-Walrasian states. It depends on the structure and level of the constraints of the economy at the temporary equilibria.

Continuity in parameters is one of the weakest conditions imposed on demands in the literature. Since no one has adop'ted this hypothesis, the implicit consensus seems to be that demands are discontinuous at the KE/IE border in this model. Most authors posit a stationary real wage locus something like the following:

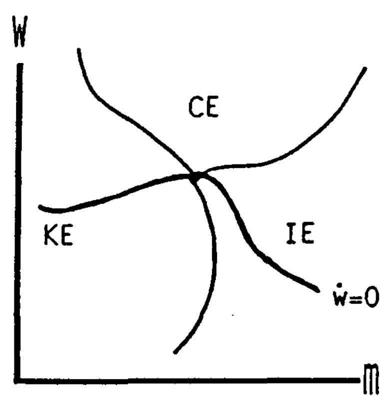


Figure 5 Stationary wage locus II

Combining this with the equilibrium savings locus, **it** is not at all clear that a quasi-equilibrium path exists at all, and existence in turn does not imply stability. Here are some of the simpler possibilities:

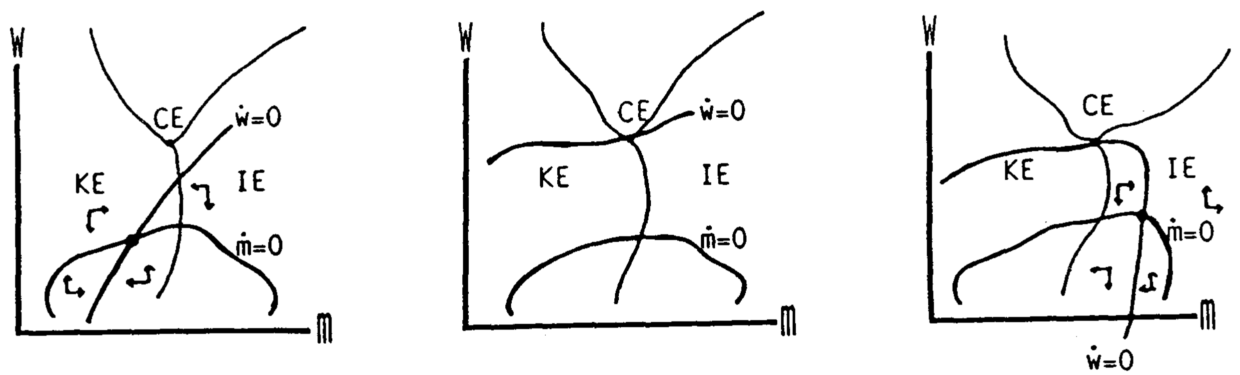


Figure 6 Money and price dynamics in inventoryless model

The lack of restrictions on the w=0 locus leads to an unmanageable proliferation of possibilities. We have not even drawn any cases where the locus $\dot{w}=0$ is not monotone in the state space. The problem is that the LSD, while having decisive predictive force in the CE region (and in the general model, the UE) possesses <u>no</u> power of resolution in the KE and IE regions. However, we can reveal what factors control the shape of the stationary wage locus in this space.

Increased money balances have two conflicting effects on the excess demands that determine real wage movements. By increasing wealth they increase demand for the good, and thus higher money balances tend to depress the real wage. But they also provide a substitute for labor, and thus increase the wage required to hire a given volume of labor. **If** we assume that this second factor dominates the first to a large enough extent, a unique quasi-equilibrium in the KE region results as in diagram 6(a) above. The phase diagram indicates stability for appropriate initial states.

We can develop a graphical tool to illustrate the factors determining the type of steady state that will be reached. Based on our findings above, we focus on steady states of excess supply in all markets.

For the linear price dynamics we have:

(34) $\dot{w} = w(h_{21}L^{f+} - h_{22}L^{h+} - h_{11}Y^{h+} + h_{12}Y^{f+}),$

$\dot{m} = g - r(x) - m(h_{11}Y^{h+} - h_{12}Y^{f+}).$

We solve the money equilibrium equation first since it contains only goods, supplies, and demands. It can be rewritten as:

(35) $h_{12}Y^{f+} - h_{11}Y^{h+} = [r(x) - g]/m,$

or

$$Y^{f+} = [r(x) - g]/mh_{12} + (h_{11}/h_{12})Y^{h+}.$$

This defines a line in (Y^{h+},Y^{f+}) space; any point along this line defines demand and supply of the good consistent with equilibrium in real money supply. We are interested in highlighting what factors might cause part (or all) of these points to give excess supply in effective demands--those points lying above the 45 degree line.

To start out with the strongest case, if the slope is greater than one and the intercept positive, then the entire locus lies above the 45° line and only excess supply in the goods market may prevail in steady states. The slope is given by:

(36) $h_{11}/h_{12}.$

For this to be greater than 1, means that the price of the goods is more sensitive to demand factors than to supply factors. This complements the nominal wage stickiness we encounter below in affirming that Keynesian outcomes are associated with "supply stickiness" in inter-period dynamics.

The intercept is:

(37) $[r(x) - g]/h_{12}m.$

So positivity means:

(38) $r(x) - g > 0.$

This reflects two factors associated with KE. Low levels of autonomous demand and lower wages (hence higher profits r) tend to produce KE in atemporal models, and this is the extension to the dynamic setting of these ideas. If these two conditions hold, our diagram appears as follows:

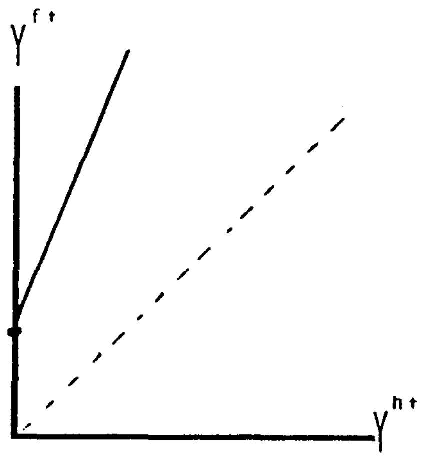


Figure 7 Excess supply of labor as the only possible steady state

Solving the equilibrium wage equation will give us similar conditions for excess supply in the goods market. We substitute in (35) to supress the demands and supplies for the goods market in the following derivation:

(39) $\dot{w} = w(h_{21}L^{f+} - h_{22}L^{h+} - h_{11}Y^{h+} + h_{12}Y^{f+})$

$= w[h_{21}L^{f+} - h_{22}L^{h+} - h_{11}Y^{h+} + h_{12}(r(x) - g]/mh_{12} + (h_{11}/h_{12})Y^{h+}] = 0.$

This leads to:

(40) $h_{21}L^{f+} - h_{22}L^{h+} - h_{11}Y^{h+} + [r(x) - g]/m + h_{11}Y^{h+} = 0.$

So our line in (L^{h+}, L^{f+}) space is:

(41) $$L^{h+} = (h_{21}/h_{22})L^{f+} + [r(x) - g]/mh_{22}.$$

Again a positive intercept and a slope greater than unity will give excess supply of labor in effective demands as the only steady states for the real wage. The interpretation of the intercept condition is the same as in the goods market above, and the slope condition here is the classical Keynesian case, in which wages are inelastic in supply factors.

These are sufficient conditions for <u>all</u> steady states to exhibit excess supply in both markets (KE). It is easy to see that under more relaxed conditions, KE could prevail. Further the precise point selected on both line loci is determined simultaneously in a general equilibrium that cannot be illustrated here. But the conditions that can give rise to Keynesian steady states are clear; we can safely study their stability without worrying that we are examining a vacuous case.

Having seen that the existence of Keynesian steady states is not a rarity, we now state Theorem I. (For proof, see appendix).

<u>V. Theorem I</u>

Under the basic assumptions made about the inventoryless economy, KE steady state equilibria are stable.

This proof is an improvement over earlier attempts because no ad hoc assumptions beyond the basic structure of the model are necessary for the stability proof. Although the proof is tedious, it involves only elementary techniques.

The system will have imaginary roots and oscillate (stably, unstably, or critically, depending on the sign of the real part of the characteristic roots) if, and only if:

(42) $$(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) < 0.$$

It is difficult to pin down the sign of this expression without gratuitous and economically meaningless asssumptions. Some authors (Malinvaud [1977], Honkapohja [1979], and Bohm [1978]) have posited the plausibility of cycling in this model along these lines. Other authors have argued for cycling from the existence of various saddle-point equilibria discussed above. Blad and Zeeman (1982) have constructed a stochastic model with expectations based on past observations that produces cycling between the KE and IE regions. Unfortunately they require extended assumptions that we are wary of making and their modeling of expectations introduces undesirable controversies. Sneessens, in estimating a variant of the model for the Belgian economy, found that the model cycled between KE and IE states in the 1970s.

Returning to the stability of the inventoryless model, we briefly examine KE in the case in which we remove our restriction on the wage level and admit the possibility of periods of CE. If we retain all other assumptions, we have the same outcome.

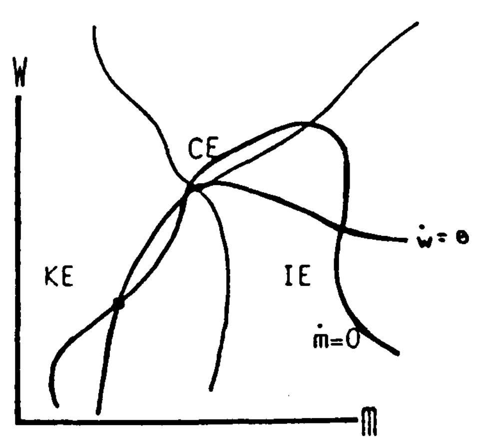


Figure 8 KE in the inventoryless model with unrestricted wage level

The slope of the equilibrium savings locus in the CE region cannot be signed unambiguously, but since the equilibrium wage locus does not enter the CE region, this is not so disturbing. The phase diagram suggests stability as it did when we restricted the level of w.

It is bothersome that our stability results are either indicated only by phase diagrams or hold only for a linearized version of the system, and so at best, establish local stability. The most common tool in demonstrating . stability for general (nonlinear) systems of differential equations is Lyapunov's Second Theorem. We discuss what must be done to apply this technique to our model with regime switching.

Assume a Lyapunov function V proves the stability of a system $x=g_1(x)$ at x^*. Now let a new system $x=g_2(x)$ of differential equations be defined over the same region. Assume:

- x^* is also an equilibrium of the new system g_2;
- x^* can be shown stable with the same Lyapunov function that gives stability for system g_1.

Now define a combination of the two systems in the same phase space:

$$\text{(43)} \qquad x = g_1(x) \quad : x \in S_1$$

$$x = g_2(x) \quad : x \; S_2$$

where

$$S_1 \cup S_2 = \text{entire phase space.}$$

We must further assume that:

$$\text{(44)} \qquad g_1(x) = g_2(x): x \in [S_1 \cap S_2].$$

Then we can trivially apply Lyapunov's Theorem to show the stability of x^* in the hybrid system. Further, the extension to many regimes with the same assumptions applied to each addition is straightforward.

The price of putting the theorem to such direct use is exorbitant in terms of sufficient assumptions. It assumes a unique and common equilibrium point to the separately defined systems; this would be a fluke in our model. Further, it requires equality of the systems on their border, which is stronger than the continuity assumption that we utilize. Eckalbar (1980) developed and applied the theorem to a much simpler economy than ours:

- no stocks,
- labor supply exogenously fixed, and
- the price adjustment equation takes a special form gratuitous to applying the theorem.

It does not appear that the theorem can be extended to economies like ours.

Honkapohja and Ito (1983) present Fillipov's method as a more powerful tool for solving problems with regime switching. This generalization of Lyapunov's method permits the solution to ignore behavior of the system on any set of measure zero, like the boundaries of our system. Thus, the method can be extended to the more general case in which even discontinuity is permitted on the borders between regimes. However, it is applied to an economy similiar to the one Eckalbar studied with his straightforward Lyapunov function and cannot be used to solve our sets of differential equations.

Thus, although attempts have been made to strengthen the conclusions of dynamic analysis of disequilibrium models by applying more powerful mathematical methods, these studies haven't reached fruition. There is still no elegant approach to the regime-switching problem. In light of the sharply decreasing marginal returns to the use of the more sophisticated mathematical tools, the rest of our dynamic studies sticks to basic methods.

Perhaps simulations of these economies over a broad range of parameter sets will provide more convincing evidence of their dynamic tendencies.

Malinvaud (1980) is the only author we have read who has pursued this avenue of inquiry. Keynesian outcomes abound in his simulations, although his model is much different than ours, and he makes some very specific assumptions that might not be necessary.

Model without Money

We now allow the firm a stock variable (inventories) and remove money from the household (and government) sectors. Our procedures are parallel to the case with only money.

This model requires some changes in our framework, since the government deficits/surpluses cannot be financed without the debt instrument money. We could allow any level of government expenditure and replace money with inventories. A balanced budget would have g equa'l to the hypothetical profits of the firm, with the government taxing all of these inventory profits. When the government ran a deficit, it would expropriate the required amount of the good from the firm's normal inventories; a surplus would be managed by the firm retaining 'excess' profits in the form of higher inventories. But there would exist levels of g that could not be financed (depending on stocks and production of the good), so we would have to restrict the size of the government deficit/surplus. To avoid these complications, we instead let the level of profits define the size of (now always balanced) government expenditures. We still tax profits 100 percent, but permit no deficits. We no longer need money; all transactions are barters.

Since the household has no stock-variable decisions to make, it simply maximizes utility by choice of the desired level of work (which immediately implies consumption, since there is no storage). Thus in the moneyless model, it is households that cannot conceivably be constrained in both the good and the labor market, and CE cannot occur in this model. On the other hand, inventory desires might lead to situations where the firm is constrained in both labor purchases and good sales, so UE reappear. Of course KE and IE remain.

Our parameter space is now $x=(i,w)$. The division between the three possible states can be most easily seen by collapsing the CE region out of the diagram in (i,w) space for the general model in figure 1(c).

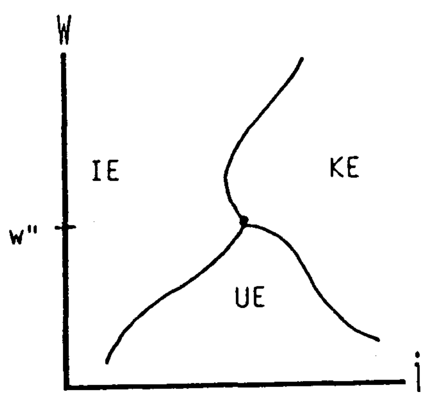


Figure 9 Division of parameter space in moneyless model

Since the real wage increases unambiguously in the UE region, we know that there cannot be even a quasi-equilibrium there; we again restrict the domain of the wage, this time bounding it below by w" so that we do not have to consider the UE region in our first examination of this model. The monotonicity of the KE/UE and IE/UE borders assures us of this.

The stationary inventory locus is derived as in the previous model with money. The implicit function theorem, applied to the first order equilibrium condition, shows that the i=0 locus slopes downward in the IE region and upward in the Keynesian region for (i,w) space.

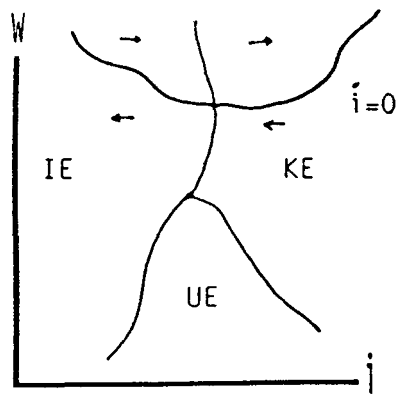


Figure 10 Stationary inventory locus in moneyless model

This partial model, like the previous one, immediately suggests that KE are the most likely candidates for stable quasi-equilibria. Again this is contingent on our assumption that the i=0 locus is continuous on the KE/IE border.

Our general comments on price dynamics will not be repeated. If we accept the continuity of excess demands across regimes, we must have that the w=0 locus is the KEIIE border. The phase diagram indicates that an equilibrium along the KE/IE boundary will be oscillatory, and stability is not clear.

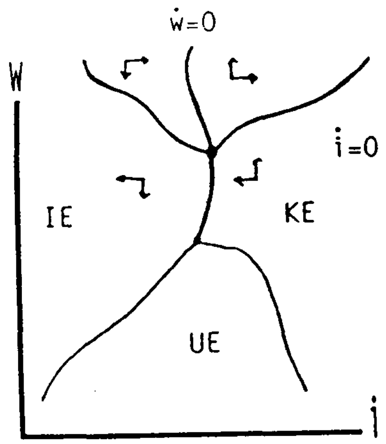


Figure 11 Oscillatory equilibria in moneyless model under stationary wage locus I

If we reject this version of the equilibrium real wage locus and posit a more general form, we again only know that the ẇ=0 locus lies in the KE and IE regions and goes through the WE point. This case may yield an oscillatory KE. As in the previous model, there is an abundance of other possibilities.

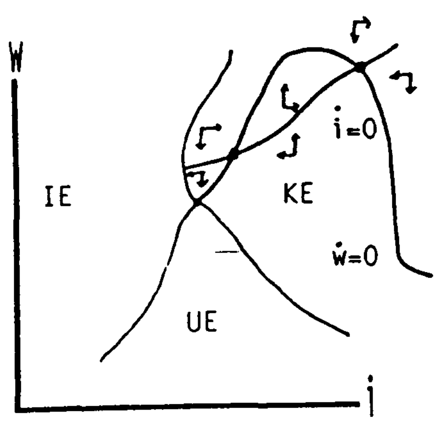


Figure 12 Dynamics of moneyless model with wage dynamics II

We can heuristically argue when the system has an oscilllatory equilibria. Increased inventories (cetaris paribus) decrease the demand for labor in a given period and thus depress the real wage. However they also tend to raise the desired sales of the firm, placing downward pressure on prices and tending to push up the real wage. If the second influence dominates the first over an appropriate range of inventory levels, we have an oscillatory Keynesian equilibrium as in figure 12.

Trying to establish the local stability of this KE by analyzing the linearized system about the point proved unenlightening. Too many of the signs are indeterminate, and stability cannot be directly established as in the previous case. We note, however, that instability is no more apparent than stability when the linearized system is examined.

Finally, if we remove the artificial restriction on the real wage level, so that the dynamic path may move through the UE region we gain little information. The slope of the $\dot{i}=0$ locus is indeterminate in the UE region, but this doesn't affect any of our qualitative re'sults.

The General Model

We will now examine the stability of KE in the general model with both money and inventories. Unfortunately our 'main tool--the phase diagram--will be unavailable to us. With one state variable (one first order equation), a phase diagram trivially gives the stability of any equilibrium point. In two-dimensional systems, it is not always clear, but it does help illustrate general tendencies. But as with all graphic tools it is almost completely useless in three dimensions.

Of course the modified Fillipov and Lyapunov techniques are even less helpful here than they were in the simpler cases. Thus, for the general model, we follow the suggestions of our analysis above and posit the existence of a KE quasi-equilibrium and examine its local stability by studying the linearized version of the system about the point.

Our generic form for the differential system in the general model is:

(45) $\dot{m} = g - r(x) - mh_1(x) = A_1(x),$

$\dot{w} = w[h_1(x) - h_2(x)] = A_2(x),$

$\dot{i} = F(L) - Y = A_3(x),$

Translating to the origin and taking the linear approximation of the system we have:

(46) $\dot{m} = [\partial A_1/\partial m]m + [\partial A_1/\partial w]w + [\partial A_1/\partial i]i,$

$\dot{w} = [\partial A_2/\partial m]m + [\partial A_2/\partial w]w + [\partial A_2/\partial i]i,$

$\dot{i} = [\partial A_3/\partial m]m + [\partial A_3/\partial w]w + [\partial A_3/\partial i]i.$

We can denote this system by $x = Ax$ or more explicitly:

(47)
$$x = \begin{bmatrix} a_{11} a_{12} a_{13} & m \\ a_{21} a_{22} a_{23} & w \\ a_{31} a_{32} a_{33} & i, \end{bmatrix}$$

where the coefficients of the system are given by:

$a_{11} = \partial A_1/\partial m = -\partial r/\partial m + mh_{,1}\partial Y^{h+}/\partial m \; mh_{12}\partial Y^{h+}/\partial m,$

$a_{12} = \partial A_1/\partial w = -\partial r/\partial w + mh_{11}\partial Y^{h+}/\partial w - mh_{12}\partial Y^{h+}/\partial w,$

$a_{13} = \partial A_1/\partial i = 0.$

(48) $a_{21} = \partial A_2/\partial m = w[h_{21}\partial L^{f+}/\partial m - h_{22}\partial L^{h+}/\partial m - h_{11}\partial Y^{h+}/\partial m - h_{12}\partial Y^{f+}/\partial m],$

$a_{22} = \partial A_2/\partial w = w[h_{21}\partial L^{f+}/\partial w - h_{22}\partial L^{h+}/\partial w - h_{11}\partial Y^{h+}/\partial w - h_{12}\partial Y^{f+}/\partial w],$

$a_{23} = \partial A_2/\partial i = 0.$

$$a_{31} = \partial A_3/\partial m = 0,$$

$$a_{32} = \partial A_3/\partial w = (F')\partial L^{h+}/\partial w - \partial Y^{f+}/\partial w,$$

$$a_{33} = \partial A_3/\partial i = (F')\partial L^{f+}/\partial i - \partial Y^{h+}/\partial i.$$

We can then prove:

Theorem II

Keynesian equilibria of the general model are stable. See appendix for proof.

VI. Summary and Conclusions

The essence of Walrasian equilibrium theory is that prices clear markets. The essence of non-Walrasian equilibrium theory is that they do not; quantities adjust faster than prices, and some agents are rationed.

Both approaches have developed rigorous atemporal models proving the existence of equilibrium. Although Walrasian static models are more elegant, they agree less with the stylized facts of the world. We have seen that unemployment is natural in non-Walrasian worlds. However, unemployment must be forced into Walrasian models with *ad hoc* specifications on information, utility functions, technology shocks, or other areas. At least these extensive efforts to coax employment swings out of equilibrium models show that Walrasian theorists realize the existence of unemployment. But we believe non-Walrasian models capture a greater slice of the reality of markets and the causes of unemployment.

Simple Walrasian dynamics (which have not progressed beyond the tatonnement) similarly cannot explain the persistent unemployment modern economies experience, while we have seen that our model can exhibit Keynesian unemployment as a steady state. Again equilibrium models can exhibit prolonged unemployment with various modifications, but we find it at least as plausible to postulate disequilibrium as to impose some derivative restrictions on an equilibrium model.

However, equilibrium analysis and comparative statics (for Walrasian or fixprice worlds) are applicable only if the dynamics of a model are stable. More emphasis should be placed on dynamics, whether equilibrium or disequilibrium. The assumption of stability, like the assumption of market-clearing prices, is justified as a necessary simplification in developing tractable models. But both issues are crucial to the results of stable flexprice models, and neither is theoretically or empirically clear. This paper has explored discarding both assumptions.

Appendix

Proof of Theorem ▮

From the real money stock and wage differential equation (34), we have

(49) $$w = \dot{w}[h_{21}(D^{l+}) - h_{22}(S^{l+}) - h_{11}(D^{Y+}) + h_{12}(S^{Y+})],$$

$$\dot{m} = g - r(x) + (m)[h_{11}(D^{Y+}) - h_{12}(S^{Y+})].$$

We translate the posited KE to the origin and take a linear approximation of the dynamic system to rewrite our differential equations as:

(50) $$\dot{m} = a_{11}m + a_{12}w,$$

$$\dot{w} = a_{21}m + a_{22}w,$$

or in the shorthand:

(51) $$\dot{x} = Ax,$$

where our coefficients in the matrix A are given by:

(52) $$a_{11} = -\partial r/\partial m + mh_{11}\partial Y^{h+}/\partial m - mh_{12}\partial Y^{h+}/\partial m,$$

$$a_{12} = -\partial r/\partial w + mh_{11}\partial Y^{h+}/\partial w - mh_{12}\partial Y^{h+}/\partial w,$$

$$a_{21} = w[h_{21}\partial L^{f+}/\partial m - h_{22}\partial L^{h+}/\partial m - h_{11}\partial Y^{h+}/\partial m - h_{12}\partial Y^{f+}/\partial m]$$

$$a_{22} = w[h_{21}\partial L^{f+}/\partial w - h_{22}\partial L^{h+}/\partial w - h_{11}\partial Y^{h+}/\partial w - h_{12}\partial Y^{f+}/\partial w]$$

Then the characteristic equation is derive'd from the determinant of A–XI:

(53) $$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}),$$

and the stability of the system depends on the negativity of the roots of this polynomial.

Instead of directly solving the quadratic equation for λ, we make use of the equivalent Routh–Hurwicz conditions for the stability of linear systems. In this case we have stability **if**:

(54) $a_{11} + a_{22} < 0,$

$a_{11}a_{22} - a_{12}a_{21} > 0.$

Under KE we have the following forms for the components of the dynamic equations:

(55) real profits: $r(x)/p = Y^{h+} + g - wL^{f+},$

excess supply in L: $Z^{l} = L^{h+} - L^{f+} < 0,$

excess supply in Y: $Z^{Y} = y + - y^{h}+ < 0.$

We examine each term of the matrix A and try to pin down a sign; we do not try to compare quantities in establishing equilibrium, since our model is entirely qualitative:

(56) $a_{11} = -\partial r/\partial m + mh_{11}\partial Y^{h+}/\partial m - mh_{12}\partial Y^{h+}/\partial m$

$= mh_{11}\partial Y^{h+}/\partial m - mh_{12}\partial Y^{h+}/\partial m$

$= (-p + mh_{11} - m_{h12})\partial Y^{h+}/\partial m - (w)\partial L^{f+}/\partial m,$

because (6) implies

(57) $\partial r/\partial m = (p)\partial Y^{h+}/\partial m - (w)\partial L^{f+}/\partial m.$

By (9) we have

(58) $\partial Y^{h+}/\partial m > 0; \quad (w)\partial L^{f+}/\partial m = 0.$

We will show $a_{11} > 0$ by demonstrating that:

(59) $(-p + mh_{11} - m_{h12}) < 0.$

Define:

(60) $H = h_{11} + h_{12}.$

Then from our linear price dynamics in equation (32) we have:

(61) $h_{11} = (p/p + HY^{f+})/(Y^{h+} + Y^{f+}),$

$h_{12} = (HY^{h} - p/p)/(Y^{h+} + Y^{f+}).$

Then (59) can be expressed as:

(62) $$-p + m[(p/p + HY^{f+})/(Y^{h+} + Y^{f+})] - m[(HY^{h} - p/p)/(Y^{h+} + Y^{f+})]$$

$$= [-p(Y^{h+} + Y^{f+}) + mHY^{f+} - mHY^{h+}]/(Y^{h+} + Y^{f+}),$$

and since in KE we have $Y^{h+} < Y^{f+}$ we immediately have:

(63) $$a_{11} < 0$$

(64) $$a_{12} = -\partial r/\partial w + mh_{11}\partial Y^{h+}/\partial w - mh_{12}\partial Y^{f+}/\partial w.$$

From (6) we have:

(65) $$\partial r/\partial w < 0$$

trivially. Assumption (9) gives:

(66) $$\partial Y^{h+}/\partial w > 0; \quad \partial Y^{f+}/\partial w < 0,$$

and so we have:

(67) $$a_{12} > 0,$$

(68) $$a_{,1} = w[h_{21}\partial L^{f+}/\partial m - h_{22}\partial L^{h+}/\partial m - h_{11}\partial Y^{h+}/\partial m - h_{12}\partial Y^{f+}/\partial m].$$

"Since:

(69) $$\partial L^{f+}/\partial m = 0; \quad \partial Y^{f+}/\partial m < 0,$$

we will show that:

(70) $$- h_{22}\partial L^{h+}/\partial m - h_{11}\partial Y^{h+}/\partial m < 0,$$

to demonstrate that a_{21} is positive. This inequality can be rewritten as:

(71) $$h_{11}\partial Y^{h+}/\partial m > -h_{22}\partial L^{h+}/\partial m.$$

Integrating with respect to m yields:

(72) $$h_{11}Y^{h+} > -h_{22}L^{h+}.$$

Since Y, L, and the h's are positive, this confirms the inequality, and we have:

(73) $a_{21} < 0,$

(74) $a_{22} = w[h_{21}\partial L^{f+}/\partial w - h_{22}\partial L^{h+}/\partial w - h_{11}\partial Y^{h+}/\partial w - h_{12}\partial Y^{f+}/\partial w].$

Under assumption (9) we can sign each term as follows:

(75) $\partial L^{f+}/\partial w < 0; \quad \partial L^{h+}/\partial w > 0; \quad \partial Y^{h+}/\partial w > 0; \quad \partial Y^{f+}/\partial w < 0.$

So we have:

(76) $a_{22} < 0.$

The basic model, then, qualitatively satisfies the Routh-Hurwicz conditions for stability:

(77) (i) $a_{11} + a_{22} = (-) + (-) < 0;$

(ii) $a_{11}a_{22} - a_{12}a_{21} = (-)(-) - (-)(+) = (+) - (-) > 0,$

so we have stability for all KE. This completes the proof of Theorem I.

<u>Proof of Theorem II</u>

From our linearized inventoryless model we have:

(78) $a_{11} < 0; \quad a_{12} > 0,$

$a_{21} < 0; \quad a_{22} < 0.$

We also have:

(79) $a_{13} = \partial A_1/\partial i = 0,$

$a_{23} = \partial A_2/\partial i = 0,$

$a_{,,} = \partial A_3/\partial m = 0.$

Then the only unsigned terms are a_{32} and a_{33}; they are easily signed:

(80) $a_{32} = (F')\partial L^{h+}/\partial w - \partial Y^{f+}/\partial w.$

From assumptions (2) and (6) we have:

(81) $F' > 0;\ \partial L^{h+}/\partial w > 0;\ \partial Y^{f+}/\partial w < 0,$

and so we have:

(92) $a_{32} > 0.$

(83) $a_{33} = (F')\partial L^{h+}/\partial i - \partial Y^{h+}/\partial i.$

Under assumptions (2) and (6) we have:

(84) $F' > 0;\ \partial L^{h+}/\partial i = 0;\ \partial Y^{f+}/\partial i < 0,$

and thus:

(85) $a_{33} < 0.$

Then qualitatively our matrix of coefficients A for the linearized system is:

(86)
$$A = \begin{bmatrix} (-) & (+) & (0) \\ (-) & (-) & (0) \\ (0) & (+) & (-) \end{bmatrix}$$

It is then easy to show that this linear system is stable. We demonstrate that the real part of each eigenvector of the matrix must be negative by showing that A is negative definite. For any vector

$z = (z_1, z_2, z_3)$ we have qualitatively:

(87) $z'Az = (z'A)z = [-z_1 - z_2,\ z_1 - z_2 + z_3,\ -z_3][z_1, z_2, z_3]'$

$= -z_1^2 - z_1z_2 + z_1z_2 - z_2^2 + z_2z_3 - z_1z_3 - z_2z_3 - z_3^2)$

$= -z_1^2 - z_2^2 - z_3^2 - z_1z_3$

It is then sufficient to show:

(88) $\left|z_1^2 + z_3^2\right| > \left|z_1z_3\right|,$

to prove negative definiteness. But this inequality is trivial; squaring both sides yields the result immediately. So we have shown that when it exists, the linearized version of our dynamic system at a Keynesian equilibrium will be stable.

References

Arrow, Kenneth, J., and F. H. Hahn. General Competitive Analysis. San Francisco: Holden Day, 1971.

Arrow, Kenneth J. "Towards a Theory of Price Adjustment," in A. Abramovitz, ed., The Allocation of Economic Resources. Stanford: Stanford University Press, 1959.

________, H. Block, and L. Hurwicz. "On the Stability of the Competitive Equilibrium II," Econometrica, 27:1 (January 1959), pp. 82-109.

________, and L. Hurwicz. "On the Stability of the Competitive Equilibrium I," Econometrica 26:1 (October 1958), pp. 522-52.

Blad, Michael C , and E. Christopher Zeeman. "Oscillations between Repressed Inflation and Keynesian Equilibria Due to Inertia in Decision Making," Journal of Economic Theory, vol. 28, no. 1 (October 1982), pp. 165-82.

Bohm, Volker. "Disequilibrium Dynamics in a Simple Macroeconomic Model," Journal of Economic Theory, vol. 17, no. 2 (April 1978), pp. 179-99.

Debreu, Gerard. The Theory of Value. Cowles Foundation Monograph No. 17, New Haven: Yale University Press, 1959.

Dehez, P., and J. Jaskold Gabszewicz. "On the Convergence of Sequences of Disequilibria," CORE Discussion Paper No. 7701 (1977).

Eckalbar, John C. "Stable Quantities in Fixed Price Disequilibrium," Journal of Economic Theory, vol. 25, no. 2 (October 1981), pp. 302-13.

________. "The Stability of Non-Walrasian Processes: Two Examples," Econometrica, 48:2 (March 1980) pp. 371-86.

Fisher, Franklin M. Disequilibrium Foundations of Equilibrium Theory. Econometric Society Monograph in Pure Theory, No. 6, New York: Cambridge University Press, 1984.

Grandmont, Jean-Michel. "Temporary General Equilibrium Theory," in Kenneth J Arrow and M. Intriligator, eds., The Handbook of Mathematical Economics: vol. II. Amsterdam: North-Holland Publishing Company, 1982, pp. 879-922.

Hansen, Bent. A Survey of General Equilibrium Systems. New York: McGraw-Hill, 1970.

Honkapohja, Seppo. "On the Dynamics of Disequilibria in a Macro Model with Flexible Wages," in Masanao Aokio, and A. Marzollo, eds., New Trends in Dynamic System Theory and Economics. New York, NY: Academic Press, 1979.

_______, and Takatoshi Ito. "Stability with Regime Switching," Journal of Economic Theory, vol. 29, no. 1 (February 19831, pp. 22-48.

_______. "Inventory Dynamics in a Simple Disequilibrium Model," Scandanavian Journal of Economics, vol. 82, no. 2 (19801, pp. 184-98.

_______. "Non-trivial Equilibrium in an Economy with Stochastic Rationing," NBER Working Paper No. 322 (February 19791.

Jordan, Dominic, and P. Smith. Nonlinear Ordinary Differential Equations. Oxford, UK: Oxford University Press, 1979.

Malinvaud, Edmond. Profitability and Unemployment. Cambridge, UK: Cambridge University Press, 1980.

_______. The Theory of Unemployment Reconsidered. Blackwell: Oxford University Press, 1977.

Samuelson, Paul Anthony. Foundations of Economic Analysis. New York, NY: Atheneum, 1979.

Shapley, Lloyd, and Martin Shubik. "Trade Using One Commodity as a Means of Payment," Journal of Political Economy, vol. 85, no. 5 (1977) pp. 937-68.

Sneessens, Henri, R. "Theory and Estimation of Macroeconomic Rationing Models," in M. Beckmann and H.P. Kunzi, eds., Lecture Notes in Economics and Mathematical Systems, vol. 191, Berlin: Springer-Verlag, 1981.

Tobin, James. Asset Accumulation and Economic Activity. Chicago: The University of Chicago Press, 1980.

van den Heuvel, Paul. "The Stability of an Economic System with Quantity Constraints," Lecture Notes in Economics and Mathematical Systems, vol. 211, Berlin: Springer-Verlag, 1983.

Veendorp, E. C. H. "Stable Spillovers among Substitutes," Review of Economic Studies, vol. 42, no. 3 (September 1975), pp. 445-56.