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Firm Size, Market Power, and Competition in a Frictional Market*

Bharadwaj Kannan, Roberto Pinheiro, and Jan-Peter Siedlarek†

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ABSTRACT

We study how firm size and market concentration drive price dispersion and margins in a market with information frictions. We start from the noisy-search framework and tie firm size to the number of establishments owned by a firm. Consumers sample establishments rather than firms, allowing multi-establishment firms to capture even informed consumers. This benefit comes at a cost to the firm, since establishments cannibalize each other's demand. We show that uniform pricing across a firm's establishments is optimal. The unique equilibrium involves price distributions with supports connected and nested by firm size: larger firms post prices over a wider support, but earn lower profit per establishment. Finally, mergers can increase concentration while reducing expected transaction prices and markups when the merger does not create a new market leader. When the merger creates a new largest firm, its effect can instead reverse, depending on the relative strength of the concentration and largest-firm channels. Thus, the effect of greater concentration depends critically on how ownership changes the firm size distribution.

Keywords: Search Frictions, Market Power, Firm Heterogeneity
JEL classification: D43, D83, L13

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1 Introduction

The deleterious effects of market concentration and the potential increase in market power over the last few decades (De Loecker, Eeckhout, and Unger (2020)) have been extensively discussed in the recent literature. However, how the interaction of firm size distribution, market competition, and market frictions impacts concentration and market power is still unclear. For example, there is disagreement in the empirical literature about how the patterns of market concentration across national and local markets interact with each other (Rossi-Hansberg, Sarte, and Trachter, 2021; Smith and Ocampo, 2025). The roles of industry leaders and superstar firms seem key in explaining such effects (Autor et al., 2020). Furthermore, the interaction between firm size distribution and market competition seems to create stark differences in market concentration across market sizes (see Hershbein, Macaluso, and Yeh (2019)). Finally, there is significant disagreement even about which measures of market concentration or market power to focus on (Rossi-Hansberg, Sarte, and Trachter (2021) and Eeckhout (2021)). Hence, a clear theoretical framework that takes these features into account is necessary in order to interpret the available evidence.

In this paper, we consider how firm size and market concentration affect price dispersion and market power in a market with information frictions. We introduce differences in firm size through the number of establishments owned by a firm. This is a simplification, but it is a useful modeling device for three reasons. First, the number of establishments gives us a clear empirical counterpart. Second, it gives the firm a concrete way to influence how many consumers it is able to capture by adding establishments within the market. Finally, it creates a natural way of measuring both the benefits and the costs of market power: more establishments give the firm access to more captive consumers, but they also compete with each other for the same demand.

We model information frictions using a noisy-search framework as in Burdett and Judd (1983) but with multi-establishment firms. A key distinction of our model is that consumers sample price quotes from establishments rather than firms. As a result, even a consumer who observes multiple price quotes may receive quotes from the same firm. This creates the main trade-off in the paper. By operating more establishments in a local market, a multi-establishment firm increases the probability that a consumer who searches ends up seeing only that firm's price. In this sense, the firm endogenously increases its captive customer base. At the same time, operating more establishments creates sales cannibalization. When two establishments of the same firm are sampled by the same consumer, the firm does not make two sales. Instead, its establishments compete with each other for that consumer. The same organizational choice that increases the firm's ability to capture consumers therefore also creates a within-firm negative externality between establishments. The equilibrium in

this search market is shaped by the balance between these two forces and, importantly, by how many other multi-establishment firms are competing in the same market.

Our model delivers several key results. First, we show that it is optimal for multi-establishment firms to post identical prices across its multiple locations. This is in line with the empirical evidence presented by DellaVigna and Gentzkow (2019). The intuition is simple — the presence of multiple establishments in the same market by itself creates intra-firm quantity cannibalization, regardless of the prices posted by each establishment. If the multi-establishment firm deviated from common pricing and allowed within-firm establishments to price independently, it would introduce additional price cannibalization, since the consumer would always buy from the cheaper establishment, which would further reduce firm profits.

Second, our model delivers a unique pricing equilibrium in mixed strategies that we fully characterize through firms' optimal pricing distributions. Moreover, the number of establishments owned by a firm — that is, its size — determines its optimal pricing strategy. We show that the supports of the optimal mixed-strategy pricing distributions of all firms are connected and nested by firm size, in that larger firms' price distributions fully overlap with and extend above those of smaller firms. As a result, firms of different size share a common floor price in their equilibrium pricing distributions, and if one firm is smaller than another, the upper limit of the smaller firm's price distribution is strictly below that of the larger firm. Upper limits are equal only if the larger firm is the unique largest firm in the market, in which case it has a mass point in its pricing distribution at the consumers' reservation price, and the supports of the price distributions of the next-largest firms have an upper limit at the reservation. If there are multiple firms in the largest size class, all price distributions have a continuous support without mass points.

Third, our model allows us to evaluate the effect of changes in the firm establishment concentration. We consider mergers that keep the total number of establishments constant, while changing the firm size distribution and, thus, the market concentration. Our analysis shows that results hinge on whether the merger creates a new largest firm or not. If the merger creates a larger firm but not a market leader, competition between firms intensifies. As a result, ex ante expected transaction prices and markups fall, even though concentration, measured by the Herfindahl-Hirschman Index (HHI), increases. Moreover, with an interior reservation price below the maximum consumer valuation, the reservation price and the common floor price also drop. The merged firm's profit per establishment strictly falls relative to the establishment-weighted pre-merger benchmark, while the profits for other competitors also drop if the common floor price declines. By contrast, if the merger creates a new industry leader, the direction of the effects depends on the parameters. In particular, outcomes depend on the impact of the merger on (i) the expected number of distinct price quotes received

by consumers, (ii) the change in market concentration, and (iii) the new leader's pricing support range. The interaction between these three measures yields distinct parameter regions that allow us to evaluate how the merger impacts expected transaction prices, markups, and reservation and common floor prices. In general, if changes in concentration are small relative to the change in the leader's pricing support range, expected transaction price and markups tend to go up with the merger, and decline otherwise.

Finally, our model allows us to evaluate the entry of new firms in the local market, as well as changes in concentration, through comparative statics. In particular, the entry of new firms drives down expected transaction prices. In contrast, an increase in the number of establishments per firm increases expected transaction prices. As a result, our model allows us not only to evaluate the impact of entry in a local market, but also the mechanism through which the new entrant decides to access the market. As our model shows, opening a single establishment, opening multiple establishments, acquiring a local competitor, or acquiring and merging multiple local competitors impacts the firm size distribution in starkly different ways.

Our model contributes to several strands of the literature. First, our model emphasizes the key distinction between firms and establishments and shows the optimality of uniform pricing across a firm's establishments. DellaVigna and Gentzkow (2019) document a high degree of uniform pricing within US retail chains, while Hazell et al. (2023) find an important role for national wage setting. This type of cross-establishment coordination has also motivated work on the transmission of local shocks and policy. Butters, Sacks, and Seo (2022), for example, study how national firms respond to local cost shocks, while Wu (2024) examines how the geographic organization of retail chains affects monetary transmission. We show that uniform pricing across establishments can be optimal in a frictional market setting and ask how it affects competition when consumers search locally.

Second, our model helps clarify how changes in national and local concentration measures are related to the way in which a national leader decides to enter a new market, as well as the ownership structure in the market before and after the entry decision. As a result, our model speaks to the conflicting results found by Rossi-Hansberg, Sarte, and Trachter (2021) and Rinz (2022) who show local and national concentration moving in opposite directions, and Smith and Ocampo (2025) who find that retail concentration increased at both the national and the local levels. Moreover, the model highlights the important role played by the largest firms in determining equilibrium prices in a search setting, revealing a superstar firm effect that echoes themes in Autor et al. (2020): the reservation price, the common floor, and the expected transaction price depend on the ownership distribution only through the largest network size and a single concentration index. Moreover, our results on merger analysis highlight how starkly different results can be depending on whether the merger creates the new market leader or

not.

Third, our model contributes to arguments that measured firm size concentration and market power should not be treated as interchangeable objects. Our merger analysis and comparative statics results indicate that these measures can behave in quite different ways and represent qualitatively distinct objects. Furthermore, our model delivers intuitive closed-form formulas for both the revenue-based HHI and markups. As emphasized by Eeckhout (2021), concentration measures such as the HHI can be difficult to interpret as measures of market power. Market definition is difficult, and changes in population or market size can mechanically affect local concentration. A natural alternative is to focus on markups more directly, following the logic in Hall (1988). But markup measures do not tell us why firms have market power. Menzio (2026) stresses that this distinction matters: a markup generated by an information friction has a different economic interpretation from a markup generated by the standard monopolistic-competition mechanism in Dixit and Stiglitz (1977). Our framework adds ownership structure to this search-theoretic view of markups. The same search friction can generate different price and markup distributions depending on whether establishments are spread across many firms or concentrated within a few owners.

Finally, our model presents a contribution to the established search literature. Our noisy-search framework is closest to Burdett and Smith (2010) and Menzio and Trachter (2015), both of which study an environment with one large firm. Burdett and Smith (2010) assume that small firms are strategic followers to the large firm in a Stackelberg sense. We instead let all firms set prices simultaneously in a Nash equilibrium. This strategic agnosticism allows us to separate the role of firm size from the large firm's first-mover advantage and, more importantly, makes it feasible to integrate competition among several large firms into the model. Menzio and Trachter (2015) consider sequential search. Their large firm can use low prices to induce consumers to stop searching, which produces a distinctive pattern of periodic "sales." In our noisy-search model in a comparable setting with only one multi-establishment firm, the large firm's price distribution instead first-order stochastically dominates the distribution of the single-establishment firms and places its additional probability mass at the reservation price. When there are several largest multi-establishment firms, competition among them eliminates this atom altogether. The comparison shows that the form of the information friction matters for the shape of equilibrium price dispersion, as does the competitive environment.

At its core, our paper presents a generalization of Burdett and Judd (1983) in which a flexible ownership structure adds three margins that are useful for thinking about market power. First, it endogenizes part of the captive-consumer share. In Varian (1980) and in asymmetric sales models such as Myatt and Ronayne (2026), captive, or loyal, consumer shares are primitive features of the environment. Here, the share of effectively captive consumers arises

from the distribution of establishments across owners and the noisy-search setting. Second, it introduces a cost of capture through within-firm cannibalization. A firm can increase the probability that it is the only owner observed by a consumer, but doing so requires a larger local establishment network whose branches sometimes compete for the same consumer. Third, it allows us to speak directly about how ownership structure shapes markup distributions, adding an ownership margin to the search-theoretic analysis in Menzio (2026).

The main message is therefore not simply that large firms have more market power. The way a large firm becomes large matters. A firm with more establishments is more likely to capture a consumer, but it also creates more cannibalization inside the firm. A market with more multi-establishment firms has more large competitors, and competition among them disciplines prices in ways that cannot be reproduced by simply adding more establishments to an incumbent. By separating the number of firms from the number of establishments, the model provides a tractable way to study how ownership structure interacts with search frictions to determine price dispersion, markups, and the prices consumers actually pay.

2 The Model

Consider a model of consumer search in the spirit of Burdett and Judd (1983) and Burdett and Smith (2010), in which several firms each own a non-atomistic share of the establishments in the market. Specifically, there are $N_s > 1$ single-establishment firms and $N_m > 1$ multi-establishment firms, where N_s and N_m are integers. Each multi-establishment firm operates the same integer number $m > 1$ of establishments. Let

$$T \equiv N_s + N_m m \qquad \text{and} \qquad N \equiv N_s + N_m \tag{1}$$

denote, respectively, the total number of establishments and the total number of firms in the market. Firms sell a homogeneous, indivisible good, and marginal production cost is constant and normalized to zero. In order to recover results for a positive marginal cost κ, consider p as the unit margin and $\widetilde{p} = p + \kappa$ as the posted price. The model is then solved in margin units. The reservation price R below then represents a "reservation margin," with posted reservation price recovered as $\widetilde{R} = R + \kappa$. We will revisit the issue of non-zero marginal costs in our discussion of markups in Section 3.2. Firms post prices simultaneously, and every establishment controlled by a given multi-establishment firm posts the same price. As we show in more detail later, common pricing is optimal, since within-firm establishment price competition would decrease firm profits.

Consumers demand one unit of the good. At cost $c > 0$, a consumer visits the market and receives one price quote with probability $1 - \alpha$ and two price quotes with probability α,

where $0 < \alpha < 1$. The first establishment is sampled uniformly from the T establishments. Conditional on receiving two quotes, the second establishment is sampled uniformly from the remaining $T - 1$ establishments. Thus the two sampled establishments are distinct. Prices posted by different firms are independently randomized, whereas prices observed at two establishments controlled by the same multi-establishment firm are perfectly correlated. Unlike in Moraga-González, Sándor, and Wildenbeest (2017), two quotes may therefore come from the same firm. A consumer who purchases at price p obtains utility $z - p$. We maintain exogenous participation in the market-search stage and interpret z as the highest price at which a transaction can occur. Section 2.4 discusses how the boundary case changes if consumers may instead decline to enter the market.

The demand equations below are written from the perspective of an establishment receiving a quote contact. Among a measure n of consumers, one-quote consumers generate $(1-\alpha)n$ contacts and two-quote consumers generate $2\alpha n$ contacts. Hence the fraction of quote contacts generated by one-quote consumers, γ, is distinct from the primitive probability $1 - \alpha$ that a consumer receives one quote:

$$\gamma \equiv \frac{(1-\alpha)n}{(1-\alpha)n + 2\alpha n} = \frac{1-\alpha}{1+\alpha} \in (0, 1). \tag{2}$$

In such a noisy-search setting, multi-establishment firms may have a larger set of captured consumers, since even "well-informed" consumers observing two prices can end up with all quotes from the same multi-establishment firm. As a result, multi-establishment firms in this setting have greater market power than single-establishment firms.

Note that in our model we depart from the sequential order of price setting in Burdett and Smith (2010). In our model, all firms announce their prices simultaneously. This approach allows us to isolate the contribution of firm size and firm competition, rather than strategic first-mover advantage, to the equilibrium price distribution. The simultaneous pricing assumption should not be confused with the sequential consumer-search protocol studied by Menzio and Trachter (2015).

We start analyzing the model by setting out the profit maximization problem for single-establishment firms, before extending the problem to allow for multiple establishments and solving for optimal pricing strategies. We then study the consumer problem and characterize the equilibrium for the model.

2.1 Single-Establishment Firm Demand

A single-establishment firm i faces two types of consumers at its establishment: (i) consumers who observed only its price, and (ii) consumers who observed its price along with a second

price quoted by a competing establishment. Consumers of type (i) buy from the establishment whenever its posted price does not exceed their reservation price. Consumers of type (ii) buy from the establishment whenever its posted price does not exceed both their reservation price and the competing price they observed. Let $F_{s,i}$, $i \in \{1, \ldots, N_s\}$ denote the price distribution posted by single-establishment firm i, and let $F_{m,j}$, $j \in \{1, \ldots, N_m\}$ denote the price distribution posted by multi-establishment firm j.

We allow $F_{s,i}$ and $F_{m,j}$ to have mass points, so that two or more establishments may post the same price with positive probability. When such a tie occurs at a price no greater than the reservation price R, we assume that each tied establishment sells the good with probability $\frac{1}{2}$. To accommodate mass points and this tie-breaking rule, we define the midpoint CDF

$$\widetilde{F}(p) \equiv F(p^-) + \frac{1}{2}\Delta F(p), \qquad \Delta F(p) \equiv F(p) - F(p^-). \tag{3}$$

Hence, the demand faced by single-establishment firm i when it posts price p is:

$$D_{s,i}(p) = \gamma + \frac{1-\gamma}{N_s + N_m m - 1}\left\{\sum_{k \neq i}\left[1 - \widetilde{F}_{s,k}(p)\right] + m\sum_{\ell=1}^{N_m}\left[1 - \widetilde{F}_{m,\ell}(p)\right]\right\},$$

where γ is the contact-weighted probability defined in (2). Recalling the notation $T \equiv N_s + N_m m$ from (1), collecting terms and simplifying notation yields

$$D_{s,i}(p) = 1 - \frac{1-\gamma}{T-1}\left[\sum_{k \neq i}\widetilde{F}_{s,k}(p) + m\sum_{\ell=1}^{N_m}\widetilde{F}_{m,\ell}(p)\right]. \tag{4}$$

Notice that the framework is sufficiently general to allow both mass points in the posting-price distributions and within-type asymmetry in pricing strategies.

2.2 Multi-Establishment Firm Demand

We derive the demand faced by each establishment of multi-establishment firm j. Consider a two-quote consumer whose first sampled establishment belongs to firm j. With probability $\frac{m-1}{T-1}$ the second sampled establishment also belongs to firm j; because every establishment of firm j posts the same price, the two quotes are then identical, and the tie-breaking rule of Section 2.4[1] awards the sale to the sampled establishment with probability $\frac{1}{2}$. Otherwise, the second sampled establishment belongs to a competing firm, and the consumer buys from establishment j only if firm j's price does not exceed the competing price. Combining these

[1]This tie is guaranteed by the ownership structure and is distinct from the mass-point ties governed by the midpoint CDF $\widetilde{F}$.

cases with the one-quote consumers, analogous to firm i's demand above, the demand faced by an establishment of multi-establishment firm j is given by:

$$D_{m,j}(p) = \gamma + \frac{(1-\gamma)(m-1)}{2(T-1)} + \frac{1-\gamma}{T-1}\left\{\sum_{k=1}^{N_s}\left[1-\widetilde{F}_{s,k}(p)\right] + m\sum_{\ell\neq j}\left[1-\widetilde{F}_{m,\ell}(p)\right]\right\}.$$

Collecting terms yields

$$D_{m,j}(p) = 1 - \frac{(1-\gamma)(m-1)}{2(T-1)} - \frac{1-\gamma}{T-1}\left[\sum_{k=1}^{N_s}\widetilde{F}_{s,k}(p) + m\sum_{\ell\neq j}\widetilde{F}_{m,\ell}(p)\right]. \tag{5}$$

The fact that consumers may sample two quotes from the same firm creates intra-firm quantity cannibalization, since two contacts generate a single sale, irrespective of the price posted. If the multi-establishment firm deviated from common pricing and allowed within-firm establishments to price independently, it would additionally introduce price cannibalization, since the consumer would always buy from the cheaper establishment. Price cannibalization would reduce firm profits further. Proposition A.1 and Corollary A.1 formally show the optimality of common pricing.

2.3 The Firm Problem and Optimal Pricing Choice

For a fixed reservation price R, each firm simultaneously chooses a mixed pricing strategy to maximize expected profit. The resulting game is the pricing subgame conditional on R. Formally, the problem of single-establishment firm i is given by:

$$\Pi_{s,i} \equiv \max_{F_{s,i}\in\mathcal{F}[0,R]}\int_0^R \pi_{s,i}(p)\,dF_{s,i}(p) = \max_{F_{s,i}\in\mathcal{F}[0,R]}\int_0^R D_{s,i}(p)p\,dF_{s,i}(p) \tag{6}$$

where $\mathcal{F}[0,R]$ is the set of admissible cumulative distribution functions over prices and $D_{s,i}(p)$ is the demand function faced by a single-establishment firm i when setting price p. We restrict attention to the price interval $p \in [0,R]$, where R is the consumers' reservation price. We will show later that consumers will indeed find it optimal to use a reservation price strategy and, given that, prices outside this interval can never be optimal.

The problem of multi-establishment firm j is analogously given by:

$$\Pi_{m,j} \equiv \max_{F_{m,j}\in\mathcal{F}[0,R]} m\int_0^R D_{m,j}(p)p\,dF_{m,j}(p)$$

For the multi-establishment firms, this setup reflects the restriction that a multi-establishment

firm chooses a single optimal pricing strategy for all its establishments. That is, all establishments owned by the same firm set the same price.

For $\ell \in \{s, m\}$, let $\pi_{\ell,i}(p) \equiv D_{\ell,i}(p)p$ denote the profit per establishment earned by firm i of type ℓ from posting price p, let $F_{\ell,i}(\cdot)$ denote its equilibrium price distribution, and let $\pi^*_{\ell,i}$ denote its equilibrium profit per establishment. In a mixed-strategy equilibrium, optimality requires that firm i be indifferent among all prices to which $F_{\ell,i}$ assigns positive probability, and that no price yield strictly higher profit. Thus, in equilibrium,

$$\pi_{\ell,i}(p) = \begin{cases} \pi^*_{\ell,i} & \text{if } p \in \operatorname{supp}\{F_{\ell,i}\}, \\ \leq \pi^*_{\ell,i} & \text{otherwise,} \end{cases}$$

where $\operatorname{supp}\{F_{\ell,i}\}$ is the support of $F_{\ell,i}$.

We are now able to show some important intermediate results. First, we will show that there is no equilibrium with within-type asymmetric strategies. In other words, while single-establishment and multi-establishment firms may optimally choose different pricing strategies, it is not optimal for either single-establishment or multi-establishment firms to have different pricing strategies among themselves.

Proposition 1 (Endogenous type symmetry)**.** *Suppose firms of the same type have identical primitives and action sets, price ties are divided equally, and $1 - \gamma > 0$. In any mixed-strategy Nash equilibrium,*

$$F_{s,i} = F_{s,k} \qquad \textit{for every } i, k \in \{1, \ldots, N_s\},$$

and

$$F_{m,j} = F_{m,\ell} \qquad \textit{for every } j, \ell \in \{1, \ldots, N_m\}.$$

Hence every Nash equilibrium is type-symmetric.

As a result, we can focus on type-symmetric equilibria. Moreover, following the usual arguments presented by Burdett and Judd (1983), we can show the following standard but essential properties for this particular payoff structure: equilibrium strategies are atomless, and the union of their supports has no gaps.

Lemma 1. *If $N_s > 1$ and $N_m > 1$, $F_m(\cdot)$ and $F_s(\cdot)$ have no mass points in their support.*

Lemma 2. *Let $\underline{p}$ and $\overline{p}$ be the minimum and maximum of $\operatorname{supp}\{F_s\} \cup \operatorname{supp}\{F_m\}$. Then every $p \in [\underline{p}, \overline{p}]$ belongs to this union; equivalently, the union of the two supports is connected.*

However, it is important to highlight that Lemma 1 relies on $N_m > 1$. As we show later, in the case $N_m = 1$, $F_m(p)$ presents a mass point in R.

Given Proposition 1 and Lemma 1, any pricing equilibrium in the case of $N_s > 1$ and $N_m > 1$ must be type symmetric without mass points in the pricing strategies. As a result, we

can focus on strategies that are symmetric and continuous within firm types. Hence, we can rewrite (4) as:

$$D_s(p) = \left\{ 1 - (1-\gamma)\left[\tfrac{N_s-1}{T-1}F_{-s}(p) + \tfrac{N_m m}{T-1}F_m(p)\right] \right\} \tag{7}$$

and, similarly, rewrite (5) as:

$$D_m(p) = \left\{ \begin{array}{c} 1 - (1-\gamma)\frac{(m-1)}{2(T-1)} \\ -(1-\gamma)\left[\frac{N_s}{T-1}F_s(p) + \frac{(N_m-1)m}{T-1}F_{-m}(p)\right] \end{array} \right\}. \tag{8}$$

Based on these simplified expressions, we can show the following auxiliary result.

Lemma 3. *If p_1 and p_2 are prices posted by both single- and multi-establishment firms in equilibrium, we must have either $D_s(p_1) \geq D_m(p_1)$ and $D_s(p_2) \geq D_m(p_2)$ or $D_s(p_1) \leq D_m(p_1)$ and $D_s(p_2) \leq D_m(p_2)$.*

Given Lemma 3, from equations (8) and (7), we have:

$$D_m(p) - D_s(p) = \frac{(1-\gamma)}{(T-1)} \left\{ \begin{array}{c} -\frac{(m-1)}{2} \\ -\left[N_s F_s(p) + (N_m - 1)mF_m(p)\right] \\ +\left[(N_s - 1)F_s(p) + N_m m F_m(p)\right] \end{array} \right\} \tag{9}$$

simplifying it:

$$D_m(p) - D_s(p) = \frac{(1-\gamma)}{T-1}\left\{ mF_m(p) - \frac{(m-1)}{2} - F_s(p) \right\} \tag{10}$$

We now characterize the unique pricing equilibrium conditional on reservation price R. A detailed proof with several intermediate results can be found in the appendix.

Theorem 1 (Pricing equilibrium conditional on R)**.** *Fix a reservation price $R \in (0, z]$, $0 < \gamma < 1$, and integers $N_s, N_m, m \geq 2$. The pricing subgame has a unique Nash equilibrium. It is symmetric within firm type. Single-establishment firms set prices according to the following distribution:*

$$F_s(p) = \frac{T - 1 + \alpha(N-1)}{2\alpha(N-1)}\left(1 - \frac{\underline{p}}{p}\right), \qquad p \in [\underline{p}, \overline{p}_s]. \tag{11}$$

with support $\operatorname{supp}\{F_s\} = [\underline{p}, \overline{p}_s]$*, where:*

$$\underline{p} = \frac{T - 1 - \alpha(T-m)}{T - 1 + \alpha(T-m)} R \tag{12}$$

and

$$\overline{p}_s = \left[\frac{T - 1 - \alpha(T-m)}{T - 1 + \alpha(T-m)}\right]\left[\frac{T - 1 + \alpha(N-1)}{T - 1 - \alpha(N-1)}\right] R \tag{13}$$

while the multi-establishment firms' posted price distribution is given by:

$$F_m(p) = \begin{cases} \dfrac{T-1+\alpha m(N-1)}{2\alpha m(N-1)}\left(1-\dfrac{\underline{p}}{p}\right), & p \in [\underline{p}, \overline{p}_s], \\ 1-\dfrac{T-1-\alpha(T-m)}{2\alpha(N_m-1)m}\left(\dfrac{R}{p}-1\right), & p \in (\overline{p}_s, R]. \end{cases} \tag{14}$$

with support $\operatorname{supp}\{F_m\} = [\underline{p}, R]$.

The lower limit of the support of the price distributions of both single- and multi-establishment firms is strictly greater than zero if either $m > 1$ or $\alpha < 1$. In other words, as long as there is some market power either from ownership concentration or from search frictions, even the lowest prices a consumer might see are strictly above the marginal cost of production.

Given the characterization of the equilibrium distributions in Theorem 1, we obtain the following stochastic dominance result:

Corollary 1 (Stochastic dominance of posted prices)**.** *In the equilibrium characterized by Theorem 1, prices posted by multi-establishment firms are first-order stochastically higher than prices posted by single-establishment firms. In particular,*

$$F_m(p) \leq F_s(p) \qquad \text{for all } p,$$

with strict inequality for every $p \in (\underline{p}, R)$.

Expected posted prices by multi-establishment firms are therefore higher than expected posted prices by single-establishment ones.

2.4 Consumer Behavior

The consumer problem is to decide whether to purchase the good at one of the prices previously observed or to continue searching. When the consumer considers purchasing, the lowest observed price maximizes consumer utility, so the consumer problem reduces to deciding between the lowest available price and the expected benefit from another round of search. Let P_R denote the random variable describing the lowest price obtained on a market visit when firms price according to the pricing equilibrium conditional on reservation price R. The distribution of P_R is a mixture of the one- and two-quote events as described below. With repeated search and exogenous market participation, the consumer's stationary value satisfies

$$V = -c + \mathbb{E}\big[\max\{z - P_R, V\}\big]. \tag{15}$$

Define the consumer's reservation price by $Q \equiv z - V$. Subtracting V from both sides of (15) yields the best-response condition

$$c = \mathbb{E}\big[(Q - P_R)_+\big], \qquad (x)_+ \equiv \max\{x, 0\}. \tag{16}$$

The truncation $(\cdot)_+$ sets the integrand to zero whenever the sampled price P_R exceeds Q: in that event the consumer does not purchase and instead continues searching, so the draw contributes no surplus relative to the value of continued search.

Let $\nu_{i\bullet}$, $i \in \{s, m\}$ denote the probability that the establishment sampled by a one-quote consumer is of type i. Let ν_{ij}, $i, j \in \{s, m\}$ denote the probability that a two-quote consumer's two sampled establishments consist of one type-i and one type-j establishment (with ν_{ss} and ν_{mm} denoting the probabilities that both are of type s and both are of type m, respectively). Let G_{ss}, G_{sm}, and G_{mm} denote the CDFs of the minimum of two price quotes drawn from distinct firms of the indicated types. Then (16) is

$$\begin{aligned} c = {} & (1-\alpha)\nu_{m\bullet} \int (Q-p)_+ \, dF_m(p \mid R) + (1-\alpha)\nu_{s\bullet} \int (Q-p)_+ \, dF_s(p \mid R) \\ & + \alpha\nu_{ss} \int (Q-p)_+ \, dG_{ss}(p \mid R) + \alpha\nu_{sm} \int (Q-p)_+ \, dG_{sm}(p \mid R) \\ & + \alpha\nu_{mm} \frac{m-1}{N_m m - 1} \int (Q-p)_+ \, dF_m(p \mid R) \\ & + \alpha\nu_{mm} \frac{(N_m - 1)m}{N_m m - 1} \int (Q-p)_+ \, dG_{mm}(p \mid R). \end{aligned} \tag{17}$$

The distributions and support in (17) depend on the firms' conjectured cutoff R, which they take as given, and not on an individual consumer's deviation cutoff price Q.

The effective cutoff faced by firms is $R = \min\{Q, z\}$. In an interior equilibrium, $Q = R < z$. Because $P_R \le R$ almost surely, (16) then reduces to

$$c = R - \mathbb{E}[P_R] = R\Omega, \qquad \Omega \equiv 1 - \frac{\mathbb{E}[P_R]}{R}. \tag{18}$$

The pricing distributions are homogeneous of degree one in R, so $\Omega = \Omega(\alpha, N_s, N_m, m)$ does not depend on the level of R. If the unconstrained cutoff exceeds z, exogenous participation implies $Q \ge z$ and the effective cutoff is $R = z$. Therefore

Theorem 2 (Reservation-price equilibrium under exogenous participation)**.** *Conditional on the pricing equilibrium in Theorem 1, the effective reservation price is*

$$R = \min\left\{z, \frac{c}{\Omega(\alpha, N_s, N_m, m)}\right\}, \tag{19}$$

where $\Omega(\alpha, N_s, N_m, m)$ *is given by:*

$$\Omega = 1 - \frac{T - 1 - \alpha(T - m)}{T - 1 + \alpha(T - m)} \left[1 + \alpha - \frac{\alpha N_m m(m-1)}{T(T-1)} \right]. \tag{20}$$

If consumers may instead decline to enter the market and receive an outside option of zero, the Bellman equation becomes $V = \max\{0, -c + \mathbb{E}[\max\{z - P_R, V\}]\}$. In that alternative specification, the boundary case $R = z$ must be accompanied by the participation condition $V \geq 0$; sufficiently high search costs can then imply no market participation in equilibrium.

Based on the proof of Theorem 2, we have the following corollary:

Corollary 2. *The expected transaction price* $\mathbb{E}[P_R]$ *is given by:*

$$\mathbb{E}[P_R] = \frac{T - 1 - \alpha(T - m)}{T - 1 + \alpha(T - m)} \left[1 + \alpha - \frac{\alpha N_m m(m-1)}{T(T-1)} \right] R. \tag{21}$$

Furthermore, if R *is interior, we have that:*

$$\mathbb{E}[P_R] = \left(\frac{1 - \Omega}{\Omega} \right) c.$$

Comparison with Burdett and Judd (1983) with noisy search. We compare the setting in this paper to noisy consumer search with only single-establishment firms, as discussed in Burdett and Judd (1983, Section 3.3). The present model reduces to that in Burdett and Judd (1983) with noisy-search parameters $q_1 = 1 - \alpha$ and $q_2 = \alpha$ when we set $m = 1$ in our environment, such that every establishment is effectively an independent firm. Equations (12) and (21) then reduce to

$$\underline{p}_{BJ} = \frac{1 - \alpha}{1 + \alpha} R, \qquad \mathbb{E}[P_{R,BJ}] = (1 - \alpha) R \tag{22}$$

How does the presence of multi-establishment firms affect these model outcomes? Market power in the sense of positive probability of consumer capture, that is, $m > 1$, creates a wedge in the expected price consumers pay relative to the $m = 1$ benchmark:

$$\mathbb{E}[P_R] - \mathbb{E}[P_{R,BJ}] = \alpha \frac{(m-1)}{T-1} \left(R + \frac{N_s}{T} \underline{p} \right) > 0 \qquad \text{for } m > 1. \tag{23}$$

This wedge is strictly positive as long as there is a positive probability that consumers encounter two prices ($\alpha > 0$) and there is at least one firm with more than one establishment. If all firms have exactly one establishment, the two settings are identical, and thus the wedge in the expected price vanishes. If $\alpha = 0$, consumers only ever observe a single price before searching again, which eliminates effective competition between firms. The expected transac-

tion price then rises to the reservation price R for all firm types, again eliminating the wedge between Burdett and Judd (1983) and the generalized model studied in this paper.

2.5 Comparative Statics and Market Structure

With equilibrium price distributions fully pinned down, we are able to evaluate the impact of changes in market structure on the equilibrium objects. We start by looking at the impact of the market structure on reservation price R and the expected transaction price $\mathbb{E}[P_R]$. Since N_s, N_m, and m are integer primitives, the derivatives below are understood as derivatives of the natural continuous extension of the equilibrium formulas; for discrete entry or expansion, the corresponding finite changes have the same local interpretation.

Proposition 2 (Comparative statics)**.** *Assume that z is large enough such that the optimal R is an interior solution and given by $R = \frac{c}{\Omega}$ in equilibrium. Then,*

$$(i).\ \frac{\partial R}{\partial N_s} < 0, \qquad (ii).\ \frac{\partial R}{\partial N_m} < 0, \qquad (iii).\ \frac{\partial R}{\partial m} > 0.$$

As additional single- or multi-establishment firms join the market, the reservation price is decreasing with competition. In contrast, the reservation price is increasing with concentration as m increases while keeping the number of firms constant. The changes in the reservation price translate into changes in the other key equilibrium variables as follows:

Corollary 3. *Assume that z is large enough such that the optimal R is an interior solution and given by $R = \frac{c}{\Omega}$ in equilibrium. Then, we have that:*

1. $\frac{\partial \mathbb{E}[P_R]}{\partial N_s} < 0, \ \frac{\partial \mathbb{E}[P_R]}{\partial N_m} < 0, \ \frac{\partial \mathbb{E}[P_R]}{\partial m} > 0.$
2. $\frac{\partial \underline{p}}{\partial N_s} < 0, \ \frac{\partial \underline{p}}{\partial N_m} < 0, \ \frac{\partial \underline{p}}{\partial m} > 0.$
3. $\frac{\partial \overline{p}_s}{\partial N_m} < 0$, *while the signs of* $\frac{\partial \overline{p}_s}{\partial N_s}$ *and* $\frac{\partial \overline{p}_s}{\partial m}$ *are ambiguous.*

The impact of market structure on $\underline{p}$ and on the expected transaction price $\mathbb{E}[P_R]$ mirrors the impact on the reservation price. In other words, greater competition decreases both the minimum price posted, $\underline{p}$, and the expected transaction price, $\mathbb{E}[P_R]$, while greater market concentration increases both.

While results for the highest price posted by single-establishment firms $\overline{p}_s$ are ambiguous for N_s and m, we are still able to evaluate how competition among single- and multi-establishment firms change with market structure. In particular, we consider the

share of prices posted by multi-establishment firms that overlaps with those posted by single-establishment firms; in other words, we evaluate how:

$$\tau_m \equiv \Pr(P_m \leq \overline{p}_s) = F_m(\overline{p}_s) \tag{24}$$

From equation (14), we have that:

$$\tau_m = \frac{T-1+\alpha m(N-1)}{2\alpha m(N-1)}\left(1-\frac{\underline{p}}{\overline{p}_s}\right) = \frac{T-1+\alpha m(N-1)}{m\left[T-1+\alpha(N-1)\right]}$$

where the second equality comes from the fact that $F_s(\overline{p}_s) = 1$ and equation (11). We are now ready to show the following result.

Lemma 4. *Assume that z is large enough such that the optimal R is an interior solution and given by $R = \frac{c}{\Omega}$ in equilibrium. Define $\tau_m \equiv \Pr(P_m \leq \overline{p}_s)$. We are able to show that:*

$$(i.)\ \frac{\partial \tau_m}{\partial N_s} > 0 \qquad (ii.)\ \frac{\partial \tau_m}{\partial N_m} < 0 \qquad (iii.)\ \frac{\partial \tau_m}{\partial m} < 0.$$

The results in Lemma 4 are quite intuitive. As competition from single-establishment firms increases – N_s increases – the cost of losing informed customers to single-establishment firms rises, and the equilibrium price distribution for multi-establishment firms puts more probability mass on prices also posted by single-establishment firms. As the number of multi-establishment firms increases with N_m, competition among multi-establishment firms increases as well, so τ_m decreases. Finally, as m increases, the likelihood that even informed customers become captive increases, which reduces competition from single-establishment firms and thus also decreases τ_m.

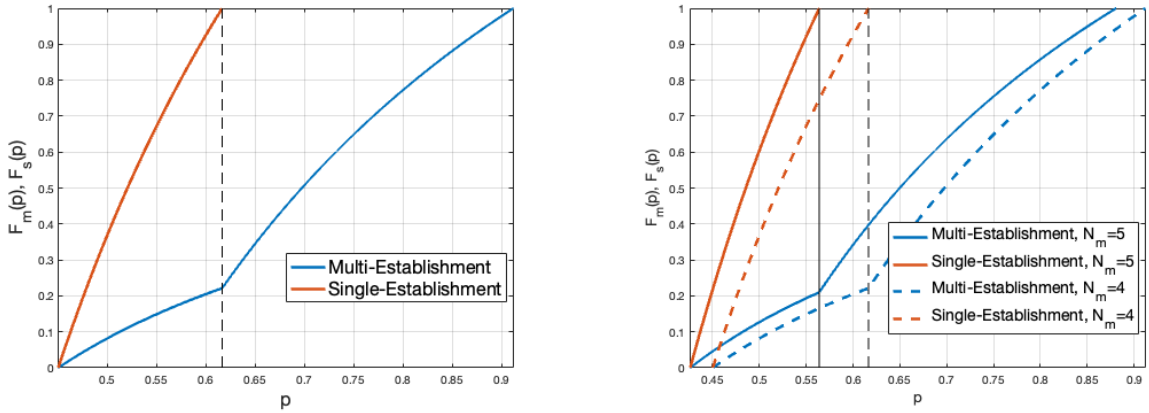
Example 1. *Consider an example that is loosely empirically disciplined by the US banking sector. Based on data from the FDIC's Summary of Deposits about typical medium to large banking markets, we assume 60 establishments in the market in total, with about 20 single-establishment institutions and about 4 branch-network institutions operating about 10 local branches each. As a result, we set $N_s = 20$, $N_m = 4$, $m = 10$.*

Moreover, according to results reported by the Consumer Financial Protection Bureau (CFPB), more 30 percent of borrowers reported not comparison shopping for their mortgage, and more than 75 percent of borrowers reported applying for a mortgage with only one lender. Similarly, in deposit markets, Kiser (2002) finds that one-third of households had never changed depository institutions. As a result, we set $\alpha = 0.4$ as a reasonable value. Finally, we normalize $z = 1$ and set $c = 0.3$, i.e., the search cost is approximately 30 percent of the normalized valuation.

Figure 1a presents the equilibrium price distributions for single- and multi-establishment firms. There is an inflection point in the multi-establishment distribution at $p = \overline{p}_s$, the highest price charged

by single-establishment firms, with much of the distribution of posted prices for multi-establishment firms concentrated at higher price levels. The reservation price in this market is $R = 0.912$.

Furthermore, because a multi-establishment firm internalizes competition among its own establishments for same-firm two-quote consumers, its equilibrium profit per establishment *is lower:* $\pi_s^* > \pi_m^*$.



(a) Single- vs. Multi-Establishment **(b)** Entry of Multi-Establishment Firm

Figure 1: Price Posting Distributions

Now consider the entry of a new multi-establishment firm. Results are presented in Figure 1b. First, the support of posted prices shifts left, from $[0.45, 0.912]$ *to* $[0.426, 0.881]$. *Second, the inflection point shifts from* 0.617 *to* 0.564. *Finally, profit per establishment drops for both single- and multi-establishment firms.*

2.6 A Single Multi-Establishment Firm – $N_m = 1$

The benchmark characterization above maintains $N_m > 1$. The boundary case $N_m = 1$ is not obtained by mechanically substituting $N_m = 1$ into Theorem 1. With a single multi-establishment firm, its own pricing distribution drops out of its demand function:

$$D_m(p) = 1 - \frac{1-\gamma}{T-1}\left[\frac{m-1}{2} + N_s \widetilde{F}_s(p)\right]. \tag{25}$$

where $T = N_s + m$, since $N_m = 1$. Consequently, there is no other multi-establishment firm whose mixing behavior disciplines the sole multi-establishment firm's pricing distribution.

Lemma 5 (Failure of atomlessness when $N_m = 1$)**.** *Let* $N_m = 1$, $N_s > 1$, *and* $m > 1$. *Lemma 1 does not extend to this case.*

The lemma establishes only that atomlessness is no longer guaranteed when $N_m = 1$. Proposition 3 below shows that this possibility is realized in equilibrium: the sole multi-establishment firm places a strictly positive atom at R.

Proposition 3 (Unique pricing equilibrium with one multi-establishment firm). *Fix $R \in (0, z]$, $0 < \gamma < 1$, $N_s \geq 2$, $m \geq 2$, and $N_m = 1$. Hence, $T = N_s + m$ and $N = N_s + 1$. Then the following strategies constitute the unique Nash equilibrium of the pricing subgame conditional on R. Each single-establishment firm uses*

$$F_s(p) = \frac{T - 1 + \alpha(N-1)}{2\alpha(N-1)} \left(1 - \frac{\underline{p}}{p}\right), \qquad p \in [\underline{p}, R) \tag{26}$$

where $\underline{p} = \frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)} R$. The sole multi-establishment firm uses

$$F_m(p) = \begin{cases} \frac{T-1+\alpha m(N-1)}{2\alpha m(N-1)} \left(1 - \frac{\underline{p}}{p}\right), & p \in [\underline{p}, R), \\ 1, & p \geq R. \end{cases} \tag{27}$$

In particular, $F_s(R^-) = 1$, so the single-establishment firm's price distribution has no atom at R, whereas

$$F_m(R^-) = \frac{T - 1 + \alpha m(N-1)}{m\,(T - 1 + \alpha(N-1))} < 1 \tag{28}$$

and the multi-establishment firm's price distribution has an atom

$$\Delta F_m(R) = \frac{(m-1)(T-1)}{m\,[T - 1 + \alpha(N-1)]}. \tag{29}$$

at R. Equilibrium profit per establishment is

$$\pi_s^* = \underline{p}, \qquad \pi_m^* = \left(1 - \frac{\alpha(m-1)}{(1+\alpha)(T-1)}\right) \underline{p}. \tag{30}$$

In terms of consumer behavior, as we show in Proposition 4, the reservation price and expected transaction price are still given by the expressions obtained in Section 2.4, while setting $N_m = 1$.

Proposition 4. *Conditional on the pricing equilibrium in Proposition 3, the effective reservation price is:*

$$R = \min\left\{z, \frac{c}{\Omega_1(\alpha, N_s, m)}\right\}, \tag{31}$$

where $\Omega_1(\alpha, N_s, m)$ is given by:

$$\Omega_1 = 1 - \frac{T - 1 - \alpha N_s}{T - 1 + \alpha N_s} \left[1 + \alpha - \frac{\alpha m(m-1)}{T(T-1)}\right]. \tag{32}$$

Moreover, the expected transaction price $\mathbb{E}[P_R]$ *is given by:*

$$\mathbb{E}[P_R] = R\frac{T-1-\alpha N_s}{T-1+\alpha N_s}\left[1+\alpha-\frac{\alpha m(m-1)}{T(T-1)}\right]. \tag{33}$$

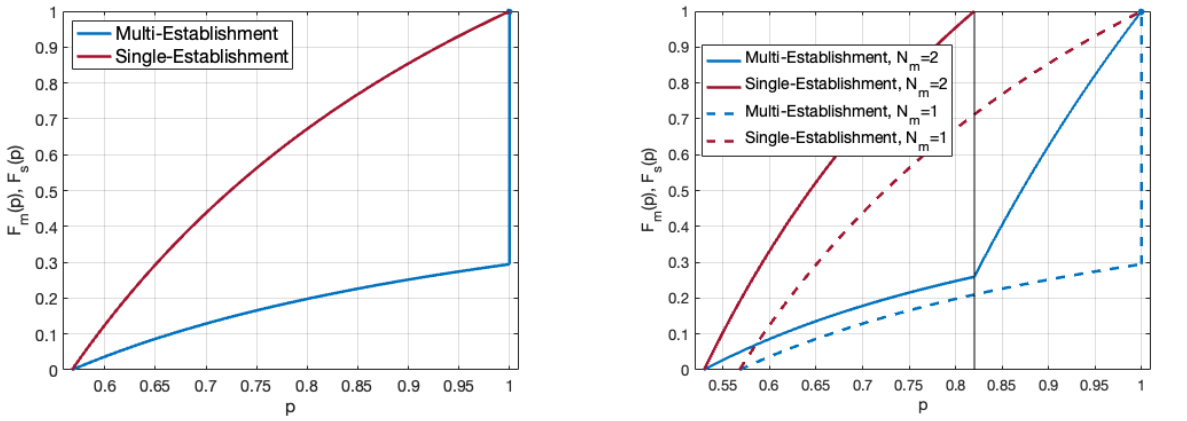
Furthermore, given $N_m = 1$, (32) *and* (33) *are identical to* (20) *and* (21)*, respectively.*

Consequently, given Proposition 4, the comparative statics relative to N_s and m for the case of a single multi-establishment firm are the same as the general case.

Finally, given the results presented in Propositions 3 and 4, following the same argument presented in Corollary 1, we can show that $F_m(\cdot)$ stochastically dominates in first order $F_s(\cdot)$ in the case of $N_m = 1$ as well.

Example 2. *To illustrate the* $N_m = 1$ *case, we consider the same parameters as in Example 1, but set* $N_m = 1$*. Results are presented in Figure 2a. As we can see, there is a mass point in the multi-establishment firm's price distribution. Moreover, for this particular example, the constraint* $R \leq z$ *binds and* $R = 1$.[2]

We also consider the entry of a second multi-establishment firm into the local market. Results for this particular example are presented in Figure 2b. As we can see, there is a significant decline in posted prices for both single- and multi-establishment firms. Moreover, competition among multi-establishment firms eliminates the mass point at R*.*



(a) Sole Multi-Establishment Firm **(b)** Entry of 2nd Multi-Establishment Firm

Figure 2: Price Posting Distributions – $N_m = 1$

Comparison with Menzio and Trachter (2015). The case with a single multi-establishment firm is related to Menzio and Trachter (2015), but the two models have key distinctions in their

[2]This is not a necessary result. For lower values of the search cost c, the equilibrium is interior and $R < z$.

environments. These distinctions translate into different equilibrium characteristics, even though the equilibrium price distributions share some similar features. First, search is sequential in Menzio and Trachter (2015), while we have noisy search as in Burdett and Judd (1983). More importantly, in their model, consumers observe the large seller price before they start their search. As a result, consumers' reservation price $R(x)$ and the value of searching $V(x)$ are functions of the realized price posted by the large seller. In contrast, we assume that consumers know the ex ante distribution of prices posted by firms but do not observe the realized posted prices of any firm.

These differences in consumers' information set and search environment are quite important. First, equilibrium in Menzio and Trachter (2015) exhibits ex post allocative inefficiency with positive probability. When the large seller posts a price below the upper endpoint of its support, some small sellers may post prices that are ex post above consumers' reservation price (i.e., $y > R(x)$), so mutually beneficial meetings need not result in trade. By contrast, in our model all equilibrium prices lie weakly below the common reservation price R. Conditional on entering the market, consumers therefore purchase after receiving their quotes; price comparison determines which firm makes the sale rather than whether trade occurs. Second, the economic rationale for the firm's strategies is quite distinct. In Menzio and Trachter (2015), the large seller knows that its price influences consumers' optimal search behavior. A lower large-seller price x reduces $R(x)$, inducing consumers to reject a larger set of small-seller offers, and keeps consumers searching for longer. Hence, x affects the total number of meetings and ultimately the large seller's traffic. In contrast, in our model the number of quotes is exogenous and competition is mostly through head-to-head price comparison. Firm size changes the composition of demand by increasing the probability that a consumer's quotes all come from establishments owned by the same firm. This increases the captive demand of the multi-establishment firm, but also generates a cost of size due to within-firm establishment competition, since multiple own-establishment contacts do not generate multiple sales. This cannibalization channel is absent from Menzio and Trachter (2015). As a result, in our model the large firm balances exploiting the additional captive consumers against reducing the likelihood of losing a sale when facing competition from a single-establishment firm.

Nevertheless, the equilibrium distributions in both models share some identical features. In both models, the large or multi-establishment seller follows a mixed strategy with a continuous component and a strictly positive atom at the upper endpoint of its support, whereas the small or single-establishment sellers' distribution is atomless. Furthermore, in both models, the two distributions have the same upper end of the distribution support.

That said, the distributions in the two models differ in meaningful ways. First, while in our model the lower bound in the support of the two distributions is the same $\underline{p}$, in Menzio and Trachter (2015) the large seller posts prices below every price posted by the small sellers (i.e.,

$\underline{x} < \underline{y}$). Second, in our model both multi-establishment and single-establishment densities are proportional to p^{-2}, being decreasing in p. Moreover, since the multi-establishment firm places less probability on every interior lower-price interval and assigns the remaining probability to an atom at R, it first-order stochastically dominates the single-establishment firm's distribution. In contrast, in Menzio and Trachter (2015) the large seller's continuous density is decreasing and it places an atom at the common upper endpoint $\overline{x}$, whereas the small sellers have an increasing density. As a result, their large- and small-seller cumulative distributions cross: the large seller is the only seller posting the lowest prices, but it also places positive probability on the highest price. This pattern supports their interpretation of the large seller as alternating between temporary sale prices and a normal high price.

3 Market Concentration and Price-Cost Markups

3.1 Herfindahl-Hirschman Index (HHI)

While there are many criticisms of the HHI (see Eeckhout (2021) and Bresnahan (1989)), several studies that discuss market power still rely on this metric. In order to compare our results to this literature, it is worthwhile to construct this measure based on the model primitives.

We start by considering the revenue-based HHI at the firm level as the benchmark measure. This is the benchmark measure calculated by the Antitrust Division of the Department of Justice (DoJ)[3] and the most commonly used in the literature due to data availability.

In order to calculate the revenue-based HHI, notice that, given the profit equalization we observe in equilibrium, the revenue obtained by single-establishment firms is given by $r_s = \underline{p}$. Similarly, the revenue obtained by multi-establishment firms is $r_m = m\left[1 - \frac{\alpha(m-1)}{(1+\alpha)(T-1)}\right]\underline{p}$. Based on these equations, the revenue share of a single-establishment firm is:

$$s_s^{Rev} = \frac{(1+\alpha)(T-1)}{N_s(1+\alpha)(T-1) + N_m m\left[T-1+\alpha(T-m)\right]} \tag{34}$$

while the revenue share of a multi-establishment firm is:

$$s_m^{Rev} = \frac{m\left[T-1+\alpha(T-m)\right]}{N_s(1+\alpha)(T-1) + N_m m\left[T-1+\alpha(T-m)\right]} \tag{35}$$

[3]Find details of the DoJ's discussion of market shares and concentration at https://www.justice.gov/atr/merger-guidelines/tools/market-shares.

As a result, given that $HHI^{Rev} \equiv N_s(s_s^{Rev})^2 + N_m(s_m^{Rev})^2$, we have:

$$HHI^{Rev} = \frac{N_s(1+\alpha)^2(T-1)^2 + N_m m^2 \left[T-1+\alpha(T-m)\right]^2}{\left\{N_s(1+\alpha)(T-1) + N_m m \left[T-1+\alpha(T-m)\right]\right\}^2}. \tag{36}$$

Notice that we can easily see the impact of the ownership structure in the HHI formula. If we do not take into account the ownership structure and treat each establishment independently, we would obtain the following HHI_{Est}^{Rev}

$$HHI_{Est}^{Rev} = \frac{N_s(1+\alpha)^2(T-1)^2 + N_m m \left[T-1+\alpha(T-m)\right]^2}{\left\{N_s(1+\alpha)(T-1) + N_m m \left[T-1+\alpha(T-m)\right]\right\}^2}. \tag{37}$$

Since $m \geq 2$, we can easily see that $HHI^{Rev} > HHI_{Est}^{Rev}$. Hence, disregarding the ownership structure would yield an underestimate of the degree of market concentration.

We now consider how HHI^{Rev} varies as we change the market structure, pinned down by N_s, N_m, and m. Proposition 5 summarizes the key results.

Proposition 5 (Comparative statics of the revenue-share HHI)**.** *Suppose $N_s \geq 2$, $N_m \geq 1$, $m \geq 2$, and $\alpha \in (0,1)$. Treating N_s, N_m, and m as continuous in the comparative-static derivatives,*

$$\frac{\partial HHI^{Rev}}{\partial N_s} < 0, \qquad \frac{\partial HHI^{Rev}}{\partial m} > 0,$$

whereas

$$\frac{\partial HHI^{Rev}}{\partial N_m} \quad \textit{is ambiguous.}$$

Results in Proposition 5 show that HHI^{Rev} decreases as more single-establishment firms enter the market and increases with concentration, while the impact of more multi-establishment firms in the market is ambiguous. It is also related to the discussion in Rossi-Hansberg, Sarte, and Trachter (2021) and Smith and Ocampo (2025) of the potentially diverging pattern of concentration in local vs. national markets. If a national chain enters a local market by opening a single establishment and behaves as a single establishment, as Proposition 5 shows, HHI^{Rev} would decline. Moreover, even if we still assume common pricing, the single establishment of the large firm would have a lower revenue than the incumbent single-establishment firms, reducing local HHI^{Rev}. Nevertheless, at the national level we are increasing m, and Proposition 5 shows that national HHI^{Rev} should go up. These results are in line with the rationale presented by Rossi-Hansberg, Sarte, and Trachter (2021) of diverging patterns in concentration for local vs. national patterns. However, this result relies on two key assumptions. First, it depends on a "timid" expansion of national chains to local markets, with the opening of a single establishment. Larger expansions, in

which a chain opens multiple establishments in the local market, or the entry of multiple multi-establishment firms into the same local market, would likely break the diverging pattern, moving closer to the result of Smith and Ocampo (2025). Second, the result depends on a proper definition of a local market, which is a difficult task as shown by Eeckhout (2021).

3.2 Markups

In this section, we focus on markups, which is the preferred measure of market power in most of the literature. Because firms use mixed pricing strategies, the realized markup is a random variable. As a result, we focus on the *ex ante markup distribution* and the *ex ante expected markup*.

Moreover, as mentioned in Section 2, while in most papers we use the unit margin p, for this section we focus on the posted price $\widetilde{p} = p + \kappa$, where $\kappa > 0$ is the marginal cost of producing the good. Then, following Menzio (2026), we define gross markup as:

$$\mu(p) \equiv \frac{\widetilde{p}}{\kappa} = 1 + \frac{p}{\kappa}. \tag{38}$$

Notice that, since $\mu(p)$ is a strictly increasing affine transformation on p, we can readily map the model to an equivalent model in which firms post markups directly and all results carry over. In particular, let H_i denote the cumulative distribution of gross markups posted by i-establishment firms, with $i \in \{s, m\}$, $H_i(\mu) = F_i(\kappa(\mu - 1))$, $i \in \{s, m\}$. As expected, the corresponding supports are $\left[1 + \frac{p}{\kappa}, 1 + \frac{\overline{p}_s}{\kappa}\right]$ for H_s and $\left[1 + \frac{p}{\kappa}, 1 + \frac{R}{\kappa}\right]$ for H_m, respectively.

Given one-to-one correspondence from posted prices to posted markups, the following proposition is a straightforward application of previously presented results.

Proposition 6 (Ownership and markups)**.** *Under the equilibrium characterized above:*

(i) The multi-establishment markup distribution first-order stochastically dominates the single-establishment markup distribution:

$$H_m(\mu) \leq H_s(\mu)$$

for every markup in the union of their supports, with strict inequality at every interior markup at which both distributions are nondegenerate.

(ii) The expected gross markup paid by consumers is

$$\mathbb{E}[\mu^{tr}] = 1 + \frac{\mathbb{E}[P_R]}{\kappa}. \tag{39}$$

Using equation (21),

$$\mathbb{E}[\mu^{tr}] = 1 + \frac{R}{\kappa}\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\left[1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)}\right]. \tag{40}$$

If the reservation price is interior, so that $R = c/\Omega$*, then*

$$\mathbb{E}[\mu^{tr}] = 1 + \frac{c}{\kappa}\frac{1-\Omega}{\Omega}. \tag{41}$$

Consequently,

$$\frac{\partial \mathbb{E}[\mu^{tr}]}{\partial N_s} < 0, \qquad \frac{\partial \mathbb{E}[\mu^{tr}]}{\partial N_m} < 0, \qquad \frac{\partial \mathbb{E}[\mu^{tr}]}{\partial m} > 0.$$

This distinction is also useful for interpreting the model in light of Menzio (2026). Menzio (2026) emphasizes that search frictions can generate heterogeneous markups and flexible relationships between seller size and markups even when firms sell a homogeneous product. Our model identifies a specific ownership mechanism behind markup heterogeneity. Consumer contacts occur at the establishment level, while competition occurs at the owner level. A larger establishment network is therefore more likely to generate redundant same-owner contacts, which shifts the firm's equilibrium markup distribution. At the same time, conditional on network size, the firm's realized sales decline endogenously in response to higher markups. The relationship between firm size and markups therefore depends on whether size refers to an establishment network or to sales generated by a particular price realization.

4 Noisy Search with Heterogeneous Firm Sizes

The paper so far has considered a market with two classes of firms: single-establishment firms and multi-establishment firms of a common size m. This two-class setting keeps the focus on the key feature of multi-establishment competition and yields the compact formulas of Theorem 1 and Theorem 2. Nonetheless, the model generalizes to the case with multiple classes of multi-establishment firms of different sizes, and we can characterize pricing and consumer search for this general case. The equilibrium retains the structure of the two-class case, with each size class acting as a separate strategic group.

We introduce some additional notation. Let

$$1 = q_1 < q_2 < \cdots < q_K$$

denote the distinct network sizes present in the market, and let n_k be the number of firms of size q_k. The class $q_1 = 1$ is the single-establishment fringe, with $n_1 = N_s \geq 2$, and the total

number of establishments is

$$T \equiv \sum_{k=1}^{K} n_k q_k. \tag{42}$$

Let F_k denote the common pricing distribution used by firms in size class k.

Demand per customer visit for each establishment of a class-k firm follows (8), adjusted for the additional competing network sizes:

$$D_k(p) = 1 - \frac{1-\gamma}{T-1}\left[\frac{q_k - 1}{2} + (n_k - 1)q_k \widetilde{F}_k(p) + \sum_{\ell \neq k} n_\ell q_\ell \widetilde{F}_\ell(p)\right]. \tag{43}$$

Much of the earlier structure carries over with limited adjustment. First, Proposition 1 on within-type symmetry extends directly.

Proposition 7 (Symmetry within establishment-size class)**.** *In every Nash equilibrium of the pricing game, firms with the same number of establishments use the same pricing distribution.*

Arguments mirroring Lemmas 1, 2, and 5 lead to the following properties of the equilibrium price distributions, which extend their counterparts in the two-class setting.

Lemma 6 (Firm price distribution atoms and supports)**.** *In the heterogeneous-size pricing game:*

(i) No firm price distribution has an atom at a price strictly below R.

(ii) There is at most one firm price distribution that has an atom at R.

(iii) The price distribution for any size class containing at least two firms is atomless.

(iv) The union of supports of all pricing distributions is connected and has upper endpoint R.

Proposition 8 (Nested supports of equilibrium price distributions)**.** *Let $\underline{p}_i$ and $\overline{p}_i$ denote the lower and upper endpoints of firm i's support, and let q_i be its number of establishments. In every equilibrium of the heterogeneous-size pricing game:*

(i) All firms' price distributions share a common lower endpoint

$$\underline{p}_i = \underline{p} \qquad \forall i.$$

(ii) The upper endpoints of a pair of firm price distributions are weakly increasing in the number of establishment holdings of each firm:

$$q_i < q_j \quad \Longrightarrow \quad \overline{p}_i \leq \overline{p}_j.$$

(iii) The firm price distribution of every establishment-size class has a connected support.

As a result of (i) to (iii), the equilibrium supports are nested as follows:

$$[\underline{p}, \overline{p}_1] \subseteq [\underline{p}, \overline{p}_2] \subseteq \cdots \subseteq [\underline{p}, \overline{p}_K].$$

and $\overline{p}_K = R$.

Proposition 8 establishes that every equilibrium has a common lower support endpoint and connected supports that are nested by establishment holdings, matching the structure of Theorem 1 and Proposition 3, which showed that in the two-class setting $[\underline{p}, \overline{p}_s] \subseteq [\underline{p}, \overline{p}_m] = [\underline{p}, R]$.

Theorem 3 characterizes the unique equilibrium with heterogeneous network sizes. We introduce some additional notation to simplify the results and emphasize the recursive structure of the result: For $r = 1, \ldots, K$, let Q_r be the number of firms of size at least q_r and C_r the total establishments held by firms of size at least q_r:

$$Q_r \equiv \sum_{k=r}^{K} n_k, \qquad C_r \equiv \sum_{k=r}^{K} n_k q_k. \tag{44}$$

Whenever $Q_r \geq 2$, let

$$G_r \equiv \frac{1 - \dfrac{1-\gamma}{2(T-1)}(C_r - 1)}{Q_r - 1}, \qquad c_{kr} \equiv \frac{1}{2} + \frac{(T-1)G_r}{(1-\gamma)q_k}, \qquad k \geq r. \tag{45}$$

Let

$$r^\star \equiv \begin{cases} K, & n_K \geq 2, \\ K-1, & n_K = 1, \end{cases} \tag{46}$$

and for $r \leq r^\star$ define

$$x_r \equiv \frac{1 + \dfrac{T - C_r}{q_r(Q_r - 1)}}{c_{rr}}. \tag{47}$$

Theorem 3 (Unique equilibrium with heterogeneous network sizes)**.** *Suppose* $K \geq 2$, $n_1 \geq 2$, *and* $0 < \gamma < 1$. *Then*

$$0 < x_1 < \cdots < x_{r^\star} < 1.$$

Define

$$\underline{p} \equiv (1 - x_{r^\star})R, \qquad p_r \equiv \frac{\underline{p}}{1 - x_r}, \quad r = 1, \ldots, r^\star, \qquad p_0 \equiv \underline{p}. \tag{48}$$

Thus $p_{r^\star} = R$.

There exists a class-symmetric pricing equilibrium in which every size class begins mixing at $\underline{p}$ and smaller size classes leave the support first. On a segment $p \in [p_{r-1}, p_r]$, the active classes are $r, r+1, \ldots, K$, and every active class $k \geq r$ uses

$$F_k(p) = c_{kr}\left(1 - \frac{\underline{p}}{p}\right) - \frac{T - C_r}{q_k(Q_r - 1)}. \tag{49}$$

For $k \leq r^\star$, class k has support $[\underline{p}, p_k]$ and $F_k(p) = 1$ above p_k.

If $n_K \geq 2$, the largest size class reaches one continuously at R and all distributions are atomless. If $n_K = 1$, the unique largest network satisfies

$$F_K(R^-) < 1 \tag{50}$$

and places the residual atom $1 - F_K(R^-)$ at R.

Equilibrium profit per establishment in size class k is

$$\pi_k^* = \left[1 - \frac{(1-\gamma)(q_k - 1)}{2(T-1)}\right]\underline{p}. \tag{51}$$

This is the unique Nash equilibrium of the heterogeneous-size pricing game.

Setting $K = 2$, $q_1 = 1$, $n_1 = N_s$, $q_2 = m$, and $n_2 = N_m$, the recursive construction reduces to Theorem 1 when $N_m \geq 2$ and to Proposition 3 when $N_m = 1$.

4.1 Consumer Behavior with Heterogeneous Firm Sizes

The consumer problem is unchanged from Section 2.4: consumers use a reservation-price strategy, the effective cutoff is $R = \min\{Q, z\}$, and in an interior equilibrium the best-response condition (18) still reads $c = R - \mathbb{E}[P_R] = R\,\Omega$. What changes is the mapping from market structure to the constant Ω, since the expected transaction price must now aggregate over the K size classes. The next proposition gives the heterogeneous-size analogue of Theorem 2.

Proposition 9 (Reservation price with heterogeneous firm sizes)**.** *Conditional on the pricing equilibrium in Theorem 3, the effective reservation price is*

$$R = \min\left\{z, \frac{c}{\Omega\left(\alpha, \{n_k, q_k\}_{k=1}^K\right)}\right\},$$

where, writing q_K for the largest network size and

$$S \equiv \sum_{k=1}^{K} n_k q_k (q_k - 1) = \sum_{k=1}^{K} n_k q_k^2 - T, \tag{52}$$

the constant Ω is given by

$$\Omega = 1 - \frac{T - 1 - \alpha(T - q_K)}{T - 1 + \alpha(T - q_K)} \left[1 + \alpha - \frac{\alpha S}{T(T-1)} \right]. \tag{53}$$

Equivalently, the lower endpoint of the common support and the expected transaction price are

$$\underline{p} = \frac{T - 1 - \alpha(T - q_K)}{T - 1 + \alpha(T - q_K)} R, \qquad \mathbb{E}[P_R] = \left[1 + \alpha - \frac{\alpha S}{T(T-1)} \right] \underline{p}, \tag{54}$$

so that $\Omega = 1 - \mathbb{E}[P_R]/R$. If R is interior, $\mathbb{E}[P_R] = \left(\frac{1-\Omega}{\Omega}\right) c$.

The quantity

$$\pi_{\text{same}} \equiv \frac{\alpha S}{T(T-1)} = \alpha \sum_f \frac{q_f}{T} \frac{q_f - 1}{T - 1}$$

is the probability that a two-quote consumer draws both quotes from the *same* firm, summed over firms f. Such quotes carry a single price and generate no competition, so only *effective* second quotes, with probability $\alpha - \pi_{\text{same}}$, raise the expected transaction price above $\underline{p}$. Consider the expected number of distinct price quotes received by a consumer,

$$(1 - \alpha) + \alpha \sum_{k=1}^{K} \frac{q_k}{T} \frac{q_k - 1}{T - 1} n_k + \alpha \left[1 - \sum_{k=1}^{K} \frac{q_k}{T} \frac{q_k - 1}{T - 1} n_k \right] 2$$

where $\sum_{k=1}^{K} \frac{q_k}{T} \frac{q_k - 1}{T-1} n_k$ is the probability that two quotes are sampled from the same firm. Rearranging it, we have

$$1 + \alpha - \alpha \sum_{k=1}^{K} \frac{q_k}{T} \frac{q_k - 1}{T - 1} n_k$$

since $S = \sum_{k=1}^{K} n_k q_k (q_k - 1)$, we define the expected number of distinct price quotes received by a consumer as

$$B(S) \equiv 1 + \alpha - \frac{\alpha S}{T(T-1)} \tag{55}$$

as a result, we can rewrite $\mathbb{E}[P_R] = B(S)\underline{p}$, i.e., the expected transaction price is given by the product of the price floor $\underline{p}$ and the expected number of distinct price quotes.

Two features stand out. First, the price floor depends on the size distribution only through

the largest network size q_K. Second, the competition wedge enters only through the concentration index $S = \sum_k n_k q_k^2 - T$: markets with the same T, q_K, and S share the same reservation price, expected transaction price, and support, regardless of how the remaining establishments are distributed across owners.

Finally, to facilitate our discussion of concentration measures and merger analysis, define $\phi(q)$ as

$$\phi(q) \equiv \frac{T - 1 - \alpha(T - q)}{T - 1 + \alpha(T - q)} \tag{56}$$

As a result, $\phi(q_K) = \frac{p}{R}$ is the ratio of the lower to the upper endpoint of the largest firm's support, and thus a measure of price dispersion as a function of the largest firm's size. Nevertheless, we keep $\phi(q)$ as a function of q since in the merger analysis the size and identity of the largest firm may change due to the merger.

Given $B(S)$ and $\phi(q)$, we can rewrite Ω as:

$$\Omega = 1 - \phi(q_K)B(S) \tag{57}$$

Remark 1. *Setting $K = 2$, $q_1 = 1$, $n_1 = N_s$, $q_2 = m$, and $n_2 = N_m$, expression (53) reduces to Ω in (20) when $N_m \geq 2$ and to Ω_1 in (32) when $N_m = 1$; in both cases $q_K = m$ and $S = N_m m(m-1)$.*

4.2 Revenue HHI and Markups with Heterogeneous Firm Sizes

We now extend the results presented in Section 3 for the general case of heterogeneous firm sizes.

Proposition 10 (Revenue concentration and markups with heterogeneous firm sizes)**.** *In the heterogeneous-size equilibrium:*

(i) The firm-level revenue-share HHI is

$$HHI^{Rev} = \frac{\sum_{k=1}^{K} n_k q_k^2 \left[T - 1 + \alpha(T - q_k)\right]^2}{\left\{\sum_{k=1}^{K} n_k q_k \left[T - 1 + \alpha(T - q_k)\right]\right\}^2}. \tag{58}$$

(ii) For marginal cost $\kappa > 0$, the expected gross markup paid by consumers is

$$\mathbb{E}[\mu^{tr}] = 1 + \frac{B(S)\phi(q_K)}{\kappa} \min\left\{z, \frac{c}{1 - B(S)\phi(q_K)}\right\}. \tag{59}$$

Hence expected markups are strictly increasing in the single index $B(S)\phi(q_K)$.

Proposition 10 highlights a distinction that will be important for the merger analysis below. Revenue concentration depends on the entire ownership distribution $\{n_k, q_k\}_{k=1}^K$. By contrast, for given T, α, c, z, and κ, the expected transaction markup depends on ownership only through the same two sufficient statistics that govern consumer search in Proposition 9: the concentration index S defined in equation (52) and the size q_K of the largest firm. Consequently, changes in HHI^{Rev} and changes in expected markups need not have the same sign.

4.3 Merger Analysis

We now use the heterogeneous-firm characterization to study the effects of consolidation. The key distinction is whether a merger merely creates a larger competitor or instead creates a new largest firm. Every merger increases the degree to which establishments are concentrated under common ownership, but only a merger that changes q_K activates the largest-firm channel through $\phi(q)$. With an interior reservation price, this distinction matters not only for the common price floor but also for the reservation price itself and, therefore, for the expected transaction price and markup.

The merger experiment. Two firms with establishment counts $q_a \leq q_b$ merge into a single firm of size $s \equiv q_a + q_b$, holding the total number of establishments T and the ownership of every other firm fixed. Superscripts "pre" and "post" denote objects before and after the merger. Let q_K^{nm} denote the largest firm size among firms other than the two merging firms.[4]

The merger changes the two sufficient statistics governing equilibrium prices in a particularly simple way. First,

$$q_K^{\text{post}} = \max\{s, q_K^{nm}\} \geq \max\{q_b, q_K^{nm}\} = q_K^{\text{pre}}, \tag{60}$$

so the merger creates a new largest firm if and only if $s > q_K^{\text{pre}}$.[5] Second, the merger raises ownership concentration S by

$$\Delta S \equiv S^{\text{post}} - S^{\text{pre}} = s(s-1) - q_a(q_a-1) - q_b(q_b-1) = 2q_a q_b > 0. \tag{61}$$

As a result, the expected number of distinct price quotes received by the consumer, $B(S)$ declines by

$$B(S^{\text{post}}) - B(S^{\text{pre}}) = -\frac{\alpha \Delta S}{T(T-1)}, \tag{62}$$

[4] Hence, nm refers to "non-merger."

[5] Notice that, since q_K^{nm} is the largest firm size among the non-merger firms and $q_b \geq q_a$, in order to obtain the largest pre-merger firm size, we just need to compare q_K^{nm} and q_b.

thus a merger, by increasing the concentration index, decreases the expected number of distinct price quotes received by the consumer.

For later use, define the per-establishment profit coefficient

$$b(q) \equiv 1 - \frac{(1-\gamma)(q-1)}{2(T-1)}. \tag{63}$$

Firm profits. Profit per establishment of the size-s firm is pinned down by

$$\pi^*_{a+b} = \left[1 - (1-\gamma)\frac{q_a + q_b - 1}{2(T-1)}\right] \underline{p}^{post} \tag{64}$$

Then, define:

$$\overline{b}_{ab} \equiv \frac{q_a}{q_a + q_b}\left[1 - (1-\gamma)\frac{q_a - 1}{2(T-1)}\right] + \frac{q_b}{q_a + q_b}\left[1 - (1-\gamma)\frac{q_b - 1}{2(T-1)}\right]$$

i.e., $\overline{b}_{ab}$ is the weighted average of the pre-merger profit coefficient of the two merging firms. As a result, we can show that:

$$\left[1 - (1-\gamma)\frac{q_a + q_b - 1}{2(T-1)}\right] = \overline{b}_{ab} - \frac{(1-\gamma)q_a q_b}{s(T-1)}.$$

where the last term on the right hand side shows the decline in the coefficient due to the additional within-firm cannibalization. Consequently, the merged firm's change in profit per establishment is

$$\Delta\pi_{ab} = \underbrace{\overline{b}_{ab}\Delta\underline{p}}_{\text{common-floor effect}} - \underbrace{\frac{(1-\gamma)q_a q_b}{s(T-1)}\underline{p}^{\text{post}}}_{\text{within-firm cannibalization}}, \tag{65}$$

while any non-merging firm ℓ satisfies

$$\Delta\pi_\ell = b(q_\ell)\Delta\underline{p}. \tag{66}$$

Theorem 4 (Merger effects with an interior reservation price)**.** *Suppose* $R^{\text{pre}} = \frac{c}{\Omega^{\text{pre}}} < z$ *and* $R^{\text{post}} = \frac{c}{\Omega^{\text{post}}} < z$*. Then:*

(i) Large but not largest competitor: $s \leq q_K^{\text{pre}}$*. The largest firm size is unchanged while S strictly rises. Therefore, reservation price, common price floor, expected transaction price, and expected markup fall.*

The merged firm's profit per establishment strictly falls relative to the establishment-weighted pre-merger benchmark, $\Delta\pi_{ab} < 0$*, and every non-merging firm strictly loses, i.e.,* $\Delta\pi_\ell < 0$*.*

(ii) New largest firm: $s > q_K^{\text{pre}}$. *Both q_K and S rise. The equilibrium falls into three regions:*

(a) If

$$\frac{\alpha \Delta S}{T(T-1)} < B(S^{\text{pre}}) \left[1 - \frac{\phi(q_K^{\text{pre}})}{\phi(s)} \right],$$

then the reservation price, expected transaction price, expected transaction markup, and common price floor all rise.

(b) If

$$B(S^{\text{pre}}) \left[1 - \frac{\phi(q_K^{\text{pre}})}{\phi(s)} \right] < \frac{\alpha \Delta S}{T(T-1)} < \frac{1}{\phi(q_K^{\text{pre}})} - \frac{1}{\phi(s)},$$

then the reservation price, expected transaction price, and expected transaction markup fall, even though the common price floor rises.

(c) If

$$\frac{\alpha \Delta S}{T(T-1)} > \frac{1}{\phi(q_K^{\text{pre}})} - \frac{1}{\phi(s)},$$

then the reservation price, expected transaction price, expected transaction markup, and common price floor all fall.

At the first threshold, the changes in R, $\mathbb{E}[P_R]$, and the expected transaction markup are zero, while the common price floor is still higher. At the second threshold, the common price floor is unchanged, while R, $\mathbb{E}[P_R]$, and the expected transaction markup are lower.

Every non-merging firm gains if and only if the common floor rises.

The merged firm gains if and only if

$$\frac{\underline{p}^{\text{post}}}{\underline{p}^{\text{pre}}} > 1 + \frac{(1-\gamma) q_a q_b}{(T-1)s \left[1 - (1-\gamma) \frac{s-1}{2(T-1)} \right]}.$$

Thus a profitable merger necessarily raises the common floor and benefits every non-merging firm, but the converse need not hold.

Theorem 4 isolates two opposing forces when the merger creates a new largest firm. The increase in S lowers $B(S)$ and, holding q_K fixed, raises Ω and lowers R. By contrast, the increase in q_K raises $\phi(q_K)$ and, holding $B(S)$ fixed, lowers Ω and raises R. The same increase in $\phi(q_K)$ also lifts the common floor relative to R. Importantly, the threshold required for the floor to rise is weaker than the threshold required for R to rise. A merger can therefore compress the support from below—raising $\underline{p}$—while simultaneously lowering the reservation price and the expected transaction markup.

Corollary 4 (Merger effects with a capped reservation price)**.** *Suppose* $R^{\text{pre}} = R^{\text{post}} = z$*. Then:*

(i) If $s \leq q_K^{\text{pre}}$*, the common floor is unchanged,* $\underline{p}^{\text{post}} = \underline{p}^{\text{pre}} \equiv \underline{p}$*. Every non-merging firm is unaffected and*

$$\Delta\pi_{ab} = -\frac{(1-\gamma)q_a q_b}{s(T-1)}\underline{p} < 0. \tag{67}$$

Expected transaction prices and expected markups nevertheless fall because $B(S)$ *falls while* $\phi(q_K)$ *is unchanged.*

(ii) If $s > q_K^{\text{pre}}$*, the common floor strictly rises,* $\Delta\underline{p} = \left[\phi(s) - \phi(q_K^{\text{pre}})\right] z > 0$*, so every non-merging firm strictly gains.*

The merged firm's profit change is positive if and only if

$$\frac{\phi(s)}{\phi(q_K^{\text{pre}})} > 1 + \frac{(1-\gamma)q_a q_b}{(T-1)s\left[1 - \frac{(1-\gamma)(s-1)}{2(T-1)}\right]}. \tag{68}$$

The effect on the expected transaction markup remains ambiguous because the increase in $\phi(q_K)$ *competes with the decline in* $B(S)$*.*

The profit decomposition in (65) does not depend on whether the reservation price is capped or interior. If a merger moves the economy from one reservation-price regime to the other, the same profit identities continue to apply after substituting the appropriate equilibrium value of $\underline{p}$.

Revenue concentration and markups. The merger effects on concentration and market power can now be stated directly. For the revenue calculations, let

$$\lambda \equiv \frac{\alpha}{(1+\alpha)(T-1)}.$$

Corollary 5 (Revenue HHI and expected markups under mergers)**.** *Let*

$$\mathcal{R}^{\text{pre}} \equiv \sum_f q_f \left[1 - \lambda(q_f - 1)\right] = T - \lambda S^{\text{pre}}, \qquad H^{\text{pre}} \equiv HHI^{Rev,\text{pre}}.$$

Then:

(i) The exact change in the firm-level revenue-share HHI is

$$\Delta HHI^{Rev} = \frac{2q_a q_b}{\left(\mathcal{R}^{\text{pre}} - 2\lambda q_a q_b\right)^2}\Big[1 + (2-3s)\lambda + \left(2s^2 - q_a q_b - 3s + 1\right)\lambda^2 + 2\lambda H^{\text{pre}}\left(\mathcal{R}^{\text{pre}} - \lambda q_a q_b\right)\Big]. \tag{69}$$

Therefore (69) provides a necessary and sufficient condition for a merger to raise HHI^{Rev}. A distribution-free sufficient condition is

$$\lambda \le \frac{2}{3s - 2 + \sqrt{s^2 + 4q_a q_b}}. \tag{70}$$

In particular, either of the following simpler conditions is sufficient:

$$s \le \frac{4T-2}{3+\sqrt{2}} \qquad \text{for any } \alpha \in (0,1], \tag{71}$$

or

$$\alpha \le \frac{2(T-1)}{(1+\sqrt{2})T}, \tag{72}$$

in which case every feasible merger raises HHI^{Rev}.

(ii) For any reservation-price regime, expected transaction markups rise if and only if

$$\left[B(S^{\text{pre}}) - \frac{\alpha \Delta S}{T(T-1)}\right]\phi(q_K^{\text{post}}) > B(S^{\text{pre}})\phi(q_K^{\text{pre}}). \tag{73}$$

Hence, if $s \le q_K^{\text{pre}}$, expected markups strictly fall. If $s > q_K^{\text{pre}}$, the sign is ambiguous. On the interior branch, condition (73) is equivalent to condition (A.110).

Corollary 5 yields a particularly sharp concentration–market-power comparison. Under either sufficient condition in (71) or (72), a merger that does not create a new largest firm necessarily satisfies

$$HHI^{Rev,\text{post}} > HHI^{Rev,\text{pre}}, \qquad \mathbb{E}[\mu^{tr}]^{\text{post}} < \mathbb{E}[\mu^{tr}]^{\text{pre}}.$$

Thus a conventional revenue concentration measure can rise while the expected markup paid by consumers falls. If the merger creates a new largest firm, the same increase in HHI^{Rev} may instead be accompanied by either an increase or a decrease in expected markups, depending on (73).

This result connects directly to the preceding discussion of concentration and market

power. Rossi-Hansberg, Sarte, and Trachter (2021) emphasize that concentration can evolve differently at the national and local levels as large firms expand geographically, while Eeckhout (2021) stresses that concentration measures should not be interpreted mechanically as measures of market power. The model puts these observations in a common framework. The HHI–markup disconnect here does not require a change in geographic market definition, population, or the number of establishments: T and the market boundary are held fixed. Instead, the disconnect arises because consumers sample establishments whereas firms set prices at the ownership level. A merger always concentrates ownership, but its effect on expected markups depends on whether consolidation also changes the size of the largest local network. In this sense, the model provides a structural reason why changes in concentration alone need not identify changes in market power.

Remark 2 (One largest firm versus several)**.** *Conditional on q_K and S, the number n_K of firms attaining the largest size has no additional effect on $\underline{p}$, R, $\mathbb{E}[P_R]$, or equilibrium profit levels. It does affect the shape of the top-class price distribution. When $n_K \geq 2$ the largest size class is atomless and reaches one continuously at R, whereas when $n_K = 1$ the unique largest firm places a residual atom at R. Thus any level effect accompanying a merger-induced change in n_K is already captured by the simultaneous changes in q_K and S.*

4.3.1 A Numerical Example: Creating a Large Competitor versus a New Leader

Example 3. *Let $\alpha = 0.4$, $c = 0.15$, $z = 1$, and suppose the pre-merger market contains 20 single-establishment firms, two firms with 3 establishments each, and two firms with 10 establishments each. Thus, $T = 46$, $q_K^{\text{pre}} = 10$, $n_K^{\text{pre}} = 2$, and $S^{\text{pre}} = 192$. The reservation price is interior before and after both mergers considered below. We compare two experiments:*

- Merger A (large but not largest): *the two size-3 firms merge, so $s = 6 < q_K^{\text{pre}} = 10$.*
- Merger B (new largest firm): *the two size-10 firms merge, so $s = 20 > q_K^{\text{pre}} = 10$.*

For this market, $\lambda = 0.00635$ and $\alpha = 0.4 < \frac{2(T-1)}{(1+\sqrt{2})T}$. Hence Corollary 5 guarantees that HHI^{Rev} rises after either merger; this is not merely a numerical feature of the calibration. For the markup calculations, set marginal cost $\kappa = 1$, so that $\mu = 1 + p$ and $\mathbb{E}[\mu^{tr}] = 1 + \mathbb{E}[P_R]$. Table 1 reports the equilibrium outcomes.

Merger A: creating a large but not largest competitor. *The merger raises S from 192 to 210 but leaves $q_K = 10$ and $n_K = 2$. The ownership-concentration channel therefore operates without a largest-firm effect. The reservation price falls from 0.5035 to 0.5005, the common floor falls from 0.2594 to 0.2578, and expected transaction markup falls from 1.3535 to 1.3505. At the same time, HHI^{Rev} rises from 0.1074 to 0.1161, or by 87 points on the conventional 10,000-point scale. Thus*

Table 1: Merger examples: Prices, concentration, markups, and profits

	Pre-merger	Merger A $(3+3 \to 6)$	Merger B $(10+10 \to 20)$
Firm-size classes	$(1^{20}, 3^2, 10^2)$	$(1^{20}, 6, 10^2)$	$(1^{20}, 3^2, 20)$
Largest size q_K	10	10	20
Firms in top class n_K	2	2	1
Concentration index S	192	210	392
Reservation price R	0.5035	0.5005	0.8673
Price floor $\underline{p}$	0.2594	0.2578	0.5417
Expected transaction price $\mathbb{E}[P_R]$	0.3535	0.3505	0.7173
Concentration measure			
HHI^{Rev}	0.1074	0.1161	0.1832
Gross markups, $\kappa = 1$			
Markup support	$[1.2594, 1.5035]$	$[1.2578, 1.5005]$	$[1.5417, 1.8673]$
Expected transaction markup $\mathbb{E}[\mu^{tr}]$	1.3535	1.3505	1.7173
Merged-firm profit per establishment			
Weighted pre-merger profit $\overline{\pi}^{\text{pre}}$	—	0.2561	0.2446
Post-merger profit $\pi(s)$	—	0.2497	0.4764
$\Delta\pi_{ab}$	—	-0.0064	$+0.2318$
Common-floor effect	—	-0.0015	$+0.2662$
Cannibalization	—	0.0049	0.0344
Floor ratio $\underline{p}^{\text{post}}/\underline{p}^{\text{pre}}$	—	0.9940	2.0884

Notes: $\alpha = 0.4$, $c = 0.15$, $z = 1$, and $T = 46$. Superscripts on firm sizes denote the number of firms in the corresponding size class. HHI^{Own} uses establishment ownership shares, HHI^{Rev} uses firm revenue shares, and HHI^{Rev}_{Est} treats establishments as independent units. For the markup rows only, marginal cost is normalized to $\kappa = 1$. Entries are computed from unrounded values.

this experiment delivers the general divergence identified in Corollary 5: revenue concentration rises while expected markups fall. The merged firm's profit per establishment also falls, from the weighted pre-merger benchmark 0.2561 *to* 0.2497.

*Figure 3 shows the corresponding equilibrium price distributions. The distributions shift slightly to the left. The size-*3 *class disappears and is replaced by a size-*6 *firm, while the two size-*10 *firms remain co-largest and continue to mix atomlessly up to* R.

Merger B: creating a new largest firm. *The two size-*10 *firms merge into a size-*20 *firm. Both ownership concentration and the largest-firm size rise sharply. In this calibration,* $\frac{\alpha\Delta S}{T(T-1)} = 0.03865$, *while the reservation-price threshold in* (A.110) *is approximately* 0.23873 *and the common-floor threshold in* (A.112) *is approximately* 0.34002. *Thus* $\frac{\alpha\Delta S}{T(T-1)}$ *lies below both thresholds, so Theorem 4 predicts that both the reservation price and the common floor rise. Indeed, the floor increases from* 0.2594 *to* 0.5417, *the reservation price rises from* 0.5035 *to* 0.8673, *and the expected transaction markup rises*

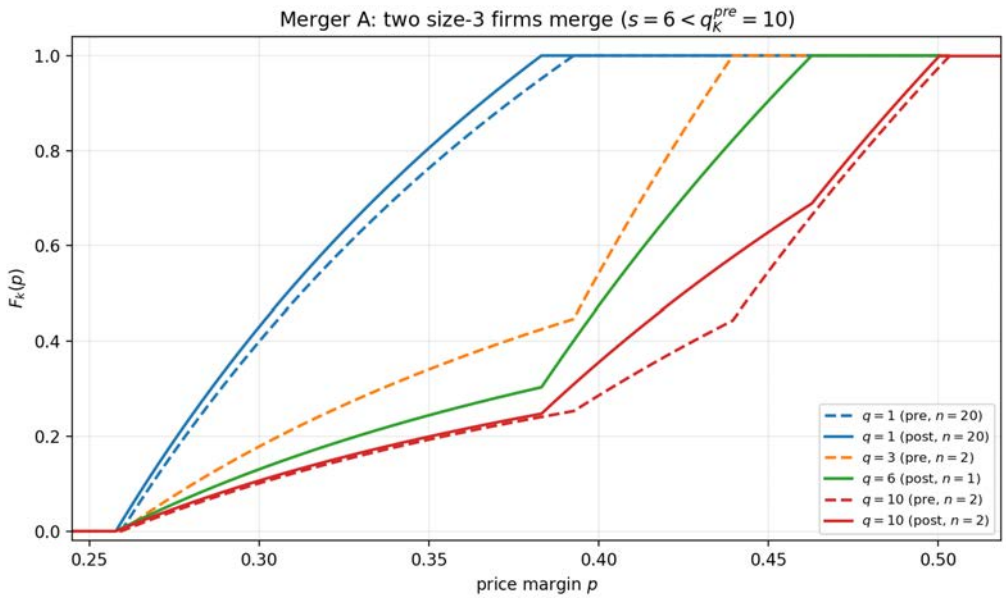


Figure 3: Price posting distributions when the merger creates a large but not largest competitor. Dashed lines are pre-merger distributions and solid lines are post-merger distributions.

from 1.3535 *to* 1.7173. HHI^{Rev} *also rises, from* 0.1074 *to* 0.1832, *an increase of* 758 *HHI points. The floor ratio,* 2.0884, *exceeds the profitability hurdle shown in* (68), *so merged-firm profit per establishment rises from* 0.2446 *to* 0.4764.

*Figure 4 shows the much larger rightward shift in equilibrium pricing. Before the merger there are two co-largest size-*10 *firms and the top class is atomless. After the merger, the size-*20 *firm is the unique largest firm and places a residual atom at* R.

The two experiments summarize the distinction between concentration and market power in the model. Both mergers raise owner concentration and revenue concentration. Nevertheless, expected markups move in opposite directions. What differs across the two experiments is not the sign of the HHI change, but whether consolidation changes q_K and thereby activates the largest-firm channel. When the reservation price is interior, that largest-firm channel also determines whether the merger raises or lowers R, not merely whether it raises or lowers the common floor.

5 Conclusion

We study how the establishment ownership structure shapes price dispersion and market power in a noisy-search market. Because consumers sample establishments rather than firms, a larger establishment network raises the chance that even a well-informed consumer sees only one owner's price, but it also makes the firm's own establishments compete for the same

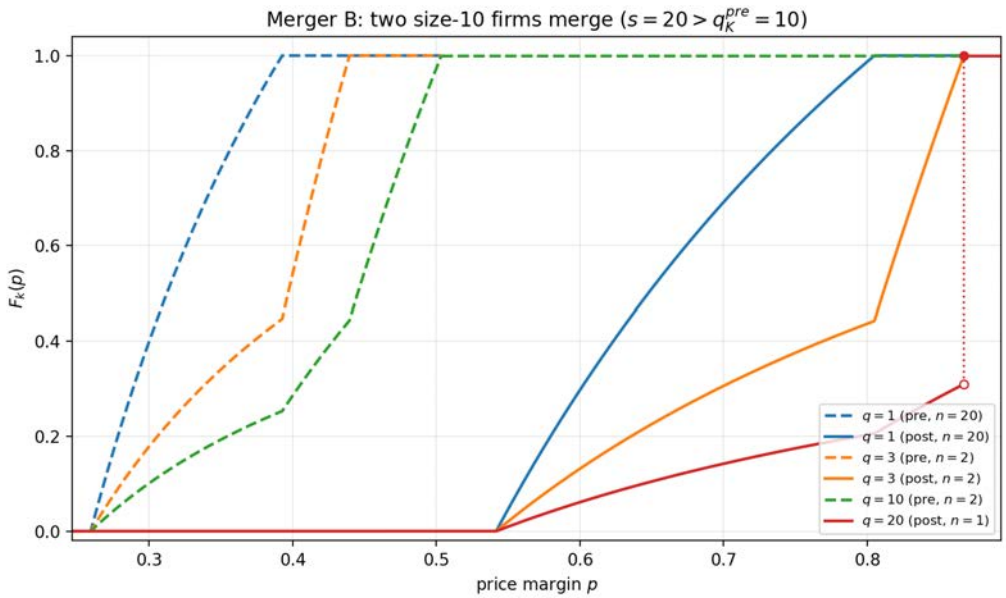


Figure 4: Price posting distributions when the merger creates a new unique largest firm. Dashed lines are pre-merger distributions and solid lines are post-merger distributions. The open and filled circles mark the atom of the size-20 firm at the reservation price.

consumer. The equilibrium balances this endogenous capture against within-firm cannibalization.

We emphasize three main results of the analysis. First, uniform pricing across a firm's establishments is optimal in our setting. Second, the equilibrium pricing distributions are unique and share a common price floor. Their supports are ordered by firm size, with each larger firm's support containing those of all smaller firms, while extending the upper limit of the price distribution. We also show that the effect of firm consolidation by merger hinges on whether the merger creates a new largest firm: when it does not, expected transaction prices and markups fall even as concentration rises, whereas when it does, the largest-firm channel can reverse the sign, so that a merger raising the common price floor can still lower the reservation price and expected markup.

Two features of the analysis stand out. First, concentration and markups are distinct objects in the model. Revenue concentration responds to the full ownership distribution, while the expected transaction markup depends on ownership only through the largest network size and a single concentration index, so the two can move in opposite directions. Second, how a firm grows matters as much as how large it becomes. Opening one establishment, opening several, or acquiring rivals reshapes the size distribution differently and therefore such structural changes have different price effects. By separating the number of firms from the number of establishments, the model yields closed-form expressions for both concentration and markups from the same primitives and traces how ownership structure determines the

prices consumers pay.

References

Autor, David, David Dorn, Lawrence F Katz, Christina Patterson, and John Van Reenen (2020). "The Fall of the Labor Share and the Rise of Superstar Firms." *The Quarterly Journal of Economics* 135.2, pp. 645–709. DOI: 10.1093/qje/qjaa004.

Bresnahan, Timothy F. (1989). "Empirical Studies of Industries with Market Power." In: *Handbook of Industrial Organization*. Vol. 2. Elsevier, pp. 1011–1057. DOI: 10.1016/S1573-448X(89)02005-4.

Burdett, Ken and Eric Smith (2010). "Price Distributions and Competition." *Economics Letters* 106.3, pp. 180–183. DOI: 10.1016/j.econlet.2009.11.014.

Burdett, Kenneth and Kenneth L. Judd (1983). "Equilibrium Price Dispersion." *Econometrica* 51.4, p. 955. DOI: 10.2307/1912045. JSTOR: 1912045.

Butters, R. Andrew, Daniel W. Sacks, and Boyoung Seo (2022). "How Do National Firms Respond to Local Cost Shocks?" *American Economic Review* 112.5, pp. 1737–1772. DOI: 10.1257/aer.20201524.

De Loecker, Jan, Jan Eeckhout, and Gabriel Unger (2020). "The Rise of Market Power and the Macroeconomic Implications." *The Quarterly Journal of Economics* 135.2, pp. 561–644. DOI: 10.1093/qje/qjz041.

DellaVigna, Stefano and Matthew Gentzkow (2019). "Uniform Pricing in U.S. Retail Chains." *The Quarterly Journal of Economics* 134.4, pp. 2011–2084. DOI: 10.1093/qje/qjz019.

Dixit, Avinash K. and Joseph E. Stiglitz (1977). "Monopolistic Competition and Optimum Product Diversity." *American Economic Review* 67.3, pp. 297–308. JSTOR: 1831401.

Eeckhout, Jan (2021). "Comment on 'Diverging Trends in National and Local Concentration'." *NBER Macroeconomics Annual* 35, pp. 151–166. DOI: 10.1086/712318.

Hall, Robert E. (1988). "The Relation between Price and Marginal Cost in U.S. Industry." *Journal of Political Economy* 96.5, pp. 921–947. DOI: 10.1086/261570.

Hazell, Jonathon, Christina Patterson, Heather Sarsons, and Bledi Taska (2023). *National Wage Setting*. IZA Discussion Paper 16493.

Hershbein, Brad, Claudia Macaluso, and Chen Yeh (2019). *Concentration in U.S. Local Labor Markets: Evidence from Vacancy and Employment Data*. 2019 Meeting Papers. Society for Economic Dynamics, 1336.

Kiser, Elizabeth K. (2002). "Household Switching Behavior at Depository Institutions: Evidence from Survey Data." *The Antitrust Bulletin* 47.4, pp. 619–640. DOI: 10.1177/0003603X0204700404.

Menzio, Guido (2026). "Markups: A Search-Theoretic Perspective." *Journal of Political Economy*, pp. 2871–2925. DOI: 10.1086/742420.

Menzio, Guido and Nicholas Trachter (2015). "Equilibrium Price Dispersion with Sequential Search." *Journal of Economic Theory* 160, pp. 188–215. DOI: 10.1016/j.jet.2015.09.004.

Moraga-González, José Luis, Zsolt Sándor, and Matthijs R. Wildenbeest (2017). "Nonsequential Search Equilibrium with Search Cost Heterogeneity." *International Journal of Industrial Organization* 50, pp. 392–414. DOI: 10.1016/j.ijindorg.2016.06.009.

Myatt, David P. and David Ronayne (2026). "Asymmetric Models of Sales." *American Economic Journal: Microeconomics* 18.1, pp. 281–311. DOI: 10.1257/mic.20240437.

Rinz, Kevin (2022). "Labor Market Concentration, Earnings, and Inequality." *Journal of Human Resources* 57.S, S251–S283. DOI: 10.3368/jhr.monopsony.0219-10025R1.

Rossi-Hansberg, Esteban, Pierre-Daniel Sarte, and Nicholas Trachter (2021). "Diverging Trends in National and Local Concentration." *NBER Macroeconomics Annual* 35, pp. 115–150. DOI: 10.1086/712317.

Smith, Dominic A. and Sergio Ocampo (2025). "The Evolution of US Retail Concentration." *American Economic Journal: Macroeconomics* 17.1, pp. 71–101. DOI: 10.1257/mac.20220249.

Varian, Hal R. (1980). "A Model of Sales." *American Economic Review* 70.4, pp. 651–659. JSTOR: 1803562.

Wu, Edson (2024). *Rationally Inattentive Seller: The Geographic Dispersion of US Retail Chains and Its Aggregate Implications*. Working Paper. Kilts Center at Chicago Booth Marketing Data Center.

A Proofs

A.1 Auxiliary Results

Optimality of Common Pricing

Fix a multi-establishment firm j with $m \geq 2$ establishments. Allow the firm to choose an arbitrary joint distribution for the vector of establishment prices

$$(P_{m,j,1}, \ldots, P_{m,j,m}) \in [0, R]^m.$$

The establishment prices need not be independent, identically distributed, or atomless. Let $F_{m,j,r}$ denote the marginal cumulative distribution of $P_{m,j,r}$ for establishment $r \in \{1, \ldots, m\}$. Denote $\Pi^{ES}_{m,j}$ the profit under this establishment-specific pricing strategy.

Proposition A.1 (Common pricing globally dominates establishment-specific pricing)**.** *Suppose $N_s \geq 1$, $N_m \geq 1$, $m \geq 2$, and $0 < \alpha < 1$. Fix the reservation price R and the pricing strategies of all other firms. Consider any, possibly asymmetric and correlated, establishment-specific pricing strategy $(P_{m,j,1}, \ldots, P_{m,j,m})$ for multi-establishment firm j. Construct a common-pricing strategy for firm j by drawing one price and posting that same realization at all m establishments, where the common price has cumulative distribution*

$$F^{CP}_{m,j}(p) = \frac{1}{m} \sum_{r=1}^{m} F_{m,j,r}(p), \qquad p \in [0, R]. \tag{A.1}$$

Then the common-pricing strategy yields weakly greater expected firm profit than the original establishment-specific strategy for every strategy profile of the other firms, i.e., $\Pi^{CP}_{m,j} \geq \Pi^{ES}_{m,j}$. Moreover, the inequality is strict whenever the original establishment-specific strategy sometimes assigns different prices to two establishments of firm j, $\Pr(P_{m,j,r} \neq P_{m,j,s}) > 0$, for some $r \neq s$. Hence every genuinely establishment-specific pricing strategy is strictly dominated by a common-pricing strategy. Equality can hold only when $P_{m,j,1} = \cdots = P_{m,j,m}$ almost surely, so the original strategy is already a common-pricing strategy.

Proof. The proof separates the possible sampling events.

Events in which no establishment of firm j is sampled. The pricing strategy of firm j is irrelevant for these consumers, so expected profit is identical under the two strategies.

Events in which exactly one establishment of firm j is sampled. This includes both a one-quote consumer who samples firm j and a two-quote consumer who samples one establishment of firm j and one establishment belonging to another firm.

Because establishments are sampled uniformly, conditional on exactly one establishment of firm j being sampled, each of its m establishments is equally likely to be the sampled establishment. Therefore, under the original establishment-specific strategy, the distribution of the price observed from firm j is

$$\frac{1}{m}\sum_{r=1}^{m} F_{m,j,r}(p).$$

By construction, this is exactly the distribution of the firm's quote under the common-pricing strategy in (A.1). Thus the common-pricing transformation leaves unchanged the distribution of firm j's quote in every sampling event in which exactly one of its establishments is observed. Since the strategies of all rival firms are held fixed, it also leaves unchanged the joint distribution of firm j's quote and any rival quote. Consequently, expected revenue, winning probabilities, and expected profit are identical under the two strategies for all such events. This conclusion continues to hold in the presence of atoms and price ties.

Events in which two establishments of firm j are sampled. Conditional on this event, the two sampled establishments are uniformly drawn without replacement from the m establishments of firm j. Under the original establishment-specific strategy, if establishments r and s are sampled, the consumer purchases one unit from the lower-priced establishment. Firm-level revenue from this consumer is therefore $\min\{P_{m,j,r}, P_{m,j,s}\}$. Averaging over all ordered pairs of distinct establishments, expected firm revenue is

$$\frac{1}{m(m-1)}\sum_{r\neq s}\mathbb{E}\left[\min\{P_{m,j,r}, P_{m,j,s}\}\right]. \tag{A.2}$$

Under the common-pricing strategy, both sampled establishments post the same realized price. The firm therefore makes one sale at that common price. From (A.1), expected revenue conditional on this sampling event is

$$\frac{1}{m}\sum_{r=1}^{m}\mathbb{E}[P_{m,j,r}]. \tag{A.3}$$

For any two prices,

$$\min\{P_{m,j,r}, P_{m,j,s}\} = \frac{P_{m,j,r} + P_{m,j,s} - |P_{m,j,r} - P_{m,j,s}|}{2}.$$

Substituting this identity into (A.2) gives

$$\begin{aligned}
&\frac{1}{m(m-1)}\sum_{r\neq s}\mathbb{E}\left[\min\{P_{m,j,r},P_{m,j,s}\}\right] \\
&\quad=\frac{1}{m}\sum_{r=1}^{m}\mathbb{E}[P_{m,j,r}]-\frac{1}{2m(m-1)}\sum_{r\neq s}\mathbb{E}\left[|P_{m,j,r}-P_{m,j,s}|\right]. \qquad \text{(A.4)}
\end{aligned}$$

Comparing (A.4) with (A.3), the gain from common pricing conditional on two establishments of firm j being sampled is therefore

$$\frac{1}{2m(m-1)}\sum_{r\neq s}\mathbb{E}\left[|P_{m,j,r}-P_{m,j,s}|\right]\geq 0. \qquad \text{(A.5)}$$

The inequality is strict if

$$\Pr(P_{m,j,r}\neq P_{m,j,s})>0$$

for at least one pair $r\neq s$.

For a two-quote consumer, the probability that both sampled establishments belong to firm j is $\frac{m(m-1)}{T(T-1)}$. Hence, with a measure n of consumers, the difference in total expected firm profit is

$$\begin{aligned}
\Pi^{CP}_{m,j}-\Pi^{ES}_{m,j}&=\alpha n\frac{m(m-1)}{T(T-1)}\frac{1}{2m(m-1)}\sum_{r\neq s}\mathbb{E}\left[|P_{m,j,r}-P_{m,j,s}|\right] \\
&=\frac{\alpha n}{2T(T-1)}\sum_{r\neq s}\mathbb{E}\left[|P_{m,j,r}-P_{m,j,s}|\right]\geq 0. \qquad \text{(A.6)}
\end{aligned}$$

All other sampling events generate exactly the same expected profit under the two strategies. Therefore (A.6) is the entire profit effect of replacing the arbitrary establishment-specific strategy with the common-pricing strategy.

Since $\alpha>0$ and every pair of establishments of firm j is sampled with positive probability, the inequality is strict whenever at least one pair of establishment prices differs with positive probability. If equality holds, then $\mathbb{E}\left[|P_{m,j,r}-P_{m,j,s}|\right]=0$ for every $r\neq s$, which implies $P_{m,j,1}=\cdots=P_{m,j,m}$ almost surely. Thus any establishment-specific strategy that is not already common pricing is strictly dominated by the common-pricing strategy constructed in (A.1). □

Corollary A.1 (Common pricing is without loss in the pricing subgame)**.** *For every fixed reservation price R and every $N_m\geq 1$, any Nash equilibrium of the pricing subgame in which multi-establishment firms are allowed to choose arbitrary joint establishment-specific pricing strategies must satisfy $P_{m,j,1}=\cdots=P_{m,j,m}$ almost surely for every multi-establishment firm j.*

Proof. Suppose instead that, in a candidate Nash equilibrium, some multi-establishment firm j uses an establishment-specific strategy for which $\Pr(P_{m,j,r} \neq P_{m,j,s}) > 0$ for some $r \neq s$. Proposition A.1 constructs a feasible common-pricing strategy that leaves the firm's expected payoff unchanged in every event in which at most one of its establishments is sampled and strictly raises its expected payoff when two of its establishments are sampled. The original strategy therefore cannot be a best response, contradicting Nash equilibrium. □

Consequently, the common-pricing restriction imposed in the main text is without loss of generality, since it follows from firm-level profit maximization rather than having to be maintained as an exogenous restriction on the strategy space.

A.2 Proof of Main Results

Proof of Proposition 1

Proof. Consider two single-establishment firms i and j, and define the signed bounded-variation function

$$H(p) \equiv F_{s,i}(p) - F_{s,j}(p), \qquad p \in [0, R].$$

A unilateral change in firm i's mixing distribution does not change its demand conditional on posting a given pure price p. Optimality of $F_{s,i}$ relative to the feasible strategy $F_{s,j}$ therefore implies

$$\int_{[0,R]} p D_{s,i}(p)\, dH(p) \geq 0.$$

Applying the analogous inequality to firm j and adding gives

$$\int_{[0,R]} p \left[D_{s,i}(p) - D_{s,j}(p) \right] dH(p) \geq 0. \tag{A.7}$$

From the firm-specific demand equations,

$$D_{s,i}(p) - D_{s,j}(p) = \frac{1-\gamma}{T-1} \left[\widetilde{F}_{s,i}(p) - \widetilde{F}_{s,j}(p) \right].$$

then:

$$\begin{aligned} \widetilde{F}_{s,i}(p) - \widetilde{F}_{s,j}(p) &= F_{s,i}(p^-) + \tfrac{1}{2}\Delta F_{s,i}(p) - \left[F_{s,j}(p^-) + \tfrac{1}{2}\Delta F_{s,j}(p) \right] \\ &= \left[F_{s,i}(p^-) - F_{s,j}(p^-) \right] + \tfrac{1}{2} \left[\Delta F_{s,i}(p) - \Delta F_{s,j}(p) \right] \end{aligned}$$

notice that:

$$\begin{aligned} \Delta F_{s,i}(p) - \Delta F_{s,j}(p) &= F_{s,i}(p) - F_{s,i}(p^-) - \left[F_{s,j}(p) - F_{s,j}(p^-) \right] \\ &= F_{s,i}(p) - F_{s,j}(p) - \left[F_{s,i}(p^-) - F_{s,j}(p^-) \right] \\ &= H(p) - H(p^-) \end{aligned}$$

substituting it back, we have:

$$\begin{aligned}\widetilde{F}_{s,i}(p) - \widetilde{F}_{s,j}(p) &= H(p^-) + \tfrac{1}{2}\left[H(p) - H(p^-)\right] \\ &= \tfrac{1}{2}\left[H(p) + H(p^-)\right]\end{aligned}$$

therefore

$$D_{s,i}(p) - D_{s,j}(p) = \frac{1}{2}\left(\frac{1-\gamma}{T-1}\right)\left[H(p) + H(p^-)\right]$$

For a right-continuous function of bounded variation, the Stieltjes product rule gives

$$d(H^2)(p) = \left[H(p) + H(p^-)\right] dH(p).$$

Substituting into (A.7),

$$0 \leq \frac{1-\gamma}{2(T-1)} \int_{[0,R]} p \, d(H^2)(p).$$

Because both cumulative distribution functions equal one at R, $H(R) = 0$; and because both are zero to the left of the action set, $H(0^-) = 0$. Integration by parts with the continuous function p yields

$$\int_{[0,R]} p \, d(H^2)(p) = RH(R)^2 - \int_0^R H(p)^2 \, dp = -\int_0^R H(p)^2 \, dp.$$

Hence $H = 0$ Lebesgue-almost everywhere. Right continuity then implies $H(p) = 0$ for every p, so $F_{s,i} = F_{s,j}$.

Now consider two multi-establishment firms j and z and define

$$M(p) \equiv F_{m,j}(p) - F_{m,z}(p).$$

The same pair of optimality inequalities, now accounting for total firm profit $mpD_{m,j}(p)$, gives

$$0 \leq m \int_{[0,R]} p\left[D_{m,j}(p) - D_{m,z}(p)\right] dM(p).$$

Since

$$D_{m,j}(p) - D_{m,z}(p) = \frac{m(1-\gamma)}{2(T-1)}\left[M(p) + M(p^-)\right],$$

we obtain

$$0 \leq \frac{m^2(1-\gamma)}{2(T-1)} \int_{[0,R]} p \, d(M^2)(p) = -\frac{m^2(1-\gamma)}{2(T-1)} \int_0^R M(p)^2 \, dp.$$

Thus $M = 0$ everywhere by the same right-continuity argument. Because the pairs of firms were arbitrary, equilibrium strategies are symmetric within each type. □

Proof of Lemma 1

Proof. By Proposition 1, all firms of a given type use the same CDF. Suppose first that F_s has an atom of size $\delta_s > 0$ at p_0. Because $\gamma > 0$, a price of zero cannot be optimal: any sufficiently small positive price earns strictly positive profit from one-quote consumers. Hence $p_0 > 0$.

Let $\delta_m \geq 0$ denote the size of any atom of F_m at the same price. The demand of a single-establishment firm at p_0 uses the midpoint CDFs. If the firm instead posts a sequence of prices converging to p_0 from below, its limiting demand is

$$D_s(p_0^-) = D_s(p_0) + \frac{1-\gamma}{2(T-1)}\left[(N_s - 1)\delta_s + N_m m \delta_m\right] > D_s(p_0),$$

because $N_s > 1$. The price loss from undercutting vanishes as the deviation approaches p_0, whereas the demand gain remains strictly positive. Thus prices sufficiently close to p_0 from below yield strictly more profit than p_0, contradicting the positive probability assigned to p_0.

A zero-price atom is ruled out for a multi-establishment firm by the same positive-profit argument. If instead F_m has an atom of size $\delta_m > 0$ at $p_0 > 0$, undercutting gives the limiting demand gain

$$D_m(p_0^-) - D_m(p_0) = \frac{1-\gamma}{2(T-1)}\left[N_s \delta_s + (N_m - 1)m\delta_m\right] > 0,$$

where the strict inequality uses $N_m > 1$. The same argument rules out the atom. Therefore both equilibrium CDFs are atomless. □

Proof of Lemma 2

Proof. Suppose, toward a contradiction, that the union of the two supports has a nonempty open gap. Choose a maximal gap $(a, b) \subset (\underline{p}, \overline{p})$. By Lemma 1, neither F_s nor F_m has a mass point at either endpoint, and both distributions are constant on (a, b).

Let type $t \in \{s, m\}$ be a type whose support contains a. The pure-price payoff of type t is continuous, so a is a best response. Since both distributions are constant on the gap, $D_t(p) = D_t(a)$ for every $p \in (a, b)$. Moreover, $D_t(a) \geq \gamma > 0$. Therefore every $p \in (a, b)$ satisfies

$$pD_t(p) = pD_t(a) > aD_t(a),$$

a profitable deviation from the support price a. This contradiction proves that the union of the supports is connected. □

Proof of Lemma 3

Proof. For any p_1 and p_2 in the support of s and m we must have that for each type of firm they generate the same payoff and thus:

$$\frac{D_s(p_1)p_1}{D_m(p_1)p_1} = \frac{D_s(p_2)p_2}{D_m(p_2)p_2}$$

simplifying it:

$$\frac{D_s(p_1)}{D_m(p_1)} = \frac{D_s(p_2)}{D_m(p_2)} \tag{A.8}$$

Toward a contradiction, assume that p_1 and p_2 are each on one side of a crossing between $D_s(\cdot)$ and $D_m(\cdot)$. Without loss of generality, assume that $D_s(p_1) > D_m(p_1)$ and $D_s(p_2) < D_m(p_2)$. But then (A.8) cannot be satisfied. □

Proposition A.2. *Denote the support for $F_i(\cdot)$ by $supp\{F_i\}$. Let $\underline{p}_i \equiv \min supp\{F_i\}$ and $\overline{p}_i \equiv \max supp\{F_i\}$ for $i \in \{m, s\}$. In equilibrium, we cannot have $\underline{p}_s \leq \underline{p}_m$ and $\overline{p}_s \geq \overline{p}_m$.*

Proof. Toward a contradiction, assume that $\underline{p}_s \leq \underline{p}_m$. Then, from equation (10), we have that:

$$D_m(\underline{p}_m) - D_s(\underline{p}_m) = \frac{(1-\gamma)}{N_s + N_m m - 1}\left\{-\frac{(m-1)}{2} - F_s(\underline{p}_m)\right\} \leq \frac{(1-\gamma)}{N_s + N_m m - 1}\left\{-\frac{(m-1)}{2}\right\} < 0 \tag{A.9}$$

In contrast, at the upper end, we have:

$$D_m(\overline{p}_m) - D_s(\overline{p}_m) = \frac{(1-\gamma)}{N_s + N_m m - 1}\left\{m - \frac{(m-1)}{2} - F_s(\overline{p}_m)\right\} \geq \frac{(1-\gamma)}{N_s + N_m m - 1}\frac{(m-1)}{2} > 0. \tag{A.10}$$

From Lemma 1, F_s and F_m are continuous. Since the demand functions are affine in F_s and F_m, both D_s and D_m are continuous in p, as well as the respective profit functions. Then, from Lemma 2 and a continuity argument, we must have that $\underline{p}_m$ and $\overline{p}_m$ are also optimal prices for single-establishment firms.

Then, Lemma 3 implies that $\underline{p}_m$ and $\overline{p}_m$ cannot be on opposite sides of the single crossing, concluding the contradiction. □

Given Proposition A.2, we can rule out equilibria in which $F_s(\cdot)$ and $F_m(\cdot)$ are either the same or share the same support. This also rules out many other combinations of supports.

Proposition A.3. *Let $\underline{p}_i \equiv \min supp\{F_i\}$ and $\overline{p}_i \equiv \max supp\{F_i\}$ for $i \in \{m, s\}$. In any equilibrium, we must have $\underline{p}_s = \underline{p}_m = \underline{p}$.*

Proof. Toward a contradiction, we consider two cases.

First, suppose that $\underline{p}_s > \underline{p}_m$. Given Lemma 2, $p \in [\underline{p}_m, \underline{p}_s]$ are in the support of F_m. Hence, from equation (8) and the fact that $F_m(\underline{p}_m) = F_s(\underline{p}_m) = F_s(\underline{p}_s) = 0$, we have that:

$$F_m(p) = \frac{N_s + N_m m - 1}{(1-\gamma)(N_m - 1)m} \left\{ \begin{array}{c} 1 - (1-\gamma)\frac{0.5(m-1)}{N_s + N_m m - 1} \\ - \left[1 - (1-\gamma)\frac{0.5(m-1)}{N_s + N_m m - 1}\right] \frac{\underline{p}_m}{p} \end{array} \right\}$$

considering the single-establishment firm's incentive to post a price $p \in [\underline{p}_m, \underline{p}_s)$. p is an optimal deviation if:

$$\left\{ N_s - 1 + N_m m - (1-\gamma) N_m m F_m(p) \right\} p \geq \left\{ N_s - 1 + N_m m - (1-\gamma) N_m m F_m(\underline{p}_s) \right\} \underline{p}_s$$

notice that:

$$(1-\gamma) N_m m F_m(p) = \frac{N_m}{(N_m - 1)} \left\{ \begin{array}{c} N_s + N_m m - 1 - (1-\gamma)0.5(m-1) \\ - \left[N_s + N_m m - 1 - (1-\gamma)0.5(m-1)\right] \frac{\underline{p}_m}{p} \end{array} \right\}$$

substituting it back:

$$\begin{array}{c} \left\{ \begin{array}{c} (N_s + N_m m - 1) \\ -\frac{N_m}{(N_m - 1)} \left\{ \begin{array}{c} N_s + N_m m - 1 - (1-\gamma)0.5(m-1) \\ - \left[N_s + N_m m - 1 - (1-\gamma)0.5(m-1)\right] \frac{\underline{p}_m}{p} \end{array} \right\} \end{array} \right\} p \\ \geq \left\{ \begin{array}{c} (N_s + N_m m - 1) \\ -\frac{N_m}{(N_m - 1)} \left\{ \begin{array}{c} N_s + N_m m - 1 - (1-\gamma)0.5(m-1) \\ - \left[N_s + N_m m - 1 - (1-\gamma)0.5(m-1)\right] \frac{\underline{p}_m}{\underline{p}_s} \end{array} \right\} \end{array} \right\} \underline{p}_s \end{array}$$

simplifying it:

$$\begin{array}{c} \left\{ -(N_s - 1) - \gamma N_m m - N_m (1-\gamma) 0.5(m+1) \right\} p \\ \geq \left\{ -(N_s - 1) - \gamma N_m m - N_m (1-\gamma) 0.5(m+1) \right\} \underline{p}_s \end{array} \tag{A.11}$$

Since $N_s \geq 1$, we have that $-(N_s - 1) - \gamma N_m m - N_m(1-\gamma)0.5(m+1) < 0$ and $p < \underline{p}_s$, the inequality presented in (A.11) is satisfied and single-establishment firms have an optimal deviation.

Second, assume that $\underline{p}_s < \underline{p}_m$. From Proposition A.2, we must have that $\overline{p}_m > \overline{p}_s$. Notice that $\overline{p}_m = R$; otherwise a firm posting $\overline{p}_m$ has an optimal deviation. Notice that $p \in [\underline{p}_s, \underline{p}_m)$ is posted only by single-establishment firms. From Lemma 2, any $p \in [\underline{p}_s, \underline{p}_m)$ must be in the

support of $F_s(\cdot)$. From the profit-maximization condition, we have:

$$F_s(p) = \frac{N_s - 1 + N_m m}{(1-\gamma)(N_s - 1)}\left[1 - \frac{\underline{p}_s}{p}\right]$$

Then, multi-establishment firms do not deviate to a price $p < \underline{p}_m$ if:

$$\left\{ \begin{array}{c} 1 - (1-\gamma)\frac{0.5(m-1)}{N_s+N_m m-1} \\ -(1-\gamma)\frac{N_s}{N_s+N_m m-1}F_s(p) \end{array} \right\} p < \left\{ \begin{array}{c} 1 - (1-\gamma)\frac{0.5(m-1)}{N_s+N_m m-1} \\ -(1-\gamma)\frac{N_s}{N_s+N_m m-1}F_s(\underline{p}_m) \end{array} \right\} \underline{p}_m$$

since:

$$(1-\gamma)\frac{N_s}{N_s - 1 + N_m m}F_s(p) = \frac{N_s}{N_s - 1}\left[1 - \frac{\underline{p}_s}{p}\right]$$

we have:

$$\left\{ \begin{array}{c} -\left[\frac{1}{N_s-1} + (1-\gamma)\frac{0.5(m-1)}{N_s+N_m m-1}\right] p \\ +\frac{N_s}{N_s-1}\underline{p}_s \end{array} \right\} < \left\{ \begin{array}{c} -\left[\frac{1}{N_s-1} + (1-\gamma)\frac{0.5(m-1)}{N_s+N_m m-1}\right] \underline{p}_m \\ +\frac{N_s}{N_s-1}\underline{p}_s \end{array} \right\}$$

simplifying we get $p > \underline{p}_m$, which is a contradiction.

Hence, we must have $\underline{p}_s = \underline{p}_m = \underline{p}$. □

Corollary A.2. *In any equilibrium, $\overline{p} = \overline{p}_m > \overline{p}_s$.*

Proof. By Proposition A.3, $\underline{p}_s = \underline{p}_m = \underline{p}$. From Proposition A.2, we showed that we could not have $\underline{p}_s = \underline{p}_m$ and $\overline{p}_s \geq \overline{p}_m$ in equilibrium. Hence, $\overline{p}_m > \overline{p}_s$. Since $\overline{p} = \max \text{supp}\{F_s\} \cup \text{supp}\{F_m\}$, $\overline{p} = \overline{p}_m$. □

Corollary A.3. *If $p \in supp\{F_s\} \cap supp\{F_m\}$, we must have $D_m(p) < D_s(p)$*

Proof. From Proposition A.3, we have that $\underline{p}_s = \underline{p}_m = \underline{p}$. At the lower endpoint, $F_s(\underline{p}) = F_m(\underline{p}) = 0$. Using equation (10), we have that $D_m(\underline{p}) < D_s(\underline{p})$. Then, from Lemma 3, the sign cannot switch across common-support prices. Therefore $D_m(p) < D_s(p)$ throughout the common support. □

Lemma A.1 (Common support affine form)**.** *If $p \in supp\{F_s\} \cap supp\{F_m\}$, then $F_s(p)$ and $F_m(p)$ can be rewritten as:*

$$F_s = C_s\left(1 - \frac{\underline{p}}{p}\right) \qquad \text{and} \qquad F_m(p) = C_m\left(1 - \frac{\underline{p}}{p}\right)$$

where

$$C_s = \frac{\frac{T-1}{1-\gamma} - N_m\frac{m-1}{2}}{N-1} \qquad \text{and} \qquad C_m = \frac{\frac{T-1}{1-\gamma} + (N_s-1)\frac{m-1}{2}}{m(N-1)}$$

Hence, on the common support, both distributions are affine functions of $\frac{1}{p}$.

Proof. $D_m(p)$ and $D_s(p)$ are given by:

$$D_m(p) = 1 - \frac{1-\gamma}{T-1}\left\{\left(\frac{m-1}{2}\right) + N_s F_s(p) + (N_m - 1)mF_m(p)\right\} \tag{A.12}$$

and

$$D_s(p) = 1 - \frac{1-\gamma}{T-1}\left[(N_s - 1)F_s(p) + N_m m F_m(p)\right] \tag{A.13}$$

Defining π_i^* as the equilibrium profit for an i-type firm, since p is in the support of both types, both profit equalizations must hold. Moreover, from Proposition A.3, we have that $\underline{p} \in suppF_i$, for $i \in \{s, m\}$. Therefore, we have that:

$$\pi_s^* = \underbrace{D_s(\underline{p})}_{=1}\underline{p} = \underline{p} \tag{A.14}$$

consequently, for $p \in \sup\{F_s\}$, we have:

$$D_s(p)p = \pi_s^* = \underline{p} \Rightarrow D_s(p) = \frac{\underline{p}}{p} \tag{A.15}$$

from (A.13) and (A.15), we have:

$$\left(1 - \frac{\underline{p}}{p}\right)\frac{T-1}{1-\gamma} = (N_s - 1)F_s(p) + N_m m F_m(p) \tag{A.16}$$

similarly, for multi-establishment firms, we have:

$$\pi_m^* = D_m(\underline{p})\underline{p} = \left[1 - \frac{1-\gamma}{T-1}\left(\frac{m-1}{2}\right)\right]\underline{p} \tag{A.17}$$

from (A.12) and (A.17), we have:

$$1 - \frac{1-\gamma}{T-1}\left\{\frac{m-1}{2} + N_s F_s(p) + (N_m - 1)mF_m(p)\right\} = \left[1 - \frac{1-\gamma}{T-1}\left(\frac{m-1}{2}\right)\right]\frac{\underline{p}}{p}$$

rearranging it:

$$\left[1 - \frac{1-\gamma}{T-1}\left(\frac{m-1}{2}\right)\right]\left(1 - \frac{\underline{p}}{p}\right)\frac{T-1}{1-\gamma} = N_s F_s(p) + (N_m - 1)mF_m(p) \tag{A.18}$$

We can then solve the system of equations (A.16) and (A.18) to obtain $F_s(p)$ and $F_m(p)$. In

particular, solving the system, we obtain:

$$F_s(p) = \underbrace{\left(\frac{\frac{T-1}{1-\gamma} - N_m\left(\frac{m-1}{2}\right)}{N-1}\right)}_{\equiv C_s}\left(1 - \frac{\underline{p}}{p}\right) \qquad \text{and} \qquad F_m(p) = \underbrace{\left(\frac{\frac{T-1}{1-\gamma} + (N_s-1)\left(\frac{m-1}{2}\right)}{(N-1)m}\right)}_{\equiv C_m}\left(1 - \frac{\underline{p}}{p}\right)$$

concluding the proof. □

Moreover, since $\gamma \in (0,1)$, C_s and C_m are both positive. As a result, on common-support prices, both $F_s(p)$ and $F_m(p)$ are strictly increasing in p.

We define another auxiliary result before turning to the main result on connected supports.

Lemma A.2. *Suppose (a,b) is an open interval in the support of one type and disjoint from the support of the other type. Then, the active type's CDF is affine in $\frac{1}{p}$ on (a,b), while inactive type's* deviation payoff *is affine in p on (a,b).*

Proof. Initially, suppose that $(a,b) \cap supp\{F_m\} = \varnothing$ and $(a,b) \subseteq supp\{F_s\}$. Since F_m has no mass points and has no support in (a,b), there is a constant K_m such that $F_m(p) = K_m$, $\forall p \in (a,b)$. In contrast, since single-establishment firms post in (a,b), from profit maximization and Proposition A.3, we have that $D_s(p)p = \underline{p}$, $\forall p \in (a,b)$. Then, combining with equation (A.13) and rearranging it, we have:

$$F_s(p) = \frac{1}{N_s - 1}\left[\frac{T-1}{1-\gamma}\left(1 - \frac{\underline{p}}{p}\right) - N_m m K_m\right], \qquad p \in (a,b) \tag{A.19}$$

therefore, $F_s(p)$ is affine in $\frac{1}{p}$ on (a,b).

Now evaluate the multi-establishment deviation payoff on (a,b). From the multi-establishment firm profit function, substituting (A.19) and $F_m(p) = K_m$, after algebraic manipulations, we have:

$$\pi_m(p) = \underbrace{\left\{-\frac{1}{N_s-1} - \frac{1-\gamma}{T-1}\left(\frac{m-1}{2}\right) + \frac{1-\gamma}{T-1}\frac{(N-1)}{N_s-1}mK_m\right\}}_{\equiv S_m(K_m)} p + \underbrace{\frac{N_s}{N_s-1}\underline{p}}_{\equiv T_m}.$$

Therefore, $\pi_m(p)$ is affine in p on the open gap (a,b). Moreover,

$$S_m(K_m) = 0 \quad \Longleftrightarrow \quad K_m = \frac{\frac{T-1}{1-\gamma} + (N_s-1)\frac{m-1}{2}}{(N-1)m} = C_m.$$

Now, suppose instead that (a,b) is a gap in the single-establishment support. Then, $(a,b) \cap supp\{F_s\} = \varnothing$ and $(a,b) \subseteq supp\{F_m\}$. Since F_s has no mass points and has no support in (a,b),

there is a constant K_s such that $F_s(p) = K_s$, $\forall p \in (a,b)$. Since multi-establishment firms post in (a,b), from profit maximization and Proposition A.3, we have that

$$D_m(p)p = \left[1 - \frac{1-\gamma}{T-1}\left(\frac{m-1}{2}\right)\right]\underline{p}, \qquad p \in (a,b).$$

Then, combining with equation (A.12) and rearranging it, we have:

$$F_m(p) = \frac{1}{(N_m-1)m}\left\{\frac{T-1}{1-\gamma} - \left(\frac{m-1}{2} + N_s K_s\right) - \left(\frac{T-1}{1-\gamma} - \frac{m-1}{2}\right)\frac{\underline{p}}{p}\right\}. \tag{A.20}$$

Therefore, $F_m(p)$ is affine in $\frac{1}{p}$ on (a,b).

Now evaluate the single-establishment deviation payoff on (a,b). From the single-establishment firm profit function, substituting (A.20) and $F_s(p) = K_s$ and collecting the terms multiplying p, we obtain

$$\pi_s(p) = \underbrace{\frac{-1 + \frac{1-\gamma}{T-1}\left[N_m\frac{m-1}{2} + (N-1)K_s\right]}{N_m - 1}}_{\equiv S_s(K_s)} p + \underbrace{\frac{N_m}{N_m-1}\left[1 - \frac{1-\gamma}{T-1}\left(\frac{m-1}{2}\right)\right]\underline{p}}_{\equiv T_s}.$$

Hence $\pi_s(p)$ is affine in p on the open gap (a,b). Moreover,

$$S_s(K_s) = 0 \quad \Longleftrightarrow \quad K_s = \frac{\frac{T-1}{1-\gamma} - N_m\frac{m-1}{2}}{N-1} = C_s.$$

□

Proposition A.4 (No genuine internal open gaps)**.** *The support of equilibrium posted prices is connected for both single- and multi-establishment firms.*

Proof. In order to prove this proposition, we must show:

(a) There is **no** nonempty open interval $(a,b) \subset (\underline{p}, \overline{p})$ such that $(a,b) \cap \operatorname{supp}\{F_m\} = \varnothing$, with multi-establishment posted prices approaching a from the left and b from the right.

(b) There is **no** nonempty open interval $(a,b) \subset (\underline{p}, \overline{p})$ such that $(a,b) \cap \operatorname{supp}\{F_s\} = \varnothing$, with single-establishment posted prices approaching a from the left and b from the right.

We consider the limits to a and b in order to avoid assuming that the limits of the interval are necessarily posted by both types of firms. However, the proof is certainly valid for the case of closed supports.

We start with (a). Suppose, toward a contradiction, that multi-establishment firms have a genuine internal open gap (a,b). Thus $(a,b) \cap \operatorname{supp}\{F_m\} = \varnothing$. Moreover, there are sequences

$p_n^\ell \in \text{supp}\{F_m\}$, $p_n^r \in \text{supp}\{F_m\}$, $p_n^\ell \uparrow a$, $p_n^r \downarrow b$. From Lemma 2, every price in (a,b) is posted by the single-establishment firms, i.e., $(a,b) \subseteq P_s$.

Since F_m has no mass points and no multi-establishment probability is assigned to (a,b), there exists K_m such that $F_m(p) = K_m$, $p \in (a,b)$. By continuity of F_m, the same value is obtained as a one-sided limit at both endpoints of the gap.

First use the left endpoint a. Since $(a,b) \subseteq P_s$, for any sequence $q_n \downarrow a$ with $q_n \in (a,b)$, $D_s(q_n)q_n = \pi_s^*$. By continuity:

$$D_s(a)a = \pi_s^*. \tag{A.21}$$

Since $p_n^\ell \in P_m$ and $p_n^\ell \uparrow a$, $D_m(p_n^\ell)p_n^\ell = \pi_m^*$ for all n, and therefore continuity gives:

$$D_m(a)a = \pi_m^*. \tag{A.22}$$

Equations (A.21) and (A.22) are the two payoff-equalization equations at a. Lemma A.1 implies

$$K_m = F_m(a) = C_m\left(1 - \frac{p}{a}\right). \tag{A.23}$$

Next use the fact that the inactive type's payoff on the open gap is affine. By Lemma A.2, for $p \in (a,b)$,

$$\pi_m(p) = S_m(K_m)p + T_m.$$

The left sequence $p_n^\ell \uparrow a$ and the right sequence $p_n^r \downarrow b$ give, by payoff equalization and continuity,

$$\lim_{p \downarrow a} \pi_m(p) = \pi_m^*, \qquad \lim_{p \uparrow b} \pi_m(p) = \pi_m^*.$$

An affine function on (a,b) whose one-sided endpoint limits are equal at two distinct endpoints must be constant. Hence, we must have $S_m(K_m) = 0$. From the formula for $S_m(K_m)$ pinned down in Lemma A.2, we can show that $S_m(K_m) = 0$ is equivalent to

$$K_m = C_m. \tag{A.24}$$

Combining (A.23) and (A.24) yields

$$C_m = C_m\left(1 - \frac{p}{a}\right).$$

Since $C_m > 0$ and $a > \underline{p} > 0$, this is impossible. Thus multi-establishment firms cannot have a genuine internal open gap.

The proof of part (b) is analogous. Suppose instead that single-establishment firms have a genuine internal open gap (a,b), with single-establishment posted prices approaching a from

the left and b from the right. From Lemma 2, $(a,b) \subseteq \operatorname{supp}\{F_m\}$. Since F_s has no mass points and no single-establishment probability is assigned to (a,b), there is K_s such that $F_s(p) = K_s$, $p \in (a,b)$. At the left endpoint a, payoff equalization for multi-establishment firms along interior prices and payoff equalization for single-establishment firms along the left sequence imply, by continuity,

$$D_m(a)a = \pi_m^*, \qquad D_s(a)a = \pi_s^*.$$

Lemma A.1 therefore gives

$$K_s = F_s(a) = C_s\left(1 - \frac{p}{a}\right). \tag{A.25}$$

By Lemma A.2, the single-establishment deviation payoff is affine on (a,b), i.e., $\pi_s(p) = S_s(K_s)p + T_s$. Payoff equalization for single-establishment firms along sequences approaching both sides of the gap implies that this affine function has the same one-sided endpoint limit at a and at b. Therefore it is constant, so $S_s(K_s) = 0$.

By Lemma A.2, this is equivalent to

$$K_s = C_s. \tag{A.26}$$

Combining (A.25) and (A.26) yields

$$C_s = C_s\left(1 - \frac{p}{a}\right),$$

which is impossible because $C_s > 0$ and $a > \underline{p} > 0$. Thus single-establishment firms also cannot have a genuine internal open gap. □

Finally, we are ready to pin down the equilibrium given that the support of equilibrium price distributions is connected for both single- and multi-establishment firms.

A.3 Proof of Theorem 1

Proof. From Proposition A.3 we must have $\underline{p}_s = \underline{p}_m = \underline{p}$. Moreover, from Corollary A.2, $\overline{p} = \overline{p}_m > \overline{p}_s$. Finally, from Proposition A.4, we have that supports for F_s and F_m are connected. It is easy to see that $\overline{p} = R$, since otherwise a firm posting $\overline{p}$ has a profitable deviation to R. Hence, the supports for F_s and F_m are given by:

$$\operatorname{supp}\{F_s\} = \left[\underline{p}, \overline{p}_s\right] \qquad \text{and} \qquad \operatorname{supp}\{F_m\} = \left[\underline{p}, R\right], \text{ where } \underline{p} < \overline{p}_s < R \tag{A.27}$$

We start by considering multi-establishment posted prices $p \in [\overline{p}_s, R]$. From profit maxi-

mization, we have:

$$\left\{ \begin{array}{c} T-1-(1-\gamma)0.5(m-1) \\ -(1-\gamma)\left[N_s+(N_m-1)mF_m(p)\right] \end{array} \right\} p = \left\{\gamma(T-1)+(1-\gamma)0.5(m-1)\right\} R$$

then:

$$F_m(p) = \frac{1}{(1-\gamma)(N_m-1)m} \left\{ \begin{array}{c} \gamma N_s + N_m m - 1 - (1-\gamma)0.5(m-1) \\ -\left[\gamma(T-1)+(1-\gamma)0.5(m-1)\right]\frac{R}{p} \end{array} \right\}$$

rearranging it:

$$F_m(p) = 1 - \left[\frac{\gamma(T-1)+(1-\gamma)0.5(m-1)}{(1-\gamma)(N_m-1)m}\right]\left(\frac{R}{p}-1\right) \tag{A.28}$$

Next, consider the common support segment $\left[\underline{p}, \overline{p}_s\right]$. From Lemma A.1, for $p \in \left[\underline{p}, \overline{p}_s\right]$ we have:

$$F_s(p) = \left(\frac{\frac{T-1}{1-\gamma} - N_m\frac{m-1}{2}}{N-1}\right)\left(1-\frac{\underline{p}}{p}\right) \tag{A.29}$$

and

$$F_m(p) = \left(\frac{\frac{T-1}{1-\gamma} + (N_s-1)\frac{m-1}{2}}{m(N-1)}\right)\left(1-\frac{\underline{p}}{p}\right) \tag{A.30}$$

From $F_s(\overline{p}_s) = 1$, we have:

$$\overline{p}_s = \left(\frac{T-1-(1-\gamma)\frac{N_m(m-1)}{2}}{\gamma(N_s-1)+N_m m-(1-\gamma)\frac{N_m(m+1)}{2}}\right)\underline{p} \tag{A.31}$$

Then, from profit equalization $D_m(\underline{p})\underline{p} = D_m(R)R$, using equation (8), we have:

$$\underline{p} = \left[\frac{\gamma\,(T-1)+\frac{(1-\gamma)}{2}\,(m-1)}{N_s+N_m m-1-\frac{(1-\gamma)}{2}\,(m-1)}\right] R \tag{A.32}$$

We now consider the incentive of single-establishment firms to deviate. A single-establishment firm will not deviate and post $p \in (\overline{p}_s, R]$ if:

$$\left\{ \begin{array}{c} \gamma(N_s-1)+N_m m \\ -(1-\gamma)N_m m F_m(p) \end{array} \right\} p < \left\{ \begin{array}{c} \gamma(N_s-1)+N_m m \\ -(1-\gamma)N_m m F_m(\overline{p}_s) \end{array} \right\} \overline{p}_s$$

since:

$$(1-\gamma)N_m m F_m(p) = \frac{N_m}{(N_m-1)} \left\{ \begin{array}{c} \gamma(N_s-1)+N_m m-(1-\gamma)0.5(m+1) \\ -\left[\gamma(T-1)+(1-\gamma)0.5(m-1)\right]\frac{R}{p} \end{array} \right\}$$

substituting back, we have:

$$\left\{\begin{array}{c} -[\gamma(N_s-1)+N_m m-(1-\gamma)0.5N_m(m+1)] \\ +N_m\left[\gamma(T-1)+(1-\gamma)0.5(m-1)\right]\frac{R}{p} \end{array}\right\} p$$
$$< \left\{\begin{array}{c} -[\gamma(N_s-1)+N_m m-(1-\gamma)0.5N_m(m+1)] \\ +N_m\left[\gamma(T-1)+(1-\gamma)0.5(m-1)\right]\frac{R}{\overline{p}_s} \end{array}\right\} \overline{p}_s$$

simplifying it:

$$\begin{array}{c} [\gamma(N_s-1)+N_m(m-(1-\gamma)0.5m-(1-\gamma)0.5)]p \\ > [\gamma(N_s-1)+N_m(m-(1-\gamma)0.5m-(1-\gamma)0.5)]\overline{p}_s \end{array} \tag{A.33}$$

notice that:

$$m-(1-\gamma)0.5m-(1-\gamma)0.5=\gamma m+(1-\gamma)0.5(m-1)>0.$$

Then, given that $p > \overline{p}_s$, inequality in (A.33) is always satisfied. Therefore, single-establishment firms have no incentive to deviate.

In contrast, multi-establishment profitable deviations inside the support are ruled out by profit equalization on their full support. It is easy to see that deviations to $p < \underline{p}$ or $p > R$ are trivially sub-optimal.

Substituting the contact-share formula $\gamma = (1-\alpha)/(1+\alpha)$ from (2) gives the following equivalent representation in terms of the primitive sampling probability α.

$$\underline{p}=\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}R$$

and

$$\overline{p}_s=\left[\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right]\left[\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right]R.$$

while the distributions become:

$$F_s(p)=\frac{T-1+\alpha(N-1)}{2\alpha(N-1)}\left(1-\frac{\underline{p}}{p}\right), \qquad p\in[\underline{p},\overline{p}_s].$$

and

$$F_m(p)=\begin{cases}\dfrac{T-1+\alpha m(N-1)}{2\alpha m(N-1)}\left(1-\dfrac{\underline{p}}{p}\right), & p\in[\underline{p},\overline{p}_s], \\ 1-\dfrac{T-1-\alpha(T-m)}{2\alpha(N_m-1)m}\left(\dfrac{R}{p}-1\right), & p\in(\overline{p}_s,R].\end{cases}$$

The preceding support results apply to every equilibrium: the two types have the common lower endpoint, connected supports, and the multi-establishment support reaches R while the single-establishment support ends below it. On the common-support segment, the two

payoff-equalization equations uniquely determine F_s and F_m; on $(\overline{p}_s, R]$, multi-establishment payoff equalization uniquely determines the second branch of F_m. The condition $F_s(\overline{p}_s) = 1$ uniquely determines $\overline{p}_s$ relative to $\underline{p}$, while the endpoint profit equality $D_m(\underline{p})\underline{p} = D_m(R)R$ uniquely determines $\underline{p}$ relative to R. Hence the equilibrium just constructed is the unique Nash equilibrium conditional on R. □

A.4 Proof of Theorem 2

Proof. To pin down the consumer's reservation price R, we use equation (18). Hence, we must pin down $\mathbb{E}[P_R]$. In particular, $\mathbb{E}[P_R]$ is defined by:

$$\mathbb{E}[P_R] = \left\{ \begin{array}{c} \left[(1-\alpha)\nu_{m\bullet} + \alpha\nu_{mm}\frac{m-1}{N_m m-1}\right]\mathbb{E}[P_m] + (1-\alpha)\nu_{s\bullet}\mathbb{E}[P_s] \\ +\alpha\nu_{ss}\mathbb{E}[\min\{P_{s1}, P_{s2}\}] + \alpha\nu_{sm}\mathbb{E}[\min\{P_s, P_m\}] \\ +\alpha\nu_{mm}\frac{(N_m-1)m}{N_m m-1}\mathbb{E}[\min\{P_{m,1}, P_{m,2}\}] \end{array} \right\} \tag{A.34}$$

where:

- $\nu_{m\bullet} \equiv \frac{N_m m}{T}$: Probability the consumer observes the price from a multi-establishment firm
- $\nu_{s\bullet} \equiv \frac{N_s}{T}$: Probability the consumer observes the price from a single-establishment firm.

Conditional on receiving two quotes,

- $\nu_{ss} \equiv \frac{N_s(N_s-1)}{T(T-1)}$: Probability the consumer observes only prices from single-establishment firms;
- $\nu_{sm} \equiv \frac{2N_s N_m m}{T(T-1)}$: Probability the consumer observes one price of each type;
- $\nu_{mm} \equiv \frac{N_m m(N_m m-1)}{T(T-1)}$: Probability the consumer observes only prices from multi-establishment firms.

Conditional on two multi-establishment quotes,

- $\frac{m-1}{N_m m-1}$: Probability the consumer observes two price quotes from the same firm,
- $\frac{(N_m-1)m}{N_m m-1}$: Probability the consumer observes two prices from distinct multi-establishment firms.

Therefore, we must pin down the expected price in each case presented in equation (A.34). Then, the expected price from one single-establishment quote is:

$$\mathbb{E}[P_s] = \int_{\underline{p}}^{\overline{p}_s} p\, f_s(p)dp = \overline{p}_s - \left(\frac{T-1+\alpha(N-1)}{2\alpha(N-1)}\right)(\overline{p}_s - \underline{p}) + \left(\frac{T-1+\alpha(N-1)}{2\alpha(N-1)}\right)\underline{p}\ln\left(\frac{\overline{p}_s}{\underline{p}}\right)$$

notice that:

$$\begin{aligned}\overline{p}_s - \underline{p} &= \left[1 - \left(\tfrac{T-1-\alpha(N-1)}{T-1+\alpha(N-1)}\right)\right]\overline{p}_s \\ &= \tfrac{2\alpha(N-1)}{T-1+\alpha(N-1)}\overline{p}_s\end{aligned}$$

hence:

$$\overline{p}_s - \left(\frac{T-1+\alpha(N-1)}{2\alpha(N-1)}\right)(\overline{p}_s - \underline{p}) = 0$$

as a result:

$$\mathbb{E}[P_s] = \left(\frac{T-1+\alpha(N-1)}{2\alpha(N-1)}\right)\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}R\ln\left(\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right) \tag{A.35}$$

Next, consider the expected price from one multi-establishment quote:

$$\mathbb{E}[P_m] = \int_{\underline{p}}^{R} p\, f_m(p)dp$$

since

$$F_m(p) = \begin{cases} \dfrac{T-1+\alpha m(N-1)}{2\alpha m(N-1)}\left(1 - \dfrac{\underline{p}}{p}\right), & p \in [\underline{p}, \overline{p}_s], \\ 1 - \dfrac{T-1-\alpha(T-m)}{2\alpha(N_m-1)m}\left(\dfrac{R}{p} - 1\right), & p \in (\overline{p}_s, R]. \end{cases}$$

As a result, we must split the integration in two before solving through integration by parts:

$$\int_{\underline{p}}^{\overline{p}_s} p\, f_m(p)dp = \left\{\begin{array}{c} \overline{p}_s\left(\frac{T-1+\alpha m(N-1)}{2\alpha m(N-1)}\right)\left(1 - \frac{\underline{p}}{\overline{p}_s}\right) \\ -\left(\frac{T-1+\alpha m(N-1)}{2\alpha m(N-1)}\right)(\overline{p}_s - \underline{p}) \\ +\left(\frac{T-1+\alpha m(N-1)}{2\alpha m(N-1)}\right)\underline{p}\ln\left(\frac{\overline{p}_s}{\underline{p}}\right) \end{array}\right\}$$

simplifying it:

$$\int_{\underline{p}}^{\overline{p}_s} p\, f_m(p)dp = \left(\frac{T-1+\alpha m(N-1)}{2\alpha m(N-1)}\right)\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}R\ln\left(\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right)$$

and

$$\int_{\overline{p}_s}^{R} p\, f_m(p)dp = \left\{\begin{array}{c} R - \left[1 - \frac{T-1-\alpha(T-m)}{2\alpha(N_m-1)m}\left(\frac{R}{\overline{p}_s} - 1\right)\right]\overline{p}_s \\ -\left[1 + \frac{T-1-\alpha(T-m)}{2\alpha(N_m-1)m}\right](R - \overline{p}_s) \\ +\left(\frac{T-1-\alpha(T-m)}{2\alpha(N_m-1)m}\right)R\ln\left(\frac{R}{\overline{p}_s}\right) \end{array}\right\}$$

simplifying it, we have:

$$\int_{\overline{p}_s}^{R} p\, f_m(p)dp = \left(\frac{T-1-\alpha(T-m)}{2\alpha(N_m-1)m}\right) R\left[\ln\left(\frac{T-1-\alpha(N-1)}{T-1+\alpha(N-1)}\right)+\ln\left(\frac{T-1+\alpha(T-m)}{T-1-\alpha(T-m)}\right)\right]$$

putting everything together, we have:

$$\mathbb{E}[P_m] = \left\{ \begin{array}{c} \left(\frac{T-1+\alpha m(N-1)}{2\alpha m(N-1)}\right)\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\ln\left(\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right) \\ +\left(\frac{T-1-\alpha(T-m)}{2\alpha(N_m-1)m}\right)\left[\ln\left(\frac{T-1-\alpha(N-1)}{T-1+\alpha(N-1)}\right)+\ln\left(\frac{T-1+\alpha(T-m)}{T-1-\alpha(T-m)}\right)\right] \end{array} \right\} R \tag{A.36}$$

Now consider the case in which the consumer observes multiple offers. The distribution of the minimum price of two offers from single-establishment firms is:

$$G_{ss}(p|R) = 1-\left(1-F_s(p|R)\right)^2 = \left[1-\left(1-F_s(p|R)\right)\right]\left[1+\left(1-F_s(p|R)\right)\right],$$

therefore $G_{ss}(p|R) = F_s(p|R)\left[2-F_s(p|R)\right]$. Consequently,

$$\int_{\underline{p}}^{\overline{p}_s} G_{ss}(p \mid R)\, dp = R\left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right)\times\left[\begin{array}{c} \frac{[T-1+\alpha(N-1)][2\alpha(N-1)-(T-1)]}{\alpha(N-1)[T-1-\alpha(N-1)]} \\ +\frac{(T-1)^2-\alpha^2(N-1)^2}{2\alpha^2(N-1)^2}\ln\left(\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right) \end{array}\right]$$

as a result:

$$\mathbb{E}[\min\{P_{s1},P_{s2}\}] = \overline{p}_s - \int_{\underline{p}}^{\overline{p}_s} G_{ss}(p \mid R)\, dp$$

Substituting the expressions, we have:

$$\mathbb{E}[\min\{P_{s1},P_{s2}\}] = R\left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right)\left(\frac{(T-1)^2-\alpha^2(N-1)^2}{\alpha(N-1)}\right)\left[\begin{array}{c} \frac{1}{T-1-\alpha(N-1)} \\ -\frac{1}{2\alpha(N-1)}\ln\left(\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right) \end{array}\right] \tag{A.37}$$

Now consider the case in which the consumer receives offers from one single-establishment and one multi-establishment firm. In this case:

$$\mathbb{E}[\min\{P_s,P_m\}] = \int_{\underline{p}}^{\overline{p}_s} p\, dG_{sm}(p)$$

Notice that $G_{sm}(p \mid R) = 1-[1-F_s(p)][1-F_m(p)]$. Therefore, $G_{sm}(p \mid R) = F_m(p)+F_s(p)-F_m(p)F_s(p)$. Moreover, notice that the support is $[\underline{p},\overline{p}_s]$, since the highest price posted by a

single-establishment firm is $\overline{p}_s$. As a result:

$$\mathbb{E}[\min\{P_s, P_m\}] = \int_{\underline{p}}^{\overline{p}_s} p \, dG_{sm}(p) = \overline{p}_s - \int_{\underline{p}}^{\overline{p}_s} G_{sm}(p) \, dp$$

and

$$\int_{\underline{p}}^{\overline{p}_s} G_{sm}(p) \, dp = \int_{\underline{p}}^{\overline{p}_s} F_m(p) \, dp + \int_{\underline{p}}^{\overline{p}_s} F_s(p) \, dp - \int_{\underline{p}}^{\overline{p}_s} F_s(p) F_m(p) \, dp$$

solving the integrals, we have:

$$\int_{\underline{p}}^{\overline{p}_s} G_{sm}(p) \, dp = R \left(\frac{T - 1 - \alpha(T-m)}{T - 1 + \alpha(T-m)} \right) \left[\begin{array}{c} \frac{2\alpha(N-1)}{T-1-\alpha(N-1)} - \frac{T-1}{\alpha m(N-1)} \\ + \frac{(T-1)^2 - m\alpha^2(N-1)^2}{2\alpha^2 m(N-1)^2} \ln\left(\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)} \right) \end{array} \right].$$

Therefore:

$$\mathbb{E}[\min\{P_s, P_m\}] = R \left(\frac{T - 1 - \alpha(T-m)}{T - 1 + \alpha(T-m)} \right) \left[\begin{array}{c} \frac{T-1+\alpha m(N-1)}{\alpha m(N-1)} \\ - \frac{(T-1)^2 - m\alpha^2(N-1)^2}{2\alpha^2 m(N-1)^2} \ln\left(\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)} \right) \end{array} \right] \tag{A.38}$$

Finally, consider the case in which the consumer faces two distinct prices from multi-establishment firms. In this case,

$$\mathbb{E}[\min\{P_{m,1}, P_{m,2}\}] = \int_{\underline{p}}^{R} p \, dG_{mm}(p) = R - \int_{\underline{p}}^{R} G_{mm}(p) \, dp \tag{A.39}$$

where $G_{mm}(p|R) = 1 - \left(1 - F_m(p|R)\right)^2 = \left[1 - \left(1 - F_m(p|R)\right)\right] \left[1 + \left(1 - F_m(p|R)\right)\right]$. Therefore $G_{mm}(p|R) = F_m(p|R) \left[2 - F_m(p|R)\right]$. As a result,

$$\int_{\underline{p}}^{R} G_{mm}(p|R) \, dp = 2 \int_{\underline{p}}^{R} F_m(p|R) \, dp - \int_{\underline{p}}^{R} F_m(p|R)^2 \, dp$$

Solving the integrals, we obtain:

$$\int_{\underline{p}}^{R} G_{mm}(p \mid R)\, dp = R\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\left\{\begin{array}{c} \frac{T-1+\alpha m(N-1)}{\alpha m^2(N-1)}\left[\begin{array}{c}\frac{\alpha m(N-1)\left[T-1+2\alpha(N-1)\right]-(T-1)^2}{(T-1)^2-\alpha^2(N-1)^2} \\ +\frac{T-1-\alpha m(N-1)}{2\alpha(N-1)}\ln\left(\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right)\end{array}\right] \\ +\frac{2\alpha(T-1)(N_m-1)(m-1)}{\left[T-1-\alpha(T-m)\right]\left[T-1-\alpha(N-1)\right]} \\ -\frac{(T-1)^2-\alpha^2(T-m)^2}{4\alpha^2(N_m-1)^2m^2}\left[\begin{array}{c}\left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right)^{-1} \\ -\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)} \\ +2\ln\left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right)\end{array}\right]\end{array}\right\}.$$

as a result:

$$\mathbb{E}[\min\{P_{m,1}, P_{m,2}\}] = R\left\{\begin{array}{c}\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\left[\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right. \\ -\frac{T-1+\alpha m(N-1)}{\alpha m^2(N-1)}\left(\frac{\alpha m(N-1)\left[T-1+2\alpha(N-1)\right]-(T-1)^2}{(T-1)^2-\alpha^2(N-1)^2}\right. \\ \left.\left.+\frac{T-1-\alpha m(N-1)}{2\alpha(N-1)}\ln\left(\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right)\right)\right] \\ +\frac{\left[T-1-\alpha(T-m)\right]^2}{4\alpha^2(N_m-1)^2m^2}\left[\left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right)^{-1}\right. \\ -\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)} \\ \left.+2\ln\left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right)\right]\end{array}\right\}.$$

Putting everything together, we have:

$$\mathbb{E}[P_R] = \left\{\begin{array}{c}\left[(1-\alpha)\nu_{m\bullet}+\alpha\nu_{mm}\frac{m-1}{N_m m-1}\right]\mathbb{E}[P_m]+(1-\alpha)\nu_{s\bullet}\mathbb{E}[P_s] \\ +\alpha\nu_{ss}\mathbb{E}[\min\{P_{s1}, P_{s2}\}]+\alpha\nu_{sm}\mathbb{E}[\min\{P_s, P_m\}] \\ +\alpha\nu_{mm}\frac{(N_m-1)m}{N_m m-1}\mathbb{E}[\min\{P_{m,1}, P_{m,2}\}]\end{array}\right\} \tag{A.40}$$

Substituting the expressions and simplifying, we obtain:

$$\mathbb{E}[P_R] = R\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\left[1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)}\right]. \tag{A.41}$$

Moreover, since Ω is given by:

$$\Omega = 1 - \frac{\mathbb{E}[P_R]}{R}$$

we have that:

$$\Omega = 1 - \frac{T - 1 - \alpha(T - m)}{T - 1 + \alpha(T - m)} \left[1 + \alpha - \frac{\alpha N_m m(m-1)}{T(T-1)} \right]. \tag{A.42}$$

Thus the unconstrained interior cutoff is $Q = c/\Omega$. Since firms never transact above z, the effective cutoff faced by firms is $R = \min\{Q, z\}$. Therefore

$$R = \min\left\{ z, \frac{c}{\Omega} \right\},$$

which proves both the interior and capped branches stated in Theorem 2. □

Proof of Corollary 1

Proof. Consider first prices on the common support $p \in [\underline{p}, \overline{p}_s]$. From (11) and the first branch of (14),

$$\begin{aligned} F_s(p) - F_m(p) &= \left[\frac{T - 1 + \alpha(N-1)}{2\alpha(N-1)} - \frac{T - 1 + \alpha m(N-1)}{2\alpha m(N-1)} \right] \left(1 - \frac{\underline{p}}{p} \right) \\ &= \frac{(T-1)(m-1)}{2\alpha m(N-1)} \left(1 - \frac{\underline{p}}{p} \right). \end{aligned}$$

Since $m > 1$, this expression is strictly positive for every $p \in (\underline{p}, \overline{p}_s]$, while both cumulative distributions are equal to zero at $p = \underline{p}$.

Now consider $p \in (\overline{p}_s, R)$. Because the support of the single-establishment firms ends at $\overline{p}_s$, $F_s(p) = 1$. From the second branch of (14),

$$F_m(p) = 1 - \frac{T - 1 - \alpha(T - m)}{2\alpha(N_m - 1)m} \left(\frac{R}{p} - 1 \right) < 1,$$

where the inequality follows from $p < R$. Hence $F_m(p) < F_s(p)$ throughout $(\overline{p}_s, R)$.

Finally, for $p < \underline{p}$, both cumulative distributions are zero, while for $p \geq R$, both are equal to one. Therefore, $F_m(p) \leq F_s(p)$ for all p, with strict inequality for $p \in (\underline{p}, R)$. □

Proof of Proposition 2

Proof. We start by showing (i). Notice that:

$$\frac{\partial R}{\partial N_s} = -\frac{c}{\Omega^2} \times \frac{\partial \Omega}{\partial N_s} \tag{A.43}$$

As a result, we can focus on $\frac{\partial \Omega}{\partial N_s}$. Based on the product rule applied to equation (20), we have that:

$$\frac{\partial \Omega}{\partial N_s} = \left\{ \begin{array}{c} \frac{2\alpha(m-1)}{[T-1+\alpha(T-m)]^2}\left[1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)}\right] \\ -\left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right)\frac{\alpha N_m m(m-1)(2T-1)}{T^2(T-1)^2} \end{array} \right\} \tag{A.44}$$

Therefore, $\frac{\partial \Omega}{\partial N_s} > 0$ if:

$$\frac{2\alpha(m-1)}{[T-1+\alpha(T-m)]^2}\left[1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)}\right] > \left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right)\frac{\alpha N_m m(m-1)(2T-1)}{T^2(T-1)^2}$$

simplifying it:

$$2\left[1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)}\right] > \frac{[T-1-\alpha(T-m)][T-1+\alpha(T-m)]N_m m(2T-1)}{T^2(T-1)^2} \tag{A.45}$$

From the left-hand side of (A.45) (LHS(A.45)):

$$1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)} = 1+\alpha\left[1-\frac{N_m m(m-1)}{T(T-1)}\right] > 1 \tag{A.46}$$

since $N_m m < T$ and $m-1 < T-1$, as $T = N_s + N_m m$, $N_s > 1$, $N_m \geq 1$.

From the right-hand side of (A.45) (RHS(A.45)):

$$[T-1-\alpha(T-m)][T-1+\alpha(T-m)] = (T-1)^2 - \alpha^2(T-m)^2 < (T-1)^2.$$

Hence, for RHS(A.45), we have:

$$\frac{[T-1-\alpha(T-m)][T-1+\alpha(T-m)]N_m m(2T-1)}{T^2(T-1)^2} < \frac{N_m m(2T-1)}{T^2}$$

since $N_m m < T$:

$$\frac{N_m m(2T-1)}{T^2} < \frac{2T-1}{T} < 2.$$

Therefore, LHS(A.45)>2 and RHS(A.45)<2. As a result, the inequality in (A.45) is always satisfied. Hence $\frac{\partial \Omega}{\partial N_s} > 0$. Given (A.43), we have $\frac{\partial R}{\partial N_s} < 0$.

Now consider (ii). Based on the product rule applied to equation (20), we have that:

$$\frac{\partial \Omega}{\partial N_m} = \left\{ \begin{array}{c} \frac{2\alpha m(m-1)}{[T-1+\alpha(T-m)]^2}\left[1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)}\right] \\ +\left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right)\frac{\alpha m(m-1)[T(T-1)-N_m m(2T-1)]}{T^2(T-1)^2} \end{array} \right\} \tag{A.47}$$

notice that:

$$T(T-1) - N_m m(2T-1) = N_s(N_s-1) - (N_m m)^2$$

notice that the term in the second row of the RHS curly brackets has an ambiguous sign. If $N_s(N_s-1) - (N_m m)^2 > 0$, then both terms on the RHS are positive and we have $\frac{\partial \Omega}{\partial N_m} > 0$.

In contrast, if we have $N_s(N_s-1) - (N_m m)^2 < 0$, $\frac{\partial \Omega}{\partial N_m} > 0$ if:

$$\frac{2\alpha m(m-1)}{[T-1+\alpha(T-m)]^2}\left[1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)}\right] > \left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right)\frac{\alpha m(m-1)[N_m m(2T-1)-T(T-1)]}{T^2(T-1)^2}$$

simplifying it:

$$2\left[1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)}\right] > \frac{[(T-1)^2-\alpha^2(T-m)^2][N_m m(2T-1)-T(T-1)]}{T^2(T-1)^2} \tag{A.48}$$

notice that LHS(A.48) is the same as LHS(A.45). Consequently, we know that LHS(A.48)>2. Now, from RHS(A.48), we have:

$$\frac{[(T-1)^2-\alpha^2(T-m)^2][N_m m(2T-1)-T(T-1)]}{T^2(T-1)^2} < \frac{(2T-1)-(T-1)}{T} = 1$$

therefore LHS(A.48)>RHS(A.48), and the inequality is always satisfied.

Hence, $\frac{\partial \Omega}{\partial N_m} > 0$. Given an analogous relationship to (A.43) for N_m, we have $\frac{\partial R}{\partial N_m} < 0$.

Finally, we look into (iii). Based on the product rule applied to equation (20), we have:

$$\frac{\partial \Omega}{\partial m} = \left\{ \begin{array}{c} -\frac{2\alpha(N_s+N_m-1)}{[T-1+\alpha(T-m)]^2}\left[1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)}\right] \\ +\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\frac{\alpha N_m\left[\begin{array}{c}(2m-1)T(T-1)\\ -N_m m(m-1)(2T-1)\end{array}\right]}{T^2(T-1)^2} \end{array} \right\} \tag{A.49}$$

Notice that, if $(2m-1)T(T-1) - N_m m(m-1)(2T-1) \leq 0$, then the second term in (A.49) is non-positive. As we showed before, the first term in (A.49) is strictly negative. As a result, $\frac{\partial \Omega}{\partial m} < 0$.

Suppose instead that $(2m-1)T(T-1) - N_m m(m-1)(2T-1) > 0$. Then, $\frac{\partial \Omega}{\partial m} < 0$ if:

$$\frac{2\alpha(N_s+N_m-1)}{[T-1+\alpha(T-m)]^2}\left[1+\alpha-\frac{\alpha N_m m(m-1)}{T(T-1)}\right] > \frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\frac{\alpha N_m\left[\begin{array}{c}(2m-1)T(T-1)\\ -N_m m(m-1)(2T-1)\end{array}\right]}{T^2(T-1)^2}$$

simplifying it:

$$2(N_s + N_m - 1)\left[1 + \alpha - \frac{\alpha N_m m(m-1)}{T(T-1)}\right] > \left\{ \begin{array}{c} \frac{[T-1-\alpha(T-m)][T-1+\alpha(T-m)]}{T^2(T-1)^2} \\ \times N_m \left[\begin{array}{c} (2m-1)T(T-1) \\ -N_m m(m-1)(2T-1) \end{array} \right] \end{array} \right\} \tag{A.50}$$

As we showed before, $\left[1 + \alpha - \frac{\alpha N_m m(m-1)}{T(T-1)}\right] > 1$. As a result LHS(A.50)$> 2(N_s + N_m - 1)$. Similarly,

$$\left\{ \begin{array}{c} \frac{[T-1-\alpha(T-m)][T-1+\alpha(T-m)]}{T^2(T-1)^2} \\ \times N_m \left[\begin{array}{c} (2m-1)T(T-1) \\ -N_m m(m-1)(2T-1) \end{array} \right] \end{array} \right\} < \frac{N_m}{T^2} \left[\begin{array}{c} (2m-1)T(T-1) \\ -N_m m(m-1)(2T-1) \end{array} \right]$$

Consequently, the inequality in (A.50) is satisfied if:

$$2(N_s + N_m - 1) - \frac{N_m}{T^2} \left[\begin{array}{c} (2m-1)T(T-1) \\ -N_m m(m-1)(2T-1) \end{array} \right] > 0 \tag{A.51}$$

Using $T = N_s + N_m m$, the left-hand side of (A.51) can be expanded as

$$\begin{aligned} &N_m^2 m^2 (N_m - 1) + 2N_m N_s m(2N_m - 1) + 2N_m N_s^2 m \\ &\quad + N_m N_s (3N_s - 1) + 2N_s^2 (N_s - 1). \end{aligned} \tag{A.52}$$

Every term in (A.52) is nonnegative under $N_s \geq 1$, $N_m \geq 1$, and $m \geq 2$, and the expression is strictly positive. As a result, we must have $\frac{\partial \Omega}{\partial m} < 0$. Given that $\frac{\partial R}{\partial m} = -\frac{c}{\Omega^2} \times \frac{\partial \Omega}{\partial m}$, we have that $\frac{\partial R}{\partial m} > 0$, concluding the proof. □

Proof of Corollary 3

Proof. From equation (18), we have that, in the case of an interior R, $\mathbb{E}[P_R] = R - c$. Since c is a constant, we have that $\frac{\partial \mathbb{E}[P_R]}{\partial x} = \frac{\partial R}{\partial x}$, for all $x \in \{N_s, N_m, m\}$, concluding the proof of part (1).

Then, from (12), we have that, for $x \in \{N_s, N_m, m\}$:

$$\frac{\partial p}{\partial x} = \frac{\partial \left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right)}{\partial x} R + \frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)} \frac{\partial R}{\partial x} \tag{A.53}$$

Given that $T = N_s + N_m m$, we have that:

$$\frac{\partial \left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right)}{\partial N_s} = -\frac{2\alpha(m-1)}{[T-1+\alpha(T-m)]^2} < 0$$

and

$$\frac{\partial\left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right)}{\partial N_m} = -\frac{2\alpha m(m-1)}{[T-1+\alpha(T-m)]^2} < 0$$

finally,

$$\frac{\partial\left(\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}\right)}{\partial m} = \frac{2\alpha(N-1)}{[T-1+\alpha(T-m)]^2} > 0.$$

Combining the results from these three partial derivatives with (A.53) and the results from Proposition 2, we obtain the results in part (2).

Finally, from equations (12) and (13), we have:

$$\overline{p}_s = \left[\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right]\underline{p} \tag{A.54}$$

Therefore, the partial derivative of $\overline{p}_s$ with respect to $x \in \{N_s, N_m, m\}$ is given by:

$$\frac{\partial \overline{p}_s}{\partial x} = \frac{\partial\left[\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right]}{\partial x}\underline{p} + \left[\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right]\frac{\partial \underline{p}}{\partial x} \tag{A.55}$$

then, notice that:

$$\frac{\partial\left[\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right]}{\partial N_m} = -\frac{2\alpha(N_s-1)(m-1)}{[T-1-\alpha(N-1)]^2} < 0.$$

since $N_s > 1$. Then, combining the result above, (A.55) and result 2, we have that $\frac{\partial \overline{p}_s}{\partial N_m} < 0$. In contrast:

$$\frac{\partial\left[\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right]}{\partial N_s} = \frac{2\alpha N_m(m-1)}{[T-1-\alpha(N-1)]^2} > 0$$

combining the result above, (A.55) and result 2, we have that $\frac{\partial \overline{p}_s}{\partial N_s}$ has an ambiguous sign. In fact, we can find numerical examples that corroborate the ambiguity. Similarly,

$$\frac{\partial\left[\frac{T-1+\alpha(N-1)}{T-1-\alpha(N-1)}\right]}{\partial m} = -\frac{2\alpha N_m(N-1)}{[T-1-\alpha(N-1)]^2} < 0$$

again, combining the result above, (A.55) and result 2, we have that $\frac{\partial \overline{p}_s}{\partial m}$ has an ambiguous sign. In fact, we can find numerical examples that corroborate the ambiguity.

□

Proof of Lemma 4

Proof. Notice that:

$$\frac{\partial \tau_m}{\partial N_s} = \frac{\alpha N_m (m-1)^2}{m\left[T-1+\alpha(N-1)\right]^2} > 0$$

and

$$\frac{\partial \tau_m}{\partial N_m} = -\frac{\alpha (N_s-1)(m-1)^2}{m\left[T-1+\alpha(N-1)\right]^2} < 0$$

Finally,

$$\frac{\partial \tau_m}{\partial m} = -\frac{(T-1)}{m^2\left[T-1+\alpha(N-1)\right]} - \frac{\alpha(N-1)N_m(m-1)}{m\left[T-1+\alpha(N-1)\right]^2} < 0$$

□

Proof of Lemma 5

Proof. We first show that F_s remains atomless. Suppose that F_s has an atom of size $\delta_s > 0$ at some price $p_0 > 0$, and let $\delta_m \geq 0$ denote the size of any atom of F_m at the same price. Because ties are divided equally, demand at p_0 is evaluated using the midpoint CDFs. By undercutting p_0 by an arbitrarily small amount, a single-establishment firm obtains the limiting demand gain

$$D_s(p_0^-) - D_s(p_0) = \frac{1-\gamma}{2(T-1)}\left[(N_s-1)\delta_s + m\delta_m\right] > 0,$$

where the strict inequality follows from $N_s > 1$ and $\delta_s > 0$.

The reduction in price can be made arbitrarily small, whereas the increase in demand remains discrete. Hence prices sufficiently close to p_0 from below yield strictly greater profit than p_0, which contradicts the positive probability assigned to p_0. An atom at zero is also impossible, since a sufficiently small positive price earns strictly positive profit. Therefore, F_s is atomless.

Now suppose that the sole multi-establishment firm has an atom of size $\delta_m > 0$ at a price p_0. For a general number of multi-establishment firms, the limiting demand gain from undercutting an atom is

$$D_m(p_0^-) - D_m(p_0) = \frac{1-\gamma}{2(N_s + N_m m - 1)}\left[N_s\delta_s + (N_m-1)m\delta_m\right].$$

Setting $N_m = 1$ and using the atomlessness of F_s, so that $\delta_s = 0$, gives $D_m(p_0^-) - D_m(p_0) = 0$.

Equivalently, when $N_m = 1$, the demand faced by each establishment of the sole multi-establishment firm is

$$D_m(p) = 1 - \frac{1-\gamma}{T-1}\left[\frac{m-1}{2} + N_s\widetilde{F}_s(p)\right],$$

which does not depend on F_m. The firm's own atom therefore creates no tie-related demand loss for the firm itself. Undercutting p_0 changes the price but produces no discrete increase in demand. As the deviation approaches p_0 from below, its profit approaches $p_0 D_m(p_0)$ from below rather than exceeding it.

Consequently, the step that excludes atoms of F_m in Lemma 1 relies essentially on $(N_m - 1)m > 0$ and cannot be applied when $N_m = 1$. Atomlessness of the sole multi-establishment firm's strategy is therefore not implied by Lemma 1. □

Lemma A.3 (No pure strategy for the sole multi-establishment firm)**.** *Let $N_m = 1$, $N_s > 1$, $m > 1$, and $0 < \gamma < 1$. In a pricing equilibrium conditional on R, the sole multi-establishment firm cannot post a pure price.*

Proof. Suppose, toward a contradiction, that the multi-establishment firm posts the pure price $P \in [0, R]$. By Proposition 1, all single-establishment firms use the same CDF F_s. The standard undercutting argument rules out atoms in the single-establishment strategy: an atom at a positive price can be profitably undercut because $N_s > 1$, while a zero-price atom yields zero profit and is dominated by a small positive price.

Against the pure multi-establishment price P, a single-establishment firm's demand away from P is

$$D_s(p) = \begin{cases} 1 - \frac{(1-\gamma)(N_s-1)}{T-1} F_s(p), & p < P, \\ 1 - \frac{(1-\gamma)m}{T-1} - \frac{(1-\gamma)(N_s-1)}{T-1} F_s(p), & p > P. \end{cases} \tag{A.56}$$

If the common single-establishment profit is π_s, payoff equalization therefore implies

$$F_s(p) = \frac{T-1}{(1-\gamma)(N_s-1)} \left(1 - \frac{\pi_s}{p}\right) \quad \text{on a mixing interval below } P, \tag{A.57}$$

and

$$F_s(p) = \frac{T-1}{(1-\gamma)(N_s-1)} \left(1 - \frac{(1-\gamma)m}{T-1} - \frac{\pi_s}{p}\right) \quad \text{on a mixing interval above } P. \tag{A.58}$$

The pure price P creates a downward jump of size $\frac{(1-\gamma)m}{T-1}$ in a single-establishment firm's demand.

Moreover, keep in mind that a single-establishment firm can always deviate and post a price R and obtain a profit of at least γR. As a result, $\pi_s \geq \gamma R$.

Because $\pi_s \geq \gamma R$ pins down the fringe profit and P generates the demand jump above, the position of P relative to γR and to the induced fringe support exhausts the following four

cases. In each we exhibit a profitable deviation, either for a single-establishment firm or for the multi-establishment firm, contradicting the pure price P.

Case 1: $P \leq \gamma R$. Single-establishment firms are unable to profitably mix below P: even a price immediately below P yields at most $P \leq \gamma R$, while posting R yields γR. Its upper endpoint must be R; otherwise, once $F_s = 1$, demand is the constant γ and a higher price is profitable. Hence $\pi_s = \gamma R$. Equation (A.58) then gives the support of $F_s(\cdot)$ as:

$$\left[\frac{(T-1)\gamma R}{T-1-(1-\gamma)m}, R\right]. \tag{A.59}$$

Since $\frac{(T-1)}{T-1-(1-\gamma)m}\gamma R > \gamma R \geq P$, the multi-establishment firm faces the same demand at every price between P and $\frac{(T-1)}{T-1-(1-\gamma)m}\gamma R$. Raising its price toward $\frac{(T-1)}{T-1-(1-\gamma)m}\gamma R$ is therefore strictly profitable, a contradiction.

Case 2: $\gamma R < P < \frac{T-1}{T-1-(1-\gamma)(N_s-1)}\gamma R$.

Start by assuming that all single-establishment firms post a price above P. Hence, we must have that $\overline{p}_s = R$ and $\pi_s = \gamma R$. Again, by the proof of Lemma 5, F_s is continuous without mass points. As a result, the support of the $F_s(\cdot)$ is again given by equation (A.59) and the multi-establishment firm has an optimal deviation, as discussed in Case 1.

Then consider the case in which the entire support of $F_s(\cdot)$ is below P. In this case, at $\overline{p}_s$, we have that $D_s(\overline{p}_s) = 1 - \frac{(1-\gamma)(N_s-1)}{T-1}$, so profits are increasing in $\overline{p}_s$. So we must have $\overline{p}_s = P^-$. But then, profits are given by:

$$\left[1 - \frac{(1-\gamma)(N_s-1)}{T-1}\right] P^- < \underbrace{\left[1 - \frac{(1-\gamma)(N_s-1)}{T-1}\right] \frac{T-1}{T-1-(1-\gamma)(N_s-1)}}_{=1} \gamma R$$

where the inequality comes from $P < \frac{T-1}{T-1-(1-\gamma)(N-1)}\gamma R$ in Case 2. But then, $\pi_s < \gamma R$ and single-establishment firms have an optimal deviation.

Consequently, we must have that the support of $F_s(\cdot)$ must be such that $\underline{p} < P < \overline{p}_s$. As a result, we must have $\overline{p}_s = R$ and $\pi_s = \gamma R$. But then, a single-establishment firm posting $\underline{p}$ sells the good for sure. As a result, from profit equalization we must have $\underline{p} = \gamma R$. But then, as shown in equation (A.56), $D_s(\cdot)$ has a downward discontinuity at P. Since all posted prices in the support of $F_s(\cdot)$ must deliver the same profit, we must have a gap around P. In other words, we should have a gap between $\overline{p}_{s_1}$ and $\overline{p}_{s_2}$. It is easy to see that $\overline{p}_{s_1} = P^-$; otherwise, single-establishment firms would have an optimal deviation. $\overline{p}_{s_2}$ is pinned down by:

$$\left[1 - \frac{(1-\gamma)m}{T-1} - \frac{(1-\gamma)(N_s-1)}{T-1}\frac{T-1}{(1-\gamma)(N_s-1)}\left(1 - \frac{\gamma R}{P}\right)\right]\overline{p}_{s_2} = \gamma R$$

simplifying it, we have:

$$\overline{p}_{s_2} = \frac{\gamma R(T-1)}{\gamma R(T-1) - (1-\gamma)mP} P \tag{A.60}$$

hence $\overline{p}_{s_2} > P$. Therefore, there is a gap $(P, \overline{p}_{s_2})$ in the support of $F_s(\cdot)$. Hence, on the gap $(P, \overline{p}_{s_2})$, F_s is constant. Since the multi-establishment firm's demand depends on F_s but not on its own price, it can raise its price from P to any price in $(P, \overline{p}_{s_2})$ without losing demand. This is a strictly profitable deviation, so P cannot be an optimal strategy.

Case 3: $P > \frac{T-1}{T-1-(1-\gamma)(N_s-1)}\gamma R$. Notice that at $P > \frac{T-1}{T-1-(1-\gamma)(N_s-1)}\gamma R$, a single-establishment firm posting a price $p = P^-$ makes a profit strictly larger than γR. As a result, $F_s(\cdot)$'s support should have $\overline{p}_s \leq P^-$. Hence, $F_s(\cdot)$ would have a support $[\underline{p}, P^-]$, where $F_s(\underline{p}) = 0$. From (A.57) and profit equalization, we have:

$$\underline{p} = \left(\gamma + \frac{(1-\gamma)m}{T-1}\right) P^-$$

Consider the problem of the multi-establishment firm. Posting P, its profit is given by:

$$\pi_m = \left(\gamma + \frac{(1-\gamma)(m-1)}{2(T-1)}\right) P \tag{A.61}$$

Now consider the deviation in which the multi-establishment firm posts $\underline{p}^-$. In this case, multi-establishment profit is given by:

$$\pi_m^d = \left(\gamma + \frac{(1-\gamma)(m-1)}{2(T-1)} + \frac{1-\gamma}{T-1} N_s\right) \underline{p}^-$$

the benefit of the deviation is given by:

$$\pi_d - \pi = \left(\gamma + \frac{(1-\gamma)(m-1)}{2(T-1)}\right) (\underline{p} - P) + \frac{1-\gamma}{T-1} N_s \underline{p}$$

substituting $\underline{p}$ and rearranging, we have:

$$\pi_d - \pi = \frac{1-\gamma}{2(T-1)^2}\Big[N_s m + N_s + m - 1 - \gamma(N_s-1)(m-1)\Big] P^- > 0$$

where the bracketed term is strictly positive because $\gamma < 1$ implies $\gamma(N_s-1)(m-1) < (N_s-1)(m-1) = N_s m - N_s - m + 1$, so the bracket exceeds $2N_s + 2m - 2 > 0$. Therefore the deviation is optimal.

Case 4: $P = \frac{T-1}{T-1-(1-\gamma)(N_s-1)}\gamma R$. To make the argument clearer, we separated the boundary case. In this case, if all single-establishment firms post prices below P, single-establishment profit

is given by $\pi_s = \gamma R$, given the formula for P.[6] As a result, R could also be an optimal price for the single establishment firms, even though we already rule out that R could be a mass point. That said, the deviation constructed in Case 3 remains valid, so we can rule out this case as well.

All possible pure prices fall into one of the four cases. Therefore the sole multi-establishment firm cannot use a pure strategy in equilibrium. □

Proof of Proposition 3

Proof. We start by showing that the distributions presented in Proposition 3 pin down an equilibrium. For $p < R$, the candidate distributions are continuous, so the type-symmetric demand functions specialize to

$$D_s(p) = 1 - \frac{1-\gamma}{T-1}\left[(N_s - 1)F_s(p) + mF_m(p)\right], \tag{A.62}$$

$$D_m(p) = 1 - \frac{1-\gamma}{T-1}\left[\frac{m-1}{2} + N_s F_s(p)\right]. \tag{A.63}$$

At the lower endpoint, both CDFs are zero. Hence a single-establishment firm earns $\underline{p}$, while the multi-establishment firm earns $\left(1 - \frac{(1-\gamma)(m-1)}{2(T-1)}\right)\underline{p}$ per establishment.

Substituting (26)–(27) into (A.62)–(A.63) and substituting (2), we have:

$$D_s(p) = \frac{\underline{p}}{p}, \qquad D_m(p) = \frac{T-1+\alpha N_s}{(1+\alpha)(T-1)}\frac{\underline{p}}{p}.$$

Therefore:

$$\pi_s(p) = \underline{p}, \qquad \pi_m(p) = \frac{T-1+\alpha N_s}{(1+\alpha)(T-1)}\underline{p}$$

Thus both types are indifferent over every price in the continuous mixing region, since profits do not depend on p.

Furthermore, from (26) and the expression for $\underline{p}$, we have:

$$F_s(p) = \frac{T-1+\alpha N_s}{2\alpha N_s}\left(1 - \frac{T-1-\alpha N_s}{T-1+\alpha N_s}\frac{R}{p}\right)$$

since $T = N_s + m$ and $N = N_s + 1$. Hence:

$$F_s(R^-) = \frac{T-1+\alpha N_s}{2\alpha N_s}\left(1 - \frac{T-1-\alpha N_s}{T-1+\alpha N_s}\right) = 1$$

[6] It is arbitrarily close to γR, since $\overline{p}_s$ approaches P from below.

Similarly, for the multi-establishment firm, we have:

$$F_m(R^-) = \frac{T-1+\alpha m N_s}{2\alpha m N_s}\left(1 - \frac{T-1-\alpha N_s}{T-1+\alpha N_s}\right) = \frac{T-1+\alpha m N_s}{m\left[T-1+\alpha N_s\right]}$$

which yields (28), since $N = N_s + 1$. Moreover, $1 - F_m(R^-)$ delivers equation (29). The expression is strictly positive because $m > 1$ and $\alpha > 0$.

We are now ready to show that an atom at R is optimal for the multi-establishment firm. Since $F_s(R) = 1$ and F_s has no atom at R, from $D_m(R)$ and (2), we have that:

$$D_m(R) = \frac{T-1-\alpha N_s}{(1+\alpha)(T-1)}$$

From the expression for $\underline{p}$, we have:

$$\underline{p} = \frac{T-1-\alpha N_s}{T-1+\alpha N_s} R \Rightarrow R = \frac{T-1+\alpha N_s}{T-1-\alpha N_s}\underline{p}$$

Consequently:

$$\pi_m(R) = D_m(R)R = \frac{T-1-\alpha N_s}{(1+\alpha)(T-1)}\frac{T-1+\alpha N_s}{T-1-\alpha N_s}\underline{p} = \frac{T-1+\alpha N_s}{(1+\alpha)(T-1)}\underline{p} = \pi_m^*.$$

In contrast, a single-establishment firm does not place mass at R. The tie-adjusted CDF of the multi-establishment firm at that price is

$$\widetilde{F}_m(R) = F_m(R^-) + \frac{1}{2}\Delta F_m(R).$$

Relative to the limit from below, the single-establishment firm's demand falls by $\frac{(1-\gamma)m}{2(T-1)}\Delta F_m(R)$. Hence

$$RD_s(R) = \underline{p} - \frac{(1-\gamma)m}{2(T-1)}\Delta F_m(R)R < \underline{p}.$$

Thus R is not assigned positive probability by the single-establishment firms. It remains a limit point of their continuous mixing region; because the opponent has an atom there, the single-firm payoff is discontinuous at that endpoint.

Finally, for $p < \underline{p}$, both CDFs are zero, so a single-establishment firm earns $p < \underline{p}$ and the multi-establishment firm earns $\left(1 - \frac{(1-\gamma)(m-1)}{2(T-1)}\right)p < \pi_m^*$. Prices above R are rejected by consumers. Therefore neither type has a profitable deviation, and the proposed strategies form a Nash equilibrium.

It remains to show uniqueness. By Proposition 1, all single-establishment firms use the same cumulative distribution. With a single multi-establishment firm, symmetry for that type

is vacuous. At prices at which the relevant distributions are continuous, pure-price profits per establishment are

$$u_s(p) = p\left\{1 - \frac{1-\gamma}{T-1}\left[(N_s-1)F_s(p) + mF_m(p)\right]\right\}, \tag{A.64}$$

$$u_m(p) = p\left[1 - \frac{(1-\gamma)(m-1)}{2(T-1)} - \frac{(1-\gamma)N_s}{T-1}F_s(p)\right]. \tag{A.65}$$

By Lemma 5, F_s is atomless. We proceed in steps.

Step 1: F_m has no atom below R. Suppose F_m has an atom of size $\delta > 0$ at some $x < R$. Since F_s is atomless, the single-establishment pure-price payoff has the strict downward jump

$$u_s(x^-) - u_s(x^+) = \frac{(1-\gamma)m}{T-1}x\delta > 0.$$

Because $u_s(x^-) \le \pi_s^*$, it follows that $u_s(x^+) < \pi_s^*$. Consequently, there is a right neighborhood $(x, x+\varepsilon)$ containing no single-establishment support prices. On that neighborhood F_s is constant, so (A.65) shows that the multi-establishment firm's demand is constant and its profit is strictly increasing in price. Since the atom makes x a multi-establishment support price, raising the price slightly above x is profitable, a contradiction. Hence an atom of F_m cannot occur below R.

Step 2: the union of supports is $[\underline{p}, R]$. Appendix Lemma A.3 rules out a pure strategy for the multi-establishment firm. Let $\underline{p}$ and $\overline{p}$ be the minimum and maximum of the union of the two supports. If $\overline{p} < R$, both CDFs are constant immediately above $\overline{p}$. Because neither distribution has an atom at a price below R, the payoff of a type whose support reaches $\overline{p}$ is continuous there. That type can therefore raise its price slightly without changing demand, contradicting optimality. Thus $\overline{p} = R$.

The union cannot have an open gap $(x, y) \subset (\underline{p}, R)$. Both CDFs would be constant on the gap. At least one type has support prices approaching x from below, and its payoff is continuous at x. Raising its price from x to a nearby price in (x, y) leaves demand unchanged and raises profit. Hence

$$\operatorname{supp}(F_s) \cup \operatorname{supp}(F_m) = [\underline{p}, R].$$

Step 3: the two distributions have the same lower endpoint. Let $\underline{p}_s$ and $\underline{p}_m$ denote the two lower endpoints. Because the multi-establishment strategy is not pure and has no atom below R, $\underline{p}_m < R$.

If $\underline{p}_m < \underline{p}_s$, every price in $(\underline{p}_m, \underline{p}_s)$ is posted only by the multi-establishment firm. Since

$F_s = 0$ there, its demand is the constant a, so it cannot be indifferent over more than one price in that interval. This is a contradiction.

Suppose instead that $\underline{p}_s < \underline{p}_m$. On $[\underline{p}_s, \underline{p}_m)$ only the single-establishment firms post. Their equilibrium profit is $\pi_s^* = \underline{p}_s$, and payoff equalization in (A.64), with $F_m = 0$, gives

$$F_s(p) = \frac{T-1}{(1-\gamma)(N_s-1)}\left(1 - \frac{\underline{p}_s}{p}\right).$$

Substituting into (A.65) yields

$$u_m(p) = \left(1 - \frac{(1-\gamma)(m-1)}{2(T-1)} - \frac{N_s}{N_s-1}\right)p + \frac{N_s}{N_s-1}\underline{p}_s.$$

This expression is strictly decreasing because $1 - \frac{(1-\gamma)(m-1)}{2(T-1)} < 1 < N_s/(N_s-1)$. Hence the multi-establishment firm obtains strictly more by deviating to $\underline{p}_s$ than it obtains at its lower support endpoint $\underline{p}_m$, a contradiction. Therefore

$$\underline{p}_s = \underline{p}_m \equiv \underline{p}.$$

Since both CDFs vanish at this common lower endpoint,

$$\pi_s^* = \underline{p}, \qquad \pi_m^* = \left[1 - \frac{(1-\gamma)(m-1)}{2(T-1)}\right]\underline{p}.$$

Step 4: every price below R is in the single-establishment support. Suppose an open interval $I \subset (\underline{p}, R)$ contains no single-establishment support price. Connectedness of the union then implies that every price in I belongs to the multi-establishment support. But F_s is constant on I, so (A.65) is strictly increasing in p on I. The multi-establishment firm cannot be indifferent over all those prices. Thus

$$[\underline{p}, R) \subseteq \operatorname{supp}(F_s).$$

Because F_s is atomless, $F_s(R^-) = 1$.

Step 5: R belongs to the multi-establishment support. Suppose not, and let $q < R$ be its highest support price. Then $F_m(p) = 1$ for $p \in (q, R]$. Every $p \in (q, R)$ is a single-establishment support price, so single-establishment payoff equalization gives

$$\underline{p} = p\left\{1 - \frac{1-\gamma}{T-1}\left[(N_s-1)F_s(p) + m\right]\right\}.$$

Taking $p \uparrow R$ and using $F_s(R^-) = 1$ yields

$$\underline{p} = \gamma R.$$

The multi-establishment equilibrium payoff would therefore be $\left[1 - \frac{(1-\gamma)(m-1)}{2(T-1)}\right] \gamma R$. At R, however, its demand is

$$1 - \frac{(1-\gamma)\left(\frac{m-1}{2} + N_s\right)}{T-1} = \gamma + \frac{(1-\gamma)(m-1)}{2(T-1)},$$

and

$$\gamma + \frac{(1-\gamma)(m-1)}{2(T-1)} - \left[1 - \frac{(1-\gamma)(m-1)}{2(T-1)}\right]\gamma = \frac{(1-\gamma)(m-1)}{2(T-1)}(1+\gamma) > 0.$$

Thus posting R is a profitable deviation. Hence $R \in \operatorname{supp}(F_m)$.

Step 6: the multi-establishment support has no gap. At any multi-establishment support price p, its payoff equalization condition and (A.4) imply

$$F_s(p) = \frac{T - 1 - (1-\gamma)\left(\frac{m-1}{2}\right)}{(1-\gamma)N_s}\left(1 - \frac{\underline{p}}{p}\right). \tag{A.66}$$

Suppose that (x, y) is an open gap in the multi-establishment support, with x, y belonging to that support; the argument permits $y = R$. On the gap, $F_m(p) = K$ for some constant K. Since every price below R is in the single-establishment support, single-establishment payoff equalization gives

$$F_s(p) = \frac{T-1}{(1-\gamma)(N_s-1)}\left(1 - \frac{\underline{p}}{p}\right) - \frac{mK}{N_s - 1}, \qquad p \in (x, y). \tag{A.67}$$

Taking one-sided limits at an endpoint $z \in \{x, y\}$ and using (A.66) gives

$$K = \frac{T - 1 + (1-\gamma)(N_s-1)\left(\frac{m-1}{2}\right)}{(1-\gamma)mN_s}\left(1 - \frac{\underline{p}}{z}\right), \qquad \text{where } \frac{T - 1 + (1-\gamma)(N_s-1)\left(\frac{m-1}{2}\right)}{(1-\gamma)mN_s} > 0.$$

Applying this identity at both x and y is impossible because $x < y$. Therefore F_m has no support gap. Together with Steps 1 and 5,

$$\operatorname{supp}(F_m) = [\underline{p}, R],$$

with no atom below R and at most one atom, located at R.

Step 7: the equilibrium objects are uniquely pinned down. For every $p \in [\underline{p}, R)$, both types mix continuously. Equation (A.66) therefore holds, and substituting it into the single-establishment payoff equation gives

$$F_m(p) = \frac{T - 1 + (1-\gamma)(N_s - 1)\left(\frac{m-1}{2}\right)}{(1-\gamma)mN_s}\left(1 - \frac{\underline{p}}{p}\right).$$

Since $F_s(R^-) = 1$,

$$1 = \frac{T - 1 - (1-\gamma)\left(\frac{m-1}{2}\right)}{(1-\gamma)N_s}\left(1 - \frac{\underline{p}}{R}\right),$$

so

$$\underline{p} = \frac{\gamma(T-1) + (1-\gamma)\left(\frac{m-1}{2}\right)}{T - 1 - (1-\gamma)\left(\frac{m-1}{2}\right)}R.$$

Moreover,

$$F_m(R^-) = \frac{\frac{T-1+(1-\gamma)(N_s-1)\left(\frac{m-1}{2}\right)}{(1-\gamma)mN_s}}{\frac{T-1-(1-\gamma)\left(\frac{m-1}{2}\right)}{(1-\gamma)N_s}} = \frac{T - 1 + (1-\gamma)(N-1)\left(\frac{m-1}{2}\right)}{m(T - 1 - (1-\gamma)\left(\frac{m-1}{2}\right))}.$$

Finally,

$$\frac{T - 1 - (1-\gamma)\left(\frac{m-1}{2}\right)}{(1-\gamma)N_s} - \frac{T - 1 + (1-\gamma)(N_s - 1)\left(\frac{m-1}{2}\right)}{(1-\gamma)mN_s} = \frac{(1+\gamma)(m-1)(T-1)}{2(1-\gamma)mN_s} > 0,$$

so $F_m(R^-) < 1$ and the residual probability must be the unique atom at R:

$$\Delta F_m(R) = 1 - F_m(R^-) = \frac{(1+\gamma)(m-1)(T-1)}{2m(T - 1 - (1-\gamma)\frac{m-1}{2})}.$$

These are exactly the lower endpoint, continuous CDFs, atom, and profits stated in Proposition 3. Every equilibrium object is therefore uniquely determined. □

Proof of Proposition 4

Proof. First, we must pin down $\mathbb{E}[P_R]$. In particular, $\mathbb{E}[P_R]$ is defined by:

$$\mathbb{E}[P_R] = \left\{ \begin{array}{l} [(1-\alpha)\nu_{m\bullet} + \alpha\nu_{mm}]\,\mathbb{E}[P_m] + (1-\alpha)\nu_{s\bullet}\mathbb{E}[P_s] \\ +\alpha\nu_{ss}\mathbb{E}[\min\{P_{s1}, P_{s2}\}] + \alpha\nu_{sm}\mathbb{E}[\min\{P_s, P_m\}] \end{array} \right\} \tag{A.68}$$

where:

- $\nu_{m\bullet} \equiv \frac{m}{T}$: Probability the consumer observes the price from a multi-establishment firm
- $\nu_{s\bullet} \equiv \frac{N_s}{T}$: Probability the consumer observes the price from a single-establishment firm.

Conditional on receiving two quotes,

- $\nu_{ss} \equiv \frac{N_s(N_s-1)}{T(T-1)}$: Probability the consumer observes only prices from single-establishment firms;
- $\nu_{sm} \equiv \frac{2N_s m}{T(T-1)}$: Probability the consumer observes one price of each type;
- $\nu_{mm} \equiv \frac{m(m-1)}{T(T-1)}$: Probability the consumer observes only prices from multi-establishment firms.

Therefore, we must pin down the expected price in each case presented in equation (A.68). Then, the expected price from one single-establishment quote is:

$$\mathbb{E}[P_s] = \int_{\underline{p}}^{R} p\, f_s(p)dp = R - \left(\frac{T-1+\alpha N_s}{2\alpha N_s}\right)(R-\underline{p}) + \left(\frac{T-1+\alpha N_s}{2\alpha N_s}\right)\underline{p}\ln\left(\frac{R}{\underline{p}}\right)$$

since $\underline{p} = \frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}R$, notice that:

$$\begin{aligned} R-\underline{p} &= \left[1-\left(\tfrac{T-1-\alpha N_s}{T-1+\alpha N_s}\right)\right]R \\ &= \tfrac{2\alpha N_s}{T-1+\alpha N_s}R \end{aligned}$$

hence:

$$R - \left(\frac{T-1+\alpha N_s}{2\alpha N_s}\right)(R-\underline{p}) = 0$$

as a result:

$$\mathbb{E}[P_s] = \frac{T-1-\alpha N_s}{2\alpha N_s}R\ln\left(\frac{T-1+\alpha N_s}{T-1-\alpha N_s}\right) \tag{A.69}$$

Next, consider the expected price from one multi-establishment quote:

$$\mathbb{E}[P_m] = \int_{\underline{p}}^{R} p\, f_m(p)dp + \frac{(m-1)(T-1)}{m\left[T-1+\alpha N_s\right]}R$$

we have that:

$$\int_{\underline{p}}^{R} p\, f_m(p)dp = \left(\frac{T-1+\alpha m N_s}{m(T-1+\alpha N_s)}\right)R - \left(\frac{T-1+\alpha m N_s}{2\alpha m N_s}\right)(R-\underline{p}) + \left(\frac{T-1+\alpha m N_s}{2\alpha m N_s}\right)\underline{p}\ln\left(\frac{R}{\underline{p}}\right)$$

simplifying it:

$$\int_{\underline{p}}^{R} p\, f_m(p)dp = \left(\frac{T-1+\alpha m N_s}{2\alpha m N_s}\right)\frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)}R\ln\left(\frac{T-1+\alpha N_s}{T-1-\alpha N_s}\right)$$

Hence, we have:

$$\mathbb{E}[P_m] = \left\{ \frac{(m-1)(T-1)}{m\left[T-1+\alpha N_s\right]} + \left(\frac{T-1+\alpha m N_s}{2\alpha m N_s} \right) \frac{T-1-\alpha(T-m)}{T-1+\alpha(T-m)} \ln\left(\frac{T-1+\alpha N_s}{T-1-\alpha N_s} \right) \right\} R \quad \text{(A.70)}$$

Now consider the case in which the consumer observes multiple offers. The distribution of the minimum price of two offers from single-establishment firms is:

$$G_{ss}(p|R) = 1 - \left(1 - F_s(p|R)\right)^2 = \left[1 - \left(1 - F_s(p|R)\right)\right]\left[1 + \left(1 - F_s(p|R)\right)\right],$$

therefore $G_{ss}(p|R) = F_s(p|R)\left[2 - F_s(p|R)\right]$. Consequently,

$$\int_{\underline{p}}^{R} G_{ss}(p \mid R)\, dp = R\left(\frac{T-1-\alpha N_s}{T-1+\alpha N_s} \right) \times \left[\begin{array}{c} \frac{[T-1+\alpha N_s][2\alpha N_s-(T-1)]}{\alpha N_s[T-1-\alpha N_s]} \\ +\frac{(T-1)^2-\alpha^2 N_s^2}{2\alpha^2 N_s^2} \ln\left(\frac{T-1+\alpha N_s}{T-1-\alpha N_s} \right) \end{array} \right]$$

as a result:

$$\mathbb{E}[\min\{P_{s1}, P_{s2}\}] = R - \int_{\underline{p}}^{\overline{p}_s} G_{ss}(p \mid R)\, dp$$

Substituting the expressions and simplifying, we have:

$$\mathbb{E}[\min\{P_{s1}, P_{s2}\}] = R\left[\frac{T-1-\alpha N_s}{\alpha N_s} \right] \left[\; 1 - \frac{T-1-\alpha N_s}{2\alpha N_s} \ln\left(\frac{T-1+\alpha N_s}{T-1-\alpha N_s} \right) \; \right] \quad \text{(A.71)}$$

Now consider the case in which the consumer receives offers from one single-establishment and one multi-establishment firm. In this case:

$$\mathbb{E}[\min\{P_s, P_m\}] = \int_{\underline{p}}^{R} p\, dG_{sm}(p)$$

Notice that $G_{sm}(p \mid R) = 1 - [1 - F_s(p)][1 - F_m(p)]$. Therefore, $G_{sm}(p \mid R) = F_m(p) + F_s(p) - F_m(p)F_s(p)$. Moreover, notice that we can disregard the mass point at R for $F_m(\cdot)$, since the price for the single-establishment firm will always be lower. As a result:

$$\mathbb{E}[\min\{P_s, P_m\}] = \int_{\underline{p}}^{R} p\, dG_{sm}(p) = R - \int_{\underline{p}}^{R} G_{sm}(p)\, dp$$

and

$$\int_{\underline{p}}^{R} G_{sm}(p)\, dp = \int_{\underline{p}}^{R} F_m(p)\, dp + \int_{\underline{p}}^{R} F_s(p)\, dp - \int_{\underline{p}}^{R} F_s(p)F_m(p)\, dp$$

solving the integrals, we have:

$$\int_{\underline{p}}^{\overline{p}_s} G_{sm}(p)\,dp = R\left(\frac{T-1-\alpha N_s}{T-1+\alpha N_s}\right)\left[\begin{array}{c}\frac{2\alpha N_s}{T-1-\alpha N_s} - \frac{T-1}{\alpha m N_s} \\ +\frac{(T-1)^2 - m\alpha^2 N_s^2}{2\alpha^2 m N_s^2}\ln\left(\frac{T-1+\alpha N_s}{T-1-\alpha N_s}\right)\end{array}\right].$$

Therefore:

$$\mathbb{E}[\min\{P_s, P_m\}] = R\left(\frac{T-1-\alpha N_s}{T-1+\alpha N_s}\right)\left[\begin{array}{c}\frac{T-1+\alpha m N_s}{\alpha m N_s} \\ -\frac{(T-1)^2 - m\alpha^2 N_s^2}{2\alpha^2 m N_s^2}\ln\left(\frac{T-1+\alpha N_s}{T-1-\alpha N_s}\right)\end{array}\right] \tag{A.72}$$

Putting everything together, we have:

$$\mathbb{E}[P_R] = \left\{\begin{array}{l}[(1-\alpha)\nu_{m\bullet} + \alpha\nu_{mm}]\,\mathbb{E}[P_m] + (1-\alpha)\nu_{s\bullet}\mathbb{E}[P_s] \\ +\alpha\nu_{ss}\mathbb{E}[\min\{P_{s1}, P_{s2}\}] + \alpha\nu_{sm}\mathbb{E}[\min\{P_s, P_m\}]\end{array}\right\} \tag{A.73}$$

Substituting the expressions and simplifying, we obtain

$$\mathbb{E}[P_R] = R\frac{T-1-\alpha N_s}{T-1+\alpha N_s}\left[1+\alpha - \frac{\alpha m(m-1)}{T(T-1)}\right]. \tag{A.74}$$

Finally, since $c = R - \mathbb{E}[P_R]$ if R has an interior solution, we have:

$$R = \frac{c}{\Omega_1}$$

with

$$\Omega_1 = 1 - \frac{T-1-\alpha N_s}{T-1+\alpha N_s}\left[1+\alpha - \frac{\alpha m(m-1)}{T(T-1)}\right] \tag{A.75}$$

but notice that (A.74) and (A.75) are the same expressions as the ones presented by (21) and (20) for $N_m = 1$. Thus the unconstrained cutoff is $Q = c/\Omega_1$, and exogenous participation with the transaction cap implies

$$R = \min\left\{z, \frac{c}{\Omega_1}\right\}.$$

This establishes the boundary branch in Proposition 4 as well. □

Proof of Proposition 5

Proof. Define the ratio of the revenue share of one multi-establishment firm to the revenue

share of one single-establishment firm as

$$\rho_{\text{Rev}} \equiv \frac{m[T-1+\alpha(T-m)]}{(1+\alpha)(T-1)} = m\left[1-\frac{\alpha(m-1)}{(1+\alpha)(T-1)}\right]. \tag{A.76}$$

Dividing the numerator and denominator of HHI^{Rev} by the square of the revenue share of a single-establishment firm gives

$$HHI^{Rev} = \frac{N_s + N_m \rho_{\text{Rev}}^2}{(N_s + N_m \rho_{\text{Rev}})^2}. \tag{A.77}$$

Moreover,

$$\rho_{\text{Rev}} - 1 = \frac{(m-1)\left[T-1+\alpha(T-m-1)\right]}{(1+\alpha)(T-1)} > 0, \tag{A.78}$$

Hence $\rho_{\text{Rev}} > 1$. Moreover, we have that:

$$\frac{\partial \rho_{\text{Rev}}}{\partial N_s} = \frac{\alpha m(m-1)}{(1+\alpha)(T-1)^2} > 0. \tag{A.79}$$

Similarly,

$$\frac{\partial \rho_{\text{Rev}}}{\partial N_m} = \frac{\alpha m^2(m-1)}{(1+\alpha)(T-1)^2} > 0. \tag{A.80}$$

Finally, considering the partial derivative with respect to m, we have:

$$\frac{\partial \rho_{\text{Rev}}}{\partial m} = \frac{(T-1)[T-1+\alpha(T-m)] - \alpha m(N-1)}{(1+\alpha)(T-1)^2} > 0. \tag{A.81}$$

where the inequality comes from $T-1 > m$, $T-1 > N-1$, given $N_s \geq 1$, $N_m \geq 1$, $m \geq 2$, and $\alpha < 1$.

Now consider the derivative of HHI^{Rev} with respect to N_s. Differentiating (A.77) with respect to N_s gives

$$\frac{\partial HHI^{Rev}}{\partial N_s} = \frac{N_m \rho_{\text{Rev}}(1-2\rho_{\text{Rev}}) - N_s + 2N_s N_m(\rho_{\text{Rev}}-1)\frac{\alpha m(m-1)}{(1+\alpha)(T-1)^2}}{(N_s + N_m \rho_{\text{Rev}})^3}. \tag{A.82}$$

To sign (A.82), substitute directly the expression for $\partial \rho_{\text{Rev}}/\partial N_s$ derived above. Since

$$N_s \frac{\alpha m(m-1)}{(1+\alpha)(T-1)^2} < \frac{m[T-1+\alpha(T-m)]}{(1+\alpha)(T-1)} = \rho_{\text{Rev}},$$

because

$$\alpha N_s(m-1) < (T-1)^2 \leq (T-1)[T-1+\alpha(T-m)],$$

equation (A.82) can be signed directly as

$$\begin{aligned}\frac{\partial HHI^{Rev}}{\partial N_s} &< \frac{N_m\rho_{\text{Rev}}(1-2\rho_{\text{Rev}}) - N_s + 2N_m\rho_{\text{Rev}}(\rho_{\text{Rev}}-1)}{(N_s+N_m\rho_{\text{Rev}})^3} \\ &= -\frac{1}{(N_s+N_m\rho_{\text{Rev}})^2} < 0. \end{aligned} \tag{A.83}$$

Now consider the derivative of HHI^{Rev} with respect to m. Holding N_s and N_m fixed and differentiating (A.77),

$$\begin{aligned}\frac{\partial HHI^{Rev}}{\partial m} &= \frac{2N_m\rho_{\text{Rev}}\frac{\partial\rho_{\text{Rev}}}{\partial m}(N_s+N_m\rho_{\text{Rev}})^2}{(N_s+N_m\rho_{\text{Rev}})^4} \\ &\quad - \frac{2(N_s+N_m\rho_{\text{Rev}}^2)(N_s+N_m\rho_{\text{Rev}})N_m\frac{\partial\rho_{\text{Rev}}}{\partial m}}{(N_s+N_m\rho_{\text{Rev}})^4} \\ &= \frac{2N_sN_m(\rho_{\text{Rev}}-1)}{(N_s+N_m\rho_{\text{Rev}})^3}\frac{\partial\rho_{\text{Rev}}}{\partial m}. \end{aligned} \tag{A.84}$$

Since $N_s > 0$, $N_m > 0$, $\rho_{\text{Rev}} > 1$, and $\partial\rho_{\text{Rev}}/\partial m > 0$, it follows that $\frac{\partial HHI^{Rev}}{\partial m} > 0$.

Finally, consider the derivative of HHI^{Rev} with respect to N_m. Differentiating (A.77) with respect to N_m gives

$$\begin{aligned}\frac{\partial HHI^{Rev}}{\partial N_m} &= \frac{\left(\rho_{\text{Rev}}^2 + 2N_m\rho_{\text{Rev}}\frac{\partial\rho_{\text{Rev}}}{\partial N_m}\right)(N_s+N_m\rho_{\text{Rev}})^2}{(N_s+N_m\rho_{\text{Rev}})^4} \\ &\quad - \frac{2(N_s+N_m\rho_{\text{Rev}}^2)(N_s+N_m\rho_{\text{Rev}})\left(\rho_{\text{Rev}}+N_m\frac{\partial\rho_{\text{Rev}}}{\partial N_m}\right)}{(N_s+N_m\rho_{\text{Rev}})^4} \end{aligned} \tag{A.85}$$

$$= \frac{\rho_{\text{Rev}}\left[N_s(\rho_{\text{Rev}}-2) - N_m\rho_{\text{Rev}}^2\right] + 2N_sN_m(\rho_{\text{Rev}}-1)\frac{\alpha m^2(m-1)}{(1+\alpha)(T-1)^2}}{(N_s+N_m\rho_{\text{Rev}})^3}. \tag{A.86}$$

The two terms in the numerator of (A.86) need not have the same sign, and neither dominates globally. The ambiguity is genuine. For example, evaluating (A.86),

$$\left.\frac{\partial HHI^{Rev}}{\partial N_m}\right|_{\alpha=0.1,\, N_s=2,\, N_m=2,\, m=2} \simeq -0.07287 < 0,$$

whereas

$$\left.\frac{\partial HHI^{Rev}}{\partial N_m}\right|_{\alpha=0.1,\, N_s=20,\, N_m=2,\, m=4} \simeq 0.001491 > 0.$$

Hence

$$\frac{\partial HHI^{Rev}}{\partial N_m} \text{ is ambiguous in general.}$$

□

Proof of Proposition 6

Proof. Part (i) follows immediately from the corresponding first-order stochastic dominance result for the equilibrium margin distributions and the fact that $\mu(p) = 1 + p/\kappa$ is strictly increasing. Part (ii) follows because the gross markup paid in a transaction is $1 + P_R/\kappa$. The comparative statics then follow directly from the comparative statics of $\mathbb{E}[P_R]$ established above. □

Proof of Proposition 7

Proof. Fix a size class containing firms i and j, each with q establishments. Let $H(p) \equiv F_i(p) - F_j(p)$, $p \in [0, R]$.

A unilateral change in firm i's mixing distribution does not change its pure-price demand conditional on posting p. Optimality of F_i relative to the feasible strategy F_j, together with the analogous inequality for firm j, gives

$$0 \leq \int_{[0,R]} p\,[D_i(p) - D_j(p)]\,dH(p).$$

The two firms have the same own-size term $(q-1)/2$. Their per-establishment demand difference is

$$D_i(p) - D_j(p) = \frac{q(1-\gamma)}{T-1}\,[\widetilde{F}_i(p) - \widetilde{F}_j(p)] = \frac{q(1-\gamma)}{2(T-1)}\,[H(p) + H(p^-)].$$

Using the Stieltjes product rule

$$d(H^2)(p) = [H(p) + H(p^-)]\,dH(p),$$

we obtain

$$0 \leq \frac{q(1-\gamma)}{2(T-1)} \int_{[0,R]} p\,d(H^2)(p) = -\frac{q(1-\gamma)}{2(T-1)} \int_0^R H(p)^2\,dp.$$

Hence $H = 0$ almost everywhere, and right continuity implies $H = 0$ everywhere. Thus firms with the same establishment holdings use the same cumulative distribution function.

□

Proof of Lemma 6

Proof. A zero-price atom cannot be optimal because any sufficiently small positive price earns strictly positive profit from one-quote consumers. Consider a positive price p_0.

First, two different firms cannot both have atoms at p_0. Either firm can undercut p_0 by an arbitrarily small amount and obtain a discrete gain from strictly beating the other firm's atom. The price loss vanishes while the demand gain remains positive.

Suppose next that exactly one firm, denoted j, has an atom of size $\delta_j > 0$ at $p_0 < R$. Every rival $i \neq j$ experiences a strict downward jump in pure-price per-establishment demand when its price crosses p_0:

$$D_i(p_0^-) - D_i(p_0^+) = \frac{1-\gamma}{T-1} q_j \delta_j > 0.$$

Since the payoff immediately below p_0 cannot exceed firm i's equilibrium payoff, the payoff immediately above p_0 is strictly below its equilibrium payoff. Finiteness of the set of firms therefore gives an $\varepsilon > 0$ such that no rival assigns probability to $(p_0, p_0 + \varepsilon)$. Firm j's own pricing distribution does not enter its demand. Consequently, its demand is constant on this right neighborhood, so raising its price slightly above p_0 is strictly profitable. This contradicts the atom at p_0. Therefore no atom can occur below R.

At R, two atoms are again impossible because either atom can be undercut. A class containing at least two identical firms cannot have an atom at R: Proposition 7 would require at least two firms to place the same atom. Thus at most one firm, necessarily a singleton size class, can have an atom at R.

Now let $\overline{p}$ be the highest price in the union of supports. If $\overline{p} < R$, all rival pricing distributions are constant immediately above $\overline{p}$ and there is no atom there. A firm with support approaching $\overline{p}$ can raise its price slightly without changing demand, a contradiction. Hence $\overline{p} = R$.

Finally, suppose the union has an open gap $(a, b) \subset (\underline{p}, R)$. There are no atoms below R, so all pure-price payoffs are continuous at a. A firm with support approaching a from below earns its equilibrium payoff there. All rival pricing distributions are constant on (a, b), so that the firm's demand is unchanged throughout the gap. Raising the price into the gap is strictly profitable. Therefore the union of supports is connected. □

Proof of Proposition 8

Proof. It is convenient to work at the individual-firm level. Suppose (k, r) denotes firm $r \in \{1, \ldots, n_k\}$ in size class k, i.e., firm r owns q_k establishments and posts prices according to the

distribution $F_{k,r}$. From Proposition 7, we have that $F_{k,r} = F_k$, for all $r \in \{1, ..., n_k\}$. Moreover, at prices below R, Lemma 6 rules out atoms. Hence, the pure price demand for a firm in size category k is given by:

$$D_k(p) = 1 - \frac{1-\gamma}{T-1}\left[\frac{q_k - 1}{2} + (n_k - 1)q_k F_k(p) + \sum_{\ell \neq k} n_\ell q_\ell F_\ell(p)\right] \tag{A.87}$$

Step 1: every class size pricing distribution has the same lower endpoint.

Let

$$\underline{p} \equiv \min_k \underline{p}_k, \quad \text{for } k \in \{1, ..., K\}$$

and let

$$\mathcal{L} \equiv \{i : \underline{p}_i = \underline{p}\}$$

be the set of *firms* in the class sizes attaining the lowest support endpoint. Suppose, toward a contradiction, that $\mathcal{L}$ does not contain every firm. Define

$$\widehat{p} \equiv \min_{j \notin \mathcal{L}} \underline{p}_j > \underline{p}.$$

For every $p \in [\underline{p}, \widehat{p})$, firms outside $\mathcal{L}$ have $F_j(p) = 0$.

Then, define $Q = n(\mathcal{L})$ the cardinality of set $\mathcal{L}$, i.e., the number of firms in $\mathcal{L}$. We first note that $Q \geq 2$. If $\mathcal{L}$ contained only one firm, then on $(\underline{p}, \widehat{p})$ that firm's demand would be constant because its own price distribution does not enter its demand and every rival cumulative distribution function is zero. By connectedness of the union of supports shown in Lemma 6, that single firm would have to support an interval on which its profit is strictly increasing in price, which is ruled out by equilibrium optimality.

Assume that firm i belongs to class size i. Because there is no atom at $\underline{p}$, every $i \in \mathcal{L}$ faces demand $1 - \left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_i - 1}{2}\right)$ at the common minimum and therefore earns an equilibrium profit

$$\pi_i^* = \left[1 - \left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_i - 1}{2}\right)\right]\underline{p}.$$

Then, for any $p \in (\underline{p}, \widehat{p})$, the absence of a profitable deviation by firm $i \in \mathcal{L}$ implies

$$\left\{1 - \frac{1-\gamma}{T-1}\left[\frac{q_i - 1}{2} + (n_i - 1)q_i F_i(p) + \sum_{\ell \neq i} n_\ell q_\ell F_\ell(p)\right]\right\} p \leq \left[1 - \left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_i - 1}{2}\right)\right]\underline{p}$$

Rearranging it

$$\frac{1-\gamma}{T-1}\left[\sum_{\ell=1}^{K} n_\ell q_\ell F_\ell(p) - q_i F_i(p)\right] \geq \left[1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_i-1}{2}\right)\right]\left(1-\frac{\underline{p}}{p}\right) \tag{A.88}$$

Now, for $p \in (\underline{p}, \widehat{p})$, only firms in $\mathcal{L}$ have $F_i(p) \neq 0$. As a result, $\sum_{\ell=1}^{K} n_\ell q_\ell F_\ell(p)$ only contain firms in $\mathcal{L}$. As a result, summing up (A.88) across *firms*, we have:

$$\frac{1-\gamma}{T-1}(Q-1)\sum_{i\in\mathcal{L}} q_i F_i(p) \geq \sum_{i\in\mathcal{L}}\left[1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_i-1}{2}\right)\right]\left(1-\frac{\underline{p}}{p}\right) \tag{A.89}$$

where the summation over i is with respect to firms and not class sizes.

Now consider a firm $j \notin \mathcal{L}$. Since $F_j(p) = 0$ below $\widehat{p}$, its deviation payoff at such a price is

$$u_j(p) = p\left\{1-\left(\frac{1-\gamma}{T-1}\right)\left[\left(\frac{q_j-1}{2}\right)+\sum_{i\in\mathcal{L}} q_i F_i(p)\right]\right\}.$$

adjusting and substituting into (A.89), we have

$$u_j(p) \leq p\left\{1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_j-1}{2}\right)\right\} - \frac{p}{Q-1}\sum_{i\in\mathcal{L}}\left[1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_i-1}{2}\right)\right]\left(1-\frac{\underline{p}}{p}\right) \tag{A.90}$$

rearranging it:

$$u_j(p) \leq \underline{p}\left[1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_j-1}{2}\right)\right]+\left\{\left[1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_j-1}{2}\right)\right]-\frac{\sum_{i\in\mathcal{L}}\left[1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_i-1}{2}\right)\right]}{Q-1}\right\}\left(p-\underline{p}\right)$$

It is easy to see that $\left[1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_j-1}{2}\right)\right] < 1$. Furthermore, notice that:

$$\sum_{i\in\mathcal{L}}\left[1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_i-1}{2}\right)\right] = Q-\left(\frac{1-\gamma}{2(T-1)}\right)\left(\sum_{i\in\mathcal{L}} q_i - Q\right) > Q-1$$

where the last inequality comes from the fact that $T > \sum_{i\in\mathcal{L}} q_i$, since some firms do not belong to $\mathcal{L}$ and as we showed before $Q > 1$. Hence, we have that:

$$\left[1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_j-1}{2}\right)\right]-\frac{\sum_{i\in\mathcal{L}}\left[1-\left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_i-1}{2}\right)\right]}{Q-1} < 0$$

as a result:

$$u_j(p) < \underline{p}\left[1 - \left(\frac{1-\gamma}{T-1}\right)\left(\frac{q_j - 1}{2}\right)\right].$$

Taking $p \uparrow \widehat{p}$, the equilibrium payoff obtained along firm j's support prices approaching $\widehat{p}$ from above is no larger than the left-limit payoff: any rival atom at $\widehat{p}$ can only reduce firm j's demand when the price crosses the atom, while firm j's own CDF does not enter its demand. Hence

$$\pi_j^* < \underline{p}\left[1 - \frac{1-\gamma}{T-1}\frac{q_j - 1}{2}\right].$$

But posting $\underline{p}$ is feasible for firm j and yields exactly the payoff on the right-hand side, a contradiction. This proves

$$\underline{p}_i = \underline{p} \qquad \text{for every } i.$$

Consequently,

$$\pi_i^* = \left[1 - \frac{1-\gamma}{T-1}\frac{q_i - 1}{2}\right]\underline{p} \qquad \text{for every firm } i. \tag{A.91}$$

Step 2: upper endpoints are ordered by class size.

Consider firms i and j with $q_i < q_j$. Suppose, toward a contradiction, that $\overline{p}_j < \overline{p}_i$. Choose a support price p of firm i satisfying $p > \overline{p}_j$. At that price, $\widetilde{F}_j(p) = 1$, $\widetilde{F}_i(p) \le 1$

Subtracting the two demand equations gives

$$\begin{aligned} D_j(p) - D_i(p) &= \frac{1-\gamma}{T-1}\left[-\frac{q_j - q_i}{2} + q_j\widetilde{F}_j(p) - q_i\widetilde{F}_i(p)\right] \\ &\ge \frac{1-\gamma}{T-1}\frac{q_j - q_i}{2} > 0. \end{aligned} \tag{A.92}$$

Since p belongs to firm i's support,

$$pD_i(p) = \pi_i^* = \left[1 - \frac{1-\gamma}{T-1}\frac{q_i - 1}{2}\right]\underline{p}.$$

Equation (A.92) therefore implies

$$pD_j(p) > \left[1 - \frac{1-\gamma}{T-1}\frac{q_i - 1}{2}\right]\underline{p}.$$

But $q_j > q_i$ implies $\left[1 - \frac{1-\gamma}{T-1}\frac{q_j-1}{2}\right] < \left[1 - \frac{1-\gamma}{T-1}\frac{q_i-1}{2}\right]$, and hence

$$\pi_j^* = \left[1 - \frac{1-\gamma}{T-1}\frac{q_j - 1}{2}\right]\underline{p} < \left[1 - \frac{1-\gamma}{T-1}\frac{q_i - 1}{2}\right]\underline{p}.$$

Firm j would profitably deviate to p, a contradiction. Thus $q_i < q_j \Longrightarrow \overline{p}_i \leq \overline{p}_j$.

Step 3: The support of every firm price distribution is connected.

From **Step 1**, all firms share the same lower support endpoint $\underline{p}$, and firm i, which owns q_i establishments, earns

$$\pi_i^* = \left[1 - \frac{1-\gamma}{T-1}\frac{q_i - 1}{2}\right]\underline{p}.$$

Hence, for every price $p < R$, optimality implies

$$\left[1 - \frac{1-\gamma}{T-1}\frac{q_i - 1}{2}\right]\left(1 - \frac{\underline{p}}{p}\right) \leq \frac{1-\gamma}{T-1}\left[\sum_{\ell} q_\ell F_\ell(p) - q_i F_i(p)\right], \tag{A.93}$$

with equality whenever $p \in \operatorname{supp}(F_i)$. As above, the summation is over firms rather than size classes.

We prove connectedness by descending induction on firm size. The argument first shows that a gap in a firm's support can only be filled by strictly larger firms; connectedness then holds for the largest class outright, and the induction rules out gaps for each smaller class in turn by comparing how fast the aggregate distribution must grow across the gap against what firm i's own indifference condition allows. Suppose firm i has a gap (a, b) in its support, where $\underline{p} \leq a < b \leq \overline{p}_i$, prices in firm i's support approach both endpoints, and F_i is constant on (a, b). All CDFs below R are continuous by Lemma 6, so $F_i(a) = F_i(b)$ if $b < R$. If $b = R$, all expressions below are understood as left limits. In that case, firm i can re-enter only through an atom at R; by Lemma 6, it is then the unique atom holder. Since a firm's own CDF does not enter its demand, the equality for firm i at R is the left-limit version of (A.93).

We first show that no firm j with $q_j < q_i$ can post a price in (a, b). At the right endpoint of the gap, equality for firm i and optimality for firm j imply

$$q_j F_j(b) \leq q_i F_i(b) + \frac{q_j - q_i}{2}\left(1 - \frac{\underline{p}}{b}\right), \tag{A.94}$$

where left limits are used when $b = R$. By contrast, if some $p \in (a, b)$ belongs to $\operatorname{supp}(F_j)$, equality for firm j and optimality for firm i imply

$$q_j F_j(p) \geq q_i F_i(p) + \frac{q_j - q_i}{2}\left(1 - \frac{\underline{p}}{p}\right). \tag{A.95}$$

Because $F_i(p) = F_i(b)$ on the gap, $q_j - q_i < 0$, and $1 - \underline{p}/p < 1 - \underline{p}/b$, the right-hand side of (A.95) is strictly larger than the right-hand side of (A.94). Thus

$$q_j F_j(p) > q_j F_j(b),$$

contradicting monotonicity of F_j. Firms with $q_j = q_i$ cannot fill the gap either, because Proposition 7 implies that they use the same distribution as firm i. Consequently, every price in (a, b) must be posted by firms strictly larger than firm i.

This observation already proves connectedness for the largest size class: there is no larger firm that could fill a gap, whereas Lemma 6 says that the union of supports is connected. Now proceed downward in firm size and suppose that the supports of all firms strictly larger than firm i have already been shown to be connected. Since they all have the common lower endpoint $\underline{p}$, each of those supports is an interval from $\underline{p}$ to its upper endpoint. It follows that, inside $(a, \overline{b})$, the identity of the larger firms that mix can change only at their upper endpoints. There are finitely many firms, so (a, b) can be divided into finitely many open subintervals on each of which the same Q larger firms mix.

On any such subinterval, $Q \geq 2$. If only one firm mixed there, every rival CDF would be constant on that subinterval, while the mixing firm's own CDF does not enter its demand. Its demand would therefore be constant and its profit strictly increasing in price, contradicting mixing.

For every firm j that mixes on one of these subintervals, (A.93) holds with equality throughout the subinterval. Taking Stieltjes differences gives

$$\left[1 - \frac{1-\gamma}{T-1}\frac{q_j - 1}{2}\right] d\left(1 - \frac{\underline{p}}{p}\right) = \frac{1-\gamma}{T-1}\left[\sum_{\ell} q_\ell \, dF_\ell(p) - q_j \, dF_j(p)\right].$$

Summing this equality over the Q firms that mix on the subinterval, while all other CDFs are constant there, yields

$$\frac{1-\gamma}{T-1}\sum_{\ell} q_\ell \, dF_\ell(p) = \frac{1}{Q-1}\sum_{j:\, j \text{ mixes at } p}\left[1 - \frac{1-\gamma}{T-1}\frac{q_j - 1}{2}\right] d\left(1 - \frac{\underline{p}}{p}\right). \tag{A.96}$$

Because firm i is absent from the gap, the Q firms in this sum own strictly fewer than T establishments. Hence

$$\sum_{j:\, j \text{ mixes at } p}\left[1 - \frac{1-\gamma}{T-1}\frac{q_j - 1}{2}\right] = Q - \frac{1-\gamma}{2(T-1)}\left(\sum_{j:\, j \text{ mixes at } p} q_j - Q\right),$$

and

$$\frac{1-\gamma}{2(T-1)}\left(\sum_{j:\, j \text{ mixes at } p} q_j - Q\right) < \frac{T-Q}{2(T-1)} < 1.$$

Therefore the sum in (A.96) is strictly larger than $Q-1$, so

$$\frac{1-\gamma}{T-1}\sum_{\ell} q_\ell \, dF_\ell(p) > d\left(1-\frac{p}{=}{p}\right) \tag{A.97}$$

on every subinterval. Summing (A.97) over the finitely many subintervals of the gap, using continuity at their common endpoints and left limits if $b = R$, gives

$$\frac{1-\gamma}{T-1}\sum_{\ell} q_\ell \left[F_\ell(b) - F_\ell(a)\right] > \left(1-\frac{p}{=}{b}\right) - \left(1-\frac{p}{=}{a}\right). \tag{A.98}$$

But equality in (A.93) at the two endpoints of firm i's gap, together with $F_i(b) = F_i(a)$, implies

$$\frac{1-\gamma}{T-1}\sum_{\ell} q_\ell \left[F_\ell(b) - F_\ell(a)\right] = \left[1-\frac{1-\gamma}{T-1}\frac{q_i-1}{2}\right]\left[\left(1-\frac{p}{=}{b}\right) - \left(1-\frac{p}{=}{a}\right)\right] \tag{A.99}$$

$$\leq \left(1-\frac{p}{=}{b}\right) - \left(1-\frac{p}{=}{a}\right), \tag{A.100}$$

where the last inequality follows from $q_i \geq 1$. This contradicts (A.98). Therefore firm i cannot have a gap. The descending induction proves that every firm's support is connected.

By Proposition 7, all firms in a given size class use the same distribution, so every size class has a connected support. Combining this result with **Step 1** and the upper-endpoint ordering in **Step 2** yields the nested supports stated in the proposition. □

Derivation of price distributions

By Proposition 8 all firms share the lower endpoint $\underline{p}$ and the class supports are connected and nested by size. On a segment $[p_{r-1}, p_r]$, classes $1, \ldots, r-1$ have already reached one and classes $r, \ldots, K$ remain active. Define

$$a_k \equiv 1 - \frac{1-\gamma}{2(T-1)}(q_k - 1), \qquad S_r \equiv \sum_{k=r}^{K} n_k a_k, \qquad X_r(p) \equiv \sum_{k=r}^{K} n_k q_k F_k(p).$$

Because all CDFs vanish at the common lower endpoint, the equilibrium profit per establishment of class k is

$$\pi_k^* = a_k \underline{p}.$$

For an active class $k \geq r$, payoff equalization and (43) imply

$$a_k \frac{p}{=} = a_k - \frac{1-\gamma}{T-1}\left[(T - C_r) + X_r(p) - q_k F_k(p)\right]. \tag{A.101}$$

Multiplying by n_k and summing over active classes yields

$$\frac{1-\gamma}{T-1} X_r(p) = \frac{S_r\left(1 - \frac{p}{=}\right) - \frac{1-\gamma}{T-1} Q_r (T - C_r)}{Q_r - 1}. \tag{A.102}$$

Substituting (A.102) into (A.101) gives

$$F_k(p) = \frac{T-1}{(1-\gamma)q_k}\left[\frac{S_r}{Q_r - 1} - a_k\right]\left(1 - \frac{p}{=}\right) - \frac{T - C_r}{q_k(Q_r - 1)}. \tag{A.103}$$

Using

$$S_r = Q_r\left(1 + \frac{1-\gamma}{2(T-1)}\right) - \frac{(1-\gamma)C_r}{2(T-1)}, \qquad a_k = 1 + \frac{1-\gamma}{2(T-1)} - \frac{(1-\gamma)q_k}{2(T-1)},$$

we obtain

$$\frac{S_r}{Q_r - 1} - a_k = \frac{(1-\gamma)q_k}{2(T-1)} + \frac{1 - \frac{1-\gamma}{2(T-1)}(C_r - 1)}{Q_r - 1} = \frac{(1-\gamma)q_k}{2(T-1)} + G_r.$$

Equation (A.103) therefore reduces to (49). Its density is

$$f_k(p) = c_{kr}\frac{p}{=}{p^2} > 0. \tag{A.104}$$

Ordering of support cutoffs

Lemma A.4. *For every $r \leq r^\star$, $x_r \in (0, 1)$. On segment r, class r reaches one before every larger active class. The CDF formulas for the remaining classes are continuous when class r leaves the active set. Consequently,*

$$0 < x_1 < \cdots < x_{r^\star} < 1.$$

Proof. Positivity is immediate. To show $x_r < 1$, it is enough to verify

$$c_{rr} > 1 + \frac{T - C_r}{q_r(Q_r - 1)}.$$

After multiplication and substitution of G_r, this condition is implied by

$$\frac{2(T-1)}{1-\gamma} > q_r(Q_r - 1) + 2T - C_r - 1.$$

Because $C_r \geq q_r Q_r$, the right side is at most $2T-2$, and $\frac{2(T-1)}{1-\gamma} > 2T-2$ since $0 < \gamma < 1$. Hence $x_r < 1$.

For $k > r$,

$$F_r(p) - F_k(p) = \left(\frac{1}{q_r} - \frac{1}{q_k}\right)\left[\frac{T-1}{1-\gamma} G_r\left(1 - \frac{p}{\overline{p}}\right) - \frac{T-C_r}{Q_r - 1}\right]. \tag{A.105}$$

At the price where $F_r = 1$, the bracket is positive. Indeed, the required inequality reduces to

$$\frac{T-1}{1-\gamma} G_r > \frac{T-C_r}{Q_r - 1},$$

which follows from

$$\frac{T-1}{1-\gamma} > \frac{T-1}{2},$$

i.e., $\gamma < 1$. Since $q_k > q_r$, equation (A.105) implies $F_k < 1$ when class r reaches one.

At the transition, the surviving CDF values solve the same linear indifference system whether class r is treated as active with $F_r = 1$ or as a completed class included in C_{r+1}. The solution is unique, so the formulas join continuously. The surviving classes have strictly positive densities and are below one at p_r, so their later cutoffs satisfy $p_{r+1} > p_r$, equivalently $x_{r+1} > x_r$. □

A.5 Proof of Theorem 3

Proof. Lemma A.4 makes the proposed CDFs well-defined and monotone. Equation (A.101) shows that every active class k earns

$$pD_k(p) = a_k \underline{p} = \pi_k^*$$

on every segment of its support.

For $p < \underline{p}$, all rival CDFs are zero, and class k earns $a_k p < a_k \underline{p}$. It remains to rule out deviations above a class's upper support.

Consider a class $i < r$ that has already left the support and a segment on which the active classes are $r, \ldots, K$. A representative firm in class i faces completed opponent establishment mass $(T - C_r) - q_i$. Using (A.102), its pure-price payoff can be written

$$u_i(p) = s_{ir} p + \frac{S_r \underline{p}}{Q_r - 1}, \tag{A.106}$$

where

$$s_{ir} = \frac{(1-\gamma)q_i}{2(T-1)} - G_r + \frac{(1-\gamma)(T-C_r)}{(T-1)(Q_r-1)}. \tag{A.107}$$

This slope is strictly negative. To see this, note that active sizes satisfy $q_r \geq q_i + 1$, so

$$C_r \geq Q_r(q_i + 1).$$

This implies

$$(T - C_r) + q_i(Q_r - 1) \leq T - Q_r - q_i.$$

Therefore

$$T + (T - C_r) - 1 + q_i(Q_r - 1) \leq 2T - Q_r - q_i - 1 \leq 2T - 4 < \frac{2(T-1)}{1-\gamma},$$

which is equivalent to $s_{ir} < 0$.

At the cutoff where class i leaves, its payoff equals π_i^*. It then decreases strictly on every subsequent segment. Hence no class benefits from deviating above its support.

If $n_K \geq 2$, the terminal class contains at least two firms and reaches one continuously at R. If $n_K = 1$, the recursion stops when the penultimate class reaches one at R. Lemma A.4 gives $F_K(R^-) < 1$. The largest network's own CDF does not enter its demand, so posting R yields the same payoff as the limit from below; the residual atom is therefore optimal. Any other firm posting exactly R faces the midpoint of this atom and obtains a strictly lower payoff than its left limit. Thus no additional atom or deviation is profitable.

The proposed strategies are consequently a Nash equilibrium.

Finally, Proposition 8 implies that every equilibrium must have the common lower endpoint and the weakly nested support ordering used above. Starting from any maximal interval on which the set of active classes is fixed, the indifference equations uniquely determine (49). The cutoff conditions $F_r(p_r) = 1$ then imply the same x_r sequence derived above; since $0 < x_1 < \cdots < x_{r^\star} < 1$, coincident cutoffs across distinct size classes are ruled out ex post. The terminal condition $p_{r^\star} = R$ uniquely determines $\underline{p}$. Proposition 7 rules out asymmetry within a size class. Hence the constructed profile is the unique Nash equilibrium. □

Proof of Proposition 9

Proof. Because all pricing distributions are homogeneous of degree one in R, so is $\mathbb{E}[P_R]$, and hence $\Omega = 1 - \mathbb{E}[P_R]/R$ is independent of the level of R; it remains to compute $\mathbb{E}[P_R]$ and $\underline{p}$.

Lower endpoint. The largest class K is active on the whole support $[\underline{p}, R]$, so its equilibrium profit per establishment satisfies $D_K(\underline{p})\underline{p} = D_K(R)R$. At $\underline{p}$ every rival CDF equals zero, so from (43), $D_K(\underline{p}) = 1 - \frac{(1-\gamma)(q_K-1)}{2(T-1)}$, which is the profit coefficient in (51). At R every rival CDF equals one and firm K's own distribution drops out of its demand—the coefficient on $\widetilde{F}_K$ in (43) is $(n_K - 1)q_K$, which vanishes when $n_K = 1$ and is otherwise multiplied by $F_K(R) = 1$ net of

the atom, leaving the same value in both regimes. Hence

$$D_K(R) = 1 - \frac{1-\gamma}{T-1}\left[\frac{q_K - 1}{2} + (T - q_K)\right] = 1 - \frac{(1-\gamma)(2T - q_K - 1)}{2(T-1)}.$$

Solving $D_K(\underline{p})\underline{p} = D_K(R)R$ and substituting $1 - \gamma = \frac{2\alpha}{1+\alpha}$ yields the expression for $\underline{p}$ in (54).

Expected transaction price. In an interior equilibrium every consumer purchases one unit, so the expected transaction price equals total industry revenue per consumer. Each consumer generates $(1-\alpha) + 2\alpha = 1 + \alpha$ quote contacts, allocated symmetrically across the T establishments, so each establishment receives $(1+\alpha)/T$ contacts. A class-k establishment earns profit $\pi_k^* = \left[1 - \frac{(1-\gamma)(q_k-1)}{2(T-1)}\right]\underline{p}$ per contact by (51). Summing establishment revenue over all firms,

$$\mathbb{E}[P_R] = \frac{1+\alpha}{T}\sum_{k=1}^{K} n_k q_k \left[1 - \frac{(1-\gamma)(q_k-1)}{2(T-1)}\right]\underline{p} = (1+\alpha)\left[1 - \frac{(1-\gamma)\,S}{2T(T-1)}\right]\underline{p},$$

using $\sum_k n_k q_k = T$ and the definition of S in (52). Substituting $(1+\alpha)\frac{1-\gamma}{2} = \alpha$ gives the expression for $\mathbb{E}[P_R]$ in (54). Combining the two pieces yields $\Omega = 1 - \mathbb{E}[P_R]/R$ as in (53), and the interior best-response condition $c = R\Omega$ delivers $R = c/\Omega$, capped at z by exogenous participation. □

Proof of Proposition 10

Proof. For a firm of size q_k, equilibrium profit per establishment is

$$\pi_k^* = b(q_k)\underline{p}, \qquad b(q_k) = 1 - \frac{(1-\gamma)(q_k-1)}{2(T-1)}.$$

Using $1 - \gamma = 2\alpha/(1+\alpha)$,

$$b(q_k) = \frac{T - 1 + \alpha(T - q_k)}{(1+\alpha)(T-1)}.$$

Hence total revenue of a size-q_k firm is proportional to $q_k b(q_k)\underline{p}$. The common factors $\underline{p}/[(1+\alpha)(T-1)]$ cancel from revenue shares, so

$$HHI^{Rev} = \frac{\sum_{k=1}^{K} n_k \left[q_k\{T - 1 + \alpha(T - q_k)\}\right]^2}{\left[\sum_{k=1}^{K} n_k q_k\{T - 1 + \alpha(T - q_k)\}\right]^2},$$

which is (58).

For expected transaction markups,

$$\mathbb{E}[\mu^{tr}] = 1 + \frac{\mathbb{E}[P_R]}{\kappa} = 1 + \frac{B(S)\phi(q_K)R}{\kappa}.$$

Proposition 9 gives

$$R = \min\left\{z, \frac{c}{1 - B(S)\phi(q_K)}\right\},$$

which yields (59). Finally, writing $x \equiv B(S)\phi(q_K)$, the markup term above one is

$$\frac{x}{\kappa}\min\left\{z, \frac{c}{1-x}\right\}.$$

On the capped branch its derivative with respect to x is $z/\kappa > 0$; on the interior branch it is $c/[\kappa(1-x)^2] > 0$. The two expressions agree at the regime boundary. Thus expected transaction markups are strictly increasing in $B(S)\phi(q_K)$. □

Proofs for the Merger Analysis

Proof of Theorem 4. We first establish the merger accounting and the profit identities used in the theorem.

The merger removes firms of sizes q_a and q_b and replaces them with one firm of size $s = q_a + q_b$. Therefore

$$q_K^{\text{post}} = \max\{s, q_K^{nm}\}.$$

Since $s > q_b \geq q_a$,

$$q_K^{\text{post}} \geq \max\{q_b, q_K^{nm}\} = q_K^{\text{pre}},$$

with strict inequality if and only if $s > q_K^{\text{pre}}$.

Only the two merging firms change their contribution to S. Hence

$$\begin{aligned}\Delta S &= s(s-1) - q_a(q_a-1) - q_b(q_b-1)\\ &= (q_a+q_b)^2 - q_a^2 - q_b^2\\ &= 2q_aq_b.\end{aligned}$$

Since

$$B(S) = 1 + \alpha - \frac{\alpha S}{T(T-1)},$$

it follows that

$$B(S^{\text{post}}) = B(S^{\text{pre}}) - \frac{2\alpha q_a q_b}{T(T-1)} = B(S^{\text{pre}}) - \frac{\alpha \Delta S}{T(T-1)}.$$

Next,

$$\phi(q) = \frac{T - 1 - \alpha(T - q)}{T - 1 + \alpha(T - q)}$$

satisfies

$$\phi'(q) = \frac{2\alpha(T-1)}{[T - 1 + \alpha(T - q)]^2} > 0.$$

Thus ϕ is strictly increasing.

For profits, from (64),

$$b(q) = \left[1 - \frac{(1-\gamma)(q-1)}{2(T-1)}\right].$$

The establishment-weighted pre-merger coefficient satisfies

$$\begin{aligned}\overline{b}_{ab} - b(s) &= \frac{q_a b(q_a) + q_b b(q_b)}{s} - b(s) \\ &= \frac{1-\gamma}{s(T-1)} q_a q_b.\end{aligned}$$

Therefore

$$\begin{aligned}\Delta\pi_{ab} &= b(s)\underline{p}^{\text{post}} - \overline{b}_{ab}\underline{p}^{\text{pre}} \\ &= \overline{b}_{ab}(\underline{p}^{\text{post}} - \underline{p}^{\text{pre}}) - (\overline{b}_{ab} - b(s))\underline{p}^{\text{post}} \\ &= \overline{b}_{ab}\Delta\underline{p} - \frac{(1-\gamma)q_a q_b}{s(T-1)}\underline{p}^{\text{post}},\end{aligned}$$

which is (65). For a non-merging firm q_ℓ is unchanged, so

$$\Delta\pi_\ell = b(q_\ell)\Delta\underline{p}.$$

Moreover,

$$\Delta\pi_{ab} > 0$$

if and only if

$$b(s)\underline{p}^{\text{post}} > \overline{b}_{ab}\underline{p}^{\text{pre}},$$

or equivalently

$$\frac{\underline{p}^{\text{post}}}{\underline{p}^{\text{pre}}} > \frac{\overline{b}_{ab}}{b(s)} = 1 + \frac{(1-\gamma)q_a q_b}{(T-1)s\, b(s)} = 1 + \chi_{ab}.$$

We now prove part (i). Suppose $s \leq q_K^{\text{pre}}$. Then q_K and $\phi(q_K)$ are unchanged, while $B(S)$

falls by $\frac{\alpha\Delta S}{T(T-1)}$. Hence

$$\Omega^{\text{post}} = 1 - [B(S^{\text{pre}}) - \frac{\alpha\Delta S}{T(T-1)}]\phi(q_K) = \Omega^{\text{pre}} + \frac{\alpha\Delta S}{T(T-1)}\phi(q_K) > \Omega^{\text{pre}}.$$

Since the reservation price is interior,

$$R = \frac{c}{\Omega},$$

so $R^{\text{post}} < R^{\text{pre}}$. Because $\phi(q_K)$ is unchanged, the common floor falls as well. More explicitly,

$$\begin{aligned}\Delta\underline{p} &= c\phi(q_K)\left[\frac{1}{\Omega^{\text{post}}} - \frac{1}{\Omega^{\text{pre}}}\right] \\ &= -\frac{c\phi(q_K)^2\frac{\alpha\Delta S}{T(T-1)}}{\Omega^{\text{pre}}\Omega^{\text{post}}}.\end{aligned} \tag{A.108}$$

On the interior branch,

$$\mathbb{E}[P_R] = B(S)\phi(q_K)R = \frac{cB(S)\phi(q_K)}{1 - B(S)\phi(q_K)} = R - c.$$

Therefore the expected transaction price moves one-for-one with R. Since the expected gross markup is

$$\mathbb{E}[\mu^{tr}] = 1 + \frac{\mathbb{E}[P_R]}{\kappa},$$

the expected transaction markup has the same sign of change. Thus both fall in part (i).

Since $\Delta\underline{p} < 0$, (66) implies $\Delta\pi_\ell < 0$ for every non-merging firm. In (65), both the common-floor term and the cannibalization term are strictly negative, so $\Delta\pi_{ab} < 0$. Finally,

$$\frac{\Delta\pi_\ell}{\pi_\ell^{\text{pre}}} = \frac{\Delta\underline{p}}{\underline{p}^{\text{pre}}},$$

and

$$\underline{p}^{\text{pre}} = \frac{c\phi(q_K)}{\Omega^{\text{pre}}},$$

which together with (A.108) gives

$$\frac{\Delta\pi_\ell}{\pi_\ell^{\text{pre}}} = -\frac{\phi(q_K)\frac{\alpha\Delta S}{T(T-1)}}{\Omega^{\text{post}}}.$$

We now prove part (ii). Suppose $s > q_K^{\text{pre}}$. Then $q_K^{\text{post}} = s$. Using $B(S^{\text{post}}) = B(S^{\text{pre}}) - \frac{\alpha\Delta S}{T(T-1)}$,

$$\Omega^{\text{post}} - \Omega^{\text{pre}} = \frac{\alpha\Delta S}{T(T-1)}\phi(s) - B(S^{\text{pre}})\left[\phi(s) - \phi(q_K^{\text{pre}})\right]. \tag{A.109}$$

Because $R = c/\Omega$,

$$R^{\text{post}} > R^{\text{pre}} \quad \Longleftrightarrow \quad \Omega^{\text{post}} < \Omega^{\text{pre}}.$$

Using (A.109), this is equivalent to

$$R^{\text{post}} > R^{\text{pre}} \quad \Longleftrightarrow \quad \frac{\alpha\Delta S}{T(T-1)} < B(S^{\text{pre}})\left[1 - \frac{\phi(q_K^{\text{pre}})}{\phi(s)}\right]. \tag{A.110}$$

On the interior branch,

$$\mathbb{E}[P_R] = B(S)\phi(q_K)R = R - c.$$

Hence expected transaction prices move in the same direction as R, and so do expected gross markups. Therefore

$$\text{sign}\left(\mathbb{E}[P_R]^{\text{post}} - \mathbb{E}[P_R]^{\text{pre}}\right) = \text{sign}\left(\mathbb{E}[\mu^{tr}]^{\text{post}} - \mathbb{E}[\mu^{tr}]^{\text{pre}}\right) = \text{sign}\left(R^{\text{post}} - R^{\text{pre}}\right). \tag{A.111}$$

For the common floor,

$$\underline{p} = \phi(q_K)R = \frac{c\phi(q_K)}{1 - B(S)\phi(q_K)} = \frac{c}{1/\phi(q_K) - B(S)}.$$

All denominators are positive. Therefore

$$\underline{p}^{\text{post}} > \underline{p}^{\text{pre}} \quad \Longleftrightarrow \quad \frac{\alpha\Delta S}{T(T-1)} < \frac{1}{\phi(q_K^{\text{pre}})} - \frac{1}{\phi(s)}. \tag{A.112}$$

It remains to compare the reservation-price and common-floor thresholds. Their difference is

$$\begin{aligned}
&\left[\frac{1}{\phi(q_K^{\text{pre}})} - \frac{1}{\phi(s)}\right] - B(S^{\text{pre}})\left[1 - \frac{\phi(q_K^{\text{pre}})}{\phi(s)}\right] \\
&\quad = \frac{\phi(s) - \phi(q_K^{\text{pre}})}{\phi(s)}\left[\frac{1}{\phi(q_K^{\text{pre}})} - B(S^{\text{pre}})\right] \\
&\quad = \frac{\phi(s) - \phi(q_K^{\text{pre}})}{\phi(s)\phi(q_K^{\text{pre}})}\Omega^{\text{pre}} > 0.
\end{aligned}$$

Equivalently,

$$B(S^{\text{pre}})\left[1-\frac{\phi(q_K^{\text{pre}})}{\phi(s)}\right] < \frac{1}{\phi(q_K^{\text{pre}})} - \frac{1}{\phi(s)}. \tag{A.113}$$

The strict inequality follows from $s > q_K^{\text{pre}}$, strict monotonicity of ϕ, and $\Omega^{\text{pre}} > 0$. Combining (A.110), (A.111), (A.112), and (A.113) yields the three regions stated in part (ii).

Finally, since $b(q_\ell) > 0$ for every non-merging firm, (66) implies

$$\Delta\pi_\ell > 0 \quad \Longleftrightarrow \quad \Delta\underline{p} > 0.$$

The merged firm is profitable relative to the establishment-weighted pre-merger benchmark if and only if

$$\frac{\underline{p}^{\text{post}}}{\underline{p}^{\text{pre}}} > 1 + \chi_{ab}, \qquad \chi_{ab} \equiv \frac{(1-\gamma)q_a q_b}{(T-1)s\, b(s)} > 0. \tag{A.114}$$

Because $1 + \chi_{ab} > 1$, a profitable merger requires $\underline{p}^{\text{post}} > \underline{p}^{\text{pre}}$, which in turn implies that every non-merging firm gains. □

Proof of Corollary 4. If $R = z$ before and after the merger, then

$$\underline{p} = \phi(q_K)z.$$

If $s \le q_K^{\text{pre}}$, q_K is unchanged and therefore $\Delta\underline{p} = 0$. Equation (66) then implies that every non-merging firm's profit is unchanged, while (65) reduces to (67).

Although the floor is unchanged, the expected transaction price is

$$\mathbb{E}[P_R] = B(S)\phi(q_K)z.$$

Since the merger lowers $B(S)$ and leaves $\phi(q_K)$ unchanged, $\mathbb{E}[P_R]$, and hence the expected gross markup, strictly falls.

If $s > q_K^{\text{pre}}$, then $q_K^{\text{post}} = s > q_K^{\text{pre}}$. Since ϕ is strictly increasing,

$$\Delta\underline{p} = [\phi(s) - \phi(q_K^{\text{pre}})]z > 0.$$

Every non-merging firm therefore gains. The merged-firm condition follows from (A.114) after substituting

$$\frac{\underline{p}^{\text{post}}}{\underline{p}^{\text{pre}}} = \frac{\phi(s)}{\phi(q_K^{\text{pre}})}.$$

Finally,

$$\mathbb{E}[P_R] = B(S)\phi(q_K)z,$$

so when a new largest firm is created the fall in $B(S)$ and the rise in $\phi(q_K)$ work in opposite directions. The markup effect is therefore generally ambiguous. □

Proof of Corollary 5. For the revenue-share HHI, define the normalized revenue coefficient of a size-q firm by

$$w(q) \equiv q[1 - \lambda(q-1)].$$

The common factor $\underline{p}$ cancels from all revenue shares. Let

$$w_a = w(q_a), \qquad w_b = w(q_b), \qquad w_s = w(s).$$

Because $s = q_a + q_b$,

$$w_s = w_a + w_b - 2\lambda q_a q_b. \tag{A.115}$$

Also,

$$\mathcal{R}^{\text{pre}} = \sum_f w(q_f) = T - \lambda S^{\text{pre}},$$

and therefore

$$\mathcal{R}^{\text{post}} = \mathcal{R}^{\text{pre}} - 2\lambda q_a q_b.$$

Let

$$N^{\text{pre}} \equiv \sum_f w(q_f)^2.$$

Then

$$H^{\text{pre}} = \frac{N^{\text{pre}}}{(\mathcal{R}^{\text{pre}})^2}.$$

Using (A.115),

$$\begin{aligned} N^{\text{post}} - N^{\text{pre}} &= w_s^2 - w_a^2 - w_b^2 \\ &= 2w_a w_b - 4\lambda q_a q_b (w_a + w_b) + 4\lambda^2 q_a^2 q_b^2. \end{aligned}$$

Hence

$$\begin{aligned} \Delta HHI^{Rev} &= \frac{N^{\text{post}}}{(\mathcal{R}^{\text{post}})^2} - \frac{N^{\text{pre}}}{(\mathcal{R}^{\text{pre}})^2} \\ &= \frac{N^{\text{post}} - N^{\text{pre}} + H^{\text{pre}}\left[2\mathcal{R}^{\text{pre}}(2\lambda q_a q_b) - (2\lambda q_a q_b)^2\right]}{(\mathcal{R}^{\text{pre}} - 2\lambda q_a q_b)^2}. \end{aligned}$$

Direct substitution of

$$w(q) = q[1 - \lambda(q-1)]$$

and collection of terms gives

$$\frac{N^{\text{post}} - N^{\text{pre}}}{2q_a q_b} = 1 + (2 - 3s)\lambda + (2s^2 - q_a q_b - 3s + 1)\lambda^2.$$

Substituting this identity into the preceding expression yields (69).

The last term inside the square brackets in (69) is strictly positive. Therefore it is sufficient that

$$1 + (2 - 3s)\lambda + (2s^2 - q_a q_b - 3s + 1)\lambda^2 \geq 0. \tag{A.116}$$

The coefficient on λ^2 is positive for $s \geq 2$. The smaller root of the quadratic in (A.116) is

$$\frac{2}{3s - 2 + \sqrt{s^2 + 4q_a q_b}},$$

which proves (70).

Since $q_a q_b \leq s^2/4$,

$$\sqrt{s^2 + 4q_a q_b} \leq \sqrt{2}\, s.$$

Thus the stronger but simpler condition

$$\lambda \leq \frac{2}{(3 + \sqrt{2})s - 2}$$

is sufficient. Since $\alpha \leq 1$ implies

$$\lambda = \frac{\alpha}{(1 + \alpha)(T - 1)} \leq \frac{1}{2(T - 1)},$$

condition (71) guarantees the preceding inequality and therefore guarantees $\Delta HHI^{Rev} > 0$ for any $\alpha \in (0, 1]$.

Alternatively, to guarantee the result for every feasible merger, use $s \leq T$. It is sufficient that

$$\frac{\alpha}{(1 + \alpha)(T - 1)} \leq \frac{2}{(3 + \sqrt{2})T - 2}.$$

Rearranging gives

$$\alpha \leq \frac{2(T - 1)}{(1 + \sqrt{2})T},$$

which is (72).

Finally, expected transaction markups are strictly increasing in

$$B(S)\phi(q_K).$$

After the merger,

$$B(S^{\text{post}}) = B(S^{\text{pre}}) - \frac{\alpha \Delta S}{T(T-1)}.$$

Therefore expected markups rise if and only if

$$[B(S^{\text{pre}}) - \frac{\alpha \Delta S}{T(T-1)}]\phi(q_K^{\text{post}}) > B(S^{\text{pre}})\phi(q_K^{\text{pre}}),$$

which is (73). If $s \leq q_K^{\text{pre}}$, the two ϕ terms are equal while $\frac{\alpha \Delta S}{T(T-1)} > 0$, so expected markups strictly fall. If $s > q_K^{\text{pre}}$, ϕ rises while B falls, and the sign is generally ambiguous.

On the interior branch,

$$R = \frac{c}{1 - B(S)\phi(q_K)}$$

is strictly increasing in the same index $B(S)\phi(q_K)$. Hence, in the new-largest-firm case, the markup condition (73) is exactly equivalent to the reservation-price condition (A.110). □