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Aggression of the Weak^{*}

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Abstract

This paper compares equilibrium bidding strategies in a first-price sealed-bid auction with two asymmetric bidders who have dependent private valuations. Modeling bidder strength via a stochastic asymmetry condition – where one bidder is systematically more likely to hold the maximum valuation – I show that the globally weak bidder, facing a competitive disadvantage at all top valuation levels, bids more aggressively at every valuation. Conversely, the stronger bidder wins more frequently, generalizing Maskin and Riley (2000) to dependent private-value environments. When the stochastic asymmetry flips once, so that each bidder is stronger over a different range, the bid distributions cross once, with each bidder bidding more aggressively where their competitive disadvantage is most acute. In both environments, allocative inefficiency is an inherent feature of the equilibrium, driven by the stronger bidder’s heightened incentives to shade bids relative to their valuation. The results translate to procurement auctions and wage competition between employers, and extend to interdependent-value auctions and the war of attrition under modified conditions.

Keywords: Asymmetric Auctions, Bid Shading, First Price Auctions, Dependent Valuations

JEL Classification: D44, C72, D82

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1 Introduction

First-price sealed-bid auctions are among the most widely studied mechanisms in economic theory, yet their analysis becomes substantially more complex when bidders are asymmetric and their valuations are dependent. The existing literature addresses each of these complications in isolation. The literature on asymmetric independent private values characterizes equilibrium bidding when bidders draw from different distributions but valuations are independent. The central finding, established by [Maskin and Riley \(2000\)](#), is that under reverse hazard rate dominance the stochastically weaker bidder bids more aggressively at every valuation, and the stronger bidder wins more often despite shading more. On the other hand, the affiliated values literature, originating with [Milgrom and Weber \(1982\)](#), allows for dependent valuations but restricts attention to symmetric bidders. Strategic bid shading in the asymmetric and dependent case is not covered in either strand, although in practice the two often come together. Bidders differ in capacity, technology, or informational advantage, but their stakes are correlated: they are exposed to similar market conditions, such as shared demand or common input costs, which makes the intersection of the asymmetry and the dependence of valuations the empirically relevant case.

This paper bridges these two literatures by characterizing equilibrium bidding behavior in a first-price sealed-bid auction with two asymmetric bidders whose private valuations are dependent. The central conceptual contribution is to argue that when valuations are dependent, bidder strength is an inherently joint property of the valuation distribution — it concerns not who tends to draw higher valuations marginally, but the systematic tendency for one bidder to draw higher and the other lower simultaneously. I formalize this idea with the *joint stochastic asymmetry (JSA)* order, which posits that the strong bidder is more likely to hold the top valuation, at every level of that top valuation. Since the auction allocates the object to the top bidder, the relevant primitive concerns who owns the top valuation. Under independence, JSA reduces exactly to the reverse hazard rate dominance condition of [Maskin and Riley \(2000\)](#), so it is the natural generalization of that condition to dependent environments. Although it implies first-order stochastic dominance of the marginals, it is strictly stronger, because it further disciplines the dependence structure in a way that FOSD alone does not.

The second primitive of the analysis is a mild regularity condition on the cop-

ula that governs how the conditional quantile distributions of the two bidders relate to each other. This condition is broadly satisfied because it holds, for example, for the independence copula; for all symmetric copulas with positive dependence; for the strict Archimedean family, which accommodates positive dependence, negative dependence, and independence as special cases; and for the Gaussian copula at all imperfect correlation levels. This flexibility matters for some applications. In timber auctions, bidders with different harvesting technologies may have positively correlated costs. Negative dependence is also possible in a two-bidder scenario. In local procurement auctions, if one bidder secures scarce low-cost inputs, such as skilled labor or raw materials, the remaining supply available to the other is depleted, potentially generating negative dependence in costs. The framework accommodates both environments without placing any sign restriction on the correlation between valuations.

The main results characterize equilibrium bidding under two cases of asymmetry. In the first case, one bidder generally values the object more highly, that is, JSA dominates globally. When this holds, the weaker bidder bids more aggressively at every valuation, but the stronger bidder wins the auction with strictly greater probability despite bidding less at every valuation. These results directly generalize the canonical findings of [Maskin and Riley \(2000\)](#) to dependent-valuation structures. The copula construct reduces the key comparisons to properties of the conditional distributions alone, which yields clean conditions for comparing bid shading across bidders.

In the second case, the valuation distributions cross once, so that neither bidder dominates globally. This configuration arises naturally when one bidder is better informed about the value of the object being auctioned — for instance, an incumbent firm with proprietary information about an asset’s quality, facing an entrant firm with broader but less precise beliefs. The incumbent may be more likely to have very high or very low valuations, while the entrant’s valuations are more concentrated in the middle of the support.¹ It can also arise when one bidder cares disproportionately about a certain characteristic of the object that determines its general desirability, for example, quality. A quality-sensitive bidder may value high-quality objects more while valuing low-quality objects less than their opponent.

I show that under this single-crossing condition, the equilibrium bid distributions

¹This case includes second-order stochastic dominance, but bidders need not have the same average valuations in this scenario.

also cross exactly once, so that each bidder holds the competitive advantage over a range that aligns with their area of strength. When a local version of the JSA order holds in addition, any crossing of the bidding strategies themselves is unique and occurs at an interior valuation, with each bidder outbidding the other precisely where their disadvantage is most severe, so the aggression of the weak reappears in local form.

Finally, I consider six related economic environments in which optimal behavior has a similar structure, and ask what stochastic asymmetry implies in each. Results remain intact in procurement auctions and duopsonies: the less efficient firm quotes the thinner margin, and the less productive employer posts the higher wage conditional on productivity. Interdependent value auctions require an additional restriction on how bidders weigh their own signal against their rival's, without which asymmetry in signals need not translate into asymmetry in bidding. Second-price auctions prescribe smaller earning margins to the weaker bidder, though this is a mechanical consequence of stochastic asymmetry rather than a strategic one. All-pay auctions strip the dependence structure out of the optimality conditions altogether, so that any ordering of efforts must come from the marginals alone and JSA has no role to play. The war of attrition retains the aggression of the weak, but the condition that delivers it is JSA stated on the survival copula rather than on the copula itself.

The paper is organized as follows. Section 2 sets up the model, introduces the copula framework, defines the dominance ordering, and derives the differential equation system that characterizes the equilibrium. Section 3 models asymmetry among bidders and discusses the main results on equilibrium bid shading and winning probabilities in each case. Section 4 discusses the applicability of the results to related economic environments. Section 5 concludes the paper. All proofs are shown in the Appendix.

2 Model

2.1 Environment

There are two bidders, $i \in \{1, 2\}$, competing for a single indivisible object in a first-price sealed-bid auction. Each bidder privately observes a valuation $X_i \in [0, \omega]$, where $\omega < \infty$. Valuations are *private*: bidder i 's payoff from winning at price b is $X_i - b$,

regardless of the opponent's valuation.

2.2 Joint Distribution and Copula

Let $F_i : [0, \omega] \rightarrow [0, 1]$ denote the marginal cumulative distribution function (CDF) of bidder i 's valuation, with associated density $f_i \equiv F_i' > 0$ on the interior of its support. We maintain the following regularity condition throughout.

Assumption 1. *For each $i \in \{1, 2\}$, F_i is twice continuously differentiable on $(0, \omega)$ with $f_i(x) > 0$ for all $x \in (0, \omega)$, $F_i(0) = 0$, and $F_i(\omega) = 1$.*

Valuations need not be independent. Their joint distribution is represented via a copula. By Sklar's theorem, there exists a unique copula $C : [0, 1]^2 \rightarrow [0, 1]$ such that the joint CDF of (X_1, X_2) satisfies

$$\Pr(X_1 \leq x_1, X_2 \leq x_2) = C(F_1(x_1), F_2(x_2)).$$

I assume that the copula that describes the dependence structure is well behaved, i.e., it is smooth enough to differentiate twice and places strictly positive density everywhere on the interior of the unit square, so that no probability mass concentrates on a curve. The partial derivatives of $C(u_1, u_2)$ are denoted $C_1 \equiv \partial C / \partial u_1$, $C_2 \equiv \partial C / \partial u_2$, and $C_{12} \equiv \partial^2 C / \partial u_1 \partial u_2 = C_{21}$, the cross-partials coinciding by continuity of the second derivatives.

Assumption 2. *The copula C is symmetric, $C(u, v) = C(v, u)$ for all $(u, v) \in [0, 1]^2$, twice continuously differentiable on $(0, 1)^2$, with $C_{12}(u, v) > 0$ for all $(u, v) \in (0, 1)^2$.*

Symmetry implies $C_2(u, v) = C_1(v, u)$, so the two bidders are distinguished only by their marginals and not by the dependence structure itself. Strict positivity of C_{12} means the joint distribution admits a positive density, and implies $C_1, C_2 > 0$ on $(0, 1)^2$ since $C_1(u, v) = \int_0^v C_{12}(u, t) dt$. It also rules out ties, $\Pr(X_1 = X_2) = 0$, so the tie-breaking rule plays no role in what follows.

I impose the following condition on the dependence structure.

Assumption 3. *For all $(u, v) \in (0, 1)^2$:*

$$u \leq v \iff C_1(u, v) \geq C_2(u, v)$$

Assumption 3 regulates the behavior of conditional quantile distributions of the two valuations. $C_1(u, v) \equiv \Pr(X_2 \leq F_2^{-1}(v) | X_1 = F_1^{-1}(u))$ gives the probability that bidder 2's valuation is below their v^{th} quantile given bidder 1's valuation is exactly at their u^{th} quantile. $C_2(u, v)$ is defined analogously. The condition says that whichever bidder draws from a lower quantile is more likely to have the opponent fall below their (higher) quantile threshold — capturing the idea that being at a disadvantage in terms of one's own draw makes the opponent's draw look relatively more favorable.

Assumption 3 effectively disciplines the nature of the dependence between valuations, but this is not a major restriction as I demonstrate next. It is satisfied by the independent copula and by all symmetric copulas that feature positive dependence, i.e., when a higher draw for one bidder makes a high draw for the other more likely in the first-order stochastic dominance sense. Formally, a distribution (V) is stochastically increasing in another (U) if $\Pr(V > v | U = u)$ is nondecreasing in u . Equivalently, a copula C is *stochastically increasing* (SI) if $C_1(u, v)$ is nonincreasing in u for each v , i.e., $C_{11} \leq 0$.² This is sufficient for a copula to satisfy Assumption 3 as remarked below.

Remark 1. *Assumption 3 is satisfied by the independence copula and by every symmetric copula that is stochastically increasing.*

Under independence $C(u, v) = uv$, so $C_1(u, v) = v$ and $C_2(u, v) = u$, and $C_1 \geq C_2 \iff u \leq v$. Now let C be symmetric and SI, and take $u \leq v$. Since $C_{11} \leq 0$, C_1 is nonincreasing in its first argument, and since $C_{12} > 0$, C_2 is increasing in its first argument. Hence

$$C_1(u, v) \geq C_1(v, v) = C_2(v, v) \geq C_2(u, v),$$

where the equality follows from the symmetry of C . Applying this with the roles of u and v exchanged and using $C_1(v, u) = C_2(u, v)$ gives $C_2(u, v) \geq C_1(u, v)$ whenever $v \leq u$, which supplies the converse and establishes Assumption 3. The second inequality is strict when $u < v$, so equality of C_1 and C_2 occurs only on the diagonal.

Assumption 3 does not rule out negative dependencies but it limits its shape. It is satisfied, for instance, by all copulas of the Archimedean family, which allows for a variety of distributions that display negative dependence in addition to positive dependence and independence as a special case.

²For symmetric C , the condition is direction-free, since $C_{22}(u, v) = C_{11}(v, u)$.

Definition 1 (Archimedean Copula). *A copula $C \in \mathcal{C}^A$ is Archimedean if there exists a generator $\psi : [0, 1] \rightarrow [0, \infty]$ that is continuous, strictly decreasing, strictly convex, twice differentiable on $(0, 1)$, and satisfies $\psi(1) = 0$ and $\psi(0) = +\infty$, such that*

$$C(u, v) = \psi^{-1}(\psi(u) + \psi(v)), \quad (u, v) \in [0, 1]^2.$$

Remark 2. *Assumption 3 is satisfied by all Archimedean copulas. For any Archimedean copula with generator ψ :*

$$\frac{C_1(u, v)}{C_2(u, v)} = \frac{\psi'(u)}{\psi'(v)}$$

Since ψ is strictly convex, ψ' is strictly increasing, $u \leq v$ if and only if $\psi'(u) \leq \psi'(v) < 0$, giving $\psi'(u)/\psi'(v) \geq 1$, i.e., $C_1(u, v) \geq C_2(u, v)$ if and only if $u \leq v$.

Another commonly used formulation is the bivariate normal distribution, which satisfies Assumption 3 for all (imperfect) correlation levels as the following remark formalizes.

Remark 3. *The Gaussian copula satisfies Assumption 3 for all $\rho \in (-1, 1)$.*

Letting (X_1, X_2) be jointly normal with correlation $\rho \in (-1, 1)$, the partial derivatives are:

$$C_1(u_1, u_2) = \Phi\left(\frac{z_2 - \rho z_1}{\sqrt{1 - \rho^2}}\right), \quad C_2(u_1, u_2) = \Phi\left(\frac{z_1 - \rho z_2}{\sqrt{1 - \rho^2}}\right),$$

where $z_i = \Phi^{-1}(u_i)$. Since Φ is strictly increasing, $C_1(u_1, u_2) \geq C_2(u_1, u_2)$ if and only if: $z_2 - \rho z_1 \geq z_1 - \rho z_2$, or, equivalently, $(1 + \rho)(z_2 - z_1) \geq 0$. Since $\rho > -1$, this holds if and only if $z_2 \geq z_1$, i.e., $u_2 \geq u_1$. Hence $C_1(u_1, u_2) \geq C_2(u_1, u_2) \iff u_1 \leq u_2$, establishing Assumption 3.

2.3 Bidder Asymmetry

To characterize how asymmetry between bidders shapes equilibrium bidding and the probabilities of winning the auctioned object, I introduce the following stochastic ordering.

Definition 2 (Joint Stochastic Asymmetry). *Bidder 2 JSA dominates bidder 1 at $x \in (0, \omega)$, written $X_2 \succ_{JSA}^x X_1$, if*

$$C_1(F_1(x), F_2(x)) f_1(x) < C_2(F_1(x), F_2(x)) f_2(x),$$

and $X_1 \succ_{JSA}^x X_2$ if the reverse strict inequality holds. If $X_2 \succ_{JSA}^x X_1$ for every $x \in (0, \omega)$, then bidder 2 JSA-dominates globally, denoted by $X_2 \succ_{JSA} X_1$.

Global JSA designates bidder 2 as one with the higher valuation of the auctioned object if, for any top valuation $x = \max\{X_1, X_2\}$, bidder 2 is more likely than bidder 1 to be the one with that valuation. To see this in terms of density ratios, note that $\Pr(\max\{X_1, X_2\} \leq x) = C(F_1(x), F_2(x))$. Differentiating splits the density of the higher valuation according to which bidder holds it:

$$\frac{d}{dx} C(F_1(x), F_2(x)) = \underbrace{C_1(F_1(x), F_2(x)) f_1(x)}_{\text{bidder 1 holds it}} + \underbrace{C_2(F_1(x), F_2(x)) f_2(x)}_{\text{bidder 2 holds it}},$$

because $C_1(F_1(x), F_2(x)) = \Pr(X_2 < x \mid X_1 = x)$ is the chance that bidder 1's draw of x is the larger of the two. Relative to the total density of top valuations at x , the density of x belonging to bidder 2 is greater than a half if and only if they are JSA-dominant at x :

$$\frac{C_2 f_2}{C_1 f_1 + C_2 f_2} > \frac{1}{2} \iff C_1 f_1 < C_2 f_2,$$

where arguments are suppressed for convenience. Requiring this at every x makes bidder 2 the globally stronger bidder.

Further intuition can be gained from the following related condition for $x < z$:

$$\Pr(X_1 < z, X_2 < x) < \Pr(X_1 < x, X_2 < z),$$

which compares the probability that bidder 1 draws a high value and bidder 2 draws a low value, against the probability of the reverse configuration. Requiring the latter to exceed the former for all $x < z$ in $(0, \omega)$ systematically makes bidder 2 the stronger one in a joint distributional sense. This requirement implies $C_1 f_1 \leq C_2 f_2$, a weaker version of JSA dominance.³ Yet, it is more restrictive at the same time because JSA

³To see this, subtract $\Pr(X_1 < x, X_2 < x)$ from both sides of the inequality, divide by $(z - x)$, and let $z \rightarrow x$.

dominance restricts the dependence structure only along the diagonal, $z \simeq x$, whereas the condition above is a requirement on all $\{x, z\}$ pairs.

Note also that the JSA restriction is on the joint law rather than on the marginals alone, since the conditional probabilities C_1 and C_2 carry the dependence structure. Under independence $C_1 = F_2$ and $C_2 = F_1$, and the condition collapses to $f_2/F_2 > f_1/F_1$, the reverse hazard rate dominance used by [Maskin and Riley \(2000\)](#), so JSA is the natural generalization of that condition to environments where bidders' valuations are dependent.

More generally, global JSA dominance is a stronger condition than first-order stochastic dominance (FOSD) in dependent environments under Assumptions 1–3 as the following lemma establishes:

Lemma 1. *Under Assumptions 1–3, $X_2 \succ_{JSA} X_1 \implies F_2 \succ_{FOSD} F_1$, with $F_2(x) < F_1(x)$ for all $x \in (0, \omega)$.*

Next, I define the auction game and the associated bidding strategies.

2.4 Bidding Strategies and Payoffs

A *bidding strategy* for bidder i is a measurable function $\beta_i : [0, \omega] \rightarrow \mathbb{R}_+$. The object is awarded to the highest bidder; ties are broken by an arbitrary deterministic rule. Bidder i with valuation x who submits bid b earns

$$\Phi_i(b; x) = (x - b) \cdot \Pr(\beta_j(X_j) \leq b \mid X_i = x), \quad j \neq i.$$

I characterize equilibria in strictly increasing pure strategies, i.e., $\beta'_i(x) > 0$ for all $x \in (0, \omega)$.⁴ Given strict monotonicity, the inverse bid function $\phi_i \equiv \beta_i^{-1} : [\beta_i(0), \beta_i(\omega)] \rightarrow [0, \omega]$ is well defined and strictly increasing, with $\phi'_i(b) \equiv d\phi_i/db > 0$. Define the distribution of bids submitted by bidder i as

$$H_i(b) \equiv F_i(\phi_i(b)), \quad h_i(b) \equiv H'_i(b) = f_i(\phi_i(b)) \phi'_i(b).$$

$H_i(b)$ is the probability that bidder i submits a bid below b , and $h_i(b)$ is the associated density.

⁴Conditions under which such equilibria exist, and under which strict monotonicity obtains as a derived property, are explored in the literature ([Athey, 2001](#); [Reny and Zamir, 2004](#); [Olszewski et al., 2026](#)). I take the existence of such an equilibrium as given and characterize its properties.

2.5 Equilibrium Bid Strategies

Since β_j is strictly increasing, bidder i beats bidder j if and only if $X_j < \phi_j(b)$. Using the copula representation, the winning probability for bidder i with valuation x submitting bid b is

$$P_i(b; x) = C_i(u_1, u_2) \quad u_i = F_i(x), \quad u_j = F_j(\phi_j(b)), \quad i \neq j, \quad i, j \in \{1, 2\}, \quad (1)$$

where C_i denotes the partial derivative of C with respect to its i -th argument, evaluated with $u_i = F_i(x)$ and $u_j = F_j(\phi_j(b))$ for $j \neq i$. Under independence, $P_i(b; x) = H_j(b)$ since x carries no information about the opponent's valuation.

Bidder i chooses b to maximize $(x - b) \cdot P_i(b; x)$. The first-order condition equates the marginal gain from raising the bid, the increased probability of winning, to its marginal cost, the higher payment conditional on winning:

$$(x - b) \frac{\partial P_i}{\partial b} = P_i(b; x). \quad (2)$$

Each bidder's winning probability depends on the other bidder's bidding strategy. Substituting for $P_i(b; x)$ from equation (1) for each bidder, gives a system of first-order nonlinear ordinary differential equations (ODEs) in inverse bid functions, $\phi_1(b)$ and $\phi_2(b)$, the solution to which characterizes the equilibrium bid strategies.

$$(\phi_1(b) - b) C_{12}(H_1(b), H_2(b)) f_2(\phi_2(b)) d\phi_2 - C_1(H_1(b), H_2(b)) db = 0, \quad (3)$$

$$(\phi_2(b) - b) C_{21}(H_1(b), H_2(b)) f_1(\phi_1(b)) d\phi_1 - C_2(H_1(b), H_2(b)) db = 0, \quad (4)$$

with boundary conditions $\beta_i(0) = 0$ and $\beta_i(\omega) = \bar{b} \in (0, \omega)$ for each $i \in \{1, 2\}$. The equilibrium bid functions $\beta_i = \phi_i^{-1}$ are strictly increasing and the associated winning probabilities are

$$P_1(x) = C_1(F_1(x), H_2(\beta_1(x))) \quad (5)$$

$$P_2(x) = C_2(H_1(\beta_2(x)), F_2(x)). \quad (6)$$

Note that the winning probability reduces to $H_j(b)$ under independence because bid-

der i 's own valuation carries no information about the opponent. With dependent valuations, $P_i(b; x) = C_i(F_i(x), F_j(\phi_j(b)))$ depends on x via C_i . Drawing x revises bidder i 's beliefs about the opponent's valuation, and therefore about what they will bid. Bid shading is thus driven by two forces. The first is the familiar trade-off between surplus margin and the chance of winning, present in any first-price auction. The second comes from valuation dependence. Under positive dependence, for instance, a bidder who draws a high valuation infers that the opponent is likely to have done so as well, which narrows the room for shading. This inference channel is what the copula regularity condition in Assumption 3 disciplines, and it is what makes the relevant notion of bidder strength a joint property of the valuation distribution rather than a comparison of marginals.

A key property of any equilibrium is that both bidders submit the same minimum and maximum bids at the ends of the valuation support. When bidders do not value the object, they naturally bid nothing: $\beta_i(0) = 0$. At $x = \omega$, both bid the same amount $\bar{b}$ (Lebrun, 1999). To see this, suppose instead $\beta_1(\omega) > \beta_2(\omega)$. Then bidder 1's highest type could profitably shade their bid down to just above $\beta_2(\omega)$ and still win with probability one against all types of bidder 2, contradicting optimality. The symmetric argument rules out $\beta_2(\omega) > \beta_1(\omega)$. Therefore, $\beta_1(\omega) = \beta_2(\omega) \equiv \bar{b}$ must hold in equilibrium.

Because strictly increasing strategies preserve the copula, the pair $(\beta_1(X_1), \beta_2(X_2))$ has copula C with marginals H_1, H_2 . The CDF of the winning bid is therefore $Pr(\max\{\beta_1(X_1), \beta_2(X_2)\} \leq b) = C(H_1(b), H_2(b))$. Integrating by parts on $[0, \bar{b}]$ gives expected revenue:

$$R^I = \bar{b} - \int_0^{\bar{b}} C(H_1(b), H_2(b)) db.$$

3 Equilibrium Bid Strategies under Asymmetry

I now characterize how asymmetry in the valuation distributions shapes equilibrium bid shading, analyzing two cases in particular. The first considers global asymmetry with one bidder exhibiting higher valuation of the auctioned item throughout the valuation distribution in the JSA-dominance sense. The second considers distributions that cross once, so that each bidder dominates in a specific region of the valuation distribution.

Assumption 4 (Global JSA Dominance). $X_2 \succ_{JSA} X_1$.

Assumption 5 (Single Crossing). *There exists a unique $\tilde{x} \in (0, \omega)$ such that*

$$F_2(x) < F_1(x) \iff x > \tilde{x}.$$

Assumption 6 (Local JSA Dominance). *There exists $\hat{x} \in (0, \omega)$ with $\hat{x} > \tilde{x}$ such that $X_2 \succ_{JSA}^x X_1 \iff x > \hat{x}$.*

Assumption 5 captures situations where one bidder has a comparative advantage at high valuations and the other at low valuations. Assumption 6 strengthens this by requiring the JSA ordering to also reverse at some valuation $\hat{x} > \tilde{x}$.⁵

Assumptions 5–6 together describe situations that arise naturally in auctions. One interpretation is informational. Suppose the object has an uncertain component of value, such as the timber volume or species mix on a tract, and bidder 1 is better placed to assess it. Suppose both bidders have the same expected valuation, but bidder 1’s is concentrated while bidder 2, working from coarser information, holds a more dispersed distribution. The distributions cross once at the common mean, so neither bidder is stronger globally. The informed bidder has the comparative advantage at low valuations, where their compact distribution makes them more likely to hold the higher draw, and the less-informed bidder has the comparative advantage at high valuations, where the tail of their dispersed distribution reaches further.

Alternatively, suppose the object has a quality dimension q that both bidders observe, but value differently, such as a sushi restaurant that needs top-grade fish versus a fry shop that does not, competing at a wholesale fish auction. Writing $\ln X_i = \mu + \alpha_i(q - \bar{q}) + \varepsilon_i$, where $\bar{q}$ is the average quality of the auctioned object and α_i is bidder i ’s valuation of quality, both bidders have the same average valuation μ but different variances whenever the α_i differ. Setting $\alpha_2 > \alpha_1$ makes bidder 2 more selective: they are willing to pay more for high-quality objects and less for low-quality ones than their opponent is. The distributions again cross once, and each bidder is locally stronger on their own ground — the less selective bidder at the bottom of the quality range, the more selective bidder at the top.

⁵Assumptions 5 and 6 are only compatible if $\hat{x} \geq \tilde{x}$. Since the marginals coincide at $\tilde{x}$, JSA dominance would require $f_1(\tilde{x}) < f_2(\tilde{x})$. That contradicts Assumption 5, which has F_1 crossing F_2 from below at $\tilde{x}$ and therefore requires $f_1(\tilde{x}) \geq f_2(\tilde{x})$. The strict inequality is imposed to keep $(\tilde{x}, \hat{x})$ nonempty.

The two interpretations generate the same distributional configuration, and the results below apply to either. I adopt the second, referring to this case as “asymmetric selectivity” with bidder 2 the selective bidder.

Figure 1 illustrates the marginal distributions and JSA-dominance conditions for two numerical examples that are used to illustrate equilibrium bidding behavior below. For these illustrations, I assume that each $F_i(X_i)$ is an upper-truncated log-normal distribution with mean $\mu_i = \mathbb{E}[\ln X_i]$ and variance $\sigma_i^2 = \mathbb{V}[\ln X_i]$. I model the co-dependence between the two marginal distributions with a Clayton copula, which is an Archimedean copula with the generator function $\psi(t) = (t^{-\theta} - 1)/\theta$ for $\theta > 0$. Because the Clayton copula is from the Archimedean family, it satisfies Assumption 3 (see Remark 2). I set the dependency parameter to 4 for the displays, which implies a moderately positive relationship between the two bidders’ valuations with a Kendall’s $\tau = \theta/(\theta + 2) = 2/3$.

I model the two cases of asymmetry as follows. For global dominance, I set the second bidder’s marginal distribution to have a higher mean parameter: $1 = \mu_2 > \mu_1 = 0$ but the same dispersion parameter $\sigma_1 = \sigma_2 = 1$. Bidder 2 is stronger in the sense that their valuation distribution stochastically dominates that of the competing bidder. Although $F_2 \succ_{FOSD} F_1$ does not generally imply $X_2 \succ_{JSA} X_1$, the latter is true in this particular case. Figure 1c shows the relevant ratio from Definition 2, which implies $X_2 \succ_{JSA}^x X_1$ when $C_1(F_1(x), F_2(x))f_1(x)/C_2(F_1(x), F_2(x))f_2(x)$ is below 1 and $X_1 \succ_{JSA}^x X_2$ otherwise.

For local dominance under asymmetric selectivity, I set $\mu_1 = \mu_2 = 0$ and $1 = \sigma_1 < \sigma_2 = 1.2$. This ensures that the distribution functions cross once and implies that bidder 2 has a more dispersed valuation distribution relative to bidder 1 even as bidder 2 has the same mean. Figure 1d shows the JSA dominance condition in this case. In this example, for low realizations of valuations, $X_1 \succ_{JSA}^x X_2$, while the reverse is true for high valuations.

Next, I analyze these two cases and characterize the equilibrium bids and winning probabilities.

3.1 Global Valuation Asymmetry

When competing against a bidder who generally values the object more, the weaker bidder places a more aggressive bid in order to raise their chances of winning the

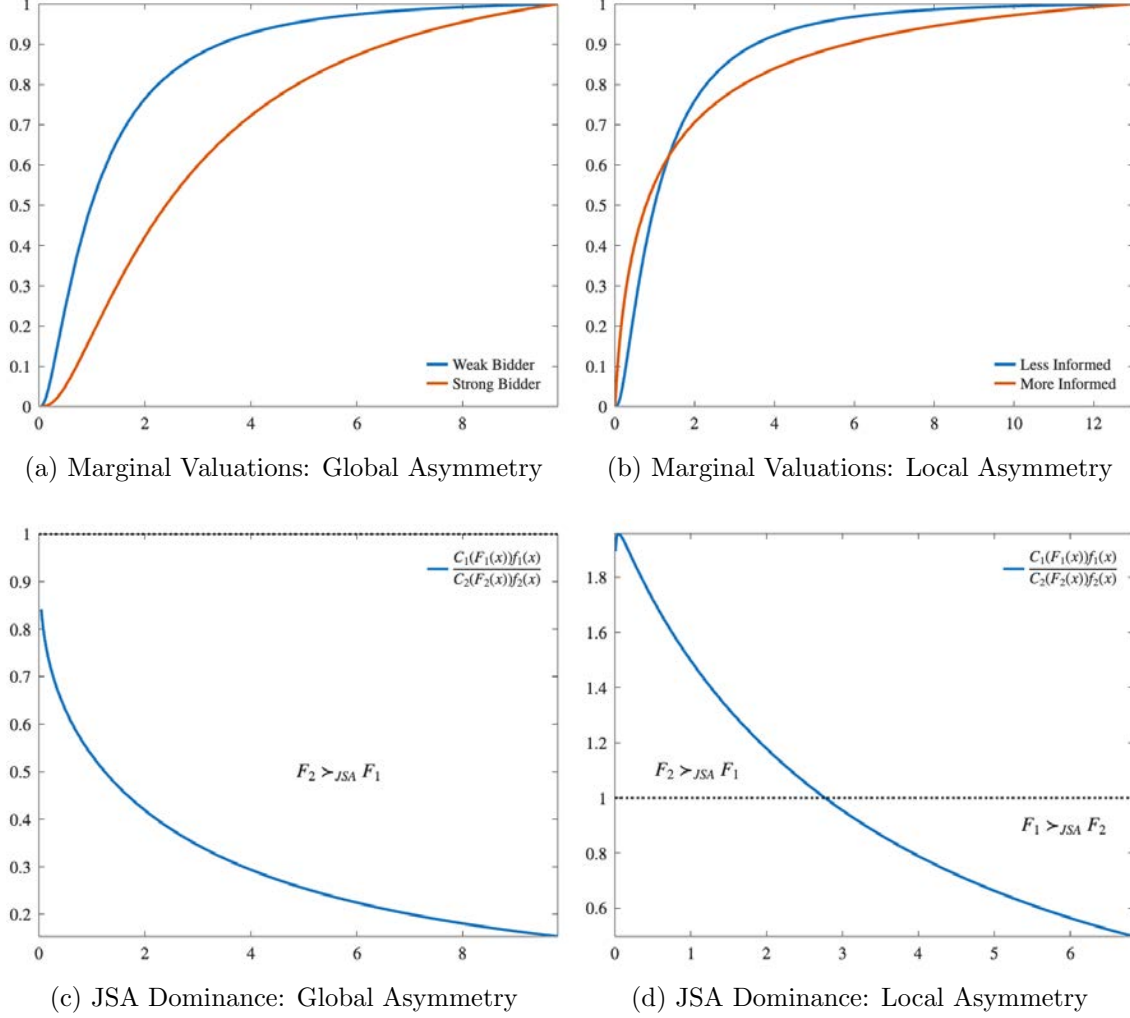


Figure 1: Marginal Distributions and Dominance under Asymmetry.

Notes.— Panels (a) and (b) plot the marginal valuation distributions $F_1(X_1)$ and $F_2(X_2)$ for the two cases of asymmetry considered in text. Panels (c) and (d) plot the corresponding JSA dominance conditions, where values below 1 indicate bidder 2 JSA dominates bidder 1. Bidders draw their valuations from a log-normal distribution either with parameters $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$ (Panels a, c) or with parameters $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$ (Panels b, d). Co-dependence is modeled by a Clayton copula with $\theta = 4$ in both cases.

auction. The stronger bidder shades their bid more, knowing that the opponent's willingness to pay for the object is relatively low. Despite the aggressive bidding by the weak bidder, the strong bidder is more likely to win the auction.

Theorem 1 (Aggression of the Weak). *Suppose Assumptions 1–3 hold. Then in equilibrium:*

(i) *If $F_2 \succ_{FOSD} F_1$, then $H_1(b) > H_2(b)$ for all $b \in (0, \bar{b})$.*

(ii) *If $X_2 \succ_{JSA} X_1$, then $\beta_1(x) > \beta_2(x)$ for all $x \in (0, \omega)$.*

Part (i) of the theorem says that bidder 2's bid distribution stochastically dominates bidder 1's, i.e., to justify any given bid, bidder 1 must draw from a higher point in their own distribution than bidder 2 does. A direct consequence is that the stochastically dominant bidder wins more often as the following corollary states.

Corollary 1. *Suppose Assumptions 1–3 hold. If $F_2 \succ_{FOSD} F_1$, then*

$$\Pi_2 > \frac{1}{2} > \Pi_1,$$

where $\Pi_i \equiv \int_0^\omega P_i(\beta_i(x); x) dF_i(x)$ denotes bidder i 's ex ante probability of winning the auction before the valuations are observed.

Part (ii) says that, at any given valuation, bidder 1, the weaker bidder, bids higher, accepting a lower surplus, $x - \beta_1(x) < x - \beta_2(x)$. In that sense, the weaker bidder competes more aggressively. The result reflects the trade-off between the two margins of bid shading. Shading is profitable when it is inconsequential, e.g., when the opponent's bid is likely to fall short anyway. Facing a stochastically weaker rival, bidder 2 is in that position, and so shades heavily. In comparison, bidder 1 knows the opponent tends to draw higher valuations and bid more, so the same shading would cost them the object too often. Consequently, bidder 1 bids closer to their valuation. The weaker bidder is more aggressive because aggression is cheap when the expected surplus is small, and expensive when it is large. This is the *aggression of the weak*.

Both parts of Theorem 1 are proved by contradiction in the Appendix, and both rely on the equilibrium requirement that the two bidders submit the same maximum bid. Were the bid distributions to ever cross, bidder 2 would have to draw on a higher valuation to submit the same bid (by FOSD), which makes their bid distribution rise

faster. The gap between the distributions would then only widen thereafter, so the two could not meet at $\bar{b}$. Similarly, at any point where the bid strategies cross, the weaker bidder's is strictly the steeper (by JSA dominance), and once bidder 1 bids more for all higher valuations, in conflict with a common bid at ω .

Note that the two parts rest on different hypotheses. Part (i), and its corollary, require only first-order stochastic ordering of the marginals, whereas Part (ii) requires the full JSA order, which disciplines the dependence structure as well. Who is likely to win is governed by the marginals, whereas how the two bidders shade their bids depends additionally on how their valuations relate to each other. Under independence, the comparison rests on the marginals alone, as in Maskin and Riley (2000), but with dependence, the relevant ordering is determined by the joint distribution.

Equilibrium bidding strategies and the allocation of the object under global valuation asymmetry are illustrated in Figure 2a. The left panel shows the bids as functions of valuations. The dashed black line represents the 45° line where bids match valuations. Both bidders shade their bids relative to their valuation, and they have identical bids at valuation boundaries. For any given valuation in the interior, the stronger bidder bids less, consistent with Theorem 1. The weak bidder understands that their competitor has a stochastically higher valuation of the object and, therefore, bids more aggressively to stay competitive.

Figure 2b shows the equilibrium allocation in a typical asymmetric equilibrium when bidder 2 has stochastically higher valuations. The axes are the quantiles of valuations for each bidder. The solid red curve shows the locus of iso-valuation pairs: $(F_1(x), F_2(x))$ and lies below the 45-degree line (dashed) because bidder 2's distribution is stochastically dominant, e.g., the median value in bidder 1's distribution is below median in bidder 2's distribution. The region above the iso-valuation locus represents points where bidder 2 values the object more than bidder 1. The likelihood of that is greater than half, because the region above the red curve contains the region above the diagonal, which has probability 1/2 since C is exchangeable.

The solid black line shows the locus of valuations that result in the same bid: (x_1, x_2) such that $\beta(x_1) = \beta(x_2)$. The region above the curve represents valuation combinations that result in the strong bidder outbidding the weak bidder. That the iso-bid curve lies below the 45-degree line, which splits the probability space into two halves, implies the strong bidder is more likely to win the auction in expectation. Because the stochastically stronger bidder wins more often, the auction allocates in

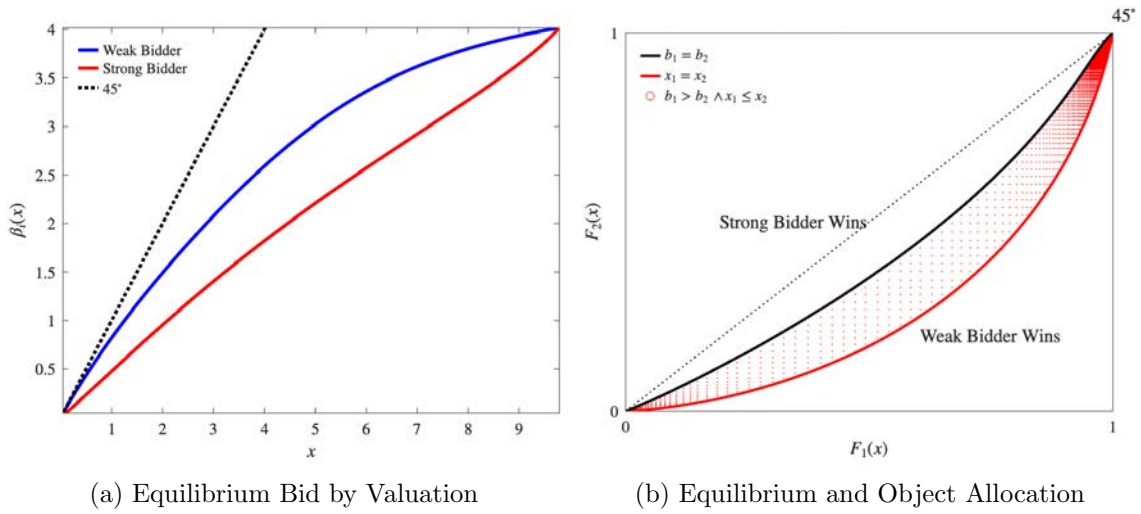


Figure 2: Equilibrium Bids and Allocation under Global Asymmetry.

Notes.— The figure plots the optimal bids against each bidder's valuation (left) and against the quantile of their valuation (right). Each bidder draws their valuation from a log-normal distribution with $\sigma = 1$ and either with $\mu = 0$ (weak) or with $\mu = 1$ (strong). Co-dependence of valuations is modeled by a Clayton copula with $\theta = 4$.

the right direction on average, even though it does so imperfectly.

Importantly, however, the dotted region between the two loci shows valuation pairs that lead to inefficient allocation: bidder 2 values the object more, yet loses to bidder 1, who bids more aggressively relative to their own valuation.

3.2 Selectivity and Local Valuation Asymmetry

I now turn to the case where neither bidder dominates globally. Bidder 2 is more selective, valuing the object more at higher valuations and less at lower valuations. Specifically, when the valuation distributions cross once, the bid distributions H_1 and H_2 also cross exactly once as the following theorem states.

Theorem 2. *If Assumptions 1- 3 and 5 hold, then there exists a unique $b^* \in [0, \bar{b})$ such that*

$$H_1(b) \geq H_2(b) \iff b \geq b^*.$$

For $b > b^*$, bidder 1's bid distribution dominates, i.e., bidder 1 requires a higher valuation to justify a high bid. For $b < b^*$, bidder 2's bid distribution dominates. The threshold b^* is determined endogenously.⁶

The proof tracks the gap between the two bid distributions, $H_1(b) - H_2(b)$, which vanishes at both ends of the bid range. Above $\tilde{b} = \beta_2(\tilde{x})$ any bid corresponds to a valuation above $\tilde{x}$ for bidder 2, and the logic from Theorem 1 implies $H_1(b) > H_2(b)$. Below $\tilde{b}$ the ordering of the marginals reverses, and bidder 1 now has the faster-rising bid distribution wherever the gap is non-negative. A gap that only widens once it turns positive can cross zero at most once, and only from below.

This result relies only on the single-crossing of marginal distributions (Assumption 5), and does not require the local JSA dominance described in Assumption 6. The latter is instrumental, however, in characterizing the equilibrium bid-shading behavior stated next.

Theorem 3. *Suppose Assumptions 1-3, 5 and 6 hold. If the equilibrium bid functions cross at some $x^* \in (0, \omega)$, then the crossing is unique, $x^* \in (\tilde{x}, \hat{x})$, and*

$$\beta_1(x) > \beta_2(x) \iff x > x^*.$$

⁶Note that $b^* = 0$ is a possibility, arising when $F_j(x)$ cross near 0 so that $X_2 \succ_{JSA}^x X_1$ for most of the range, in which case the equilibrium is akin to global valuation asymmetry, where one bidder dominates everywhere.

When the bid functions cross, they do so exactly once, at x^* . Above x^* , bidder 1 bids more aggressively; below x^* , bidder 2 does. Each bidder is more aggressive in the region where they are locally weaker, and shades more where they are locally stronger — local aggression of the weak.

The proof compares the slopes of the bidding strategies where they meet, and shows that bidder 1’s strategy is the steeper one when the crossing lies below $\hat{x}$, and the flatter one when it lies above (by JSA dominance). Crossings above $\hat{x}$ are ruled out by the logic of Theorem 1, and crossings below $\tilde{x}$ by Theorem 2. Any crossing therefore lies between the two thresholds, with bidder 1’s strategy cutting from below, and is unique.

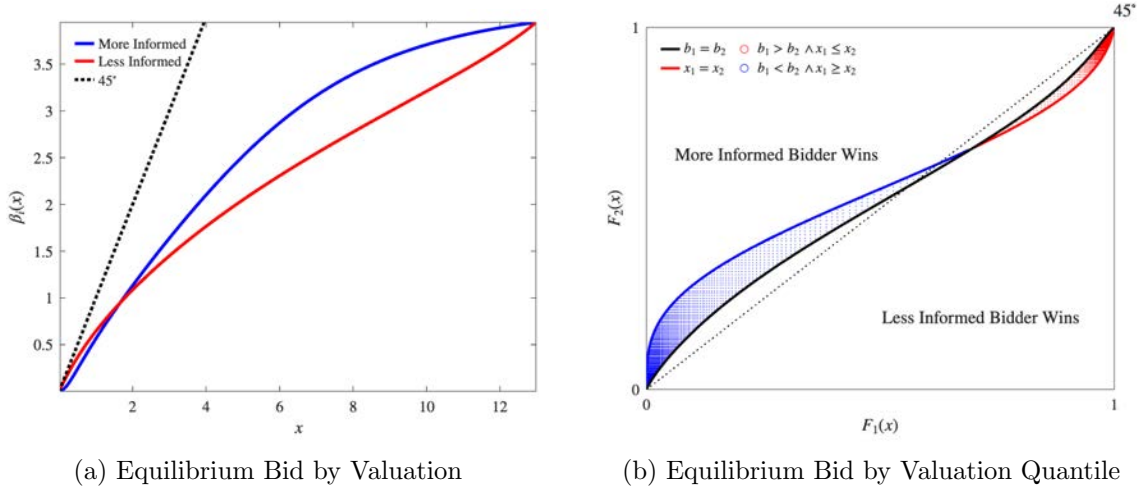


Figure 3: Equilibrium Bidding and Allocation under Asymmetric Selectivity.

Notes.— The figure plots the optimal bids against each bidder’s valuation (left) and against the quantile of their valuation (right). Each bidder draws their valuation from a log-normal distribution with $\mu = 0$ and either with $\sigma = 1$ (less selective) or with $\sigma = 1.2$ (more selective). Co-dependence of valuations is modeled by a Clayton copula with $\theta = 4$.

Figure 3a shows the equilibrium bids by valuation in this case. Once again, both bidders shade their bids relative to their valuation, and they have identical bids at valuation boundaries. Consistent with Theorem 3, the bidding functions cross once in the interior. The less selective bidder (bidder 1) has a competitive edge at low valuation levels. Because there is positive dependence between the two bidders’ valuations, conditional on a low valuation draw, there is a good chance that the

opponent also draws a low valuation. From bidder 1’s perspective, they are bidding for a low-quality object against a bidder who is selective about item quality and, therefore, unlikely to bid strongly. In this sense, bidder 1 is the stronger bidder for low-quality objects and shades their bid relatively more when their own valuation is low.

From the selective bidder’s perspective, the opposite reasoning prevails. Conditional on having a low valuation, the opponent is likely to have a low valuation as well, but that likelihood is limited by the fact that bidder 1’s possible valuations are more compactly distributed around the mean. This makes bidder 1 a stronger opponent in this region and causes the selective bidder to bid more aggressively.

An analogous logic applies conditional on having a high valuation of the object. If the opponent has a more dispersed distribution, that raises the odds of the opponent having a higher valuation, incentivizing the bidder to bid more aggressively. In other words, the less-selective bidder understands they are bidding for what they view as a high-quality object—which implies the opponent is likely to view it that way as well—and the opponent values quality more than the less-selective bidder does. As a result the less-selective bidder shades their bid by less to remain competitive in the auction. Indeed bidder 1 outbids bidder 2 at each valuation above the crossing point of bid functions in Figure 3a.

Figure 3b summarizes the allocation under asymmetric selectivity. Recall that bidder 2 has a more dispersed distribution of valuations. The red iso-valuation curve now crosses the 45-degree line (at $\tilde{x}$) where $F_1(x) = F_2(x)$ by Assumption 5. Below $\tilde{x}$, bidder 1 is more likely to have the higher valuation, whereas above $\tilde{x}$, bidder 2 is more likely to have the higher valuation. The iso-bid locus follows a similar pattern, crossing the 45-degree line from above. Below the crossing point, bidder 1 is more likely to win the object, whereas bidder 2 has a winning advantage for high realization of valuations.

The two dotted regions show valuation pairs that lead to inefficient allocation, where the bidder who values the object less wins it. In the region between the two loci below the crossing point (blue), bidder 2 wins, although bidder 1 values the object more; above the crossing point (red), bidder 1 wins, although bidder 2 values the object more.

4 Asymmetry in Related Economic Environments

This section works through six economic environments in which optimal behavior has a similar structure, and asks what stochastic asymmetry implies in each. Procurement auctions and duopsonies map to the first-price auction directly and require only a translation of terms. Common value auctions with interdependent values, second-price auctions, and the war of attrition each require a modification. All-pay auctions do not map at all, and are included because the reason they fail highlights what is special about the first-price structure. Throughout, I take as given that an equilibrium in strictly increasing strategies exists and characterize its properties. Establishing existence is not always straightforward in these environments, and has been the subject of a separate literature.⁷ The focus here is on what stochastic asymmetry under dependence delivers once such an equilibrium is in place.

The findings differ across environments. Results remain intact in procurement auctions and duopsonies. The weaker, less efficient firm quotes the thinner margin, i.e., marks down its bid or wage less, relative to productivity. Common and interdependent value auctions require an additional restriction on how bidders weigh their own signal against their rival's, without which asymmetry in signals need not translate into asymmetry in bidding. Second-price auctions prescribe smaller earning margins to the weaker bidder, though this is a mechanical consequence of the distributions rather than a strategic one. All-pay auctions strip the dependence structure out of the optimality conditions altogether, so that any ordering of efforts must come from the marginals alone and JSA has no bite. The war of attrition retains the aggression of the weak, but the condition that delivers it is JSA stated on the survival copula rather than on the copula itself.

4.1 Procurement Auctions

Public procurement is the setting in which asymmetry among bidders is best documented. Firms competing for the same contract can differ in capacity utilization, backlog, distance to the site, and access to specialized equipment, and the empirical literature routinely rejects the hypothesis that they draw costs from a common

⁷See, for instance, [Lizzeri and Persico \(2000\)](#) and [Lu and Parreiras \(2017\)](#) on interdependent valuations, and [Amann and Leininger \(1996\)](#) on all-pay auctions, where the weaker bidder may exert no effort over an interval of types.

distribution (Porter and Zona, 1993; Bajari and Ye, 2003; Flambard and Perrigne, 2006). Dependence is equally natural here as bidders on the same contract often face common input prices and share a common macroeconomic environment.

The similarity between procurement auctions and first-price auctions is well recognized. It is nonetheless informative to formally map the problems to each other to better interpret the assumptions and their implications. Suppose that firm i has a privately known cost Z_i of executing the contract, drawn from $\tilde{F}_i$ on $[0, \bar{z}]$ with joint distribution of (Z_1, Z_2) generated by a copula $\tilde{C}$. Firms simultaneously quote prices, and the lowest price wins. Firm i with cost z therefore maximizes $(p - z) \Pr(\text{rival's price} > p \mid Z_i = z)$. If the rival firm prices according to a strictly increasing rule π_j with inverse ψ_j , the rival's price exceeds p when its cost exceeds $\psi_j(p)$, so the problem is

$$\max_p (p - z) \left[1 - \Pr(Z_j \leq \psi_j(p) \mid Z_i = z) \right]. \quad (7)$$

Reversing both the cost and the price around the upper bound converts the low-price event into the high-bid event. Specifically, set

$$X_i = \bar{z} - Z_i, \quad b = \bar{z} - p, \quad \beta_i(x) = \bar{z} - \pi_i(\bar{z} - x), \quad (8)$$

and note that the payoff is preserved, $p - z = x - b$. The problem in (7) becomes

$$\max_b (x - b) \Pr(\beta_j(X_j) \leq b \mid X_i = x)$$

which is the optimal bidding problem from Section 2.5 once two objects are relabeled in the background. Valuation distributions are the reflected survival functions of the cost distributions, $F_i(x) = 1 - \tilde{F}_i(\bar{z} - x)$, so that $F_2 \succ_{FOSD} F_1$ means firm 2 draws stochastically lower costs. And the copula of (X_1, X_2) is the survival copula of the cost vector,

$$C(u, v) = u + v - 1 + \tilde{C}(1 - u, 1 - v), \quad (9)$$

representing the dependence structure of firms' cost advantages $\bar{z} - Z_i$ rather than of their costs. Regularity Assumptions 2 and 3 are imposed on C .

Global JSA dominance (Assumption 4) says the strong firm is more likely to be the low-cost firm, at every cost level, which renders it the more efficient producer.

Theorem 1 then says that the less efficient firm quotes a thinner margin for the contract, at every cost realization, while the efficient firm wins the contract more often despite charging a larger markup. The buyer therefore does not capture the full benefit of having an efficient supplier in the pool.

The single-crossing configuration of Assumptions 5–6 describes firms with different cost *structures* rather than different cost *levels*: e.g., one firm with low fixed and high variable costs, another with the reverse, so that each is the efficient producer over a range of job sizes. This is a common feature of procurement markets in which large and small contractors compete against each other (Athey et al., 2011). Theorem 3 says each contractor quotes their thinnest margins on precisely the jobs for which it is the higher-cost supplier.

In both cases, lowest-price procurement can fail to select the low-cost supplier. There is a range of cost pairs in which the buyer awards the contract to the higher-cost firm because the efficient firm uses its cost advantage to charge higher markups.

4.2 Duopsonies

Another related environment is the wage setting problem when employers compete for the same worker. Two employers may value the same worker differently because of differences in technology or organizational fit. Each knows their own valuation but not its rival’s. Competition takes the form of wage offers, and the worker accepts the higher one (Bils et al., 2025). The environment can be asymmetric when the distribution of match values differs systematically across employers. It is dependent because the worker’s general skills raise productivity at both firms.

In this setting, X_i is a worker’s productivity at firm i , with C shaping the correlation of X_i between firms. JSA dominance says firm 2 is more likely to be the high-productivity employer for this worker, at every productivity level. The single-crossing configuration (asymmetric selectivity) corresponds to situations where firm 2’s production technology is more sensitive to worker quality, which could arise, for instance, if common skills are valued more at firm 2.

Theorem 1 says the *less* productive employer posts the higher wage relative to their own productivity, surrendering some of the match surplus in order to remain competitive. The more productive employer marks the wage down further below productivity, but is still more likely to hire the worker. Theorems 2 and 3 say that

firm 2, where skill is more valuable, has a competitive advantage over their rival for a high-productivity worker, therefore applying larger markdowns if the worker is highly productive. By contrast, firm 1 marks wages down less to remain competitive for such a worker. The reverse configuration ensues for a low-productivity worker. Wage competition between asymmetric employers is a source of misallocation because the worker does not always go to the employer who values them most.

In a full-information oligopsony model with competition in employment, [Berger et al. \(2022\)](#) show that more productive firms, the stronger employers, mark down wages by more. A similar result arises in [Bils et al. \(2025\)](#), where the authors analyze a version of the model above with a continuum of workers: employers apply larger markdowns over the range of workers for whom their comparative advantage is strongest. This paper isolates the driver behind that result: comparative advantage in wage setting is governed by which employer tends to hold the higher productivity realization at all productivity levels.⁸ Consistent with the mechanism, [Bils et al. \(2025\)](#) show that employers paying higher wages on average, the natural empirical counterpart of the stronger employer under global dominance, apply the larger wage markdowns in Brazilian data.

4.3 Interdependent Value Auctions

The environments above are private-value environments, where the payoff to winning is subjective and known to the bidder. The opposite may arise in many auctions, where the payoffs are objective but unknown ([Wilson, 1977](#); [Milgrom and Weber, 1982](#)). For instance, oil has a market value, but a bidder for an offshore oil lease does not know how much oil is under the tract. Asymmetry of information is central in these settings, and signals are strongly dependent because they are signals about the same underlying quantity.⁹

Suppose each bidder observes a private signal, S_i , with marginals F_i , and copula C as in Section 2, and let the object be worth $V_i = v_i(S_1, S_2)$ to bidder i , increasing in both arguments. The two valuation functions may generally differ if the common

⁸[Bils et al. \(2025\)](#) model productivity with Archimedean copulas and posit a stronger form of dominance. JSA dominance is a strictly weaker restriction, asking only for a diagonal comparison, allowing for general distributions, including the Gaussian copula, the workhorse specification for the Roy model, among others.

⁹The canonical example concerns offshore tracts on which one bidder holds seismic data from an adjacent block and the others do not ([Hendricks and Porter, 1988](#)).

component in the object's worth is accompanied by bidder-specific extraction costs, or resale opportunities, so that two bidders holding identical information need not value the object identically. Assumptions 2-6 now discipline the joint distribution of signals and govern relative strength in informational advantage.

Let $\beta_i(s_i)$ denote i 's bid and $\phi_i(b)$ the corresponding inverse bid as before. Bidder 1 wins if and only if $S_2 < \phi_2(b)$, receiving v_1 with probability $P_1(b; s) = C_1(F_1(s), F_2(\phi_2(b)))$ and nothing otherwise. Because the density of S_2 conditional on $S_1 = s$ is $C_{12}(F_1(s), F_2(t))f_2(t)$, bidder 1 with signal s choosing bid b solves

$$\max_b \int_0^{\phi_2(b)} (v_1(s, t) - b) C_{12}(F_1(s), F_2(t)) f_2(t) dt. \quad (10)$$

The first-order condition is

$$(v_i(s, \phi_j(b)) - b) \frac{\partial P_i}{\partial b} = P_i(b; s). \quad (11)$$

Comparing equations (11) and (2), the sole change is that the margin $(x - b)$, which only depends on own valuation, becomes $(v_i(s, \phi_j(b)) - b)$, which additionally depends on the rival's signal. This is the winner's curse correction, where a bidder realizes that winning implies their signal might have been too optimistic, and shades their bid relative to their signal. The winning probability $P_i(b; s)$ is the same object, because dependence between signals is still represented by the copula.

Substituting for the winning probability in (11) from its copula definition and taking ratios of for $i = \{1, 2\}$, the optimality conditions give:

$$\frac{h_2(b)}{h_1(b)} = \frac{C_1(H_1(b), H_2(b))}{C_2(H_1(b), H_2(b))} \cdot \frac{v_2(\phi_1(b), \phi_2(b)) - b}{v_1(\phi_1(b), \phi_2(b)) - b}. \quad (12)$$

In Section 2, the latter ratio is simply $(\phi_2(b) - b)/(\phi_1(b) - b)$. Because each v_i depends on both the bidder's and the rival's signals above, Theorem 1 does not readily translate. To secure the logical progression in its proof, the v_i need to be restricted so that each bidder's own signal weighs more heavily in their own valuation than the rival's does.

Assumption 7 (Own-Signal Dominance). *The valuation functions v_1, v_2 are continuous, and for every signal pair (s_1, s_2) ,*

$$v_i(s_1, s_2) \geq v_j(s_1, s_2) \quad \text{whenever } s_i > s_j.$$

At any signal pair, the bidder holding the higher signal values the object at least as much. Note that Assumption 7 implies bidders value the object identically given identical information, even though their valuation functions may differ elsewhere.¹⁰

$$v_1(s, s) = v_2(s, s) \quad \text{for all } s. \quad (13)$$

Assumption 7, with this implication, suffices for both parts of Theorem 1.

Theorem 4. *Suppose Assumptions 1–3 and 7 hold. Then in equilibrium:*

(i) *If $F_2 \succ_{FOSD} F_1$, then $H_1(b) \geq H_2(b)$ for all $b \in (0, \bar{b})$.*

(ii) *If $S_2 \succ_{JSA} S_1$, then $\beta_1(s) > \beta_2(s)$ for all $s \in (0, \omega)$.*

The aggression of the weak therefore survives the move to interdependent values at the cost of an additional restriction. The bidder with the stochastically weaker signal still bids more aggressively at every realization, and the stronger bidder's bid distribution still dominates, albeit weakly here. Requiring strict inequality in Assumption 7 would recover strict dominance, but it would also exclude the pure common value scenario, which I turn to next.

Assumption 7 is not entirely innocuous, however. Suppose the two bidders share a valuation function $v_1 = v_2$, as they do when the object has a single common value.¹¹ Then the margins, $v_i(\phi_1(b), \phi_2(b)) - b$, in (12) coincide at every bid, implying $h_2/h_1 = C_1/C_2$ throughout. With C symmetric, the system is invariant to interchanging the bidders, so $H_1 \equiv H_2$. Under a pure common value with an exchangeable copula, asymmetry in the signal marginals is simply a relabeling of signals and, therefore, does not carry any economic content.¹² What distinguishes the bidders' behaviors in Theorem 4 is how they value that signal asymmetry in the setup above.

¹⁰Fixing s and letting $s_2 \rightarrow s_1 = s$ gives $v_2(s, s) \geq v_1(s, s)$; letting $s_1 \rightarrow s_2 = s$ gives the reverse. Equality follows by continuity.

¹¹At that boundary case, where $V_1 = V_2$, expected value is $v(s_1, s_2) = \mathbb{E}[V | s_1, s_2]$ for both bidders, and the weights on the signals depend on how informative they are rather than who receives them.

¹²Normalizing signals to their quantiles $U_i = F_i(S_i)$ makes this visible. In quantile space the marginals are uniform, the copula is still C , and the valuation function $\mathbb{E}[V | U_1, U_2]$ is common to both bidders, so the environment is symmetric outright and each bidder wins exactly half the time.

4.4 Earning Margins in the Second-Price Auction

The results so far concern bid shading, and shading is primarily a first-price phenomenon. In a second-price auction truthful bidding is dominant whatever the dependence structure, so $\beta_i(x) = x$ for both bidders; hence, the difference between valuation and bid is zero.

One might nonetheless wonder how the asymmetry shapes what the bidders earn. Since the winner pays the loser's valuation, the realized margin is $X_i - X_j$ if i wins. Writing

$$u_i^{II}(x) \equiv \mathbb{E}[(x - X_j) \mathbf{1}\{X_j < x\} \mid X_i = x]$$

for bidder i 's expected margin conditional on x , the question is whether u_1^{II} and u_2^{II} can be ordered under Section 2's assumptions. They cannot. Integrating by parts gives

$$u_1^{II}(x) = \int_0^x C_1(F_1(x), F_2(t)) dt, \quad u_2^{II}(x) = \int_0^x C_2(F_1(t), F_2(x)) dt. \quad (14)$$

Any comparison requires the copula's partial derivatives to be ordered at pairs $(F_1(x), F_2(t))$ and $(F_1(t), F_2(x))$ with $t < x$, i.e., off the diagonal of the copula, whereas JSA dominance only restricts the diagonal comparisons, where both arguments are the same. Signing the comparison therefore requires strengthening the order.

Assumption 8 (Uniform Stochastic Asymmetry). *For every $x \in (0, \omega)$ and every $z \geq x$,*

$$C_1(F_1(z), F_2(x))f_1(z) < C_2(F_1(x), F_2(z))f_2(z).$$

Setting $z = x$ recovers JSA at x , so Assumption 8 implies $X_2 \succ_{JSA} X_1$ globally, but not vice versa, making JSA a weaker condition. Under independence Assumption 8 reduces to $F_2(x)f_1(z) < F_1(x)f_2(z)$ for $z \geq x$, which is the strict reverse hazard rate dominance of Maskin and Riley (2000).

Theorem 5. *Suppose Assumptions 1, 2 and 8 hold. Then in the second-price auction*

$$f_1(x) u_1^{II}(x) < f_2(x) u_2^{II}(x) \quad \text{for all } x \in (0, \omega),$$

and, consequently, $U_1^{II} < U_2^{II}$.

The weaker bidder earns the smaller margin at every valuation, in density-weighted form. The comparison is between the two bidders' contributions to ex ante surplus

at a common valuation, which is the same comparison Theorem 1(ii) makes for bids. Integrating over x gives the overall earning comparison.

What differs here is the mechanical nature of that comparison. In the first-price auction the stronger bidder's wider margin is a strategic response to shading being relatively cheap when the probability of winning is high. In the second-price auction both bidders report truthfully, and the stronger bidder's margin is larger only because they draw higher valuations against the same opponent, so that conditional on winning earnings are higher. Therefore, the ordering of margins is common to the two formats, although only in the first-price auction is it a strategic decision.

4.5 The War of Attrition

The war of attrition replaces the bid with a length of time. Two agents contest an indivisible prize, each incurring a flow cost while the contest continues, and the first to concede loses. This describes the canonical model of exit from a declining industry, of standard-setting contests, and of legislative or bargaining stalemates (Maynard Smith, 1974; Fudenberg and Tirole, 1986; Bulow and Klemperer, 1999). Asymmetry is intrinsic because the two firms in an exit game likely differ in such business fundamentals as capacity, debt, and outside options, and dependence is natural, since they share the same demand conditions.

Let agent i value winning at $X_i \sim F_i$, with copula C , and choose a concession time t , incurring cost at unit rate until the contest ends. If the rival concedes first at $s < t$, agent i receives $x - s$; if i concedes first at t , it receives $-t$. With $P_i(t; x)$ as the conditional distribution of the rival's concession time, expected payoff is

$$\int_0^t (x - s) dP_i(s; x) - t(1 - P_i(t; x)),$$

and the first-order condition is

$$x \frac{\partial P_i}{\partial t} = 1 - P_i(t; x). \quad (15)$$

The marginal cost of waiting longer is the flow cost, incurred as long as the contest is still running, that is, with probability $1 - P_i$. Both sides of (15) therefore differ

from (2). Evaluating at $x = \phi_1(t)$ gives

$$h_2 = \frac{1 - C_1(H_1, H_2)}{\phi_1 C_{12}}, \quad h_1 = \frac{1 - C_2(H_1, H_2)}{\phi_2 C_{12}},$$

and at any valuation x_0 at which the two strategies meet,

$$\frac{\phi'_2}{\phi'_1} = \frac{1 - C_1(F_1(x_0), F_2(x_0))}{1 - C_2(F_1(x_0), F_2(x_0))} \cdot \frac{f_1(x_0)}{f_2(x_0)} < 1 \iff (1 - C_1)f_1 < (1 - C_2)f_2.$$

The structure of the argument is intact: the concession-time terms cancel where the two strategies meet, and the comparison again reduces to a single condition on the joint distribution. But the condition is not JSA. Because $1 - C_1(F_1(x), F_2(x)) = \Pr(X_2 > x \mid X_1 = x)$, the product $(1 - C_1)f_1$ is the density of the event that the *lower* of the two valuations equals x and belongs to agent 1. The relevant comparison is therefore about who holds the minimum rather than who holds the maximum. That is simply JSA stated on the survival copula.

Assumption 9 (Survival Asymmetry). *For every $x \in [0, \omega)$,*

$$(1 - C_2(F_1(x), F_2(x)))f_2(x) < (1 - C_1(F_1(x), F_2(x)))f_1(x).$$

Conditional on the lower of the two valuations being x , that value more likely belongs to agent 1, which renders agent 1 weak, mirroring Definition 2.¹³ Stating the condition at $x = 0$, where both copula terms equal one, requires $f_1(0) > f_2(0)$, which goes beyond Assumption 1, which requires strictly positive densities only on the interior.¹⁴

Theorem 6 (Aggression of the Weak in the War of Attrition). *Suppose Assumptions 1–3 and 9 hold. Then in equilibrium $\tau_1(x) > \tau_2(x)$ for all $x \in (0, \omega)$.*

The agent with less at stake holds out longer at every valuation, even though the agent with more at stake concedes later in realization because they draw from the

¹³Like JSA dominance, survival asymmetry implies first-order stochastic dominance. At any interior zero of $d \equiv F_1 - F_2$ the copula terms coincide and Assumption 9 reduces to $f_2(x) < f_1(x)$, so $d' > 0$ there. Where $d < 0$, Assumption 3 gives $1 - C_1 \leq 1 - C_2$ and the condition again yields $f_2 < f_1$. Since $d(0) = 0$, d can therefore never become negative, and $F_2 \succ_{FOSD} F_1$.

¹⁴Theorem 1 appeals to the top boundary, $\beta_1(\omega) = \beta_2(\omega) = \bar{b}$ and uses Assumption 3 to sign $\beta'_1(x)/\beta'_2(x)$ near that boundary, neither of which is available here. Theorem 6's proof instead anchors at the lower boundary, where $\tau_1(0) = \tau_2(0) = 0$ and uses $\phi'_2(0)/\phi'_1(0) = f_1(0)/f_2(0) > 1$ by Assumption 9, to rule out crossing of optimal strategies.

higher distribution. The aggression of the weak therefore survives the change of payment rule, albeit under a slightly different notion of strength. The first-price auction conditions on the strong holding the maximum, whereas the war of attrition conditions on the weak holding the minimum.

4.6 All-Pay Auctions

It is worth closing with an environment that resembles the model closely and yet does not inherit its results. In an all-pay auction, each agent's effort is sunk whether or not they win, e.g., the model of lobbying, of research races, and of contests generally (Baye et al., 1996; Krishna and Morgan, 1997; Siegel, 2009; Konrad, 2009). The asymmetric two-player case with incomplete information is characterized by Amann and Leininger (1996) as the solution to a differential system resembling (3)–(4).

With values X_i drawn from marginals F_i with copula C , and b denoting effort choice, the payoff is $x P_i(b; x) - b$ with the first-order condition

$$x \frac{\partial P_i}{\partial b} = 1. \quad (16)$$

Evaluating the ratio of bidders' optimality conditions at $x_i = \phi_i(b)$ gives:

$$\frac{h_2}{h_1} = \frac{\phi_2(b)}{\phi_1(b)}.$$

Because the marginal cost of raising effort here is one instead of the winning probability, the conditional distributions C_1 and C_2 do not appear in (16), and the copula density C_{12} cancels from the ratio. So any ordering of effort strategies has to come from conditions on the marginal distributions F_i , and JSA has no bite.

What makes JSA discipline bidding behavior in the first-price auction is that the payment is contingent on winning. The win probability then determines the marginal cost of raising the bid. Under JSA, that effect is different for the two bidders where the bidding strategies meet, permitting a ranking of bids (by way of contradiction since such crossings disrupt equilibrium requirements under our assumptions). Removing that contingency, as the all-pay rule does, prevents the stochastic asymmetry from disciplining the comparison at all.

5 Conclusion

This paper characterizes equilibrium bidding in a first-price sealed-bid auction with two asymmetric bidders whose valuations are dependent. The analysis is organized around two conditions: a stochastic ordering on the joint valuation distribution and a mild regularity condition on the copula governing the dependence structure. Under global JSA dominance, the weaker bidder bids more aggressively than the stronger bidder at every valuation and wins with strictly lower probability, directly generalizing the aggression of the weak result of [Maskin and Riley \(2000\)](#) to environments with dependence. Under single crossing of the valuation distributions, the bid distributions cross exactly once, and any crossing of the bidding strategies is unique, with each bidder bidding more aggressively precisely where they are locally disadvantaged. In both cases, the first-price auction allocates the object inefficiently. Since the stronger bidder shades more relative to valuation, there are valuation combinations where the weaker bidder wins despite valuing the object less.

Under independence, Maskin and Riley establish conditions under which the stronger bidder prefers the second-price format while the weaker bidder prefers the first-price format, building crucially on the revenue equivalence theorem. A natural question is whether the revenue and format preference results of [Maskin and Riley \(2000\)](#) extend to this setting. Revenue equivalence, however, does not hold generally when valuations are dependent: the expected payments of the two bidders in the first-price and second-price formats differ in a way that depends on the copula structure, and there is no general basis for ranking the two formats from either bidder's perspective. Establishing format preference results under dependence therefore requires a substantially different approach and remains an open question that I leave for future work.

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A Proofs

Lemma 1. *Under Assumptions 1–3, $X_2 \succ_{JSA} X_1 \implies F_2 \succ_{FOSD} F_1$, with $F_2(x) < F_1(x)$ for all $x \in (0, \omega)$.*

Proof. Let $d(x) \equiv F_1(x) - F_2(x)$, so that d is continuously differentiable on $(0, \omega)$ with $d(0) = d(\omega) = 0$ by Assumption 1. The aim is to show $d(x) > 0$ for all $x \in (0, \omega)$. To that end, consider the following result first:

Claim: If $d(x) \leq 0$, then $d'(x) < 0$.

Write $u = F_1(x)$ and $v = F_2(x)$. If $d(x) = 0$, then $u = v$, and symmetry of C gives $C_1(u, u) = C_2(u, u) > 0$. Dividing the JSA inequality at x by this common value yields $f_1(x) < f_2(x)$, that is, $d'(x) < 0$.

If $d(x) < 0$ instead, then $u < v$, so Assumption 3 gives $C_1(u, v) \geq C_2(u, v) > 0$. The JSA inequality at x then gives

$$\frac{f_1(x)}{f_2(x)} < \frac{C_2(u, v)}{C_1(u, v)} \leq 1,$$

which implies $d'(x) < 0$.

Suppose now $d(x_1) < 0$ were true for some $x_1 \in (0, \omega)$. But then d cannot vanish at any point of $[x_1, \omega)$, since a zero there would require $d' < 0$ while d approaches it from below. Hence $d < 0$ throughout $[x_1, \omega)$, and since d is strictly decreasing on that interval, so $d(x) < d(x_1) < 0$ for all $x > x_1$. Letting $x \rightarrow \omega$ and using continuity gives $d(\omega) \leq d(x_1) < 0$, contradicting $d(\omega) = 0$. Therefore $d \geq 0$ on $(0, \omega)$.

Finally, d cannot tangentially vanish in the interior, because if $d(x_0) = 0$ for some $x_0 \in (0, \omega)$, then x_0 is an interior minimum of d on $(0, \omega)$ since we established d is non-negative. At the minimum, $d'(x_0) = 0$ would have to be true, contradicting $d'(x_0) < 0$ as established above. Hence $d(x) > 0$, i.e., $F_2(x) < F_1(x)$, for all $x \in (0, \omega)$. $\square$

Theorem 1 (Aggression of the Weak). *Suppose Assumptions 1–3 hold. Then in equilibrium:*

- (i) *If $F_2 \succ_{FOSD} F_1$, then $H_1(b) > H_2(b)$ for all $b \in (0, \bar{b})$.*
- (ii) *If $X_2 \succ_{JSA} X_1$, then $\beta_1(x) > \beta_2(x)$ for all $x \in (0, \omega)$.*

Proof. Part (i). Suppose, for contradiction, that $H_1(b^*) \leq H_2(b^*)$ for some $b^* \in (0, \bar{b})$ instead. This implies two relevant inequalities.

First, because $F_2(x) < F_1(x)$ for all $x \in (0, \omega)$ by hypothesis, $H_1(b^*) = F_1(\phi_1(b^*)) \leq F_2(\phi_2(b^*)) = H_2(b^*)$ can only hold if $\phi_2(b^*) > \phi_1(b^*)$.

Second, by Assumption 3, $H_1(b^*) \leq H_2(b^*)$ also implies $C_1(H_1(b^*), H_2(b^*)) \geq C_2(H_1(b^*), H_2(b^*))$.

But the differential system described by the optimal bidding strategies implies the following equality along the equilibrium trajectory:

$$\frac{h_2(b)}{h_1(b)} = \frac{\phi_2(b) - b}{\phi_1(b) - b} \cdot \frac{C_1(H_1(b), H_2(b))}{C_2(H_1(b), H_2(b))},$$

obtained by taking the ratio of equations (3) and (4).

Evaluating this at $b = b^*$, the two inequalities above imply $h_2(b^*) > h_1(b^*)$. Since $h_i(b)$ is the derivative of $H_i(b)$, that reinforces $H_1(b) < H_2(b)$ for all $b > b^*$. Note that the argument applies at every b with $H_1(b) \leq H_2(b)$. Therefore $H_1(b) - H_2(b)$ is strictly decreasing wherever it is nonpositive and cannot return to zero, which contradicts the key equilibrium property $H_1(\bar{b}) = H_2(\bar{b})$. Hence, no such b^* can exist, and $H_1(b) > H_2(b)$ must therefore hold for all $b \in (0, \bar{b})$. **Part (ii).**

The proof proceeds by way of contradiction. First, note that if the two bidding strategies ever cross, $\phi_1(b) = \phi_2(b) = x^*$ for some $x^* \in (0, \omega)$, then from the ratio of optimality conditions (3)–(4):

$$\frac{\phi'_2(b)}{\phi'_1(b)} = \underbrace{\frac{(x^* - b)}{(x^* - b)}}_{=1} \cdot \frac{C_1(F_1(x^*), F_2(x^*))}{C_2(F_1(x^*), F_2(x^*))} \frac{f_1(x^*)}{f_2(x^*)} < 1,$$

by JSA dominance. Because $\phi_j(b)$ is the inverse of $\beta_j(x)$, $\phi'_j(b) = 1/\beta'_j(x)$. Therefore we have $\beta'_2(x) > \beta'_1(x)$ whenever $\phi_1(b) = \phi_2(b)$.

Contrary to the theorem's claim, suppose $\beta_1(z) \leq \beta_2(z)$ for some $z \in (0, \omega)$ instead. Then, for all $z' > z$, we have $\beta_2(z') > \beta_1(z')$, particularly for z' arbitrarily close to the upper bound ω since $\beta'_2(x) > \beta'_1(x)$ as established above. The two strategies cannot cross back later, because that would require $\beta'_1 \geq \beta'_2$ at that point, contradicting the inequality above. In terms of inverse bid functions, this is equivalent to $\phi_2(b) < \phi_1(b)$ for b arbitrarily close to $\bar{b}$. But then we have

$$H_2(b) = F_2(\phi_2(b)) < F_2(\phi_1(b)) < F_1(\phi_1(b)) = H_1(b),$$

because $F_2' > 0$ by definition and $F_2 \succ_{FOSD} F_1$ as implied by the assumption that $X_2 \succ_{JSA} X_1$. Since $H_2(\bar{b}) = H_1(\bar{b}) = 1$, we must have $h_2(b) > h_1(b)$ for $b \simeq \bar{b}$, i.e., $H_2(b)$ has to rise faster to catch up with $H_1(b)$ so that they meet at the termination point.

But if $h_2(b) > h_1(b)$ and $H_2(b) < H_1(b)$, then the ratio of equations (3) and (4) gives:

$$\frac{\phi_2(b) - b}{\phi_1(b) - b} = \frac{C_2(H_1(b), H_2(b)) h_2(b)}{C_1(H_1(b), H_2(b)) h_1(b)} > 1$$

by Assumption 3. The inequality above says $\phi_2(b) > \phi_1(b)$, i.e., $\beta_1(x) > \beta_2(x)$, which contradicts the supposition $\beta_1(z) \leq \beta_2(z)$. □

Corollary 1. *Suppose Assumptions 1–3 hold. If $F_2 \succ_{FOSD} F_1$, then*

$$\Pi_2 > \frac{1}{2} > \Pi_1,$$

where $\Pi_i \equiv \int_0^\omega P_i(\beta_i(x); x) dF_i(x)$ denotes bidder i 's ex ante probability of winning the auction before the valuations are observed.

Proof. Let $U \equiv F_1(X_1)$ and $V \equiv F_2(X_2)$. Because C is symmetric, (U, V) and (V, U) have the same distribution, so $\Pr(V > U) = 1/2$: absent any difference in bidding behavior, each bidder would win half the time. Bidder 2 wins whenever $V > H_2 \circ H_1^{-1}(U)$, where $H_2 \circ H_1^{-1}$ traces the iso-bid locus. Theorem 1(i) places that locus strictly below the diagonal. Bidder 2 therefore wins on a strictly larger event than $\{V > U\}$, giving $\Pi_2 > 1/2 > \Pi_1$. □

Theorem 2. *If Assumptions 1–3 and 5 hold, then there exists a unique $b^* \in [0, \bar{b})$ such that*

$$H_1(b) \geq H_2(b) \iff b \geq b^*.$$

Proof. Define $\Delta(b) \equiv H_1(b) - H_2(b) = F_1(\phi_1(b)) - F_2(\phi_2(b))$.

Assumption 5 requires $F_1(\tilde{x}) = F_2(\tilde{x})$ for some $\tilde{x}$, with $F_2(x) < F_1(x)$ for $x > \tilde{x}$. As in the proof to Theorem 1, $\Delta(b) > 0$ must hold for $b \in (\tilde{b}, \bar{b})$, where $\tilde{b} = \beta_2(\tilde{x})$. The argument is the same, though it now relies only on the local ordering: for $b > \tilde{b}$

we have $\phi_2(b) > \tilde{x}$, so $\Delta(b) \leq 0$ would give $F_1(\phi_1(b)) \leq F_2(\phi_2(b)) < F_1(\phi_2(b))$ and hence $\phi_1(b) < \phi_2(b)$, which is what the proof of Theorem 1(i) uses.

Because $\Delta(0) = \Delta(\bar{b}) = 0$ at termination points, there are two possibilities: either $\Delta(b) > 0$ for all $b \in (0, \omega)$, in which case $b^* = 0$ is the only crossing point, or $\Delta(b) < 0$ for some $b \in (0, \tilde{b})$, and, therefore, there is at least one $b^* \in (0, \tilde{b})$ with $\Delta(b^*) = 0$ by continuity.

But any such b^* must be unique, because $\Delta'(b) > 0$ for all $b \in [b^*, \tilde{b})$ enforcing $\Delta(b) > 0$ in this range as I show next. Since $\Delta(b) > 0$ already holds for $b \in (\tilde{b}, \bar{b})$, there cannot be another $b_0 > b^*$ with $\Delta(b_0) = 0$. Hence, $\Delta(b)$ crosses 0 only once, from below if $b^* > 0$.

Claim: $\Delta'(b) > 0$ for all $b \in [b^*, \tilde{b}]$.

Recall from the proof for Theorem 1(i), that the ratio of optimality conditions (3)–(4) gives:

$$\frac{h_2(b)}{h_1(b)} = \frac{\phi_2(b) - b}{\phi_1(b) - b} \cdot \frac{C_1(H_1(b), H_2(b))}{C_2(H_1(b), H_2(b))}$$

Suppose $\Delta(b) = H_1(b) - H_2(b) \geq 0$ for some $b < \tilde{b}$. Then $C_1(H_1(b), H_2(b))/C_2(H_1(b), H_2(b)) \leq 1$ by Assumption 3. This pushes $h_2(b)/h_1(b)$ below 1. Furthermore, because $\phi_2(b) < \tilde{x}$, by definition of $\tilde{b}$, $F_1(x) < F_2(x)$ when $x < \tilde{x}$, and we supposed $H_1(b) \geq H_2(b)$, we must have $F_1(\phi_1(b)) \geq F_2(\phi_2(b)) > F_1(\phi_2(b))$ for all $b < \tilde{b}$, which implies $\phi_1(b) > \phi_2(b)$ since $F'_1 > 0$. This reinforces the pressure from C_1/C_2 component and ultimately implies $h_2(b) < h_1(b)$. That is equivalent to $\Delta'(b) > 0$ whenever $\Delta(b) \geq 0$, in particular at $b = b^*$. That means $\Delta(b) = H_1(b) - H_2(b) > 0$ and $\Delta'(b) > 0$ for all $b \in (b^*, \tilde{b})$. $\square$

Theorem 3. *Suppose Assumptions 1–3, 5 and 6 hold. If the equilibrium bid functions cross at some $x^* \in (0, \omega)$, then the crossing is unique, $x^* \in (\tilde{x}, \hat{x})$, and*

$$\beta_1(x) > \beta_2(x) \iff x > x^*.$$

Proof. The proof proceeds in three steps. Claim 1 characterizes the asymmetry in slopes at any intersection of bidding strategies: bidder 1's strategy cuts bidder 2's from below when the crossing occurs below $\hat{x}$, and from above when it occurs above $\hat{x}$. Claim 2 rules out the latter: a crossing above $\hat{x}$ would leave bidder 2 bidding at least as much as bidder 1 on the whole range above it, which contradicts with the two strategies meeting at $\bar{b}$ as in the proof to Theorem 1. Claim 3 shows that crossings

below $\tilde{x}$ contradict Theorem 2. What remains is a single crossing in $(\tilde{x}, \hat{x})$, cut from below — so bidder 1 bids more aggressively above it and less aggressively below.

Claim 1 – If there is a bid b_0 such that $\phi_1(b_0) = \phi_2(b_0) = x_0$, then $\phi'_1(b_0) < \phi'_2(b_0) \iff x_0 > \hat{x}$.

Since $\phi_1(b) = \phi_2(b)$, we have, from the ratio of equations (3) and (4):

$$\frac{\phi'_2(b_0)}{\phi'_1(b_0)} = \underbrace{\frac{(x_0 - b_0)}{(x_0 - b_0)}}_{=1} \frac{C_1(F_1(x_0), F_2(x_0)) f_1(x_0)}{C_2(F_1(x_0), F_2(x_0)) f_2(x_0)},$$

which is strictly less than 1 if $x_0 > \hat{x}$ and strictly above 1 if $x_0 < \hat{x}$ by Assumption 6 and JSA dominance.

Claim 2: For all $x > \hat{x}$, $\beta_1(x) > \beta_2(x)$.

Because $\phi'_j(b) = 1/\beta'_j(x)$, the result above is equivalent to $\beta'_2(x) > \beta'_1(x)$ for $x_0 > \hat{x}$. Suppose, contrary to the claim, that $\beta_2(z) \geq \beta_1(z)$ for some $z \in (\hat{x}, \omega)$. Then for all $z' > z$, we have $\beta_2(z') \geq \beta_1(z')$. In terms of inverse bids, this is equivalent to $\phi_2(b) < \phi_1(b)$ for $b > \beta_2(z)$. But then we have:

$$H_2(b) = F_2(\phi_2(b)) < F_2(\phi_1(b)) < F_1(\phi_1(b)) = H_1(b),$$

where the second inequality is implied by Assumption 5 since $\phi_1(b) > \tilde{x}$. Since $H_2(\bar{b}) = H_1(\bar{b})$, we must have $H'_2(b) > H'_1(b)$ at some point as b approaches $\bar{b}$.

But if $H'_2(b) > H'_1(b)$ and $H_2(b) < H_1(b)$, the ratio of equations (3) and (4) implies:

$$\frac{\phi_2(b) - b}{\phi_1(b) - b} = \frac{C_2(H_1(b), H_2(b)) H'_2(b)}{C_1(H_1(b), H_2(b)) H'_1(b)} > 1,$$

implying $\phi_2(b) > \phi_1(b)$, or, $\beta_2(x) < \beta_1(x)$, which contradicts the supposition.

As a result, any crossing of ϕ_1 and ϕ_2 must occur at a valuation $x_0 < \hat{x}$. At any such crossing $\phi'_2(b_0) > \phi'_1(b_0)$ by Claim 1, so ϕ_2 crosses ϕ_1 from below. Hence there is at most one crossing, at some x^* , and $\phi_2(b) > \phi_1(b)$ — equivalently $\beta_1(x) > \beta_2(x)$ — precisely for $x > x^*$.

Claim 3: $x^* > \tilde{x}$.

Recall from Theorem 2 that $H_1(b) > H_2(b)$ for all $b > b^*$, with $\phi_2(b^*) < \tilde{x}$. Consider any bid b with $\phi_1(b) < \tilde{x}$. By Assumption 5, $F_1(\phi_1(b)) < F_2(\phi_1(b))$, so $H_1(b) > H_2(b)$ gives $F_2(\phi_2(b)) < F_1(\phi_1(b)) < F_2(\phi_1(b))$ and hence $\phi_2(b) < \phi_1(b)$. Since ϕ_2 crosses ϕ_1 from below exactly once, that crossing cannot occur below $\tilde{x}$;

therefore $x^* > \tilde{x}$.

□

Theorem 4. *Suppose Assumptions 1–3 and 7 hold. Then in equilibrium:*

- (i) *If $F_2 \succ_{FOSD} F_1$, then $H_1(b) \geq H_2(b)$ for all $b \in (0, \bar{b})$.*
- (ii) *If $S_2 \succ_{JSA} S_1$, then $\beta_1(s) > \beta_2(s)$ for all $s \in (0, \omega)$.*

Proof. The proof repeats the steps in the proof of Theorem 1, with the mark-down ratio $(\phi_2 - b)/(\phi_1 - b)$ replaced throughout by the margin ratio

$$R(b) \equiv \frac{v_2(\phi_1(b), \phi_2(b)) - b}{v_1(\phi_1(b), \phi_2(b)) - b},$$

which is well defined and positive because both margins are strictly positive along the equilibrium trajectory. Only the steps involving $R(b)$ are given.

Part (i). Suppose, for contradiction, that $H_1(b^*) \leq H_2(b^*)$ for some $b^* \in (0, \bar{b})$. As in Theorem 1, $F_2 \succ_{FOSD} F_1$ gives $\phi_2(b^*) > \phi_1(b^*)$, and Assumption 3 gives $C_1(H_1(b^*), H_2(b^*)) \geq C_2(H_1(b^*), H_2(b^*))$. At the signal pair $(\phi_1(b^*), \phi_2(b^*))$ bidder 2 holds the higher signal, so Assumption 7 gives $v_2 \geq v_1$ there and hence $R(b^*) \geq 1$. Taking the ratio of the optimality conditions (11) for $i = 1, 2$,

$$\frac{h_2(b)}{h_1(b)} = \frac{C_1(H_1(b), H_2(b))}{C_2(H_1(b), H_2(b))} \cdot R(b),$$

so $h_2(b^*) \geq h_1(b^*)$, and the same argument applies at every b with $H_1(b) \leq H_2(b)$. Therefore $H_1 - H_2$ is non-increasing wherever it is nonpositive. Combined with $H_1(\bar{b}) = H_2(\bar{b})$ this forces $H_1 \equiv H_2$ on $[b^*, \bar{b}]$, so $H_1(b) \geq H_2(b)$ for all $b \in (0, \bar{b})$. If the inequality in Assumption 7 is strict whenever $s_i > s_j$, then $R(b^*) > 1$, the gap is strictly decreasing wherever it is nonpositive and cannot return to zero, and the conclusion is strict.

Part (ii). If the two bidding strategies cross at $\phi_1(b_0) = \phi_2(b_0) = x^*$ for some $x^* \in (0, \omega)$, then $R(b_0) = 1$ by (13), and the ratio of optimality conditions gives

$$\frac{\phi'_2(b_0)}{\phi'_1(b_0)} = \underbrace{R(b_0)}_{=1} \cdot \frac{C_1(F_1(x^*), F_2(x^*))}{C_2(F_1(x^*), F_2(x^*))} \cdot \frac{f_1(x^*)}{f_2(x^*)} < 1$$

by JSA dominance, exactly as in Theorem 1. Hence $\beta'_2(s) > \beta'_1(s)$ wherever the two strategies meet.

Contrary to the claim, suppose $\beta_1(z) \leq \beta_2(z)$ for some $z \in (0, \omega)$. Then, as in Theorem 1, $\phi_2(b) < \phi_1(b)$ for b arbitrarily close to $\bar{b}$, so that

$$H_2(b) = F_2(\phi_2(b)) < F_2(\phi_1(b)) < F_1(\phi_1(b)) = H_1(b),$$

and $H_2(\bar{b}) = H_1(\bar{b}) = 1$ requires $h_2(b) > h_1(b)$ near $\bar{b}$. But $H_2(b) < H_1(b)$ delivers $C_2 \geq C_1$ by Assumption 3, so

$$R(b) = \frac{C_2(H_1(b), H_2(b))}{C_1(H_1(b), H_2(b))} \cdot \frac{h_2(b)}{h_1(b)} > 1,$$

that is, $v_2 > v_1$ at the signal pair $(\phi_1(b), \phi_2(b))$. Since $\phi_1(b) > \phi_2(b)$ there, bidder 1 holds the higher signal, and Assumption 7 requires $v_1 \geq v_2$ — a contradiction. $\square$

Theorem 5. *Suppose Assumptions 1, 2 and 8 hold. Then in the second-price auction*

$$f_1(x) u_1^{II}(x) < f_2(x) u_2^{II}(x) \quad \text{for all } x \in (0, \omega),$$

and, consequently, $U_1^{II} < U_2^{II}$.

Proof. Truthful bidding is dominant in the second-price auction under private values, so bidder i wins if and only if $X_j < X_i$ and pays X_j . Conditional on $X_1 = x$, the distribution of X_2 is $G(t) = C_1(F_1(x), F_2(t))$, which is continuous and increasing in t with $G(0) = C_1(F_1(x), 0) = 0$. Hence

$$u_1^{II}(x) = \int_0^x (x - t) dG(t) = \left[(x - t)G(t) \right]_0^x + \int_0^x G(t) dt = \int_0^x C_1(F_1(x), F_2(t)) dt,$$

the boundary terms vanishing at $t = x$ because $x - t = 0$ and at $t = 0$ because $G(0) = 0$. The same computation with the roles of the bidders exchanged gives the second expression in (14).

Fix $x \in (0, \omega)$ and let $t \in (0, x)$. Applying Assumption 8 with lower threshold t and upper threshold $z = x \geq t$,

$$C_1(F_1(x), F_2(t)) f_1(x) < C_2(F_1(t), F_2(x)) f_2(x).$$

Both sides are continuous in t on $(0, x)$, so integrating over that interval preserves the strict inequality:

$$f_1(x) u_1^{II}(x) = \int_0^x C_1(F_1(x), F_2(t)) f_1(x) dt < \int_0^x C_2(F_1(t), F_2(x)) f_2(x) dt = f_2(x) u_2^{II}(x),$$

which is the pointwise claim. Since $U_i^{II} = \int_0^\omega u_i^{II}(x) f_i(x) dx$, integrating the pointwise inequality over $x \in (0, \omega)$ gives $U_1^{II} < U_2^{II}$. $\square$

Theorem 6 (Aggression of the Weak in the War of Attrition). *Suppose Assumptions 1–3 and 9 hold. Then in equilibrium $\tau_1(x) > \tau_2(x)$ for all $x \in (0, \omega)$.*

Proof. The proof mirrors that of Theorem 1(ii), but closes at the lower rather than the upper boundary of the type space.

Suppose the two concession rules meet at some $x_0 \in (0, \omega)$, so that $\phi_1(t_0) = \phi_2(t_0) = x_0$ for some t_0 . The concession-time terms in (15) then coincide, and the ratio of the two agents' optimality conditions gives

$$\frac{\phi_2'(t_0)}{\phi_1'(t_0)} = \frac{1 - C_1(F_1(x_0), F_2(x_0))}{1 - C_2(F_1(x_0), F_2(x_0))} \cdot \frac{f_1(x_0)}{f_2(x_0)} > 1$$

by Assumption 9. Since ϕ_i is the inverse of τ_i , $\phi_i' = 1/\tau_i'$, so $\tau_1'(x_0) > \tau_2'(x_0)$: agent 1's concession rule is strictly the steeper wherever the two meet.

Type 0 concedes immediately under either rule, so $\tau_1(0) = \tau_2(0) = 0$. Evaluating the displayed ratio in the limit as $x_0 \rightarrow 0$, where $C_1(0, 0) = C_2(0, 0) = 0$, gives $f_1(0)/f_2(0) > 1$ by Assumption 9 at $x = 0$; hence, $\tau_1'(0) > \tau_2'(0)$ and $\tau_1(x) > \tau_2(x)$ on some interval $(0, \varepsilon)$.

Suppose, contrary to the claim, that $\tau_1(z) \leq \tau_2(z)$ for some $z \in (0, \omega)$. Then $z > \varepsilon$, and by continuity there is a smallest $x_0 \in (\varepsilon, z]$ with $\tau_1(x_0) = \tau_2(x_0)$, with $\tau_1 > \tau_2$ on $(0, x_0)$. Approaching that point from the left requires $\tau_1'(x_0) \leq \tau_2'(x_0)$, contradicting the steepness established above. Hence no such z exists and $\tau_1(x) > \tau_2(x)$ throughout $(0, \omega)$. $\square$