Bubbly Recessions

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Abstract

We develop a tractable rational bubbles model with financial frictions, downward nominal wage rigidity, and the zero lower bound. The interaction of financial frictions and nominal rigidities leads to a "bubbly pecuniary externality," where competitive speculation in risky bubbly assets can result in excessive investment booms that precede inefficient busts. The collapse of a large bubble can push the economy into a "secular stagnation" equilibrium, where the zero lower bound and the nominal wage rigidity constraint bind, leading to a persistent and inefficient recession. We evaluate a macroprudential leaning-against-the-bubble policy that balances the trade-off between the booms and busts of bubbles.

1 Introduction

In the recent decades, many countries in the world, including Japan, the U.S., and several European economies, have experienced episodes of rapid speculative booms and busts in asset prices, followed by declines in economic activities and in some cases persistent recessions. More generally, throughout history the collapse of large asset and credit booms tend to precede recessions and crises (e.g., Kindleberger and O'Keefe, 2001; Jordà et al., 2015). These experiences have led policymakers to be increasingly aware of the potential risks of

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asset price bubbles, leading to discussions of macroprudential regulations such as "leaning-against-the-wind" policies – preventive measures to curb the booms in asset prices in order to mitigate the eventual busts.

However, despite the recent developments in the macroeconomic literature on asset bubbles, relatively little theoretical framework has analyzed the potential efficiency trade-off between the booms and busts of risky bubbly episodes and whether preventive policies are warranted. In particular, in most rational bubble models – the workhorse models to study the macroeconomic effects of bubbles in general equilibrium – private agents correctly perceive the risk of speculating in a bubbly asset and bubbles generally improve the efficiency of the financial system (e.g., see the literature surveys in Barlevy, 2007, 2012, and more recently, Miao, 2014 and Martin and Ventura, 2017).

In this paper, we develop a tractable growth model to address the question of when and how risky rational bubbles can lead to inefficiencies and evaluate the welfare trade-off. We focus on the interaction of financial frictions and nominal rigidities during bubbly episodes. We posit an economy where entrepreneurial agents with heterogeneous productivity accumulate capital and, due to limited commitment, face financial frictions that constrain their ability to borrow from each other (as in Kiyotaki et al., 1997 or Carlstrom and Fuerst, 1997). With sufficient financial friction, speculative bubbles may arise. A rational bubble is an asset that is traded above its fundamental value; an agent purchases the overvalued asset because he or she expects to be able to sell it later. Such a bubble is inherently fragile, because it requires a coordination of beliefs across agents and time. We model the fragility by assuming that in each period the price of the bubbly asset can collapse to the fundamental value with an exogenous probability.

The possibility of trading the bubbly asset facilitates the reallocation of resources across time, because the bubbly asset can act as a savings vehicle. Trading also facilitates reallocation across agents, because the bubbly asset increases entrepreneurs' net worth and hence their ability to borrow. Thus, the boom in the price of a bubbly asset leads to a boom in credit, investment, output, wages, and consumption. When the boom finally turns into a bust, the economy simply converges back to the pre-bubble economy. Therefore, with financial frictions alone, the model so far implies that speculative bubbles help to crowd in productive investment and improve the overall efficiency of the economy, as implied in most existing expansionary bubble frameworks (e.g., Hirano et al., 2015; Hirano and Yanagawa, 2017). However, the implications will change with nominal rigidities.

We introduce nominal rigidities in the form of downward nominal wage rigidity (à-la Schmitt-Grohé and Uribe, 2016, 2018) and a zero lower bound (ZLB) on the nominal interest rate. The wage rigidity may prevent the labor market from clearing, while the ZLB may

prevent the monetary authority from setting the nominal interest rate (according to a Taylor-type feedback rule) to achieve inflation targeting. We show that the combination of these forces can significantly affect the macroeconomic dynamics after the collapse of a bubble.

When an expansionary bubble collapses, the net worth of entrepreneurial agents also falls, leading to contractions in credit and investment. Thus, the demand for labor from firms also contracts. In a flexible labor market, wages will fall to clear the labor market. However, when wages cannot flexibly fall due to the nominal wage rigidity, there is rationing in the labor market, resulting in involuntary unemployment. An increase in unemployment can in turn lead to an endogenous and protracted recession by eroding the intertemporal allocation of resources. This is because the drop in employment reduces the return to capital investment, which then lowers entrepreneurs' net worth. This further leads to a contraction in capital investment, since entrepreneurs' ability to borrow and invest depends critically on their net worth. Therefore, the future capital stock will decline, causing further downward pressure on labor demand and wages, thus reducing future capital accumulation. The vicious cycle repeats and only stops when the capital stock has fallen enough, often undershooting the bubbleless steady-state level. Then, the speed of capital decumulation slows, and eventually the declining rigid wage constraint falls below the wage level consistent with full employment. At that point, the economy exits the unemployment spell and enters a process of gradual recovery toward the bubbleless steady state.

More interestingly, we then show that the collapse of a large expansionary bubble triggers a sharp drop in the real interest rate, pushing the nominal interest rate against the ZLB. The intuition is as follows. By crowding in capital investment, the bubble leads to an investment boom. Thus, after the bubble collapses, the economy enters the post-bubble phase with a capital stock above the steady state, a situation that has been referred to as an "investment hangover" (Rognlie et al., 2014). The high capital stock implies a low marginal product of capital and a low real interest rate. The collapse of a sufficiently large bubble can thus push the real interest rate so low that the ZLB binds. We show that, under certain conditions, the post-bubble economy may fall into a liquidity trap *steady state*, or "secular stagnation," where employment and investment are persistently and inefficiently low and inflation is below target. A vicious cycle can arise from the interaction between (i) a low interest rate environment, which constrains the monetary authority from raising inflation, exacerbating the nominal wage rigidity and unemployment problem and (ii) inefficient unemployment that lowers the marginal product of capital, which in turn lowers the interest rates. In the absence of other shocks, this cycle can keep the economy in a persistent slump.

The fundamental source of inefficiencies in this paper is the interaction between financial frictions and nominal rigidities. Financial frictions are the key ingredient for creating an

environment that is fertile for bubbles, whose booms and busts can lead to booms and busts in credit and investment. But financial frictions alone are not sufficient to create a post-bubble secular stagnation. It is the combination of financial frictions and nominal rigidities that introduces a form of "bubbly pecuniary externality," as individual bubble speculators do not internalize the crowd-in effect of bubbles on investment. As aforementioned, after the bubble collapses, the excess boom in the capital stock leads to an undershooting of the real interest rate, pushing the nominal interest rate against the ZLB. The boom and bust in investment also leads to a boom and bust in real wages, which, in the presence of downward nominal wage rigidity and constrained monetary policies due to the ZLB, can push the economy into a secular stagnation with inefficient unemployment and depressed investment.

In summary, by combining nominal rigidities into a model of rational bubbles with financial frictions, *our theory identifies the booms and busts of speculative bubbly episodes as an important source of shocks that can trigger a decline into a persistent secular stagnation.* This source of shocks is very relevant in the current environments of developed economies such as the U.S., Japan, and those in the E.U., where the interest rates have been generally low, creating a fertile ground for bubbles to arise. It is complementary to other sources of shocks that have been highlighted in the literature, including but not limited to deleveraging shocks (e.g., Eggertsson and Krugman, 2012; Korinek and Simsek, 2016), shocks to inflation expectations (e.g., Schmitt-Grohé and Uribe, 2018), or idiosyncratic risk shocks (e.g., Christiano et al., 2014; Acharya and Dogra, 2017). Understanding the different possible root causes of inefficient recessions is important, as each cause implies a different set of policy responses.

In particular, our theory naturally implies that a "leaning-against-the-bubble" type of macroprudential policy intervention is warranted for excessively large bubbles. To clearly illustrate this implication, we model the macroprudential policy in a reduced form as a marginal tax on individual investment in the bubbly asset.[1] We show that, by making the individual investor internalize the general equilibrium effect of her speculative investment in the bubbly asset, an optimally chosen tax can help the economy avoid experiencing an excessively large boom in investment and the subsequent post bubble periods with inefficient unemployment. In other words, the policy helps balance the trade off between the welfare gains from the boom and the welfare losses from the bust of a bubbly episode.

The model's implications are consistent with the accounts of several speculative boom-bust cycles in history. A prominent example is the collapse of the Japanese bubble in the

[1] As in most of the literature, we implicitly assume that policymakers can observe the bubble. Of course, this is a strong assumption. Alternatively, one can interpret the macroprudential policy as imposing a tax on speculative investments in broad classes of assets that are ex ante perceived to be likely to experience bubbles, such as real estate or stocks of certain types of companies.

early 1990s and the subsequent "lost decade(s)." Both housing prices and stock prices in Japan experienced a dramatic boom in the 1980s; the Nikkei index roughly tripled, and the housing price index nearly doubled in the second half of the decade, as seen in the bottom right panel of Figure A.1. The asset prices reached the peak in 1990, when the total market value of land in Japan famously exceeded four times that in the U.S. The boom in asset prices was associated with high growth in GDP and (real and nominal) wages. However, the boom turned into the bust of the early 1990s, with asset prices beginning to fall in 1991. This coincided with the onset of a protracted period of low economic growth and high unemployment that lasted several decades, well into the 2000s. As seen in the top left panel of Figure A.1, a trend of high GDP growth abruptly ended in 1991, and the unemployment rate more than doubled from around 2% in 1991 to around 5.5% in 2002. Despite the rising unemployment rate, both nominal and real wages persisted near the peak levels of the boom, as seen in the figure's top right panel. The collapse of the asset price bubble in 1991 also coincided with significant changes in the nominal interest rate and inflation, as seen in the bottom left panel. The combination of falling asset prices, low nominal interest rates near the ZLB, disinflation, and rigid wages is a prominent feature of the onset and persistence of the Japanese lost decade.

To some extent, the more recent boom and bust of the U.S. housing and stock bubbles in the late 2000s (Figure A.2's bottom right panel) was also associated with similar movements in macroeconomic activities. As seen in the figure's top left panel, the collapse in asset prices coincided with the onset of the Great Recession. The average unemployment rate doubled from a low of about 5% in 2007 to about 10% in 2009 and remained above the pre-recession rate until 2015. The collapse in asset prices also coincided with abrupt changes in the nominal interest rate and inflation (the bottom left panel). The nominal interest rate effectively hit the ZLB between 2009 and 2015, and the economy slipped into deflation between 2009 and 2010. In the mean time, the average nominal wage continued to grow at the pre-recession trend (the top right panel).

Related literature. To the best of our knowledge, our paper is one of the first to show that the collapse of bubbles can trigger long recessions and liquidity traps. Our paper thus makes contributions to several strands of the literature.

First, we help formalize the popular notion among policymakers that the collapse of risky bubbles can trigger inefficient recessions. A large number of papers emphasize the positive aspect of (rational) bubbles in reducing dynamic inefficiencies (e.g., Samuelson, 1958; Diamond, 1965; Tirole, 1985) or reducing intratemporal inefficiencies in the allocation of resources (e.g., Farhi and Tirole, 2011; Miao and Wang, 2012, 2018; Martin and Ventura, 2012;

Ikeda and Phan, 2015; Bengui and Phan, 2016; Graczyk and Phan, 2016).[2] Other papers emphasize potential ex-ante inefficiencies of speculative bubble investment in diverting resources away from productive investment (e.g., Saint-Paul, 1992; Grossman and Yanagawa, 1993; King and Ferguson, 1993; Hirano et al., 2015), generating excessive allocations of resources in certain sectors (e.g., Cahuc and Challe, 2012; Miao et al., 2014), or generating excessive volatility (Caballero and Krishnamurthy, 2006; Ikeda and Phan, 2016). Our paper complements this literature and highlights the ex-post inefficiency of bubbles by showing that their collapse can cause persistent involuntary unemployment. As a consequence, our paper formalizes the policy-relevant trade-off between the gains during a bubble's boom and the losses during the bubble's bust.

Our framework is most related to the influential rational bubbles model with infinitely lived agents developed by Hirano et al. (2015) and Hirano and Yanagawa (2017).[3] A common theme of these papers is that they identify financial frictions as a key element in facilitating asset bubbles, and they abstract away from nominal rigidities. However, as aforementioned, financial frictions alone are not sufficient to generate a post-bubble equilibrium with a persistent secular stagnation.

By embedding downward nominal wage rigidity into a rational bubbles framework, our paper is related to our earlier work, Hanson and Phan (2017).[4] There, we developed a simple overlapping generations model based on the classic frameworks of Tirole (1985). A key difference is that the previous paper does not study the effects of the ZLB and therefore the collapse of a bubble can at most cause a transitory recession. Furthermore, a limitation of the overlapping generations framework in Hanson and Phan (2017) is that a period represents twenty or thirty years. This makes the model less appropriate for policy analyses at the business cycles frequency. In contrast, in the current paper, agents are fully forward-looking and infinitely lived, and a period can be interpreted as a quarter or a year.

Our paper is also related to the overlapping-generations model of Asriyan, Fornaro, Martin and Ventura (2016) and provides a complementary approach to explaining post-bubble liquidity traps.[5] As in our paper, they show that the collapse of a bubble can lead to a fall in the real interest rate that could push the economy into a liquidity trap. In their

[2]Besides the rational bubble literature, see Abreu and Brunnermeier (2003), Doblas-Madrid and Lansing (2014), and Barlevy (2014) for alternative approaches to modeling bubbles based on heterogeneous information or beliefs.

[3]Related frameworks with infinitely lived agents are developed by Kocherlakota (2009), Aoki and Nikolov (2015), and Miao and Wang (2018).

[4]For a complementary approach to modeling post-bubble unemployment using a search and matching model à-la Diamond-Mortensen-Pissarides, see Kocherlakota (2011) and Miao et al. (2016).

[5]For a related and emerging body of literature that analyzes the effects of monetary policies on rational bubbles, see Gali (2014, 2016), Ikeda (2016), Dong et al. (2017), and Hirano et al. (2017).

model, inefficiency arises in the liquidity trap as the holding of cash crowds out productive investment. In contrast, the inefficiency arises in our model because of the aforementioned bubbly pecuniary externality. Moreover, while they introduce a new form of nominal rigidity through the assumption that expectations about the future values of bubbly assets are set in nominal terms, we assume a nominal wage rigidity that is relatively standard in the recent New Keynesian literature.

Second, our paper is related to a large literature that investigates possible sources of shocks that trigger long recessions and liquidity traps in environments with New Keynesian frictions. Many papers have emphasized demand shocks driven by household deleveraging or tightening borrowing constraints (Eggertsson and Krugman, 2012; Christiano et al., 2015; Korinek and Simsek, 2016; Schmitt-Grohé and Uribe, 2016), long-run factors such as aging demographics or safe asset shortages (Summers, 2013; Caballero and Farhi, 2017; Eggertsson and Mehrotra, 2014; Eggertsson et al., 2016), or overinvestment of capital (Rognlie et al., 2014). By highlighting the role of rational asset bubbles, our analysis offers a complementary narrative to those in the literature. Furthermore, in our model, the collapse of bubbles reduces the net worth of borrowers and consequently leads to an endogenous tightening of borrowing constraints in equilibrium, thus giving a possible microfoundation for the deleverage shocks in, e.g., Eggertsson and Krugman (2012) and Korinek and Simsek (2016). Similarly, in our model, expansionary bubbles lead to an endogenous boom in capital investment, thus giving a microfoundation to the investment overhang in Rognlie et al. (2014).

Finally, by providing a normative analysis with macroprudential policies on speculative bubble investment, our paper complements the literature on macroprudential policies in environments with financial frictions or aggregate demand externalities (e.g., Lorenzoni, 2008; Olivier and Korinek, 2010; He and Krishnamurthy, 2011; Bianchi, 2011; Eberly and Krishnamurthy, 2014; Farhi and Werning, 2016; Bianchi and Mendoza, forthcoming).

The plan for the paper is as follows. Section 2 describes the model. Section 3 studies the bubbleless equilibrium and steady states, while Section 4 analyzes the bubbly equilibrium and steady states. Section 5 analyzes the inefficient dynamics of the post-bubble economy. Section 6 provides a policy analysis. Section 7 concludes. Detailed derivations and proofs are in the Appendix.

2 Model

Consider an economy with two types of goods: a perishable consumption good and a capital good. Time is infinite and discrete. Firms are competitive, and there exist two types of

agents, called entrepreneurs and workers, each with constant unit population. Entrepreneurs and workers have the same preferences over consumption, given by

$$E_0\left(\sum_{t=0}^{\infty}\beta^t \ln c_t^j\right)$$

where c_t^j is the consumption of an individual j in period t, $\beta \in (0,1)$ is the subjective discount factor, and $E_0(\cdot)$ is the expected value conditional on information in period 0.

2.1 Entrepreneurs

Entrepreneurs are the only producers of the capital good and face idiosyncratic productivity shocks. In each period, an entrepreneur meets either a high-productivity investment project (and becomes the *H-type*) with probability $h \in (0,1)$, or a low-productivity one (and becomes the *L-type*) with probability $1-h$. The idiosyncratic productivity shock is independent across agents and time. For stationarity, we assume that the initial ($t=0$) population measure of each type is h and $1-h$. In each period, we denote the set of H-type entrepreneurs by $\mathcal{H}_t$ and the set of L-type entrepreneurs by $\bar{\mathcal{H}}_t$, where $\mathcal{H}_t \cup \bar{\mathcal{H}}_t = [0,1]$.

After knowing the type of her investment project at the beginning of each period, an entrepreneur j produces the capital good according to the following technology:

$$k_{t+1}^j = a_t^j I_t^j,$$

where I_t^j is the investment in units of the consumption good in period t, k_{t+1}^j is the amount of the capital good produced in the subsequent period, and $a_t^j \in \{a^H, a^L\}$ is the productivity of the project, where $a^H > a^L > 0$. For tractability, we assume capital depreciates completely after each period.[6]

Financial frictions: In a frictionless world, L-type entrepreneurs would like to lend and thus delegate investment to H-type entrepreneurs. Agents can borrow and lend through one-period debt contracts. However, as in Kiyotaki et al. (1997) and Hirano and Yanagawa (2017), we assume there are frictions in the financial market so entrepreneurs can pledge at most an exogenous fraction $\theta \in [0,1]$ of the future return from capital investment to creditors (and they cannot pledge the return from bubble speculation). Thus, they face the following credit constraint:

$$R_{t,t+1} d_t^j \leq \theta q_{t+1} k_{t+1}^j, \tag{1}$$

where $R_{t,t+1}$ is the state-contingent gross interest rate between t and $t+1$, d_t^i is the amount

[6] Our result does not change qualitatively if capital depreciates more slowly.

borrowed in period t, and q_{t+1} is the price (in units of the consumption good) of capital in period $t+1$. A lower θ represents a financial market with more frictions, while $\theta = 1$ represents a frictionless credit market. Throughout the paper we assume θ is sufficiently small so that constraint (1) always binds for H-types.

Following the literature (e.g., Tirole, 1985), we introduce a (pure) bubbly asset, which is a durable and perfectly divisible asset in fixed unit supply that does not generate any dividend but can be traded at positive equilibrium prices under some conditions. Such an asset bubble is inherently fragile as it requires coordination of beliefs across agents and time. To model this fragility, we follow the literature (e.g., Weil, 1987) and assume that in each period the bubble persists with a probability $\rho \in (0,1)$ and collapses with the complementary probability $1-\rho$, where a lower ρ means a riskier bubble. Formally, let $\tilde{p}_t^b$ denote the period t price per unit of the bubbly asset in units of the consumption good and p_t^b denote the price conditional on the bubble persisting in t. Then

$$\tilde{p}_t^b = \begin{cases} p_t^b & \text{if bubble persists} \\ 0 & \text{if bubble bursts} \end{cases},$$

and

$$\begin{aligned} \Pr(\tilde{p}_{t+1}^b = 0 | \tilde{p}_t^b > 0) &= 1-\rho \\ \Pr(\tilde{p}_{t+1}^b = 0 | \tilde{p}_t^b = 0) &= 1, \forall t \geq 0. \end{aligned}$$

The first assumption states that if the bubble has not collapsed, then it will collapse in the next period with probability $1-\rho$. The second states that if the bubble has collapsed, then it is expected not to reemerge.

Let b_t^j denote a share of a bubbly asset held by entrepreneur j. Then the entrepreneur's flow budget constraint is written as

$$c_t^j + I_t^j + \tilde{p}_t^b b_t^j = q_t k_t^j + d_t^j - R_{t-1,t} d_{t-1}^j + \tilde{p}_t^b b_{t-1}^j. \tag{2}$$

The left-hand side of this budget constraint consists of expenditure on consumption, capital investment, and the purchase of bubbly assets. The right-hand side is the available funds at date t, which consists of the return from capital investment in the previous period, new borrowing minus the debt repayment, and the return from selling bubbly assets. As is standard in the literature (e.g., Martin and Ventura, 2012; Hirano and Yanagawa, 2017; Miao and Wang, 2018), we assume agents cannot invest a negative amount in the capital

stock or the bubbly asset, i.e.,[7]

$$I_t^j, b_t^j \geq 0, \forall t.$$

2.2 Workers

Workers do not have access to capital production technologies. For simplicity, we assume workers are hand to mouth, i.e.,

$$c_t^w = w_t l_t, \tag{3}$$

where w_t is the wage rate and l_t is the employment level per worker.[8]

2.3 Firms

In each period, there is a continuum of competitive firms that produce the consumption good using the standard production function:

$$y_t^i = (k_t^i)^\alpha (l_t^i)^{1-\alpha}, \ 0 < \alpha < 1,$$

where k_t^i and l_t^i are capital and labor inputs of a representative firm i. For simplicity, we have abstracted away from exogenous TFP shocks. Real competitive factor prices are given by:

$$q_t = \alpha \left(\frac{L_t}{K_t} \right)^{1-\alpha} \tag{4}$$

$$w_t = (1-\alpha) \left(\frac{K_t}{L_t} \right)^{\alpha}, \tag{5}$$

where K_t and L_t are the aggregate capital stock and employment.

[7] As otherwise, the ability to short sell would let agents borrow more despite the credit constraint.

[8] Alternatively, we can assume workers cannot borrow against their future labor income. Thus the optimization problem of workers is to maximize lifetime utility $E_0 \left(\sum_{t=0}^{\infty} \beta^t \ln c_t^w \right)$ subject to:

$$c_t^w + p_t^b b_t^w = w_t l_t + d_t^w - R_t d_{t-1}^w + p_t^b b_{t-1}^w$$

and $d_t^w \leq 0$ and $b_t^w \geq 0$. In equilibrium, it can be shown that workers will be effectively hand to mouth, i.e., $c_t^w = w_t l_t$. Intuitively, due to financial friction, the interest rate (and the returns from bubble speculation) will be too low relative to the discount factor, and thereby it will be suboptimal for workers to save or to buy the bubbly asset.

2.4 Downward nominal wage rigidity (DNWR)

Following Schmitt-Grohé and Uribe (2016, 2018), we assume that nominal wages are downwardly rigid:

$$P_t w_t \geq \gamma(1 - L_t) P_{t-1} w_{t-1}, \forall t \geq 1,$$

where the function $\gamma(1-L_t)$ governs the degree of rigidity. The condition states that nominal wage cannot fall below a certain fraction of the nominal wage in last period. For tractability, we parameterize:

$$\gamma(1 - L) \equiv \gamma_0 L^{\gamma_1}, \ \gamma_0, \gamma_1 \geq 0.$$

Note that $\gamma_0 = 0$ corresponds to an environment with flexible wages. The assumption $\gamma_1 \geq 0$ allows nominal wages to become more flexible as unemployment increases.[9] The nominal wage rigidity condition can be rewritten as:

$$w_t \geq \frac{\gamma(1 - L_t)}{\Pi_{t-1,t}} w_{t-1}, \tag{6}$$

where $\Pi_{t-1,t} \equiv \frac{P_t}{P_{t-1}}$ is the gross inflation rate between $t-1$ and t. The evidence for downward nominal wage rigidity (DNWR) has been well documented (see, e.g., Kimura and Ueda, 2001 for Japan, Holden and Wulfsberg, 2009 for the OECD, Babecký et al., 2010 for European economies, and Daly et al., 2012 for the U.S.).

The presence of rigid wages implies that the labor market does not necessarily clear. In each period, even though each worker inelastically supplies one unit of labor, the realized employment L_t per worker in equilibrium is determined by two conditions: feasibility constraint

$$L_t \leq 1, \tag{7}$$

and complementary-slackness condition

$$(1 - L_t) \left(w_t - \frac{\gamma(1 - L_t)}{\Pi_{t-1,t}} w_{t-1} \right) = 0. \tag{8}$$

These equations state that involuntary unemployment ($L_t < 1$) must be accompanied by a binding wage rigidity (6). Conversely, when (6) is slack, the economy must be in full employment ($L_t = 1$).

Remark 1. The main mechanism in our paper should hold for other forms of nominal rigidity. However, we focus on downward nominal wage rigidity due to its tractability, allowing us

[9] As shown in the subsequent section, $\gamma_1 > 0$ and (11) are the conditions for the existence of two bubbleless steady states, one of which features involuntary unemployment.

to solve the model including the transition dynamics in closed form. In a framework with sticky prices, the transitional dynamics will have to be solved numerically (see, e.g., Hirano, Ikeda, and Phan, 2017, which numerically analyzes a similar rational bubbles framework with infinitely lived agents and staggered price setting).

2.5 Monetary policy and zero lower bound (ZLB)

To close the model, we need to specify how price levels are determined. As is standard in the literature, we assume that the entrepreneurs can trade nominal government bonds, which yield an interest rate $1 + i_{t,t+1}$ and are available in net zero supply.[10] In equilibrium, L-type entrepreneurs will be indifferent between investing in the nominal bonds and lending in real terms in the credit market, leading to the following Fisher equation:

$$E_t \left[u'(c^L_{t+1}) \frac{1 + i_{t,t+1}}{\Pi_{t,t+1}} \right] = E_t \left[u'(c^L_{t+1}) R_{t,t+1} \right]. \tag{9}$$

A monetary authority sets the nominal interest rate $1+i_{t,t+1}$ between each period t and $t+1$ according to a Taylor rule subject to a ZLB on $i_{t,t+1}$:

$$1 + i_{t,t+1} = \max \left\{ 1,\ R^f_{t,t+1} \left(\Pi_{t-1,t} \right)^{\zeta} \left(\Pi^* \right)^{1-\zeta} \right\}, \tag{10}$$

where $R^f_{t,t+1}$ is the real interest rate that would prevail with full employment in $t+1$ (i.e., $L_{t+1} = 1$), $\Pi^* > 0$ is an inflation target, and $\zeta > 1$ is a constant. As is standard, the rule implies that if the monetary authority were not constrained by the ZLB, inflation would be stabilized at the target Π^*.[11]

2.6 Equilibrium

Definition. Given initial $k^j_0 = K_0$, $d^j_0 = 0$, $b^j_0 = 1$, p^b_0, a competitive equilibrium consists of prices $\{w_t, q_t, R_{t,t+1}, i_{t,t+1}, p^b_t, P_t\}_{t \geq 0}$ and quantities $\{\{I^j_t, k^j_{t+1}, c^j_t\}_{j \in \mathcal{H}_t \cup \bar{\mathcal{H}}_t}, c^w_t, K_{t+1}, L_t\}_{t \geq 0}$ such that:

- Entrepreneurs and firms optimize,

[10] For algebraic simplicity, we have abstracted away from explicitly introducing the buying and selling of nominal bonds that are in net zero supply into the budget constraints of entrepreneurs.

[11] We do not model optimal monetary policy explicitly here. This is because in our model, an increase in the inflation rate always weakly improves welfare by mitigating the wage rigidity. Thus, setting a very high inflation target to avoid involuntary unemployment and the ZLB will be optimal. Realistically, there are costs of inflation, such as the costs associated with nominal price rigidities, that are not modeled explicitly here. Also, in practice, central banks tend to follow similar Taylor rules with inflation targets.

- Workers' consumption is given by (3),
- The credit market clears: $\int_{j\in\mathcal{H}_t} d_t^j + \int_{j\in\bar{\mathcal{H}}_t} d_t^j = 0$,
- The bubble market clears: $\int_{j\in\mathcal{H}_t} b_t^j + \int_{j\in\bar{\mathcal{H}}_t} b_t^j = 1$ if $\tilde{p}_t^b > 0$,
- The consumption good market clears: $\int_{j\in\mathcal{H}_t}(c_t^j + I_t^j) + \int_{j\in\bar{\mathcal{H}}_t}(c_t^j + I_t^j) + c_t^w = K_t^\alpha L_t^{1-\alpha}$,
- Labor market conditions (6), (7), and (8) hold,
- And Fisher equation (9) and monetary policy rule (10) holds.

As usual, a steady state is an equilibrium where quantities, prices (in units of the consumption good), and inflation are time invariant.

For the rest of the paper, we assume:

$$\Pi^* > \gamma_0 > \frac{\Omega}{a^L}, \tag{11}$$

where

$$\Omega \equiv \left(\frac{h(a^H - a^L)}{1 - \frac{\theta a^H}{a^L}} + a^L \right) \beta.$$

These conditions are necessary for the existence of two steady states: one with full employment and slack ZLB, and one with involuntary unemployment and binding ZLB.

For simplicity, we also assume that in the initial period $t = 0$ the legacy wage W_{-1} is sufficiently small so that the labor market clears in $t = 0$, and the initial capital stock K_0 is sufficiently small such that $a^L \alpha K_0^{\alpha-1} > \frac{1}{\Pi^*}$, and therefore the ZLB does not initially bind.

3 Bubbleless equilibrium and multiple steady states

Let us first characterize the bubbleless equilibrium, where the price of the bubbly asset is equal to its fundamental value of zero throughout, i.e., $p_t^b = 0$ for all t. Detailed derivations are relegated to Appendix A.1. For the rest of the paper, we make the following parametric assumption:

$$\theta < \frac{(1-h)a^L}{a^H}. \tag{12}$$

This assumption states that there is sufficient financial friction (small θ) that the credit market cannot completely absorb the L-type's demand for savings. Hence, in the bubbleless

equilibrium, the L-type is making a positive capital investment (the nonnegativity constraint $I_t^j \geq 0$ is slack for the L-type):

$$I_t^j = \beta e_t^j + d_t^j, \forall j \in \bar{\mathcal{H}}_t,$$

where the net worth e_t^j is:

$$e_t^j \equiv q_t k_t^j - R_{t-1,t} d_{t-1}^j$$

and the equilibrium interest rate will be given by:

$$R_{t,t+1} = a^L q_{t+1}.$$

The price of capital q_t and wage w_t satisfy first-order conditions (4) and (5) of firms.

Turning to the H-type, their credit constraint (1) will bind, leading to the following investment equation:

$$I_t^j = \underbrace{\frac{1}{1 - \frac{\theta q_{t+1} a^H}{R_{t,t+1}}}}_{\text{leverage}} \times \beta e_t^j, \forall j \in \mathcal{H}_t.$$

Combining the investment expressions above for both types yields the following law of motion for the aggregate capital stock:

$$K_{t+1} = \Omega q_t K_t = \Omega \alpha K_t^\alpha L_t^{1-\alpha}. \tag{13}$$

The equilibrium wage w_t and employment L_t depend on whether the DNWR binds or not. Similarly, the inflation rate depends on whether the ZLB on the nominal interest rate binds or not. Formally,

$$w_t = \max\left\{(1-\alpha)K_t^\alpha,\ \gamma(1-L_t)\frac{w_{t-1}}{\Pi_{t-1,t}}\right\}$$

and

$$\frac{\max\left\{1,\ R_{t,t+1}^f \left(\Pi_{t-1,t}\right)^\zeta \left(\Pi^*\right)^{1-\zeta}\right\}}{\Pi_{t,t+1}} = R_{t,t+1},$$

where we have used the fact that there is no uncertainty in the bubbleless environment.

The kink in the Taylor rule due to the ZLB implies that there are two possible bubbleless steady states. In the "good" steady state, there is full employment ($L^* = 1$), the ZLB is slack, and inflation is at the target Π^*; the capital stock solves the steady-state version of equation (13) with $L = 1$, i.e.,

$$K^* = (\alpha\Omega)^{\frac{1}{1-\alpha}}, \tag{14}$$

and the real interest rate is given by:

$$R^* = a^L q^* = a^L \alpha K^{\alpha-1} = \frac{a^L}{\Omega}.$$

Condition (11) guarantees that the ZLB and the DNWR are slack in this steady state. The nominal interest rate is then simply given by the unconstrained Taylor rule $1 + i^* = R^* \Pi^*$.

When $\gamma_1 > 0$, there is another "bad" steady state, where there is involuntary unemployment ($L < 1$), the ZLB binds ($i = 0$) and inflation is below target. The capital stock is given by the steady-state version of (13) with $L < 1$:

$$K = (\alpha\Omega)^{\frac{1}{1-\alpha}} L < K^*.$$

The real interest rate is given by

$$R = a^L q = a^L \alpha \left(K/L\right)^{\alpha-1} = \frac{a^L}{\Omega},$$

which is the same as that in the good steady state. The inflation rate is given by the Fisher equation $R\Pi = 1$, or

$$\Pi = \frac{\Omega}{a^L} < \Pi^*.$$

The employment level L solves the binding nominal wage rigidity, which gives $1 = \frac{\gamma(1-L)}{\Pi}$, or

$$L = \left(\frac{\Omega}{\gamma_0 a^L}\right)^{\frac{1}{\gamma_1}}.$$

Assumption (11) also guarantees that there is involuntary unemployment and the ZLB binds in this steady state, i.e., $L < 1$ and $R\Pi^{\zeta}(\Pi^*)^{1-\zeta} < 1$.

4 Bubbly dynamics

We now analyze a stochastic bubbly equilibrium, where the bubble price conditional on persistence p_t^b is positive for all t. We focus on the relevant parameter range in which the DNWR and the ZLB are slack as long as the bubble persists.[12] Detailed derivations are relegated to the appendix.

Suppose the bubble persists in t, i.e., $\tilde{p}_t^b = p_t^b > 0$. As L-type entrepreneurs face a nonnegativity constraint on capital investment ($I_t^j \geq 0$), it follows that the return from

[12] This is the case when the initial capital stock K_0 and the initial bubble value p_0^b are sufficiently small, the bubble is expansionary (condition (21) below), and $R_b > 1/\Pi^*$, where R_b is given by (20).

lending must weakly dominate the return from capital investment:

$$R_{t,t+1} \geq a^L q_{t+1},$$

where the inequality must hold with equality if $I_t^j > 0$.

Furthermore, let the bubble size (relative to aggregate savings) be defined as

$$\phi_t \equiv \frac{p_t^b}{\beta(q_t K_t + p_t^b)}.$$

Then from the no-arbitrage condition for the L-type between bubble investment and lending, the bubble size evolves according to

$$\phi_{t+1} = \begin{cases} \frac{1}{\beta} \frac{\frac{1-h-\phi_t}{\rho(1-h)-\phi_t}\phi_t}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)+\frac{(1-\rho)(1-h)}{\rho(1-h)-\phi_t}\phi_t} & \text{if } \phi_t \leq \phi^* \\ \frac{\theta}{\beta} \frac{\phi_t}{\rho(1-h)-(1-\theta)\phi_t} & \text{if } \phi_t > \phi^* \end{cases}. \tag{15}$$

By using market clearing conditions, the interest rate can be derived as:

$$R_{t,t+1} = \begin{cases} q_{t+1} a^L & \text{if } \phi_t \leq \phi^* \\ q_{t+1} \frac{\theta a^H (1-\phi_t)}{1-h-\phi_t} & \text{if } \phi_t > \phi^* \end{cases}, \tag{16}$$

where threshold ϕ^* is defined as

$$\phi^* \equiv \frac{(1-h)a^L - \theta a^H}{a^L - \theta a^H} < 1.$$

Above this threshold ϕ^*, the bubble is "large," and below it, the bubble is "small." When $\phi_t \leq \phi^*$, the bubble is small in the sense that it cannot completely crowd out the L-type's (relatively inefficient) investment in capital. When this is the case, the interest rate is given by the indifference condition for the L-type between lending and capital investment ($R_{t,t+1} = a^L q_{t+1}$). However, when $\phi_t > \phi^*$, the bubble is large in the sense that it completely absorbs and crowds out the L-type's investment in capital (the Lagrange multiplier on the constraint $I_t^t \geq 0$ is strictly positive for L-types). When this is the case, the bubble raises the interest rate, making the L-type strictly prefer lending to capital investment ($R_{t,t+1} > a^L q_{t+1}$).

Similar to the bubbleless analysis, by using the credit market clearing condition, the binding credit constraint for the H-type, and the budget constraint, we can derive the following

transition dynamics for the aggregate capital stock:

$$K_{t+1} = \begin{cases} \Omega(q_t K_t + p_t^b) - a^L p_t^b & \textit{if } R_{t,t+1} = a^L q_{t+1} \\ a^H \beta(q_t K_t + p_t^b) - a^H p_t^b & \textit{if } R_{t,t+1} > a^L q_{t+1} \end{cases}.$$

The expressions above take into account the fact that some of the entrepreneurs' resources will be invested into the bubbly asset (the terms involving p_t^b). This is known as the "crowd-out" effect of bubbles on capital accumulation. In the mean time, the expressions also show how the return from bubble speculation raises entrepreneurs' aggregate net worth from $q_t K_t$ to $q_t K_t + p_t^b$. This is known as the "crowd-in" effect of bubbles. Combined with the expressions for the bubble size and the interest rate, the law of motion of the aggregate capital stock can be rewritten as

$$K_{t+1} = \begin{cases} \frac{\left(1+\frac{(a^H-a^L)}{a^L-\theta a^H}h-\phi_t\right)\beta a^L}{1-\beta\phi_t}\alpha K_t^\alpha & \textit{if } \phi_t \leq \phi^* \textit{ (small bubble)} \\ \frac{a^H\beta(1-\phi_t)}{1-\beta\phi_t}\alpha K_t^\alpha & \textit{if } \phi_t > \phi^* \textit{ (large bubble)} \end{cases}. \tag{17}$$

From (15, 16, 17), we can derive the following expressions for the bubbly steady state:

$$\phi = \begin{cases} \phi_{sb} \equiv \frac{\rho - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}{1-\frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}(1-h) & \text{if } \phi \leq \phi^* \\ \phi_{lb} \equiv \frac{\rho(1-h)}{(1-\theta)} - \frac{\theta}{\beta(1-\theta)} & \text{if } \phi > \phi^* \end{cases} \tag{18}$$

$$K_b = \begin{cases} K_{sb} \equiv \left(\frac{\left(1+\frac{a^H-a^L}{a^L-\theta a^H}\right)h}{1-\beta\rho(1-h)}\beta a^L \alpha\right)^{\frac{1}{1-\alpha}} & \text{if } \phi \leq \phi^* \\ K_{lb} \equiv \left(\frac{\beta[1-\rho(1-h)]+(1-\beta)\theta}{1-\beta\rho(1-h)}a^H\alpha\right)^{\frac{1}{1-\alpha}} & \text{if } \phi > \phi^* \end{cases} \tag{19}$$

$$R_b = \begin{cases} R_{sb} \equiv \frac{1-\beta\rho(1-h)}{\beta\left(1+\frac{a^H-a^L}{a^L-\theta a^H}\right)h} & \text{if } \phi \leq \phi^* \\ R_{lb} \equiv \frac{\theta[1-\beta\rho(1-h)]}{\beta(1-h)(1-\rho)+\theta[1-\beta(1-h)]} & \text{if } \phi > \phi^* \end{cases}. \tag{20}$$

From the analysis above, we can characterize the existence of the stochastic bubbly steady state:

Proposition 1. *A bubbly steady state exists if and only if there is sufficient financial friction:*

$$\theta < \beta\rho(1-h),$$

and the bubble is not too risky (the persistence probability is sufficiently high):

$$\rho > \frac{a^L - \theta a^H}{\beta(a^L - \theta a^H) + \beta h(a^H - a^L)}.$$

Proof. Appendix A.2.1. □

For the rest of the paper, we focus on the relevant range of parameters in which the bubble is expansionary (the crowd-in effect dominates the crowd-out effect in steady state), that is,

$$K_b > K^*, \tag{21}$$

and in which the interest rate in the bubbly steady state is sufficiently high so that the ZLB is slack, i.e.,

$$R_b > 1/\Pi^*.$$

Here, the stochastic bubbly steady-state capital stock K_b is given by (19), the good bubbleless steady state K^* is given by (14), and R_b is given by (20).

Remark 2. There exists a parameter region where the interest rate in the bubbly steady state is smaller than $1/\Pi^*$, and thus the ZLB may bind because of the bubble. However, as long as the bubble is expansionary, the DNWR will not bind, and thus the bubble economy will still achieve full employment despite binding ZLB. We do not focus on this parameter region in this paper, as the collapse of the bubble in this region does not lead to a secular stagnation.

5 Post-bubble dynamics

We now study the effect of the collapse of the bubble on the economy and establish the main results of the paper. We will show that the collapse of a large bubble can push the economy into a "secular stagnation" equilibrium, where the ZLB on the nominal interest rate constrains the monetary authority from achieving the inflation target and the DNWR binds, leading to involuntary unemployment. Under some conditions, the liquidity trap is transitory, as the economy eventually converges to the good bubbleless steady state with full employment. Interestingly, under certain conditions, because of the interaction between the ZLB and the DNWR, the post-bubble economy may never exit from the liquidity trap and instead may converge to the bad bubbleless steady state with unemployment.

5.1 Effects of DNWR

To isolate the effects of the DNWR in the post-bubble economy, it is instructive to start with a benchmark model where there is no ZLB. That is, *throughout this subsection, we assume that the monetary authority can set inflation at the target* Π^* *in all periods.* Then, it is immediate that the unique bubbleless steady state is the steady state with full employment and inflation at the target.

Suppose the economy has reached the bubbly steady state and then the bubble collapses at T (i.e., $\tilde{p}^b_{T+s} = 0,\ \forall s \geq 0$). As the expansionary effect of the bubble ends, the post-bubble capital stock and wage will decline toward the bubbleless steady state levels. However, if the downward wage rigidity constraint binds, then wage cannot flexibly fall to clear the labor market. Instead, employment is determined by the demand of firms. The rigidly high wage thus leads to involuntary unemployment. The contraction in employment has two effects on the intertemporal equilibrium dynamics: it reduces the return from capital, and it reduces entrepreneurs' net worth. Both of these effects in turn reduce entrepreneurs' accumulation of capital. The wage rigidity thus amplifies and propagates the shock of bursting bubbles.

We can characterize the post-bubble dynamics, including the depth and duration of the post-bubble unemployment episode. Let

$$s^* \equiv \min\{s \geq 0 | L_{T+s} = 1\},$$

then $T + s^*$ is the first post-bubble period when full employment is recovered. If $s^* > 0$, then we say the economy is in a *slump* between T and $T + s^* - 1$, because there is involuntary unemployment: $L_t < 1$ for all $T + 1 \leq t \leq T + s^*$.

The combination of the binding wage rigidity and the labor demand curve determines employment as:

$$L_{T+s} = \left(\frac{1-\alpha}{w_{T+s}}\right)^{\frac{1}{\alpha}} K_{T+s} < 1, \forall 0 < s < s^*. \tag{22}$$

(Note that the wage rigidity is slack at T, because capital at T is predetermined.) Based on this equality, we can characterize the post-bubble dynamics and establish a finite upper bound on the duration of the slump s^*:

Proposition 2. [Post-bubble slump] *Suppose the economy has reached the bubbly steady state and then the bubble collapses in period* T*. Then the duration of the slump is necessarily finite and bounded by:*

$$s^* \leq \max\left\{0,\ \lceil \omega(\gamma_0/\Pi^*) - 2\alpha \log_{\gamma_0/\Pi^*} K_b \rceil\right\} \tag{23}$$

where the ceiling function $\lceil x \rceil$ denotes the least integer greater than or equal to x and

$$\omega(\gamma) \equiv \frac{2\alpha}{1-\alpha} \log_{\gamma}(\alpha\Omega) - \frac{1+\alpha}{1-\alpha}.$$

The inequality holds with an equality if $\gamma_1 = 0$. For $0 < s < s^*$, the capital stock, employment, and wage are given by:

$$K_{T+s} = \alpha\Omega \left(\frac{w_{T+s-1}}{1-\alpha} \right)^{\frac{\alpha-1}{\alpha}} K_{T+s-1} \tag{24}$$

$$L_{T+s} = \left(\frac{(1-\alpha)^{1-\alpha}}{\gamma_0/\Pi^*} \left(\Omega\alpha L_{T+s-1} w_{T+s-1}\right)^{\alpha} \right)^{\frac{1}{\gamma_1+\alpha}} w_{T+s-1}^{-\frac{1}{\gamma_1+\alpha}}$$

$$w_{T+s} = \frac{\gamma_0}{\Pi^*} \left(\frac{(1-\alpha)^{1-\alpha}}{\gamma_0/\Pi^*} \left(\Omega\alpha L_{T+s-1} w_{T+s-1}\right)^{\alpha} \right)^{\frac{\gamma_1}{\gamma_1+\alpha}} w_{T+s-1}^{\frac{\alpha}{\gamma_1+\alpha}}.$$

Proof. Appendix A.2.2. □

Figure 1 illustrates the equilibrium dynamics in this result.[13] The economy begins period $t = 0$ in the bubbleless steady state with full employment. Then a large expansionary bubble unexpectedly arises in period $t = 5$ (for simplicity, we assume each entrepreneur is equally endowed with the bubbly asset in this initial bubbly period). The economy then converges to the stochastic bubbly steady state. As seen in the figure, the economy experiences a boom in capital accumulation, output, wage, and consumption.[14] Then the bubble collapses in period $t = 15$ (in the simulation, agents rationally expect that the bubble is stochastic and can burst in any period). Without nominal rigidities, the labor market would be flexible and the equilibrium wage, along with other aggregate variables in the post-bubble economy, would simply converge back to the bubbleless steady state with full employment. However, with DNWR, the post-bubble equilibrium wage may not flexibly fall to clear the labor market, leading to involuntary unemployment. The drop in employment not only reduces the economy's output, but also has important intertemporal effects. On the one hand, it reduces the net worth of entrepreneurs. On the other hand, it reduces the return rate on capital. Both of these effects depress capital accumulation. This process explains the contractions of aggregate economic activities during the slump with involuntary unemployment. Consistent

[13]The parameters chosen for this simulation are $\beta = 0.942$, $a^L = 1.6354$, $a^H = 2.0909$, $h = 0.6424$, $\theta = 0.0991$, $\alpha = 0.6742$, $\zeta = 1.5$, $\Pi^* = 1.152$, $\rho = 0.999$, $\gamma_0 = 1.1424$, and $\gamma_1 = 0.0991$.

[14]Note that the boom in consumption is more pronounced for entrepreneurs, implying that entrepreneurs tend to gain more from the bubble than workers (as the increase in net worth allows entrepreneurs to increase their investment). This asymmetry could lead to interesting political economy implications, which are absent from this model and are left for future research.

with the proposition, the figure shows that when the wage has fallen enough, the wage rigidity constraint no longer binds, the economy regains full employment and recovers toward the initial steady state.

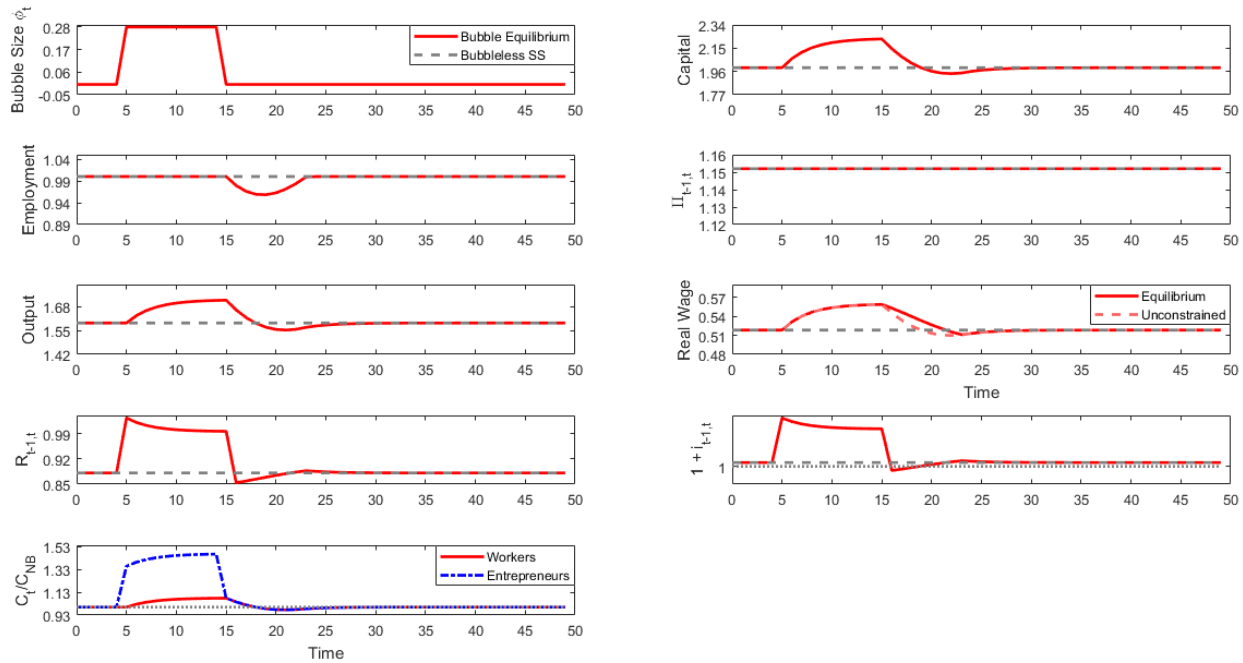

Figure 1: Equilibrium dynamics with only DNWR

5.2 Effects of the ZLB

For the rest of the paper, we assume again that the nominal interest rate is subject to the ZLB as described in Section 2.5. In contrast with the previous section, the economy may *never* exit from the unemployment trap.

We focus on the parameter region in which the bubble is large ($\phi > \phi^*$). This is because unlike a small bubble, the collapse of a large expansionary bubble has two important effects that put downward pressure on the real interest rate. First, after the large bubble collapses, the marginal capital producer of the capital good switches from the H-type to the L-type. Thus, instead of the interest rate identity $R_{T,T+1} = \frac{\theta a^H(1-\phi)}{1-h-\phi} q_{T+1}$ (associated with the H-type being the marginal capital producer) that could have prevailed if the bubble did not collapse in T, the interest rate would become $R_{T,T+1} = a^L q_{T+1} < \frac{\theta a^H(1-\phi)}{1-h-\phi} q_{T+1}$. Second, as the bubble crowds in capital accumulation, the post-bubble economy begins with a large aggregate net worth and capital stock. In other words, the post-bubble economy will follow the bubbleless dynamics as specified in Section 3 but with a large initial capital stock that is higher than that in the good steady state (recall (21)). A high capital stock leads to a low marginal

product of capital and thus a low interest rate. The combination of these two mechanisms exerts a downward pressure on the real interest rate and thus the nominal interest rate. If the bubble leads to sufficient large accumulation of capital stock, its collapse can push the interest rate against the ZLB.

Remark 3. One could think of this as corresponding to a situation of "investment hangover" at the end of an economic boom (Rognlie et al., 2014). The difference between our paper and Rognlie et al. (2014) is that the overinvestment is endogenous in our framework, while it is imposed exogenously in theirs.

The following proposition formalizes the discussion above:

Proposition 3. [Effect of bubble's collapse on real interest rate] *Suppose the economy has reached the steady state with a large expansionary bubble, and then the bubble collapses in a period denoted by T. If the bubbly steady state K_b is sufficiently large such that*

$$K_b > \bar{K} \equiv \left(a^L \Omega \Pi^*\right)^{\frac{1}{\alpha(1-\alpha)}} K^*,$$

where the good bubble-less steady state K^* is given by (14), then the Taylor rule (10) is constrained by the ZLB:

$$1 + i_{T,T+1} = 1 > R^f_{T,T+1} \left(\Pi_{T-1,T}\right)^{\zeta} \left(\Pi^*\right)^{1-\zeta}. \tag{25}$$

Proof. Appendix A.2.3. □

The next proposition shows that in the post-bubble economy, whenever the ZLB binds, the the DNWR must also bind:

Proposition 4. [ZLB implies DNWR] *For any $t \geq T+1$, if $i_{t-1,t} = 0$ then $L_t < 1$.*

Proof. Suppose on the contrary that $i_{t-1,t} = 0$ but $L_t = 1$. The DNWR constraint is slack, implying that $\frac{w^f_t}{w_{t-1}} \geq \frac{\gamma_0}{\Pi_{t-1,t}}$, or equivalently, inflation must be sufficiently high:

$$\frac{K_t^{\alpha}}{(K_{t-1}/L_{t-1})^{\alpha}} \geq \frac{\gamma_0}{\Pi_{t-1,t}}. \tag{26}$$

However, the inflation rate is determined by the Fisher equation $1 + i_{t-1,t} = R_{t-1,t}\Pi_{t-1,t}$, or[15]

$$1 = \underbrace{a^L \alpha K_t^{\alpha-1}}_{R_{t-1,t} \text{ with } L_t=1} \Pi_{t-1,t}. \tag{27}$$

[15]Recall that after the collapse of the bubble, the economy is effectively deterministic, and we can drop the stochastic discount factor from equation (9)

Substitute (27) into (26), we get a condition that the real interest rate and thus the marginal product of capital must be sufficiently low:

$$\gamma_0 a^L \alpha K_t^{\alpha-1} \leq \frac{K_t^\alpha}{(K_{t-1}/L_{t-1})^\alpha},$$

or equivalently, the capital stock must be sufficiently high:

$$K_t \geq \gamma_0 a^L \alpha (K_{t-1}/L_{t-1})^\alpha.$$

Substituting the law of motion of capital: $K_t = \Omega \alpha K_{t-1}^\alpha L_{t-1}^{1-\alpha}$ into the inequality above yields

$$L_{t-1} \geq \frac{\gamma_0 a^L}{\Omega}$$

However, as $1 \geq L_{t-1}$, it then follows that $1 \geq \frac{\gamma_0 a^L}{\Omega}$, which contradicts assumption (11). □

We say that the economy is in a *liquidity trap* in period t if the ZLB binds (implying $i_{t-1,t} = 0$) and the DNWR binds (implying $L_t < 1$). We now show that, under certain conditions, the post-bubble economy may *never* escape from the liquidity trap.[16] We will construct a post-bubble equilibrium path where $L_t < 1$ and $i_{t-1,t} = 0$ for all $t \geq T + 1$. The laws of motion of equilibrium quantities and prices K_t, L_t, and $\Pi_{t-1,t}$ are given by the bubbleless law of motion of capital (as derived in Section 3):

$$K_t = \Omega \alpha K_{t-1}^\alpha L_{t-1}^{1-\alpha}, \tag{28}$$

the binding DNWR:

$$\frac{(K_t/L_t)^\alpha}{(K_{t-1}/L_{t-1})} = \frac{\gamma(1-L_t)}{\Pi_{t-1,t}}, \tag{29}$$

and the Fisher equation:

$$\underbrace{a^L \alpha K_t^{\alpha-1} L_t^{1-\alpha}}_{R_{t-1,t}} \Pi_{t-1,t} = 1. \tag{30}$$

For the prices and quantities to indeed constitute an equilibrium, a necessary and sufficient condition is that the ZLB must bind, i.e., $R_{t-1,t}^f \left(\Pi_{t-2,t-1}\right)^\zeta \left(\Pi^*\right)^{1-\zeta} < 1$ for all t, where the real interest rate with full employment is given by $R_{t-1,t}^f = a^L \alpha K_t^{\alpha-1}$. This inequality holds if and only if K_t is sufficiently large for all t, and the equilibrium dynamics above can be solved for in closed form, as summarized in the following proposition:

[16] In reality, there can be shocks (not modeled here) that pull the economy out of the liquidity trap, such as a good technology shock or another large expansionary bubbly episode.

Proposition 5. [Persistent post-bubble slump] *Let* $\{K_{T+t}, L_{T+t}, \Pi_{T+t-1,T+t}\}$ *be defined by the following closed-form expressions:*

$$K_{T+t} = (\Omega\alpha)^{\frac{1-\alpha^t}{1-\alpha}} (K_b)^{\alpha^t} \left(\frac{\Omega}{\gamma_0 a^L}\right)^{\frac{1-\alpha}{\gamma_1}\left(\frac{1-\alpha^{t-1}}{(1-\alpha)} - \frac{1-(\alpha(1+\gamma_1))^{t-1}}{(1-\alpha(1+\gamma_1))(1+\gamma_1)^{t-1}}\right)}$$

$$L_{T+t} = \left(\frac{\Omega}{\gamma_0 a^L}\right)^{\frac{(1+\gamma_1)^t-1}{\gamma_1(1+\gamma_1)^t}}$$

$$\Pi_{T+t-1,T+t} = \frac{1}{a^L\alpha}\left(\frac{K_{T+t}}{L_{T+t}}\right)^{1-\alpha}.$$

1. *These values constitute a post-bubble equilibrium path if and only if*

   $K_t > \left(a^L\alpha\left(\frac{\Pi_{t-2,t-1}}{\Pi^*}\right)^{\zeta}\Pi^*\right)^{\frac{1}{1-\alpha}}$ *for all* $t \geq T+1$.

2. *On this equilibrium path, the economy experiences involuntary unemployment:* $L_{T+t} < 1$ *for all* $t > 0$*, and the economy converges to the bad bubbleless steady state with involuntary unemployment and below-target inflation described in Section 3.*

Proof. Appendix A.2.4. □

Figure 2 plots a simulated equilibrium path, in a manner similar to the simulation in Figure 1 (that the economy begins in the good bubbleless steady state, then a large expansionary bubble unexpectedly arises in $t = 5$; the economy reaches the bubbly steady state and then the bubble collapses in $t = 15$).[17] As seen in the figure, the collapse causes the real and nominal interest rate to fall sharply, and the nominal interest rate hits the ZLB. After the collapse, entrepreneurs cut down their investment, leading to a decline in the capital stock. The decline in the capital stock in turn causes a decline in the marginal product of labor. Wage would thus need to fall in order to clear the labor market. However, the wage floor creates a wedge that prevents labor market clearing, leading to involuntary unemployment. Inflation spikes up in the immediate aftermath of the bubble's collapse (as a consequence of the sharp decline in the real interest rate). However, as seen in the figure, the economy gradually converges to the *bad* bubbleless steady state (represented by the dashed red horizontal lines).

Why does the post-bubble economy converge to the bad bubbleless steady state? As explained previously, a large expansionary bubble can lead to a large overinvestment of capital (relative to the good bubbleless steady state). The bubble's eventual collapse will necessarily cause a sharp adjustment in market-clearing wages and a sharp decline in the

[17] The parameterization is identical to that provided in Footnote 13.

real and nominal interest rates. A large collapse can push the economy into a liquidity trap. The liquidity trap perpetuates as long as the monetary authority is constrained by the ZLB, leading to an inflation that is below the target. Low inflation in turn exacerbates the DNWR, leading to lower employment. Finally, lower employment further reduces the marginal product of capital and the interest rates, creating a vicious cycle that perpetuates the liquidity trap.

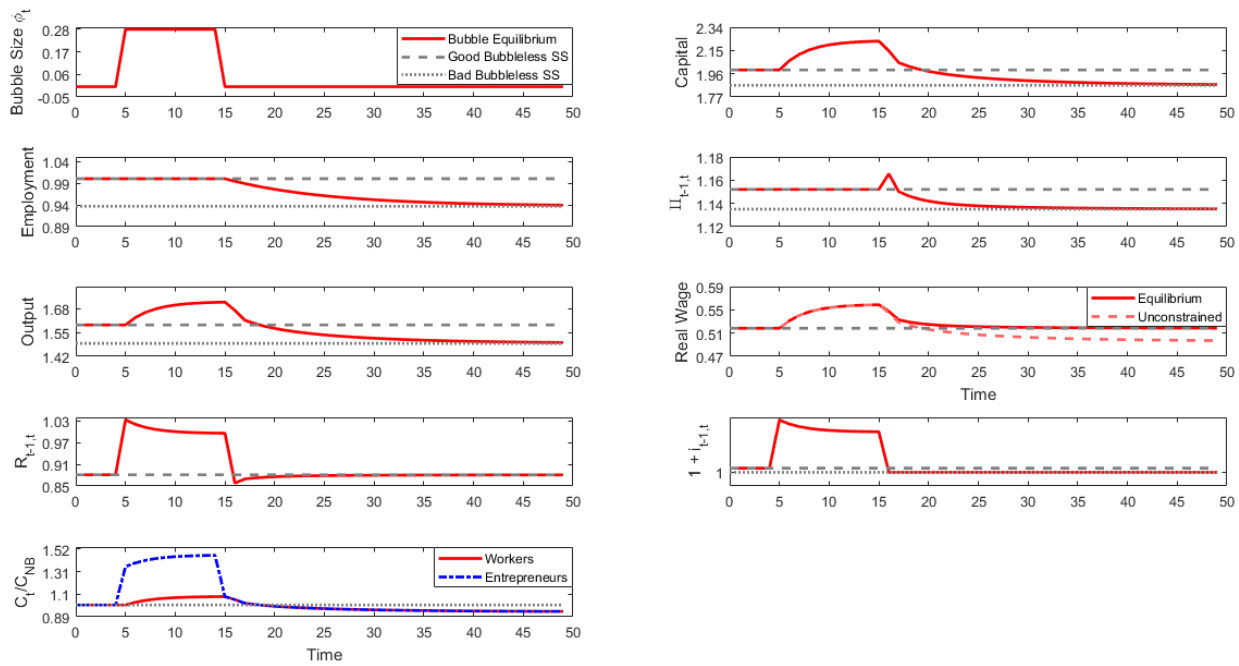

Figure 2: Equilibrium dynamics with persistent post-bubble liquidity trap and stagnation.

6 Leaning-against-the-bubble policy

We have established that the boom and bust of a bubbly episode can be a source of shocks that pushes the economy into an inefficient secular stagnation equilibrium. Under that context, policy responses are warranted. We will focus on a macroprudential policy of taxing bubble speculation, so that private agents internalize the pecuniary externality of the speculative bubble's boom and bust. As we will show, this policy has an effect of reducing the bubble size and is thus akin to the kind of "leaning-against-the-wind" policies that have been extensively discussed in the policy circle (e.g., Barlevy, 2012) and is similar to the type of tax policies often considered in the macroprudential literature (e.g., Lorenzoni, 2008; Gertler et al., 2012; and Jeanne and Korinek, 2013).

Formally, assume a policymaker (such as the regulatory arm of the central bank) levies

a macroprudential tax τ on bubble speculation. Then the budget constraint (2) becomes:

$$c_t^j + I_t^j + (1+\tau)\tilde{p}_t^b b_t^j = q_t k_t^j + d_t^j - R_{t-1,t} d_{t-1}^j + \tilde{p}_t^b b_{t-1}^j.$$

In an environment with heterogeneous agents such as ours, how tax revenues are redistributed, e.g., toward entrepreneurs or toward workers, can have disparate effects and may complicate the policy analysis. To clearly isolate the leaning-against-the-bubble effect from possible redistribution effects, we will assume that the policymaker will spend the revenue from the bubble tax in a way that does not affect the utility of entrepreneurs or workers.[18]

Given the tax, the equilibrium dynamics of the bubble size and the capital stock (conditional on the bubble surviving) are given by (derivations are relegated to Appendix A.1.4)

$$\phi_{t+1} = \begin{cases} \frac{1}{\beta} \frac{\frac{1-h-(1+\tau)\phi_t}{\rho(1-h)(1+\tau)-\phi_t}(1+\tau)\phi_t}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)+\frac{(1-\rho)(1-h)}{\rho(1-h)-(1+\tau)\phi_t}(1+\tau)\phi_t} & \text{if } \phi_t \leq \phi^* \text{(small bubble)} \\ \frac{\theta}{\beta} \frac{(1+\tau)\phi_t}{\rho(1-h)-(1-\theta)(1+\tau)\phi_t} & \text{if } \phi_t > \phi^* \text{(large bubble)} \end{cases},$$

and

$$K_{t+1} = \begin{cases} \frac{\left(1+\frac{(a^H-a^L)}{a^L-\theta a^H}h\right)\beta a^L - a^L\beta(1+\tau)\phi_t}{1-\beta\phi_t}\alpha K_t^\alpha & \textit{if } \phi_t \leq \phi^* \\ \frac{a^H\beta[1-(1+\tau)\phi_t]}{1-\beta\phi_t}\alpha K_t^\alpha & \textit{if } \phi_t > \phi^* \end{cases},$$

where the threshold ϕ^* is now defined as

$$\phi^* \equiv \frac{(1-h)a^L - \theta a^H}{(1+\tau)(a^L - \theta a^H)}.$$

The corresponding steady-state bubble size and capital stock are given by

$$\phi(\tau) = \begin{cases} \frac{\rho - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}{1-\frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}\frac{1-h}{1+\tau} & \text{if } \phi \leq \phi^* \\ \frac{\rho(1-h)}{(1-\theta)(1+\tau)} - \frac{\theta}{\beta(1-\theta)} & \text{if } \phi > \phi^* \end{cases},$$

$$K_b(\tau) = \begin{cases} \left(\frac{\left(1+\frac{a^H-a^L}{a^L-\theta a^H}\right)h}{1+\tau-\beta\rho(1-h)+\tau\frac{(1-h)(1-\rho)\beta}{\left(1+\frac{a^H-a^L}{a^L-\theta a^H}h\right)\beta-1}}\beta a^L\alpha\right)^{\frac{1}{1-\alpha}} & \text{if } \phi \leq \phi^* \\ \left((1+\tau)\frac{\beta[1-\rho(1-h)]+(1+\tau-\beta)\theta}{1+\tau-\beta\rho(1-h)}a^H\alpha\right)^{\frac{1}{1-\alpha}} & \text{if } \phi > \phi^* \end{cases} \tag{31}$$

[18] In other words, from the perspective of an individual household, the government spending is useless. See Remark 4 for a relaxation of this assumption.

Again, for the rest of the paper, we focus on the parameter space such that the bubble is large ($\phi > \phi^*$).

6.1 Maintaining full employment

Before turning to the welfare analysis, we ask a relevant question: can the ex-ante macroprudential tax help to prevent the ex-post inefficiency of involuntary unemployment? We will show that by mitigating the investment boom due to the bubbly episode, the macroprudential regulation can help maintain full employment (and thus avoid the secular stagnation) even after the bubble collapses.

Formally, we will now establish a condition on τ such that $L_t = 1$ for all $t \geq T$ along the post-bubble equilibrium path. As a direct corollary of Proposition 4, it must also be the case that the nominal interest rate is above the ZLB: $i_{t-1,t} > 0$. As a consequence, the monetary authority is unconstrained in achieving the inflation target Π^*. Thus, the desired equilibrium path features $L_t = 1$ and $\Pi_{t-1,t} = \Pi^*$ for all $t \geq T$.

In each period $t \geq T$, for the economy to achieve full employment, it must be that the DNWR constraint is slack, or

$$(1-\alpha)K_t^\alpha \geq \frac{\gamma_0}{\Pi^*} w_{t-1},$$

where the left-hand side is the real wage associated with full employment ($L_t = 1$). By substituting K_t with the law of motion of capital (28) and by substituting the equilibrium wage expression $w_{t-1} = (1-\alpha)(K_{t-1}/L_{t-1})^\alpha$, the inequality above can be rewritten as an upper bound on K_{t-1}:

$$K_{t-1} \leq \underline{K} \equiv \left(\frac{\alpha\Omega\Pi^*}{\gamma_0}\right)^{\frac{1}{1-\alpha}} \tag{32}$$

Similarly, in each t, for the nominal interest rate to be strictly positive, it must be that:

$$R^f_{t-1,t}\Pi^* > 1,$$

where recall that $R^f_{t-1,t} = a^L \alpha K_t^{\alpha-1}$ is the real interest rate associated with full employment. Again, by substituting K_t with the law of motion of capital (28), this inequality is equivalent to:

$$K_{t-1} < \tilde{K} \equiv \left(\frac{\left(a^L \alpha \Pi^*\right)^{\frac{1}{1-\alpha}}}{\alpha\Omega}\right)^{\frac{1}{\alpha}}. \tag{33}$$

Conditions (33) and (32) can be rewritten more compactly as

$$K_{t-1} \leq \min\{\underline{K}, \tilde{K}\}, \forall t \geq T.$$

Under the supposition that the economy reaches the steady state with an expansionary bubble before the bubble collapses, the maximum capital stock is given by $K_b(\tau)$. Furthermore, because of condition (11), it is straightforward to verify that $\underline{K} < \tilde{K}$. Thus, we have established the following result:

Lemma 1. *The economy can achieve full employment in a post-bubble equilibrium path ($L_t = 1$ for all $t \geq T$) if and only if:*

$$K_b(\tau) \leq \underline{K}, \tag{34}$$

where $\underline{K} \equiv \left(\frac{\alpha\Omega\Pi^*}{\gamma_0}\right)^{\frac{1}{1-\alpha}}$.

6.2 Boom-bust trade-off

An important bubble policy consideration is the trade-off between the gains during the boom and the losses following the bust. While the analysis in the previous section focused on the post-bubble phase, in this section we will analyze the boom-bust trade-off.

We suppose the policymaker has a dual mandate of maximizing the welfare of workers while maintaining the efficient level of employment. The welfare of workers is defined as the expected discounted utility of a representative worker in the stochastic bubbly steady state. We focus on the welfare of workers for tractability reasons, as only their welfare function can be characterized analytically (due to the idiosyncratic risk and intertemporal trades, the expression for entrepreneurs can only be calculated numerically).

The following lemma establishes the closed-form expression for the welfare function of a representative worker, assuming that full employment is maintained:

Lemma 2. *Assuming full employment in the post-bubble economy ($L_t = 1$ for all $t \geq T$). The expected discounted utility of a worker in the bubbly steady state is given by:*

$$\mathcal{W}(\tau) \quad = \frac{1}{1-\beta\rho}\left[\alpha\frac{1+\beta(1-\rho-\alpha)}{1-\alpha\beta}\log K_b(\tau) + \omega\right], \tag{35}$$

where $\omega \equiv \frac{1-\beta\rho}{1-\beta}\log(1-\alpha) + \frac{\beta(1-\rho)}{1-\beta}\frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\Omega)$ *is a constant.*

Proof. Appendix A.2.5. □

Lemma 2 shows that $\mathcal{W}$ is increasing in K_b. As a direct corollary, as long as the policymaker maintains full employment, to maximize the welfare of workers is the same as to maximize $K_b(\tau)$. Combined with Lemma 1, we know that the policymaker can achieve the dual mandate by setting τ such that condition (34) holds with equality.

The following result summarizes our analysis in this section. It captures in a simple way the boom-bust trade-off by establishing a macroprudential policy that maximizes the welfare gain (due to the consumption boom) during the bubbly episode while avoiding the inefficiencies of involuntary unemployment in the bust phase:

Proposition 6. *The macroprudential tax that maximizes the expected discounted utility of a worker in the bubbly steady state, subject to maintaining full employment, is given by $\tau = \tau^*$, where τ^* is the unique solution to $K_b(\tau) = \underline{K}$.*

Proof. Appendix A.2.6. □

Figure 3 shows the contrasting equilibrium dynamics between the stagnation equilibrium path and one in which a fiscal authority levies a speculation tax as outlined in Proposition 6.[19] Full employment and the inflation target are achieved over the full boom-bust cycle, at the expense of a smaller economic boom during the bubbly episode.

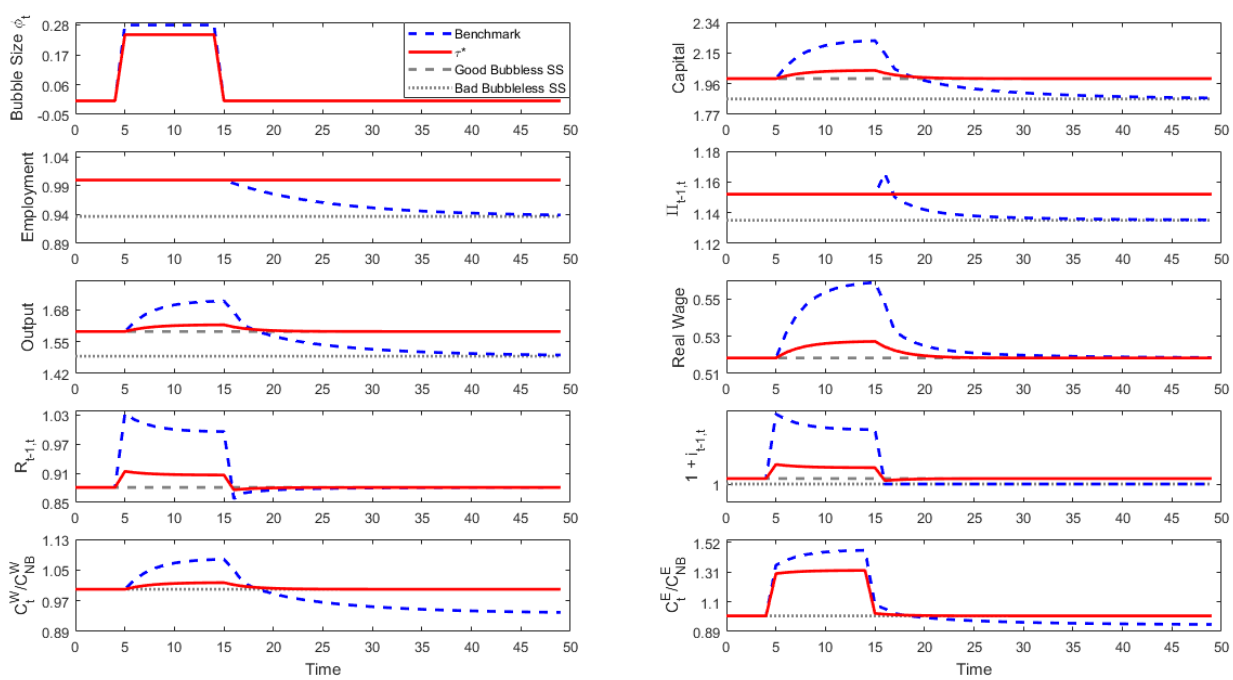

Figure 3: Effects of macroprudential policy.

[19]The level of optimal tax for the prevailing parameterization is $\tau^* = 0.0987$. Other parameters match those used in Figure 1.

Remark 4. If the tax revenue from the macroprudential policy is transferred in lump sum in every period to workers, then the welfare expression in (35) would be instead given by:

$$\hat{\mathcal{W}}(\tau) = \frac{1}{1-\beta\rho}\left[\log\left((1-\alpha) + \frac{\beta\phi_b}{1-\beta\phi_b}\alpha\tau\right) + \frac{1+\beta(1-\rho-\alpha)}{1-\alpha\beta}\alpha\log K_b(\tau) + \omega_1\right],$$

where $\omega_1 \equiv \frac{\beta(1-\rho)}{1-\beta}\left(\log(1-\alpha) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\Omega)\right)$ is a constant. The expression shows a redistribution motive, as τ shows up in the first term, capturing the consumption gain for a worker during the boom due to the transfer. The policymaker would then face an additional trade-off due to this redistribution motive.

7 Conclusion

We have developed a tractable rational bubbles model with downward wage rigidity. We show that expansionary bubbles could boost economic activities, but their collapse can push the economy into a persistent secular stagnation equilibrium with involuntary unemployment, inflation below the target and depressed investment, output, and consumption. The model's predictions are consistent with stylized features of recent bubbly episodes. The model highlights the trade-off between the economic gains during the boom due to the bubble and the loss from the bust. A macroprudential leaning-against-the-bubble policy of taxing speculative investment can help balance this boom-bust trade-off. An interesting venue for future research would be to analyze other forms of policy responses, including conventional and unconventional monetary policies (as in Hirano et al., 2017), in our framework. Another potential direction is to analyze policies when policymakers can only correctly observe bubbles in asset prices with a certain probability.

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A Appendix

A.1 Derivations

A.1.1 Bubbleless equilibrium

The equilibrium dynamics in the bubbleless environment follows once we solve for the H-type's investment function. The binding borrowing constraint of the H-types gives us their borrowing, which we can then plug directly into the budget constraint. With log utility, entrepreneurs consume a fraction $1-\beta$ of their net worth, defined as $e_t^j \equiv q_t k_t^j - R_{t-1,t} d_{t-1}^j$.

$$
\begin{aligned}
d_t^j &= \frac{\theta q_{t+1} k_{t+1}^j}{R_{t,t+1}} = \frac{\theta q_{t+1} a^H I_t^j}{R_{t,t+1}} \\
I_t^j - d_t^j &= \beta(q_t k_t^j - R_{t-1,t} d_{t-1}^j) = \beta e_t^j \\
I_t^j &= \frac{1}{1 - \frac{\theta q_{t+1} a^H}{R_{t,t+1}}} \beta e_t^j.
\end{aligned}
$$

We also note that aggregate wealth in a period is given by $\int_{j \in \mathcal{H}_t} e_t^j + \int_{j \in \bar{\mathcal{H}}_t} e_t^j = q_t K_t$. The idiosyncratic productivity shock is independent across time, which simplifies aggregation,

and we can express aggregate H-type net worth as $\int_{j\in\mathcal{H}_t} e_t^j = hq_tK_t$. Since there is sufficient financial friction, L-types will invest a portion of their savings, which will be determined from the aggregate savings in the economy:

$$\int_{j\in\mathcal{H}_t} I_t^j + \int_{j\in\bar{\mathcal{H}}_t} I_t^j = \beta q_t K_t.$$

Furthermore, the equilibrium interest rate is equal to the L-type's marginal return from investment, $R_{t,t+1} = q_{t+1}a^L$. Combining the aggregate savings, investment function, and interest rate, we are able to arrive at a law of motion for aggregate capital.

$$\begin{aligned}
K_{t+1} &= a^H \int_{j\in\mathcal{H}_t} I_t^j + a^L \int_{j\in\bar{\mathcal{H}}_t} I_t^j = a^H \int_{j\in\mathcal{H}_t} I_t^j + a^L \left[\beta q_t K_t - \int_{j\in\mathcal{H}_t} I_t^j\right] \\
K_{t+1} &= a^H \frac{h}{1-\frac{\theta a^H}{a^L}} \beta q_t K_t + a^L \left[\beta q_t K_t - \frac{h}{1-\frac{\theta a^H}{a^L}} \beta q_t K_t\right] \\
K_{t+1} &= \underbrace{\left(h\frac{a^H - a^L}{1-\frac{\theta a^H}{a^L}} + a^L\right)}_{\equiv\Omega} \beta\, q_t K_t.
\end{aligned}$$

A.1.2 Bubbly equilibrium

Capital accumulation Similar to the bubbleless environment, H-type's borrowing constraint will bind. Additionally, H-types will not hold the bubble since their return to investment is greater. However, we must consider two cases.

Case 1: $R_{t,t+1} = a^L q_{t+1}$ (small bubble). We proceed as before by solving the H-type's investment function. Net worth now reflects bubble holdings from the past period, $e_t^j \equiv q_t k_t^j + p_t^b b_{t-1}^j - R_{t-1,t} d_{t-1}^j$.

$$\begin{aligned}
I_t^j - d_t^j &= \beta(q_t k_t^j + p_t^b b_{t-1}^j - R_{t-1,t} d_{t-1}^j) = \beta e_t^j \\
I_t^j &= \frac{1}{1-\frac{\theta a^H}{a^L}} \beta e_t^j.
\end{aligned}$$

The aggregate savings will also change to reflect the presence of the bubble:

$$\int_{j\in\mathcal{H}_t} I_t^j + \int_{j\in\bar{\mathcal{H}}_t} I_t^j + p_t^b = \beta(q_t K_t + p_t^b).$$

As before, we combine the aggregate savings, investment function, and interest rate to determine the law of motion for capital:

$$K_{t+1} = a^H \frac{h}{1-\frac{\theta a^H}{a^L}} \beta(q_t K_t + p_t^b) + a^L \left[\beta(q_t K_t + p_t^b) - \frac{h}{1-\frac{\theta a^H}{a^L}} \beta(q_t K_t + p_t^b) - p_t^b \right]$$
$$K_{t+1} = \Omega(q_t K_t + p_t^b) - a^L p_t^b.$$

Case 2: $R_{t,t+1} > a^L q_{t+1}$ **(large bubble).** L-types do not invest since their return to lending and bubbles is greater than their return to investment. Therefore, all nonbubble savings are shifted to the H-type to invest, and:

$$K_{t+1} = a^H \left[\beta(q_t K_t + p_t^b) - p_t^b \right].$$

Using the definition of bubble size, $\phi_t \equiv \frac{p_t^b}{\beta(q_t K_t + p_t^b)}$, we can rewrite the above capital flows as below:

$$K_{t+1} = \begin{cases} \frac{\left(1+\frac{(a^H-a^L)}{a^L-\theta a^H}h\right)\beta a^L - a^L \beta \phi_t}{1-\beta\phi_t} \alpha K_t^\alpha & \text{if } \phi_t \leq \phi^* \\ \frac{a^H \beta[1-\phi_t]}{1-\beta\phi_t} \alpha K_t^\alpha & \text{if } \phi_t > \phi^* \end{cases}.$$

In the notation above, we define small bubbles as $\phi_t \leq \phi^*$. Small bubbles arise when the L-type is still investing, therefore the small bubble condition is equivalent to $R_{t,t+1} = q_{t+1} a^L$. On the other hand, the condition for large bubbles arising, $\phi_t > \phi^*$, is satisfied when L-types no longer invest and thus is equivalent to $R_{t,t+1} > q_{t+1} a^L$. We show the derivation for ϕ^* and the interest rate below.

Interest rate Using the definition of bubble size, the H-type's investment function, and aggregate savings, we can solve for $R_{t,t+1}$ when $R_{t,t+1} > q_{t+1} a^L$. Recall that in this case, L-types do not invest, so

$$\underbrace{h \frac{\beta(q_t K_t + p_t^b)}{1 - \frac{\theta q_{t+1} a_t^H}{R_{t,t+1}}}}_{\int_{j \in \mathcal{H}_t} I_t^j} + p_t^b = \beta(q_t K_t + p_t^b).$$

Solving for interest rate, we get the following expression for interest rate:

$$R_{t,t+1} = \begin{cases} q_{t+1} a^L & \text{if } \phi_t \leq \phi^* \\ q_{t+1} \frac{\theta a^H (1-\phi_t)}{1-h-\phi_t} & \text{if } \phi_t > \phi^* \end{cases}.$$

Here, ϕ^* is defined as the threshold bubble size that equates the two different values of interest rate:

$$a^L = \frac{\theta a^H(1-\phi^*)}{1-h-\phi^*}$$
$$\phi^* \equiv \frac{(1-h)a^L - \theta a^H}{(a^L - \theta a^H)}.$$

Bubble growth In the stochastic bubble environment, the expected returns from holding the bubble must equal the expected returns from lending. In the notation below, terms with superscript ρ represent values in the state that the bubble persists and terms with superscript $1-\rho$ represent values in the state that the bubble bursts.

$$E_t[u'(c^{i,\rho}_{t+1})\frac{p^b_{t+1}}{p^b_t}] = E_t[u'(c^{i,\rho}_{t+1})R_{t,t+1}]$$
$$\Rightarrow \rho\frac{1}{c^{i,\rho}_{t+1}}\frac{p^b_{t+1}}{p^b_t} = R_{t,t+1}\rho\frac{1}{c^{i,\rho}_{t+1}} + R^{1-\rho}_{t,t+1}(1-\rho)\frac{1}{c^{i,1-\rho}_{t+1}}$$
$$\Rightarrow \rho\frac{p^b_{t+1}}{p^b_t} = R_{t,t+1} + (1-\rho)\frac{p^b_{t+1}b^j_t}{\beta e^j_t - p^b_t b^j_t}.$$

We guess that L-types hold a portion η of their savings in the bubble, that is $\eta\beta e^j_t = p^b_t b^j_t$, and then we solve for η to get L-type bubble demand:

$$p^b_t b^j_t = \underbrace{\frac{\rho\frac{p^b_{t+1}}{p^b_t} - R_{t,t+1}}{\frac{p^b_{t+1}}{p^b_t} - R_{t,t+1}}}_{\eta}\beta e^j_t.$$

Plugging the expression for L-type bubble demand into the Euler equation above, we get the following no-arbitrage condition:

$$\frac{p^b_{t+1}}{p^b_t} = \frac{R_{t,t+1}(1-h-\phi_t)}{\rho(1-h)-\phi_t}.$$

Next, we define the evolution of wealth using the transition dynamics for aggregate

capital.

$$q_{t+1}K_{t+1}+p^b_{t+1} = \begin{cases} q_{t+1}\left[\Omega(q_tK_t+p^b_t)-a^Lp^b_t\right]+\frac{p^b_{t+1}}{p^b_t}\phi_t\beta(q_tK_t+p^b_t) & \text{if } \phi_t \le \phi^* \\ q_{t+1}\left[a^H\beta(q_tK_t+p^b_t)-a^Hp^b_t\right]+\frac{p^b_{t+1}}{p^b_t}\phi_t\beta(q_tK_t+p^b_t) & \text{if } \phi_t > \phi^* \end{cases}$$

$$\Rightarrow \frac{q_{t+1}K_{t+1}+p^b_{t+1}}{q_tK_t+p^b_t} = \begin{cases} \beta q_{t+1}\left[\frac{\Omega}{\beta}-a^L\phi_t\right]+\frac{p^b_{t+1}}{p^b_t}\phi_t & \text{if } \phi_t \le \phi^* \\ \beta q_{t+1}\left[a^H-a^H\phi_t\right]+\beta\frac{p^b_{t+1}}{p^b_t}\phi_t & \text{if } \phi_t > \phi^* \end{cases}.$$

Using the flow of bubble price, evolution of wealth, and interest rate, we characterize the evolution of the bubble below:

$$\begin{aligned} \phi_{t+1} &= \frac{\frac{p^b_{t+1}}{p^b_t}}{\frac{q_{t+1}K_{t+1}+p^b_{t+1}}{q_tK_t+p^b_t}}\phi_t \\ &= \begin{cases} \frac{1}{\beta}\frac{\frac{1-h-\phi_t}{\rho(1-h)-\phi_t}\phi_t}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)+\frac{(1-\rho)(1-h)}{\rho(1-h)-\phi_t}\phi_t} & \text{if } \phi_t \le \phi^* \\ \frac{\theta}{\beta}\frac{\phi_t}{\rho(1-h)-(1-\theta)\phi_t} & \text{if } \phi_t > \phi^* \end{cases}. \end{aligned}$$

Steady-state bubble size First, we use the above evolution of the bubble to solve for steady-state bubble size for each case of small and large bubbles:

$$\phi_{sb} \equiv \frac{\rho - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}{1-\frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}(1-h) \qquad \text{(small bubble)}$$

$$\phi_{lb} \equiv \frac{\rho(1-h)}{(1-\theta)} - \frac{\theta}{\beta(1-\theta)} \qquad \text{(large bubble).}$$

The remainder of the steady-state values follow directly from previously derived equilibrium evolution equations and the above steady-state bubble size.

A.1.3 Post-bubble equilibrium dynamics of the unemployment path with DNWR and ZLB

For notation simplicity, let us normalize the period when the bubble bursts to be period 0; that is, $T = 0$. Then, on the unemployment path $L_1, L_2, \cdots < 1$ and $i_{0,1} = i_{1,2} = \cdots = 0$. Given initial conditions of L_{lb}, K_{lb}, the unemployment path can be characterized as follows. The flow of capital is given by

$$K_t = \Omega\alpha K^{\alpha}_{t-1}L^{1-\alpha}_{t-1}.$$

Binding DNWR and ZLB provide the following two equations, respectively:

$$\Pi_{t-1,t}\frac{(K_t/L_t)^\alpha}{(K_{t-1}/L_{t-1})^\alpha} = \gamma(1-L_t)$$
$$\underbrace{a^L\alpha K_t^{\alpha-1}L_t^{1-\alpha}}_{R_{t-1,t}}\Pi_{t-1,t} = 1$$

Combining the two above equations yields

$$\frac{(K_t/L_t)^\alpha}{(K_{t-1}/L_{t-1})^\alpha} = a^L\alpha K_t^{\alpha-1}L_t^{1-\alpha}\gamma(1-L_t).$$

Rewriting the above equation by utilizing the parameterization of $\gamma(\cdot)$,

$$K_t\left(\frac{L_{t-1}}{K_{t-1}}\right)^\alpha = a^L\alpha\gamma_0 L_t^{1+\gamma_1}.$$

By substituting in the flow of capital, we find a recursive form for labor:

$$L_t = \left(\frac{\Omega}{\gamma_0 a^L}L_{t-1}\right)^{\frac{1}{1+\gamma_1}}.$$

Similarly, inflation can be expressed as a function of last period's labor and capital:

$$\begin{aligned}\Pi_{t-1,t} &= \frac{1}{R_{t-1,t}} = \frac{1}{a^L\alpha}\left(\frac{K_t}{L_t}\right)^{1-\alpha}\\ &= \frac{1}{a^L\alpha}\left(\gamma_0 a^L\alpha K_{t-1}^\alpha L_{t-1}^{\frac{\gamma_1-\alpha(1+\gamma_1)}{1+\gamma_1}}\left(\frac{\Omega}{\gamma_0 a^L}\right)^{\frac{\gamma_1}{1+\gamma_1}}\right)^{1-\alpha}.\end{aligned}$$

These expressions can be further simplified by recursively plugging in for $L_{t-1}, L_{t-2}, \ldots, L_1$. Therefore, labor, L_t, can be written as a function of $L_0 = 1$ (as shown in Appendix A.2.3, there is full employment in the period the bubble bursts) and t:

$$\begin{aligned}L_t &= \left(\frac{\Omega}{\gamma_0 a^L}L_{t-1}\right)^{\frac{1}{1+\gamma_1}}\\ &= \left(\left(\frac{\Omega}{\gamma_0 a^L}\right)^{\sum_{s=0}^{t-1}\left(\frac{1}{1+\gamma_1}\right)^s}\underbrace{L_0^{\left(\frac{1}{1+\gamma_1}\right)^{t-1}}}_{=1}\right)^{\frac{1}{1+\gamma_1}}\\ &= \left(\frac{\Omega}{\gamma_0 a^L}\right)^{\frac{(1+\gamma_1)^t-1}{\gamma_1(1+\gamma_1)^t}}.\end{aligned}$$

Similarly, using the flow of capital equation and working backward, K_t can be written as a function of K_0, t, and all past L_t:

$$\begin{aligned}
K_t &= \Omega\alpha K_{t-1}^{\alpha} L_{t-1}^{1-\alpha} \\
&= (\Omega\alpha)^{\sum_{s=0}^{t-1}\alpha^s} K_0^{\alpha^t} \left(\prod_{s=1}^{t-1} L_{t-s}^{\alpha^{s-1}}\right)^{1-\alpha} \\
&= (\Omega\alpha)^{\frac{1-\alpha^t}{1-\alpha}} K_0^{\alpha^t} \left(\frac{\Omega}{\gamma_0 a^L}\right)^{\frac{1-\alpha}{\gamma_1}\left(\frac{1-\alpha^{t-1}}{(1-\alpha)} - \frac{1-(\alpha(1+\gamma_1))^{t-1}}{(1-\alpha(1+\gamma_1))(1+\gamma_1)^{t-1}}\right)} .
\end{aligned}$$

A.1.4 Bubble dynamics with macroprudential tax

In the presence of a macroprudential tax, the equilibrium dynamics are similar to that of the bubble equilibrium. The following section updates the derivations from the bubble equilibrium as a result of the macroprudential tax.

Capital accumulation As before, the H-type's borrowing constraint will bind, and H-types will not hold the bubble since their return to investment is greater. The two cases to consider follow.

Case 1: $R_{t+1} = q_{t+1}a^L$ (small bubble). The H-type's investment function is identical as before, as the macroprudential tax does not directly affect entrepreneur net worth, as the effect is entirely through bubble holdings from the past period, $e_t^j \equiv q_t k_t^j + p_t^b b_{t-1}^j - R_t d_{t-1}^j$.

$$\begin{aligned}
I_t^j - d_t^j &= \beta(q_t k_t^j + p_t^b b_{t-1}^j - R_t d_{t-1}^j) = \beta e_t^j \\
I_t^j &= \frac{1}{1 - \frac{\theta a^H}{a^L}} \beta e_t^j .
\end{aligned}$$

The aggregate savings, however, reflects the presence of the macroprudential tax:

$$\int_{j\in\mathcal{H}_t} I_t^j + \int_{j\in\bar{\mathcal{H}}_t} I_t^j + (1+\tau)p_t^b = \beta(q_t K_t + p_t^b).$$

We combine the aggregate savings, investment function, and interest rate to determine the law of motion for capital:

$$K_{t+1} = a^H \frac{h}{1-\frac{\theta a^H}{a^L}} \beta(q_t K_t + p_t^b) + a^L \left[\beta(q_t K_t + p_t^b) - \frac{h}{1-\frac{\theta a^H}{a^L}} \beta(q_t K_t + p_t^b) - (1+\tau) p_t^b \right]$$
$$K_{t+1} = \Omega(q_t K_t + p_t^b) - a^L (1+\tau) p_t^b.$$

Case 2: $R_{t+1} > q_{t+1} a^L$ (large bubble). As L-types do not invest, all nonbubble savings are shifted to the H-type to invest. However, the nonbubble savings are updated due to the macroprudential tax:

$$K_{t+1} = a^H \left[\beta(q_t K_t + p_t^b) - (1+\tau) p_t^b \right].$$

Using the definition of bubble size, $\phi_t \equiv \frac{p_t^b}{\beta(q_t K_t + p_t^b)}$, we can rewrite the above capital flows as below:

$$K_{t+1} = \begin{cases} \frac{\left(1+\frac{(a^H-a^L)}{a^L-\theta a^H} h\right) \beta a^L - a^L \beta (1+\tau) \phi_t}{1-\beta \phi_t} \alpha K_t^\alpha & \text{if } \phi_t \leq \phi^* \\ \frac{a^H \beta [1-(1+\tau)\phi_t]}{1-\beta \phi_t} \alpha K_t^\alpha & \text{if } \phi_t > \phi^* \end{cases}.$$

The notation above is consistent to that of the bubble equilibrium, with small bubbles defined as $\phi_t \leq \phi^*$, and large bubbles defined as $\phi_t > \phi^*$. We show the derivation for ϕ^* with the macroprudential tax in the interest rate derivation below.

Interest rate Using the definition of bubble size, the H-type's investment function, and aggregate savings, we can solve for R_{t+1} when $R_{t+1} > q_{t+1} a^L$. Recall that in this case, L-types do not invest, so

$$\underbrace{h \frac{\beta(q_t K_t + p_t^b)}{1 - \frac{\theta q_{t+1} a_t^H}{R_{t+1}}}}_{\int_{j \in \mathcal{H}_t} I_t^j} + (1+\tau) p_t^b = \beta(q_t K_t + p_t^b),$$

Solving for interest rate, we get the following expression.

$$R_{t+1} = \begin{cases} q_{t+1} a^L & \text{if } \phi_t \leq \phi^* \\ q_{t+1} \frac{\theta a^H (1-(1+\tau)\phi_t)}{1-h-(1+\tau)\phi_t} & \text{if } \phi_t > \phi^* \end{cases}.$$

Here, ϕ^* is defined as the threshold bubble size that equates the two different values of interest rate:

$$a^L = \frac{\theta a^H(1-(1+\tau)\phi^*)}{1-h-(1+\tau)\phi^*}$$
$$\phi^* \equiv \frac{(1-h)a^L - \theta a^H}{(1+\tau)(a^L - \theta a^H)}.$$

Bubble growth In the stochastic bubble environment, the expected returns from holding the bubble must equal the expected returns from lending. In the notation below, terms with superscript ρ represent values in the state that the bubble persists and terms with superscript $1-\rho$ represent values in the state that the bubble bursts.

$$E_t[u'(c_{t+1}^{i,\rho})\frac{p_{t+1}^b}{(1+\tau)p_t^b}] = E_t[u'(c_{t+1}^{i,\rho})R_{t+1}]$$
$$\Rightarrow \rho\frac{1}{c_{t+1}^{i,\rho}}\frac{p_{t+1}^b}{(1+\tau)p_t^b} = R_{t+1}\rho\frac{1}{c_{t+1}^{i,\rho}} + R_{t+1}^{1-\rho}(1-\rho)\frac{1}{c_{t+1}^{i,1-\rho}}$$
$$\Rightarrow \rho\frac{p_{t+1}^b}{(1+\tau)p_t^b} = R_{t+1} + (1-\rho)\frac{p_{t+1}^b b_t^j}{\beta e_t^j - (1+\tau)p_t^b b_t^j}.$$

We guess that L-types hold a portion $\eta(\tau)$ of their savings in the bubble reflecting the macroprudential tax; that is, $\eta(\tau)\beta e_t^j = (1+\tau)p_t^b b_t^j$. Then, we solve for $\eta(\tau)$ to get L-type bubble demand.

$$(1+\tau)p_t^b b_t^j = \underbrace{\frac{\rho\frac{p_{t+1}^b}{(1+\tau)p_t^b} - R_{t+1}}{\frac{p_{t+1}^b}{(1+\tau)p_t^b} - R_{t+1}}}_{\eta(\tau)}\beta e_t^j.$$

Plugging the expression for L-type bubble demand into the Euler equation above, we get the following no-arbitrage condition:

$$\frac{p_{t+1}^b}{p_t^b} = \frac{R_{t+1}(1-h-(1+\tau)\phi_t)}{\rho(1-h)-(1+\tau)\phi_t}(1+\tau).$$

Next, the evolution of wealth is defined using the transition dynamics for aggregate

capital.

$$q_{t+1}K_{t+1}+p^b_{t+1}=\begin{cases} q_{t+1}\left[\Omega(q_tK_t+p^b_t)-a^L(1+\tau)p^b_t\right]+\frac{p^b_{t+1}}{p^b_t}\phi_t\beta(q_tK_t+p^b_t) & \text{if } \phi_t\le\phi^* \\ q_{t+1}\left[a^H\beta(q_tK_t+p^b_t)-a^H(1+\tau)p^b_t\right]+\frac{p^b_{t+1}}{p^b_t}\phi_t\beta(q_tK_t+p^b_t) & \text{if } \phi_t>\phi^* \end{cases}$$

$$\Rightarrow \frac{q_{t+1}K_{t+1}+p^b_{t+1}}{q_tK_t+p^b_t}=\begin{cases} \beta q_{t+1}\left[\frac{\Omega}{\beta}-a^L(1+\tau)\phi_t\right]+\frac{p^b_{t+1}}{p^b_t}\phi_t & \text{if } \phi_t\le\phi^* \\ \beta q_{t+1}\left[a^H-a^H(1+\tau)\phi_t\right]+\beta\frac{p^b_{t+1}}{p^b_t}\phi_t & \text{if } \phi_t>\phi^* \end{cases}.$$

Using the flow of bubble price, evolution of wealth, and interest rate, we characterize the evolution of bubble below:

$$\begin{aligned}\phi_{t+1}&=\frac{\frac{p^b_{t+1}}{p^b_t}}{\frac{q_{t+1}K_{t+1}+p^b_{t+1}}{q_tK_t+p^b_t}}\phi_t \\ &=\begin{cases} \frac{1}{\beta}\frac{\frac{1-h-(1+\tau)\phi_t}{\rho(1-h)-(1+\tau)\phi_t}(1+\tau)\phi_t}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)+\frac{(1-\rho)(1-h)}{\rho(1-h)-(1+\tau)\phi_t}(1+\tau)\phi_t} & \text{if } \phi_t\le\phi^* \\ \frac{\theta}{\beta}\frac{(1+\tau)\phi_t}{\rho(1-h)-(1-\theta)(1+\tau)\phi_t} & \text{if } \phi_t>\phi^* \end{cases}.\end{aligned}$$

Steady-state bubble size First, we use the above evolution of the bubble, to solve for steady-state bubble size for each case of small and large bubbles, as a function of macroprudential tax:

$$\phi(\tau)=\begin{cases} \frac{\rho-\frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}{1-\frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}\frac{1-h}{1+\tau} & \text{if } \phi\le\phi^* \\ \frac{\rho(1-h)}{(1-\theta)(1+\tau)}-\frac{\theta}{\beta(1-\theta)} & \text{if } \phi>\phi^* \end{cases}.$$

Steady-state capital is expressed as follows by substituting the steady-state bubble size.

$$K_b(\tau)=\begin{cases} \left(\frac{\left(1+\frac{a^H-a^L}{a^L-\theta a^H}\right)h}{1+\tau-\beta\rho(1-h)+\tau\frac{(1-h)(1-\rho)\beta}{\left(1+\frac{a^H-a^L}{a^L-\theta a^H}h\right)\beta-1}}\beta a^L\alpha\right)^{\frac{1}{1-\alpha}} & \text{if } \phi\le\phi^* \\ \left((1+\tau)\frac{\beta[1-\rho(1-h)]+(1+\tau-\beta)\theta}{1+\tau-\beta\rho(1-h)}a^H\alpha\right)^{\frac{1}{1-\alpha}} & \text{if } \phi>\phi^* \end{cases}.$$

A.2 Proofs

A.2.1 Proof of Proposition 1

Proof. Recall that the size of a large bubble in a steady state is given by

$$\phi_{lb} = \frac{\beta\rho(1-h)}{\beta(1-\theta)} - \frac{\theta}{\beta(1-\theta)}.$$

Given that a large bubble exists, its size on the saddle path must be equal to the steady-state size $\forall t$. Thus, a necessary and sufficient condition for large bubble existence is $\phi_{lb} > 0$, or equivalently

$$\theta < \beta\rho(1-h).$$

Now consider the size of a small bubble in a steady state:

$$\phi_{sb}(\tau) = \frac{\rho - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}{1 - \frac{1-\rho\beta(1-h)}{\left(1+\frac{h(a^H-a^L)}{a^L-\theta a^H}\right)\beta-\beta(1-h)}}(1-h).$$

Once again, given that a small bubble exists, its size on the saddle path must be equal to the steady state-size in all t. Thus, a necessary and sufficient condition for small bubble existence is $\phi_{sb} > 0$, or equivalently:

$$\rho > \frac{a^L - \theta a^H}{\beta(a^L - \theta a^H) + \beta h(a^H - a^L)}.$$

□

A.2.2 Proof of Proposition 2

Proof. In this proof, we conveniently define $\gamma_0^* \equiv \frac{\gamma_0}{\Pi^*}$.

Dynamics after T (the period the bubble collapses): the law of motion for capital after T is identical to that in the bubbleless environment, except that L_{T+s} may not be one:

$$A_{T+s} = q_{T+s}K_{T+s} = \alpha K_{T+s}^{\alpha} L_{T+s}^{1-\alpha}$$
$$K_{T+s+1} = ha^H \frac{\beta A_{T+s}}{1-\frac{\theta\alpha^H}{\alpha^L}} + a^L\left(\beta A_{T+s} - \frac{\beta p A_{T+s}}{1-\frac{\theta\alpha^H}{\alpha^L}}\right) = \Omega A_{T+s}.$$

From the firm's first-order conditions, we have $\left(\frac{K_{T+s}}{L_{T+s}}\right)^{\alpha} = \frac{w_{T+s}}{1-\alpha}$ so that the dynamics above

can be rewritten in terms of wage:

$$
\begin{aligned}
q_{T+s} &= \alpha \left(\frac{K_{T+s}}{L_{T+s}}\right)^{\alpha-1} = \alpha \left(\frac{w_{T+s}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} \\
A_{T+s} &= \alpha K_{T+s}^{\alpha} L_{T+s}^{1-\alpha} = \alpha \left(\frac{w_{T+s}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} K_{T+s} \\
K_{T+s+1} &= \alpha\Omega \left(\frac{w_{T+s}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} K_{T+s} \\
L_{T+s} &= \left(\frac{w_{T+s}}{1-\alpha}\right)^{-\frac{1}{\alpha}} K_{T+s}.
\end{aligned}
$$

Dynamics between T and $T+s^*-1$ (the slump): during these periods, due to nominal wage rigidity, the real wage is given by:

$$
w_{T+s} = \gamma_0^* L_{T+s}^{\gamma_1} w_{T+s-1}.
$$

The above equations can be combined to derive a recursive form for the capital stock, employment, and wage during the slump ($0 \leq s < s^*$)

$$
\begin{aligned}
K_{T+s} &= \alpha\Omega \left(\frac{w_{T+s-1}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} K_{T+s-1} \\
L_{T+s} &= \left(\frac{(1-\alpha)^{1-\alpha}}{\gamma_0^*} \left(\Omega\alpha L_{T+s-1} w_{T+s-1}\right)^{\alpha}\right)^{\frac{1}{\gamma_1+\alpha}} w_{T+s-1}^{-\frac{1}{\gamma_1+\alpha}} \\
w_{T+s} &= \gamma_0^* \left(\frac{(1-\alpha)^{1-\alpha}}{\gamma_0^*} \left(\Omega\alpha L_{T+s-1} w_{T+s-1}\right)^{\alpha}\right)^{\frac{\gamma_1}{\gamma_1+\alpha}} w_{T+s-1}^{\frac{\alpha}{\gamma_1+\alpha}}.
\end{aligned}
$$

We first consider the slump under the simple case of $\gamma_1 = 0$. Then, during the slump, the real wage is simply given by

$$
w_{T+s} = \gamma_0^{*s} w_T,
$$

and thus the capital stock and labor are

$$
\begin{aligned}
K_{T+s+1} &= \alpha\Omega \left(\frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} {\gamma_0^*}^{\frac{\alpha-1}{\alpha}s} K_{T+s} \\
L_{T+s} &= \left(\frac{\gamma^s w_T}{1-\alpha}\right)^{-\frac{1}{\alpha}} K_{T+s} = {\gamma_0^*}^{-\frac{s}{\alpha}} \frac{K_{T+s}}{K_T}.
\end{aligned}
$$

Proceeding by backward iteration:

$$
\begin{aligned}
K_{T+s+1} &= \left[\alpha\Omega\left(\frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}\right]^{s+1}\left(\prod_{i=0}^{s}{\gamma_0^*}^{\frac{\alpha-1}{\alpha}i}\right)K_T \\
&= \left[\alpha\Omega\left(\frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}\right]^{s+1}{\gamma_0^*}^{\frac{\alpha-1}{\alpha}\frac{s(s+1)}{2}}K_T \\
&= \left[\alpha\Omega K_T^{\alpha-1}\right]^{s+1}{\gamma_0^*}^{\frac{\alpha-1}{\alpha}\frac{s(s+1)}{2}}K_T.
\end{aligned}
$$

Thus, we can derive the duration of the slump s_0^* for the case of $\gamma_1 = 0$ in a closed-form expression:

$$
\begin{aligned}
s_0^* &= \min\left\{s \geq 0 \mid w_{T+s}^f \geq \gamma_0^{*s} w_T\right\} \\
&= \min\left\{s \geq 0 \mid (1-\alpha)K_{T+s}^{\alpha} \geq \gamma_0^{*s} w_T\right\} \\
&= \min\left\{s \geq 0 \mid K_{T+s}^{\alpha} \geq \gamma_0^{*s} K_T^{\alpha}\right\} \\
&= \min\left\{s \geq 0 \mid \left[\alpha\Omega K_T^{\alpha-1}\right]^s {\gamma_0^*}^{\frac{\alpha-1}{\alpha}\frac{(s-1)s}{2}} K_T \geq {\gamma_0^*}^{\frac{s}{\alpha}} K_T\right\} \\
&= \min\left\{s \geq 0 \mid s\log_{\gamma_0^*}(\alpha\Omega) + (\alpha-1)s\log_{\gamma_0^*} K_T + \frac{\alpha-1}{\alpha}\frac{(s-1)\,s}{2} \leq \frac{s}{\alpha}\right\} \\
&= \min\left\{s \geq 0 \mid (\alpha-1)\frac{(s-1)}{2} \leq 1 - \alpha\log_{\gamma_0^*}(\alpha\Omega) - \alpha(\alpha-1)\log_{\gamma_0^*} K_T\right\} \\
&= \min\left\{s \geq 0 \mid s \geq \frac{2\alpha}{1-\alpha}\log_{\gamma_0^*}(\alpha\Omega) - \frac{1+\alpha}{1-\alpha} - 2\alpha\log_{\gamma_0^*} K_T\right\}.
\end{aligned}
$$

Define

$$
\omega(\gamma) \equiv \frac{2\alpha}{1-\alpha}\log_{\gamma}(\alpha\Omega) - \frac{1+\alpha}{1-\alpha}.
$$

Then we have

$$
s_0^* = \max\left\{0, \left\lceil \omega(\gamma_0^*) - 2\alpha\log_{\gamma_0^*} K_T \right\rceil\right\},
$$

as desired.

With $\gamma_1 > 0$, as before, the length of the slump can be equivalently written as

$$
s^* = \min\left\{s \geq 0 \mid \ w_{T+s}^f(\gamma_1) \ \geq \ \gamma_0^* L_{T+s}^{\gamma_1} w_{T+s-1} = \gamma_0^{*s} w_T \textstyle\prod_{i=0}^{s-1} L_{T+s-i}^{\gamma_1} \ .\right\}.
$$

To clarify notation, we write the full employment wage as a function of $\gamma_1 > 0$. Note that

$$\gamma_0^{*s} w_T \underbrace{\prod_{i=0}^{s-1} L_{T+s-i}^{\gamma_1}}_{<1} \quad < \quad \gamma_0^{*s} w_T \quad .$$

The full employment wage is determined by the capital level at $T+s$, which in turn is a function of γ_1, as capital depends on past binding wages:

$$K_{T+s}(\gamma_1) = \alpha\Omega \left(\frac{w_{T+s-1}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} K_{T+s-1}(\gamma_1).$$

Iterating backward, we get a similar result as before, however with a new term:

$$\begin{aligned} K_{T+s}(\gamma_1) &= \alpha\Omega \left(\frac{w_{T+s-1}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} K_{T+s-1}(\gamma_1) \\ &= \underbrace{\left(\alpha\Omega \left(\frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}\right)^s \left(\prod_{i=0}^{s-1} \gamma_0^{*\frac{\alpha-1}{\alpha}i}\right) K_T}_{K_{T+s}(\gamma_1=0)} \underbrace{\prod_{j=1}^{s-1} \left(\prod_{i=j}^{s-1} L_{T+s-i}^{\gamma_1}\right)^{\frac{\alpha-1}{\alpha}}}_{>1} \\ &> K_{T+s}(\gamma_1 = 0). \end{aligned}$$

Therefore, at the end of the slump $s \geq s^*$, we have the following string of inequalities:

$$w_{T+s}^f(\gamma_1) \geq \underbrace{w_{T+s}^f(0) \quad \geq \quad \gamma_0^{*s} w_T}_{\text{end of slump with } \gamma_1=0} > \gamma_0^{*s} w_T \prod_{i=0}^{s-1} L_{T+s-i}^{\gamma_1}.$$

Ending the slump with $\gamma_1 = 0$ then implies ending the slump with $\gamma_1 > 0$; that is, $s^* \leq s_0^*$, as desired.

Once the slump has ended ($s > s^*$), there are no other external shocks to the economy. Thus, the dynamics are identical to the bubbleless environment:

$$\begin{aligned} w_{T+s} &= w_{T+s}^f = (1-\alpha) K_{T+s}^{\alpha} \\ L_{T+s} &= 1 \\ q_{T+s} &= \alpha K_{T+s}^{\alpha-1} \\ K_{T+s+1} &= \Omega A_{T+s} = \alpha\Omega K_{T+s}^{\alpha}. \end{aligned}$$

□

A.2.3 Proof of Proposition 3

First, we show that the inflation is at the target and there is full employment in the immediate aftermath of the bubble's collapse:

Lemma 3. $\Pi_{T-1,T} = \Pi^*$ *and* $L_T = 1$.

Proof. To see this, recall that, by assumption, the economy is still in the bubbly steady state in $T-1$, and therefore the nominal interest rate $i_{T-1,T}$ is determined by the unconstrained Taylor rule $1 + i_{T-1,T} = R_b\Pi^*$.

Furthermore, recall the Fisher equation (9) that equates the expected return from nominal bond holding and real lending for L-type entrepreneurs between period $T-1$ and T:

$$1 + i_{T-1,T} = \frac{\rho u'(c_b^L)R_b\Pi^* + (1-\rho)u'(c_T^L)R_{T-1,T}\Pi_{T-1,T}}{\rho u'(c_b^L) + (1-\rho)u'(c_T^L)},$$

where we have used the fact that in the good state that the bubble persists in period T (which happens with probability ρ from the information set at $T-1$), the economy continues to be in the bubbly steady state with consumption level c_b^L for the L-type, the real interest rate is R_b and inflation is Π^*. Thus the indifference condition above can be rewritten as:

$$R_b\Pi^* = \frac{\rho u'(c_b^L)R_b\Pi^* + (1-\rho)u'(c_T^L)R_{T-1,T}\Pi_{T-1,T}}{\rho u'(c_b^L) + (1-\rho)u'(c_T^L)},$$

or equivalently

$$R_b\Pi^* = R_{T-1,T}\Pi_{T-1,T}.$$

In addition, recall that the real interest rate between $T-1$ and T is given by:

$$R_{T-1,T} = \frac{a^H\beta(1-\phi)}{1-\beta\phi}\alpha\left(\frac{L_T}{K_T}\right)^{1-\alpha} = R_b L_T^{1-\alpha}.$$

Thus the equation above reduces to:

$$\Pi^* = \Pi_{T-1,T}L_T^{1-\alpha}. \tag{36}$$

Now suppose on the contrary that $L_T < 1$. Then the DNWR must bind at T, or

$$\frac{w_T}{w_{T-1}} = \frac{\gamma(1-L_T)}{\Pi_{T-1,T}}.$$

By substituting the first-order condition of firms (5), we then get:

$$L_T^{-\alpha} = \frac{\gamma(1 - L_T)}{\Pi_{T-1,T}}. \tag{37}$$

Equations (36) and (37) then imply

$$\begin{aligned} \Pi^* &= \gamma(1 - L_T)L_T \\ \Rightarrow \Pi^* &= \gamma_0 L_T^{1+\gamma_1} < \gamma_0. \end{aligned}$$

However, this violates assumption (11).

Therefore, it must be that $L_T = 1$. Equation (36) then implies $\Pi_{T-1,T} = 1$. □

Now the proof for the proposition follows straightforwardly:

Proof. Given $\Pi_{T-1,T} = \Pi^*$, it is immediate that (25) is equivalent to

$$R^f_{T,T+1}\Pi^* < 1.$$

As the bubble has collapsed in $T + 1$, the real interest rate with full employment is simply given by:

$$R^f_{T,T+1} = a^L \alpha K_{T+1}^{\alpha-1},$$

as the post-bubble economy follows the bubbleless dynamics. In addition, from the law of motion of capital, we have $K_{T+1} = \alpha\Omega K_T^\alpha L_T^{1-\alpha} = \alpha\Omega K_b^\alpha$. Therefore, $R^f_{T,T+1}\Pi^* < 1$ if and only if $a^L\alpha\left(\alpha\Omega K_b^\alpha\right)^{\alpha-1} < \frac{1}{\Pi^*}$, which is equivalent to (25). □

A.2.4 Proof of Proposition 5

The closed-form expressions in the proposition can be derived directly from equations (28), (29), and (30), and are shown in Appendix A.1.3.

For these values to constitute an equilibrium path after the collapse of the bubble in period T, the necessary and sufficient conditions are that the DNWR and the ZLB do indeed bind. From Proposition 4, we know it is sufficient to show that the ZLB binds, i.e., $R^f_{t-1,t}\left(\Pi_{t-2,t-1}\right)^\zeta\left(\Pi^*\right)^{1-\zeta} < 1$ for all t, where the real interest rate with full employment is given by $R^f_{t-1,t} = a^L\alpha K_t^{\alpha-1}$. This inequality holds if and only if $K_t > \left(a^L\alpha\left(\frac{\Pi_{t-2,t-1}}{\Pi^*}\right)^\zeta\Pi^*\right)^{\frac{1}{1-\alpha}}$ for all t, as stated in the proposition.

Finally, it is algebraically straightforward to show that $\lim_{t\to\infty} K_{T+t} = K$, $\lim_{t\to\infty} L_{T+t} = L$ and $\lim_{t\to\infty}\Pi_{T+t-1,T+t} = \Pi$, where K, L, and Π are the capital, labor, and inflation in the

bad bubbleless steady state as established in Section 3.

A.2.5 Proof of Lemma 2

Proof. With full employment, when the bubble bursts, the economy follows the bubbleless dynamics. Furthermore, investments in the burst period are made before the burst. Therefore, the capital stock in the period after the burst will be equal to the bubbly steady-state capital stock; that is, $K_{T+1} = K_T = K_b(\tau)$. The welfare of a worker in the stochastic bubbly steady state must then satisfy

$$\mathcal{W}\left(K_b(\tau)\right) = \log c_b^w + \beta\left[\rho\mathcal{W}\left(K_b(\tau)\right) + (1-\rho)\mathcal{W}_{nb}\left(K_b(\tau)\right)\right],$$

where

$$c_b^w = w_b = (1-\alpha)K_b(\tau)^\alpha,$$

and $\mathcal{W}_{nb}$ denotes the expected discounted utility of the worker in the bubbleless economy. Algebra yields

$$\mathcal{W}\left(K_b(\tau)\right) = \frac{\log c_b^w + \beta(1-\rho)\mathcal{W}_{nb}\left(K_b(\tau)\right)}{1-\beta\rho}.$$

In the bubbleless economy, a worker's welfare in any period t satisfies

$$\mathcal{W}_{nb}\left(K_t\right) = \log c_t + \beta\mathcal{W}_{nb}\left(K_{t+1}\right),$$

where

$$\begin{aligned} c_t &= w_t = (1-\alpha)K_t^\alpha \\ K_{t+1} &= \alpha\Omega K_t^\alpha. \end{aligned}$$

We guess and verify that the welfare function takes the following functional form: $\mathcal{W}_{nb}\left(K\right) = f + g\log K$. Given this guess, we have

$$f + g\log K_t = \log\left[(1-\alpha)K_t^\alpha\right] + \beta\left(f + g\log\left[\alpha\Omega K_t^\alpha\right]\right).$$

Solving for the coefficients yields

$$
\begin{aligned}
g &= \alpha + \alpha\beta g = \frac{\alpha}{1-\alpha\beta} \\
f &= \log\left[1-\alpha\right] + \beta f + \beta g \log\left[\alpha\Omega\right] \\
&= \frac{1}{1-\beta}\left(\log\left[1-\alpha\right] + \frac{\alpha\beta}{1-\alpha\beta}\log\left[\alpha\Omega\right]\right).
\end{aligned}
$$

Thus, we have verified our guess and achieve the following solution to $\mathcal{W}_{nb}$:

$$
\mathcal{W}_{nb}\left(K\right) = \frac{1}{1-\beta}\left(\log\left(1-\alpha\right) + \frac{\alpha\beta}{1-\alpha\beta}\log\left(\alpha\Omega\right)\right) + \frac{\alpha}{1-\alpha\beta}\log K.
$$

Plugging the bubble-less welfare and bubbly steady-state worker consumption into the stochastic bubbly steady state,

$$
\begin{aligned}
\mathcal{W}\left(K_b(\tau)\right) &= \frac{\log c_b^w + \beta(1-\rho)\mathcal{W}_{nb}\left(K_b(\tau)\right)}{1-\beta\rho} \\
&= \frac{1}{1-\beta\rho}\left[\log\left((1-\alpha)K_b(\tau)^{\alpha}\right)\right. \\
&\left.+\beta(1-\rho)\left(\frac{1}{1-\beta}\left(\log\left(1-\alpha\right) + \frac{\alpha\beta}{1-\alpha\beta}\log\left(\alpha\Omega\right)\right) + \frac{\alpha}{1-\alpha\beta}\log K_b(\tau)\right)\right] \\
&= \frac{1}{1-\beta\rho}\left[\alpha\frac{1+\beta(1-\rho-\alpha)}{1-\alpha\beta}\log K_b(\tau)\right. \\
&\left.+\underbrace{\frac{1-\beta\rho}{1-\beta}\log(1-\alpha) + \frac{\beta(1-\rho)}{1-\beta}\frac{\alpha\beta}{1-\alpha\beta}\log\left(\alpha\Omega\right)}_{\omega}\right].
\end{aligned}
$$

□

A.2.6 Proof of Lemma 6

Proof. From Lemma 2, we know that conditional on the maintenance of full employment, maximizing the welfare function $\mathcal{W}(\tau)$ of a worker is equivalent to maximizing the capital stock $K_b(\tau)$. Thus, the optimal policy should simply set the tax τ^* such that $K_b(\tau^*) = \underline{K}$. Recall that as the bubble is assumed to be large, the expression for $K_b(\tau)$ is

$$
K_b(\tau) = \left((1+\tau)a^H\alpha\frac{(1+\tau-\beta)\theta + \beta[1-\rho(1-h)]}{1+\tau-\beta\rho(1-h)}\right)^{\frac{1}{1-\alpha}},
$$

and recall that $\underline{K} \equiv \left(\frac{\alpha\Omega\Pi^*}{\gamma_0}\right)^{\frac{1}{1-\alpha}}$. Hence $1+\tau^*$ is simply the solution to $(1+\tau)a^H\alpha\frac{(1+\tau-\beta)\theta+\beta[1-\rho(1-h)]}{1+\tau-\beta\rho(1-h)} = \frac{\Omega\Pi^*}{a^H\gamma_0}$, or equivalently

$$\theta(1+\tau)^2 + \beta(1+\tau)(1-\theta-l\rho) - \frac{\Pi^*\Omega(1+\tau-\beta l\rho)}{a^H\gamma_0} = 0,$$

which is a quadratic equation in $1+\tau$ with one unique positive root. $\square$

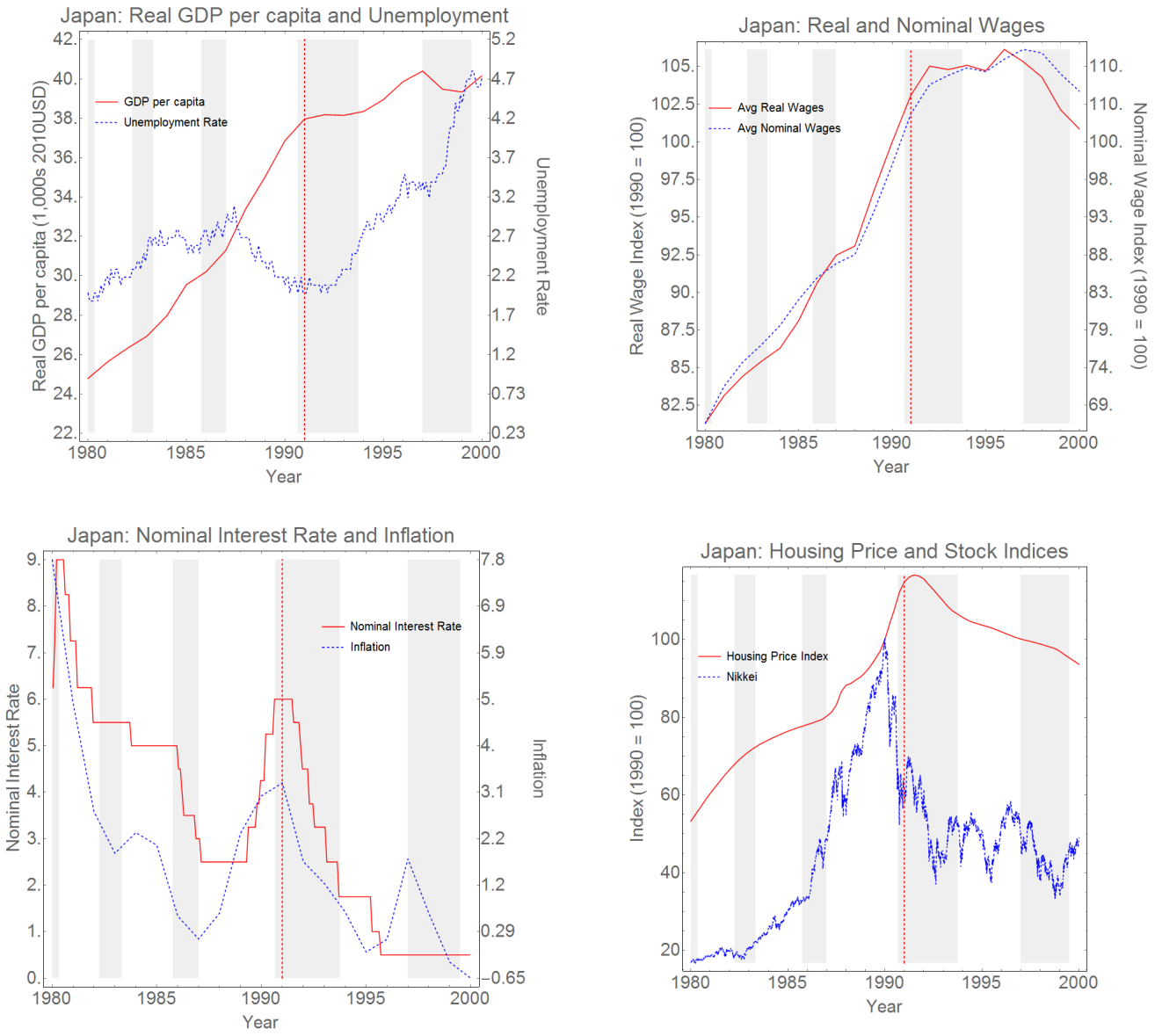

Figure A.1: Japan before and after the collapse of asset prices. Dashed vertical lines indicate the approximate beginning of the collapse in asset prices (1991). Grey bars indicate recessions, according to the OECD. Real wages are calculated from nominal wages and consumer price indices. The nominal interest rate refers to the discount rate of commercial bills and interest rates on loans secured by government bonds, specially designated securities, and bills corresponding to commercial bills. Sources: Statistics Bureau of Japan, OECD, IMF, FRB St. Louis, and Mack et al. (2011).

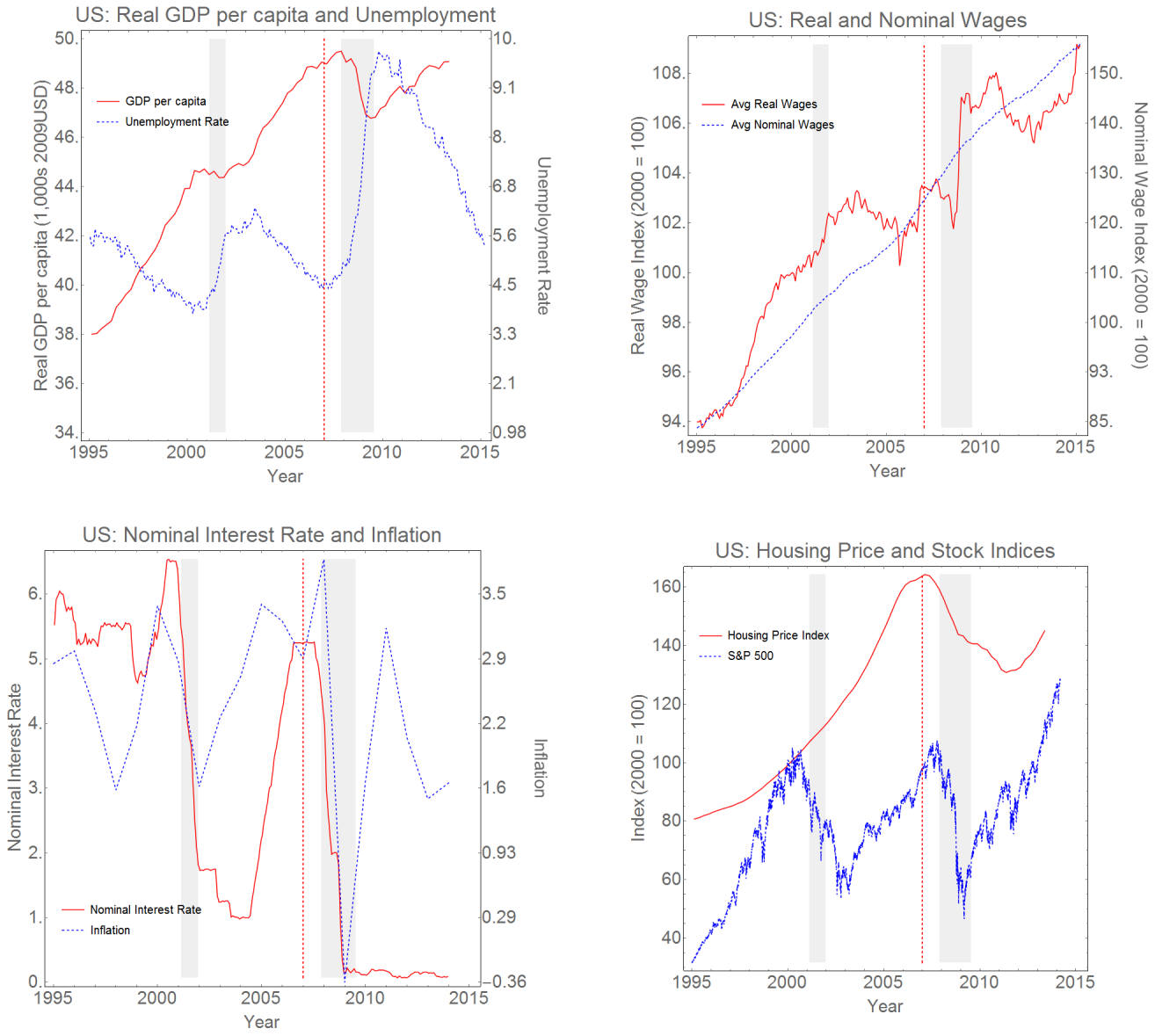

Figure A.2: U.S. before and after the collapse of asset prices. Dashed vertical lines indicate the approximate beginning of the collapse in asset prices (2007). Grey bars indicate recessions, according to the NBER. Real wages are calculated from nominal wages and consumer price indices. The nominal interest rate refers to the effective federal funds rate. Sources: NBER, OECD,US Bureau of Labor Statistics, US Bureau of Economic Analysis, S&P500, US Federal Housing Finance Agency, and FRB St. Louis.