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Equilibrium Multiplicity in Aiyagari and Krusell-Smith*

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Abstract

Repeatedly solving the Aiyagari (1994) model with random parameters, we construct hundreds of examples with multiple stationary equilibria. We never find multiplicity with risk aversion less than ≈ 1.49, depreciation less than ≈ 0.19, or income persistence less than ≈ 0.47, and multiplicity requires a disaster state for income. In cases with multiplicity, the lowest rental rate occurs near depreciation times the capital share. It is possible for the economy, without a change in fundamentals, to transition rationally from a higher-rate equilibrium to one with a lower rental rate, lower inequality, and lower welfare (for most agents). We also construct the first Krusell and Smith (1998) examples with multiple recursive competitive equilibria.

Keywords: uniqueness, multiplicity, Bewley models, Krusell-Smith

JEL codes: C6, D5, E1

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1 Introduction

Comparative statics analysis in equilibrium models is one of the cornerstones of economics. But if equilibria are not unique, comparative statics may not be useful, either because it is difficult to refine equilibria or because equilibria are unstable.[1] The Aiyagari (1994) model and its aggregate risk counterpart (Krusell and Smith, 1998) are two of the most influential models in macroeconomics.[2] Despite this influence, the literature has provided very few sufficient statistics for uniqueness in these frameworks. On the other hand, there is only one non-uniqueness example for Aiyagari (1994) (see Açıkgöz (2018)), and, to our knowledge, no one has ever provided a non-uniqueness example for Krusell and Smith (1998). In this paper, we provide many parameter sets that yield multiple Aiyagari (1994) stationary equilibria and subsequently construct two Krusell and Smith (1998) examples with multiple recursive competitive equilibria. We thus answer the longstanding question of whether multiplicity is possible in Krusell and Smith (1998) – it is, and the number of equilibria seems to equal the number of stable steady states in the corresponding Aiyagari (1994) model.

For our Aiyagari (1994) analysis, we compute stationary capital supply and demand for $24,000$ uniform draws of the model's parameters – discount factor, risk aversion, idiosyncratic labor supply persistence, idiosyncratic labor supply volatility, capital depreciation rate, and capital's share of output – and calculate the intersections of supply and demand.[3] For half of the draws, we reset the lowest labor supply realization to a "disaster" value close to zero. Two hundred and fifty of the parameter sets admit multiple stationary equilibria, and in each case there are exactly three. For the parameter ranges we consider, a probit regression suggests that all of the pa-

[1]See, for example, Kehoe (1985) and Samuelson (1941) for discussions of the connection between uniqueness, stability, and the usefulness of comparative statics.

[2]The Aiyagari (1994) model is one version of the broader "Bewley" class (Bewley, 1986; İmrohoroğlu, 1989; Huggett, 1993; Aiyagari, 1994).

[3]The Aiyagari (1994) model is macroeconomics' workhorse for quantitative studies of inequality in general equilibrium. As described in detail in Section 2, a large number of households face idiosyncratic income shocks, accumulate capital, and rent their labor and capital to a representative firm at rental and wage returns determined in equilibrium. See Cherrier et al. (2023) for a history of thought on this topic.

rameters contribute to multiplicity, except the capital share which is insignificant in our sample.[4] Lower values of the discount factor and income volatility are associated with multiplicity, whereas the reverse is true for risk aversion, income persistence, and the depreciation rate. But examining parameter ranges conditional on multiplicity reveals some starker results: we never find multiplicity with risk aversion less than ≈ 1.49, depreciation less than ≈ 0.19, or income persistence less than ≈ 0.47.

Our model runs uncover two other stark and potentially surprising patterns. First, our multiplicity examples occur *only* when there is a disaster state for income. We do not provide a proof that disasters are necessary for multiplicity, but the result does suggest a potentially important new economic mechanism we describe later. Second, whenever there is multiplicity, we find that the lowest equilibrium rental rate is in the vicinity of the surprisingly simple value $\delta\alpha$ (the depreciation rate times the capital share). Usually, the low rate is very close to $\delta\alpha$: the median distance from $\delta\alpha$ is 1 percent (*not* percentage points), and 50 percent of low rates are within 2 percent of $\delta\alpha$.

Why do disasters and higher values of risk aversion, income persistence, and depreciation lead to multiple stationary equilibria in Aiyagari (1994), and what is the role of the simple expression $\delta\alpha$? If the depreciation rate is high, the agents get a terrible return on capital, making it very difficult to weather the spells of near-zero income in the disaster state. If income is persistent and risk aversion is high, agents have a strong demand for precautionary savings. They are so afraid of the disaster state that when the rental rate falls sufficiently, agents amass capital, even though the net return is low (and perhaps even negative). Capital supply thus becomes non-monotonic in the rental rate, meaning it can cross downward-sloping capital demand from the firm more than once. Furthermore, when agents are amassing capital for precautionary reasons with low returns, they are consuming a negligible amount. In the limit with zero consumption, there is a closed-form expression for stationary capital supply. This function intersects capital demand at exactly $\delta\alpha$, which also happens to be the local minimum of the function.

[4] Although we do find that with the parameters from Açıkgöz (2018), increasing the capital share from 0.3 to 0.5 is important for constructing multiple equilibria with aggregate risk.

Conditional on being in an Aiyagari (1994) world with multiple stationary equilibria, to what extent can particular equilibria be refined away? To explore this question, we focus on our leading example with multiplicity, which has relatively non-exotic parameters (except for a high depreciation rate of ≈ 42 percent). Specifically, we attempt to calculate dynamic equilibria where agents begin in one stationary equilibrium but suddenly believe rates will transition to a different equilibrium. We conjectured that it might not be possible to reach one or two of the equilibria, but it turns out that this conjecture was incorrect in our leading example: any equilibrium can rationally transition to any other one. For example, suppose we begin in the highest rate equilibrium, which also has the most wealth inequality. We show there exists a path for the rental rate that, if expected by the agents, will be self-fulfilling and cause a transition to a different stationary equilibrium with a lower rental rate and lower inequality. There are substantial welfare losses associated with transitions to lower rates, which are not fully compensated by rising wages. But the steady states are not Pareto-ranked: very poor agents gain from falling rates (because wages rise). In our conclusion (Section 4), we discuss the potential relevance of our findings for the ongoing debate in macroeconomics about "r-star" and long-term movements in real rates.

Finally, we turn to the Krusell and Smith (1998) model, which adds aggregate risk to Aiyagari (1994). We originally conjectured that multiplicity in Aiyagari (1994) could be an artifact of fixed stationary wealth distributions and that even a small amount of aggregate risk could restore uniqueness. Indeed, there are, to our knowledge, no Krusell and Smith (1998) multiplicity examples in the literature, and the working paper by Pröhl (2018) provides a simple sufficient condition for uniqueness. But it turns out that it is possible to construct Krusell and Smith (1998) examples with multiple recursive competitive equilibria, and we offer two. Initializing distributions at their values in the separate Aiyagari (1994) equilibria with our leading parameters, the Krusell and Smith (1998) algorithm converges to three recursive competitive equilibria. Each remains in the vicinity of the corresponding Aiyagari (1994) equilibrium, suggesting they are locally stable, and they are distinct in the sense that the three ergodic rental rate distributions do not overlap.[5]

For our second Krusell and Smith (1998) multiplicity example, we use parameters similar to the ones from Açıkgöz (2018). In this case, the algorithm only converges around the low and high rate steady states. It appears that the middle steady state is unstable, and we illustrate the instability through an "MIT shock" exercise: a one-time, unanticipated increase in aggregate productivity at the middle rate steady state causes a transition to the low rate steady state.

For a recent review of uniqueness and multiplicity of competitive equilibria and the relevance for general equilibrium theory, macroeconomics, finance, and trade, see Toda and Walsh (2024). The papers most closely related to the present one are Toda (2017, 2019), Açıkgöz (2018), Light (2020, 2023), and Achdou et al. (2022). In a variant of the Huggett (1993) model with constant absolute risk aversion utility and no borrowing constraint, Toda (2017) shows how to construct examples with multiple stationary equilibria. Açıkgöz (2018) provides a single Aiyagari (1994) example with multiple stationary equilibria, noting his parameter values are "rather extreme." On the other hand, Toda (2019), Light (2020, 2023), and Achdou et al. (2022) all provide sufficient conditions for stationary equilibrium uniqueness in Bewley model variants, each requiring relative risk aversion (RRA) ≤ 1. Finally, the working paper by Pröhl (2018) gives a simple sufficient condition for uniqueness of recursive competitive equilibria in the Krusell and Smith (1998) model (and Aiyagari (1994)) if RRA > 1: $\beta(1-\delta)^{1-\gamma} < 1$, where β is the discount factor, δ is the depreciation rate, and γ is the coefficient of relative risk aversion.

Section 2 presents our Aiyagari (1994) results, Section 3 looks at the Krusell and Smith (1998) model, and in Section 4 we further discuss our findings and conclude.

[5] Multiplicity is pervasive in incomplete market economies in general, and opens the door to sunspots that randomly switch between equilibria. It is not clear how one would solve for these sunspot equilibria, because precautionary savings effects from the sunspots themselves would alter the dynamics.

2 Aiyagari

In the Aiyagari (1994) model, there is a mass one continuum of infinitely lived agents, each of which solves the recursive problem

$$v(k,e) = \max_{k' \geq 0} \left\{ \frac{\left((1+r-\delta)k + we - k'\right)^{1-\gamma}}{1-\gamma} + \beta \operatorname{E}\left[v(k',e')\,|e\right] \right\}, \tag{2.1}$$

where v is the value function, $\gamma > 0$ is RRA (the $\gamma = 1$ limit corresponds to log utility), and $\beta \in (0,1)$ is the discount factor. Each agent is maximizing the expected present value of CRRA utility over consumption $c = (1+r-\delta)k + we - k'$ by accumulating capital k, which depreciates at rate $\delta \in (0,1]$, renting capital to a representative firm at rate r, and providing labor e to the firm at wage w. The exogenous labor supply $e > 0$ follows a finite-state Markov process. To parameterize this process, we first define the AR(1) process $\tilde{e}$: $\tilde{e}' = \rho_e \tilde{e} + \sigma_e \varepsilon_e'$, where $\rho_e \in [0,1)$ is the persistence of labor income, $\sigma_e > 0$ is the volatility, and $\varepsilon_e \sim$ i.i.d. $\mathcal{N}(0,1)$. Given persistence and volatility, we take a 7-state discrete approximation using Rouwenhorst's method and define $e = \exp(\tilde{e})$. Inspired by Açıkgöz (2018), for half of our parameter draws we reset the lowest e realization to $\underline{e} = 10^{-6}$, after performing Rouwenhorst's approximation. In these cases, there is a small probability of a persistent disaster state, but $\log(e)$ otherwise approximates the $\tilde{e}$ AR(1) process.

Agents are ex ante identical in that they have the same parameters, but they are ex post heterogeneous because they receive idiosyncratic income shocks and asset markets are incomplete. The representative firm solves the problem

$$\max_{K,L} ZK^{\alpha}L^{1-\alpha} - rK - wL, \tag{2.2}$$

where $\alpha \in (0,1)$ equals capital's share of national income, K is aggregate capital demand, and L is aggregate labor demand. Z is productivity, which we normalize to one until the Krusell and Smith (1998) model in Section 3 below.

A *stationary equilibrium* consists of a value function $v^*(k,e)$, a capital decision rule $k' = a^*(k,e)$, capital demand K^*, labor demand L^*, prices w^* and r^*, and a cross-agent distribution $\Omega^*(k,e)$ such that: (i) $v^*(k,e)$ and $k' = a^*(k,e)$ solve the agent's problem (2.1) given r^* and w^*, (ii) K^* and L^* solve the firm's problem (2.2) given r^* and w^*, (iii) capital and labor markets clear: $\int a^*(k,e)\, d\Omega^* = K^*$ and $\int e d\Omega^* = L^*$, and (iv) $\Omega^*(k,e)$ is constant and generated by the decision rule $a^*(k,e)$ and process for e.

Since the labor supply processes are independent Markov chains, there is a unique stationary labor supply that determines $L^* \approx 1$. Therefore, the firm's problem (2.2) implies a downward-sloping capital demand curve $K(r) = (\alpha/r)^{1/(1-\alpha)} L^*$. And given some r, the wage that clears the labor market is then just a function of r and α. Plugging this wage into the agent's problem, we can calculate capital supply

$$A(r) = \int a(k,e;r)\, d\Omega(k,e;r),$$

where $a(k,e;r)$ denotes the decision rule conditional on r and $\Omega(k,e;r)$ is the stationary cross-agent distribution conditional on r. If capital supply $A(r)$ were everywhere upward-sloping, there could never be multiple stationary equilibria. But, as explained in Açıkgöz (2018), capital supply is not necessarily monotonic due to income effects.

To solve the model, we construct a grid for r. For each r, we solve the household's problem and construct the invariant distribution as in Young (2010). We then plot the asset supply and capital demand curves and note the intersections. Computational details are in the appendix. The parameters of the model are drawn from independent uniform distributions: $\beta \in [0,1)$ (discount factor), $\gamma \in [0,8]$ (risk aversion), $\rho_e \in [0,1]$ (income persistence), $\sigma_e \in [0,0.5]$ (income volatility), $\delta \in [0,1]$ (depreciation), and $\alpha \in [0.2,0.5]$ (capital's share of income).[6] We consider $24,000$ sets of parameters, setting $\underline{e} = 10^{-6}$ with a 50 percent probability. To be conservative, we

[6]Capital demand is always downward-sloping and Açıkgöz (2018) argues general equilibrium effects through the wage are unlikely to lead to multiple equilibria in this context. Furthermore, the known uniqueness conditions in Light (2020) and Achdou et al. (2022) do not involve α, except that it cannot exceed the production elasticity of substitution (a condition automatically satisfied by Cobb-Douglas and $\alpha < 1$). Therefore, our prior was that α in our context is not relevant for non-uniqueness.

drop 327 runs with potential accuracy issues (see Appendix A), leaving $23,673$ sets of parameters with highly accurately computed supply and demand curves. Table 1 summarizes our key findings, which we now discuss.

Two hundred and fifty parameter sets admit three stationary equilibria, and for the rest of the $23,673$ there is one equilibrium. But *all* of the multiplicity examples have the disaster state. Therefore, around 2 percent of runs with $\underline{e} = 10^{-6}$ lead to multiplicity. Figure 1 displays excess capital supply as a function of r and three stationary equilibria for our leading example parameters we will focus on for the bulk of the paper. The parameters are $\beta = 0.9702705242$ (discount factor), $\gamma = 2.3862246357$ (risk aversion), $\rho_e = 0.8637621387$ (income persistence), $\sigma_e = 0.0619142165$ (income volatility), $\delta = 0.422551207$ (depreciation), $\alpha = 0.3$ (capital share), and $\underline{e} = 10^{-6}$ (worst-case-scenario labor supply). Aside from the disaster state and high depreciation, these parameters are not particularly pathological in comparison with the existing literature, and they are much less extreme than the values used in Açıkgöz (2018).

Figure 2 plots histograms of the parameters for the 250 multiplicity examples. Comparing the histograms with the bounds of the uniform distributions reveals three striking observations. First, there are three clear cutoffs (see Panel B of Table 1): we never find multiplicity with $\gamma < 1.49$, $\rho_e < 0.47$, or $\delta < 0.19$. So for the roughly 50 percent of our runs with $\rho_e < 0.47$, there is a unique equilibrium, *regardless of the other parameters*. That low relative risk aversion makes multiplicity less likely is not surprising because lower utility curvature dampens income effects. And, as described in the introduction, several papers prove $\gamma \leq 1$ is sufficient for uniqueness.[7] Uniqueness with $\delta < 0.19$ is potentially less intuitive, although uniqueness condition in Pröhl (2018) requires low δ, and below we provide a story for why high δ contributes to multiplicity. The key role income persistence ρ_e appears to be playing is not, to our knowledge, emphasized in the literature, but we offer some intuition below.

[7] More interesting is that we do find some multiplicity examples for $\gamma < 2$. With two agents and two goods, Loi and Matta (2023) prove RRA ≤ 2 is sufficient for uniqueness with constant relative risk aversion (CRRA) utility (see also Chipman (2010)), so one may have conjectured that $\gamma = 2$ is the magic number for uniqueness.

Our second observation from Figure 2 is that income persistence and depreciation appear to have a monotonic relationship with the likelihood of multiplicity, with higher values of both clearly being associated with more non-uniqueness examples. And our third observation is that, other than the γ cutoff, it is difficult to discern a pattern for the other parameters in isolation, potentially due to the small sample size of 250. β and γ appear to have hump shapes, multiplicity seems less likely with higher volatility, and the capital share seems less relevant for multiplicity.

As the model parameters jointly interact to generate multiplicity, in Panel A of Table 1 we perform a probit regression of an indicator for multiplicity on the parameter values (restricting the sample to cases with $\underline{e} = 10^{-6}$). Confirming our graphical observations from Figure 2, multiplicity is more likely with lower values of β and σ_e and higher values of γ, ρ_e, and δ, although γ is only marginally significant. The capital share α has no clear relationship with multiplicity over the range $[0.2, 0.5]$, consistent with our initial intuition described in Footnote 6.

Pröhl (2018) shows that $\beta\,(1-\delta)^{1-\gamma} < 1$ is sufficient for uniqueness in this class of economies, and in Panel D of Table 1 we see (unsurprisingly) that the "Pröhl number" $\beta\,(1-\delta)^{1-\gamma}$ is never less than 1.25 in our multiplicity examples. Pröhl (2018) never claims the condition is necessary for uniqueness, and Panel D shows that it is not: the condition is violated in 60 percent of runs with a unique stationary equilibrium.

The intuition for the multiplicity is not difficult. Households with $\gamma > 1$ have a strong demand for precautionary savings, which is amplified by the low disaster value of $\underline{e}$. If labor productivity is very persistent, households are particularly worried about hitting $\underline{e}$ because it could last for a long time. As r falls, to satisfy this demand for precautionary savings households must hold more assets as compensation for low returns (they swap principal for interest), and high depreciation amplifies this effect. If r is small enough, this increasing demand for assets outweighs the impatience effect (driven by $\beta\,(1 + r - \delta) < 1$) and leads to a higher capital stock. As the capital supply is increasing once the rental rate is sufficiently close to $1/\beta - 1 + \delta$, supply is non-monotonic and can thus intersect demand more than once. In short, there are two distinct ways in which agents are induced to meet capital demand:

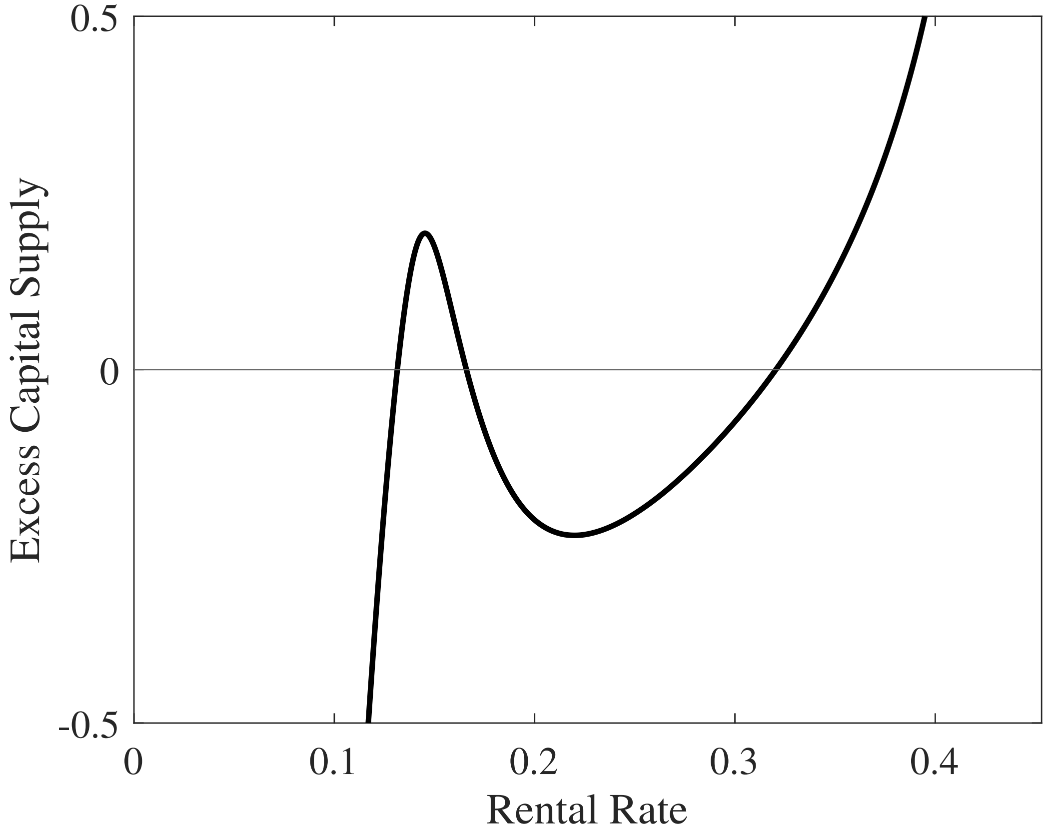

Figure 1: **Three Stationary Equilibria in Aiyagari**

Note: The figure plots excess capital supply $A(r) - K(r)$ against the rental rate r, so intersections with zero correspond to stationary equilibria. The parameters are $\beta = 0.9702705242$ (discount factor), $\gamma = 2.3862246357$ (risk aversion), $\rho_e = 0.8637621387$ (income persistence), $\sigma_e = 0.0619142165$ (income volatility), $\delta = 0.422551207$ (depreciation), $\alpha = 0.3$ (capital share), $\underline{e} = 10^{-6}$ (worst-case-scenario labor supply).

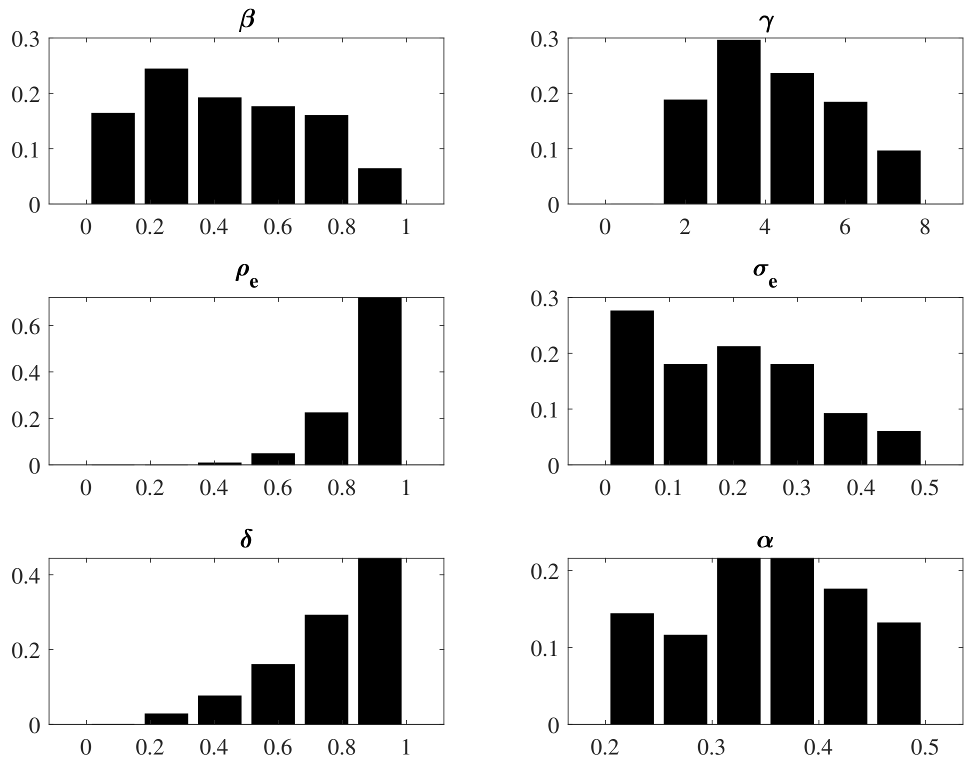

Figure 2: **Parameter Histograms Conditional on Multiplicity**

Note: The figure shows histograms of the model parameters across runs, conditional on there being multiple stationary equilibria. Parameters are drawn from independent uniform distributions on $\beta \in [0, 1)$ (discount factor), $\gamma \in [0, 8]$ (risk aversion), $\rho_e \in [0, 1]$ (income persistence), $\sigma_e \in [0, 0.5]$ (income volatility), $\delta \in [0, 1]$ (depreciation), and $\alpha \in [0.2, 0.5]$ (capital share).

high rental rates, which make the return on capital more attractive, and very low rental rates, which cause agents to amass wealth as a buffer against disasters.

The above mechanism is evident in Açıkgöz (2018), who constructs an Aiyagari (1994) economy with two stationary equilibrium. In contrast, our multiplicity examples always have exactly three stationary equilibria, with one occurring at a very low net return on capital $r - \delta < 0$. This third equilibrium was likely also present in Açıkgöz (2018), although his Figure 1 only plots capital supply and demand for net returns on saving greater than zero (he puts depreciation in the firm's problem, so his interest rate in his Figure 1 corresponds to $r - \delta$ in our analysis). We conjecture that this third equilibrium reflects a potentially new source of multiplicity, and we provide some intuition here. Our "very low rate" equilibrium occurs in flat regions of capital supply, and often capital demand appears to intersect capital supply at a local minimum ($A'(r) = 0$). See the example in Figure 3 (which uses parameters more extreme than in our leading example). And, surprisingly, the r^* that solves $A'(r) = 0$ and also constitutes an equilibrium is approximately $r^* \approx \delta\alpha$. In Panel C of Table 1, we show percentiles of $\frac{\min\{r^*\} - \delta\alpha}{\delta\alpha}$, conditional on multiplicity. The median deviation of the lowest rate from $\delta\alpha$ is 1 percent, $\min\{r^*\}$ is within 2 percent of $\delta\alpha$ 50 percent of the time, and the 1^{st} and 99^{th} percentiles are -17 percent and 21 percent. So $\min\{r^*\}$ is usually quite close to $\delta\alpha$.

How can it be that across many sets of parameters that admit multiplicity, $r^* \approx \delta\alpha$ is approximately both a stationary equilibrium *and* a local minimum of capital supply? It turns out there is a simple explanation. Suppose preference and income process parameters are such that agents are extremely cautious, and suppose the net return on savings is very low, that is, $r \ll \delta$. In this case, agents will be very wary of consuming (due to the disaster state $\underline{e}$), and from the budget constraint we will have $k' \approx we + (1 + r - \delta)k$. Integrating across agents, this condition implies the stationary capital supply $A(r) \approx \frac{wL^*}{\delta - r} = \frac{(1-\alpha)\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}} L^*}{\delta - r}$, where the second equality uses the expression for the equilibrium wage. Then since capital demand is $K(r) = (\alpha/r)^{1/(1-\alpha)} L^*$, it immediately follows that $r^* = \alpha\delta$ satisfies $A(r^*) = K(r^*)$. Furthermore, the derivative of capital supply is $A'(r) =$

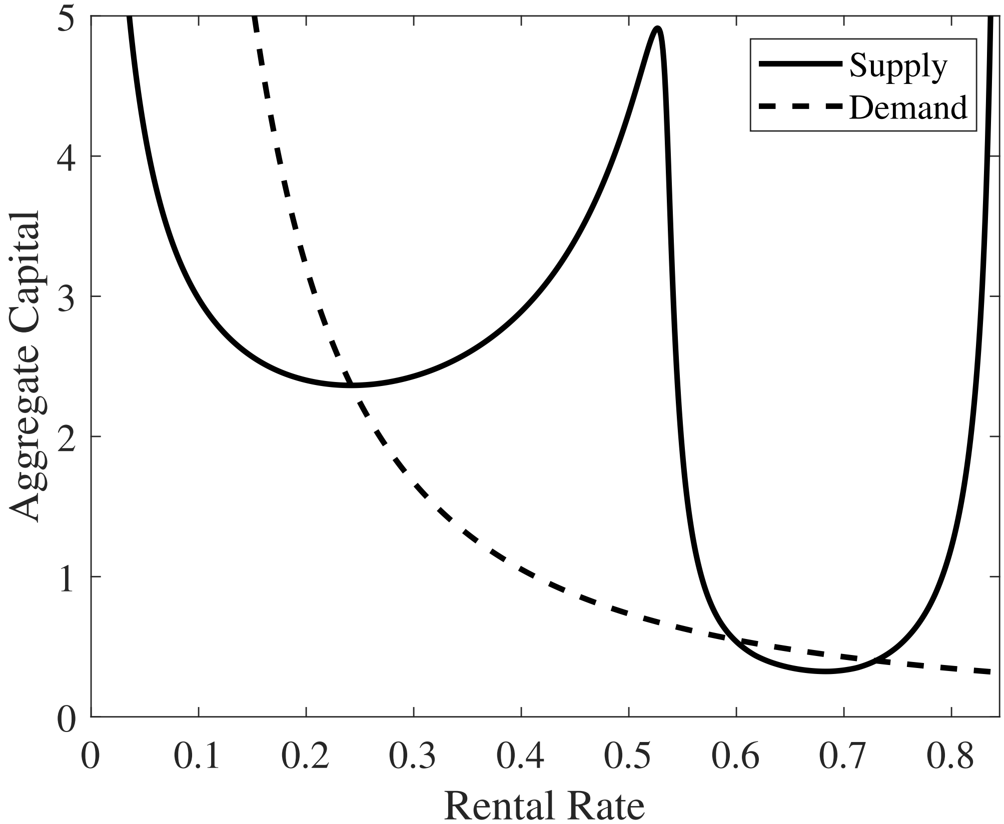

Figure 3: **Example with Stationary Equilibrium at Local Minimum**

Note: The figure plots capital supply and demand, $A(r)$ and $K(r)$, against the rental rate r, so intersections correspond to stationary equilibria. The parameters are $\beta = 0.8292273263$ (discount factor), $\gamma = 3.2492629081$ (risk aversion), $\rho_e = 0.9800171471$ (income persistence), $\sigma_e = 0.1072206969$ (income volatility), $\delta = 0.6388593264$ (depreciation), $\alpha = 0.3784194182$ (capital share), $\underline{e} = 10^{-6}$ (worst-case-scenario labor supply). The "very low rate" equilibrium occurs at $r \approx \alpha\delta = 0.24$.

$\frac{\left((1-\alpha)\frac{\alpha}{\alpha-1}\left(\frac{r}{\alpha}\right)^{\frac{1}{\alpha-1}}\frac{1}{\alpha}L^*\right)(\delta-r)+(1-\alpha)\left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}L^*}{(\delta-r)^2}$, and one can quickly confirm that $A'(\alpha\delta) = 0$.

Of course this result is only an approximation, as agents are not literally consuming zero in the stationary equilibrium. But this analysis provides intuition as to why, for some parameters, there is a tendency toward a U-shaped capital supply at low rates, and it explains the initially mysterious result that the lowest-r equilibrium occurs near $r = \alpha\delta$. Figure 4 shows supply, demand, and aggregate consumption for our leading example parameters used in Figure 1. At low rates, consumption collapses as agents become increasingly worried about getting stuck in the disaster state with such a terrible market return on capital.[8] For these parameters, the effect dissipates sufficiently quickly as rates rise to prevent the full U-shape in Figure 3, but there is still a flattening of supply around $\alpha\delta$, which is very close to the low rate equilibrium.

Given multiple stationary equilibria, it is natural for one to wonder about stability and equilibrium selection. Due to the complicated computational issues that arise when solving for models with aggregate shocks, we postpone the stability discussion (no explosive dynamics local to a particular steady state) to the next section. Here, we discuss selection: if agents believe they will be in an equilibrium with a particular steady-state interest rate, will they be right? If we confine our attention to steady states only, the answer is obviously yes; the computational approach to solving the model imposes that the agents' belief about the rental rate must be correct. But our interest is in whether the agents can shift to a different steady state purely through expectations of future rental rates. To explore this issue, we define a dynamic equilibrium in which prices are time-varying. Households at time t solve the dynamic program

$$v_t(k,e) = \max_{k'\geq 0}\left\{\frac{((1+r_t-\delta)k + w_t e - k')^{1-\gamma}}{1-\gamma} + \beta \,\mathrm{E}\left[v_{t+1}(k',e')\,|e\right]\right\}, \tag{2.3}$$

[8]Consumption is low (with a minimum value of ≈ 0.01), but it is still orders of magnitude above $w\underline{e}$.

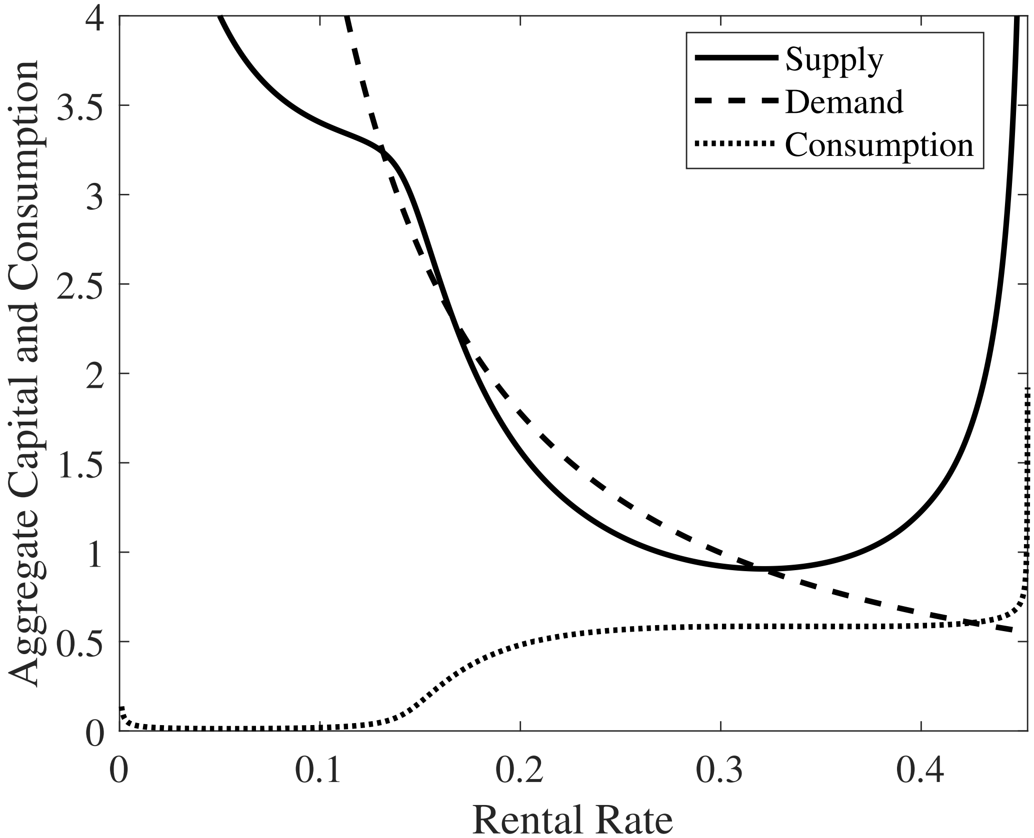

Figure 4: **Three Stationary Equilibria in Aiyagari**

Note: The figure plots capital supply and demand, $A(r)$ and $K(r)$, and aggregate consumption against the rental rate r. The parameters are the same as in Figure 1. The "very low rate" equilibrium occurs at $r \approx \alpha\delta = 0.13$.

where the value function, decision rules, prices, and distributions are indexed by t. A dynamic equilibrium is a sequence $\{r_t\}_{t=0}^{\infty}$ such that (i) household decision rules and value functions solve their dynamic program, (ii) firm factor demands satisfy their first-order conditions, and (iii) the distribution Ω_t evolves according to the policy functions chosen by the agents. Operationally, we consider an economy initially in a particular steady state, and we ask whether there exists a transition path to any other steady state; we interpret the exercise as indicating whether households, if they collectively decide that future interest rates are shifting so that the economy will be in a different steady state, can induce transitional dynamics such that they are correct.

For our leading example parameters, we find that each initial steady state has a transition path that leads to every other steady state. The dynamic equilibrium rental rate paths are plotted in Figure 5. We call the three stationary equilibria (1), (2), and (3), in descending order by rental rate, so equilibrium (1) has the highest rate $r \approx 0.32$. While transitions to equilibria (1) and (3) occur in around 20 periods, transitions to the middle equilibrium (2) take much longer. Nevertheless, in our leading example it is possible to transition rationally from any stationary equilibrium to any other.

Figure 6 describes the welfare effects of these transitions. Specifically, it shows the percent consumption compensation required by agents to be indifferent to the transition vs. the initial steady state, as a function of initial wealth (x-axis) and individual productivity (line thickness).[9] Positive numbers indicate a welfare loss, whereas negative numbers indicate a welfare gain (in this latter case, an agent would pay to undergo the transition). Intuitively, as agents have substantial precautionary savings, they nearly all suffer welfare losses from falling rental rates (which swamp the gains from rising wages). But no transition gives a Pareto gain or loss: very low income households with very low wealth actually gain from falling rates. This gain is due to rising wages, which are extremely beneficial to a household in the disaster state with no wealth; if we hold the wage fixed at the initial steady state, every household experiences a welfare loss from the transition to a lower r steady state.

[9] See the appendix for details of the welfare calculation.

Figure 7 shows the capital distribution and capital Lorenz curve for each of the three stationary equilibria. While we have seen that agents nearly all gain from transitioning to the high-rate equilibrium (1), equilibrium (1) entails more wealth inequality.

3 Krusell-Smith

The Krusell and Smith (1998) model extends Aiyagari (1994) by introducing aggregate risk in the form of randomness in the representative firm's productivity Z. Here, we assume $\log(Z') = \rho_z \log(Z) + \sigma_z \varepsilon_z'$, where $\rho_z \in [0, 1)$ and $\varepsilon_z \sim$ i.i.d. $\mathcal{N}(0, 1)$. In Krusell and Smith (1998), the evolution of idiosyncratic income e depends on Z, so employment is lower in recessions, but here we assume e and Z are uncorrelated.

While introducing aggregate risk is a small change to the Aiyagari (1994) model, doing so greatly complicates both the individual agent's problem and the computation of equilibrium: in the stationary distribution without aggregate shocks, agent optimization only requires knowledge of the (constant) wage and rental rate, or (equivalently) the aggregate capital and labor that pin down the prices. With aggregate shocks, agents must forecast price changes. Due to the Cobb-Douglas representative firm, doing so still only requires forecasts of aggregate capital and labor, but aggregate capital tomorrow depends on the cross-sectional distribution of agents Ω. Therefore, agents must keep track of Ω, an infinite dimensional object, and understand its law of motion $\Omega' = \Gamma(\Omega, Z, Z')$. Calculating equilibrium entails finding a fixed point for Γ, since it both affects and depends on the individual decision rules. *Recursive competitive equilibrium* extends the above *stationary equilibrium* concept to include the object Γ^* assumed by the agents and the "consistency condition" that Γ^* is generated by the optimal decisions rules (i.e., rational expectations). Due to the size of Ω, standard approaches to solving for the equilibrium are not feasible.

However, Krusell and Smith (1998) guessed that the evolution of aggregate capital K, the only relevant object affected by the wealth distribution (since labor supply is exogenous), depends in practice on just a small number of moments of Ω. In particular, their insight was to replace the state variable Ω with K (the mean of the

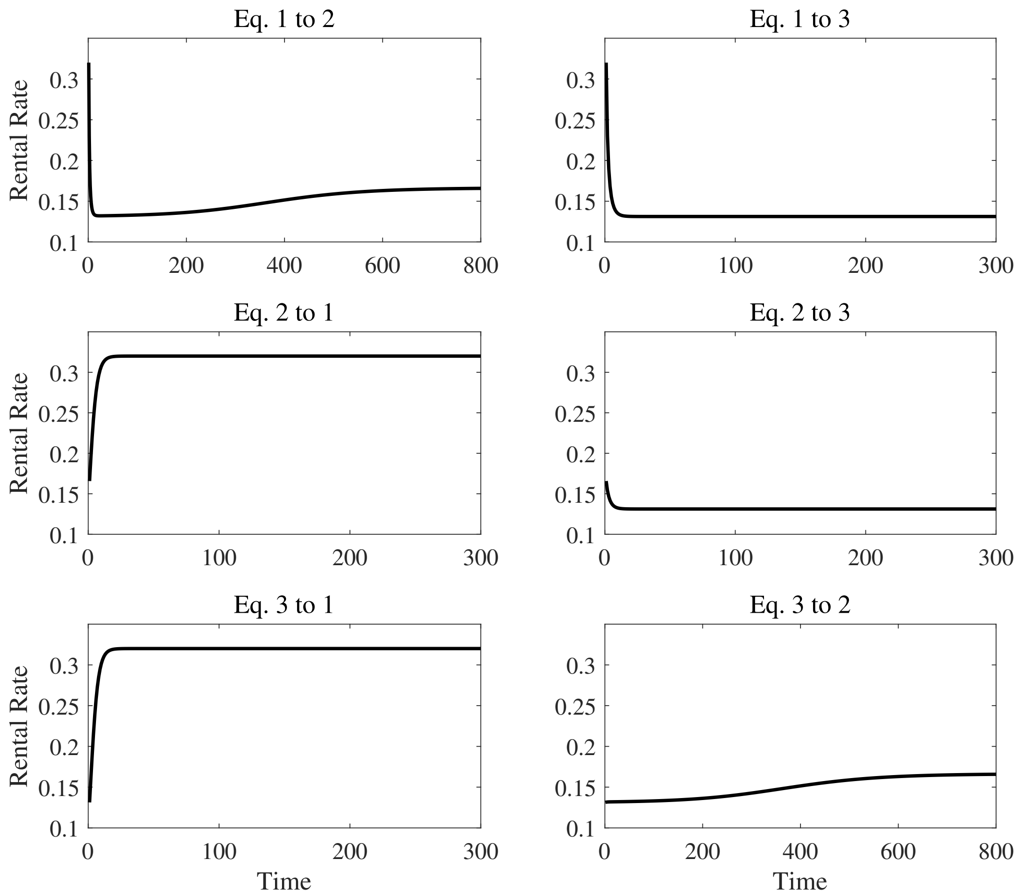

Figure 5: **Transition Paths**

Note: The figure plots dynamic rational equilibrium transitions between the three stationary equilibria in our leading example (Figure 4).

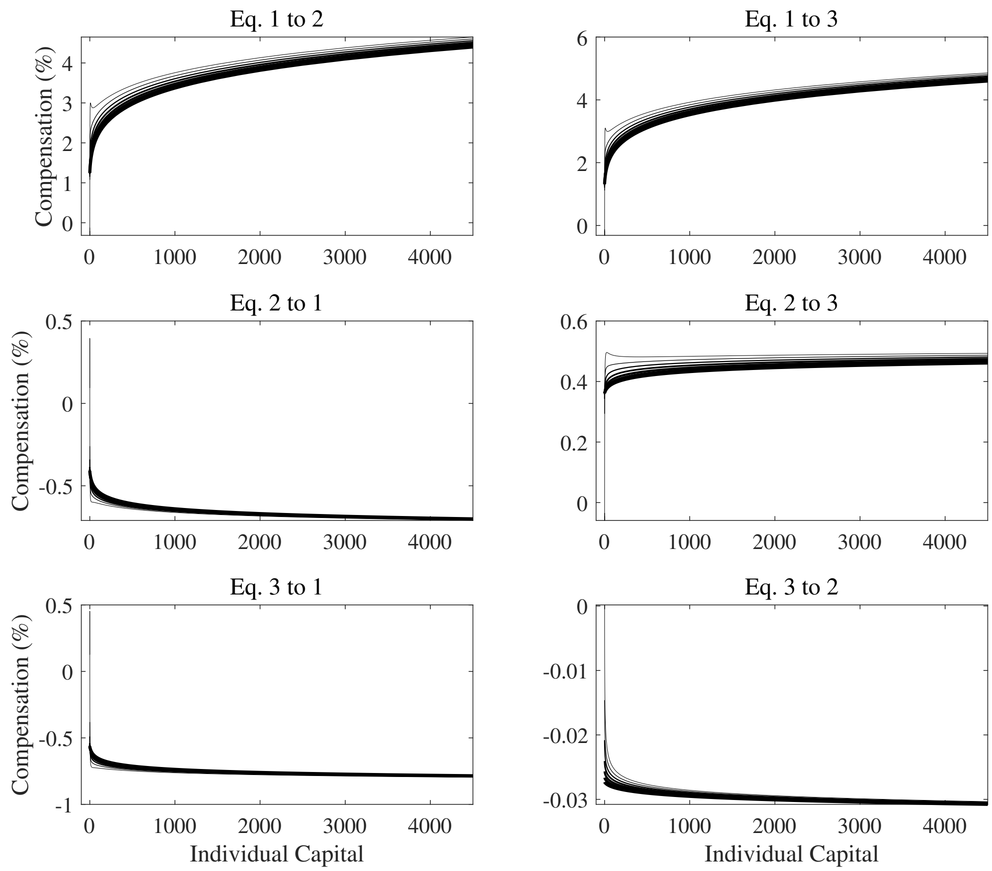

Figure 6: **Compensation Required for Transition**

Note: By initial income and wealth, the figure plots the consumption compensation an individual agent requires to be indifferent to a dynamic rational equilibrium transition between the three stationary equilibria in our leading example (Figure 4). Line thickness corresponds to initial income, so the thinnest lines represent initially being in the disaster state. Positive numbers are welfare losses. The calculation is described in the appendix.

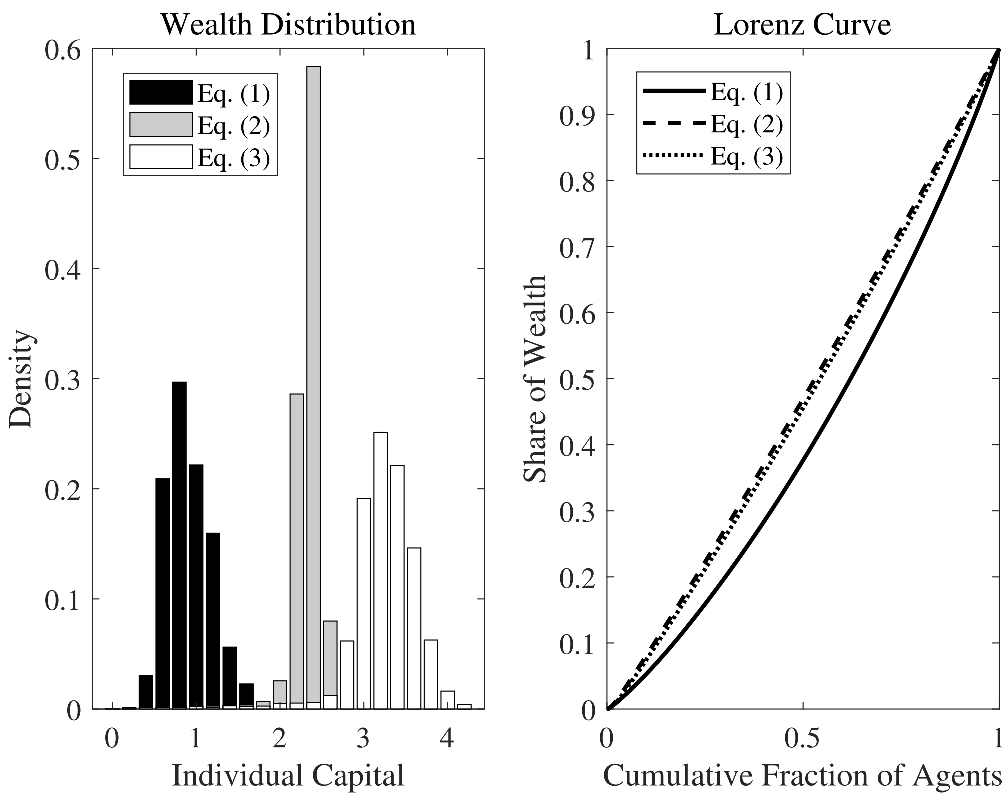

Figure 7: **Inequality in Three Stationary Equilibria in Aiyagari**

Note: The figure plots the cross-agent capital distributions and capital Lorenz curves corresponding to the three stationary equilibria with parameters from Figure 4 (our leading example).

wealth distribution) and replace Γ with some parametric function $K' = G(K, Z; B)$, where B is the parameter vector. In our setting, the agent's optimization problem becomes

$$v(k, e, K, Z) = \max_{k' \geq 0} \left\{ \begin{array}{c} \frac{1}{1-\gamma} \left((1 + r(K, Z) - \delta) k + w(K, Z) e - k' \right)^{1-\gamma} \\ +\beta \, \mathrm{E} \left[v(k', e', G(K, Z; B), Z') \, | e, Z \right] \end{array} \right\}, \tag{3.1}$$

where $r(K, Z) = \alpha Z K^{\alpha-1} L^{1-\alpha}$ and $w(K, Z) = (1-\alpha) Z K^{\alpha} L^{-\alpha}$.[10] The Krusell and Smith (1998) equilibrium consists of a value function, decision rule for k, and the parameters of the law of motion B^* such that (i) agents are optimizing given $G(K, Z; B^*)$ and (ii) B^* minimizes the distance between $G(K, Z; B)$ and the realized law of motion for K (according to OLS, for example, if G is assumed to be log-linear, as below).[11]

This equilibrium is "behavioral" in the sense that it does not require agents to completely accurately forecast the law of motion for K. However, in the original Krusell and Smith (1998) paper and in numerous subsequent papers, it turns out that $G(K, Z; B^*)$ is nearly identical to the realized law of motion (e.g., $R^2 = 0.999998$ on page 887 in the original paper). Consequently, the literature has generally accepted that this procedure yields an approximation extremely close to true competitive recursive equilibria.

To our knowledge, no one has ever provided an example in this context with multiple recursive equilibria. We construct two examples here. First, we use our leading example parameters $\beta = 0.9702705242$ (discount factor), $\gamma = 2.3862246357$ (risk aversion), $\rho_e = 0.8637621387$ (income persistence), $\sigma_e = 0.0619142165$ (income volatility), $\delta = 0.422551207$ (depreciation), $\alpha = 0.3$ (capital share), and $\underline{e} = 10^{-6}$ (worst-case-scenario) that we have already shown yield multiplicity in the corresponding Aiyagari (1994) model. Our second example uses many of the parameters from Açıkgöz (2018). Our initial conjecture was that aggregate risk might "refine away" the multiplicity, but that is not the case: all three steady states in our first

[10] Note we have omitted L as a state variable because in equilibrium it will be constant (here) or depend only on Z (as in Krusell and Smith (1998)).

[11] The expressions for r and w imply market clearing and the optimality of firm factor demand.

example are locally stable for small fluctuations driven by aggregate productivity.

We set the aggregate productivity parameters to $\rho_z = 0.95$ and $\sigma_z = 0.0076$, discretize $\log(Z)$ to 7 states via Rouwenhorst, and assume the following parametric form for the aggregate capital law of motion:

$$G(K, Z; b_0, b_1, b_2) := \exp(b_0) K^{b_1} Z^{b_2},$$

meaning $B = (b_0, b_1, b_2)$ and

$$\log(K') = b_0 + b_1 \log(K) + b_2 \log(Z). \tag{3.2}$$

To find equilibrium we follow the algorithm in Young (2010). Given a guess for B, we solve the household's problem to obtain decision rules $k' = g(k, e, K, Z)$. We then simulate the economy using these decision rules, starting from one of the stationary equilibrium distributions. We use the simulated path of the economy to update the guess for B and iterate until the coefficients converge. More details can be found in the appendix.

Using the leading example parameters from Figure 1, we find three Krusell and Smith (1998) equilibria, each surrounding one of the steady states:

$$G^{(1)}(K, Z) = \exp(-0.0279 + 0.7147 \log(K) + 0.5404 \log(Z)) \tag{3.3}$$
$$G^{(2)}(K, Z) = \exp(0.2436 + 0.7107 \log(K) + 0.4855 \log(Z)) \tag{3.4}$$
$$G^{(3)}(K, Z) = \exp(0.3464 + 0.7051 \log(K) + 0.4273 \log(Z)) \tag{3.5}$$

All R^2's are above 0.99999, and all maximum regression errors are on the order of 10^{-5}. Figure 8 shows the ergodic rental rate densities in the three recursive competitive equilibria. The distributions are non-overlapping and centered around the corresponding Aiyagari (1994) steady states.[12]

For our second example, we use $\beta = 0.1$, $\alpha = 0.5$, $\gamma = 6.5$, $\delta = 1$, $\rho_e = 0.7$, $\sigma_e = 0.2$, and $\underline{e} = 10^{-6}$, which are similar to the values in Açıkgöz (2018) and yield

[12]The middle equilibrium is very sensitive and the algorithm can easily fail to converge. It required a substantial amount of "babysitting" to find the equilibrium coefficients.

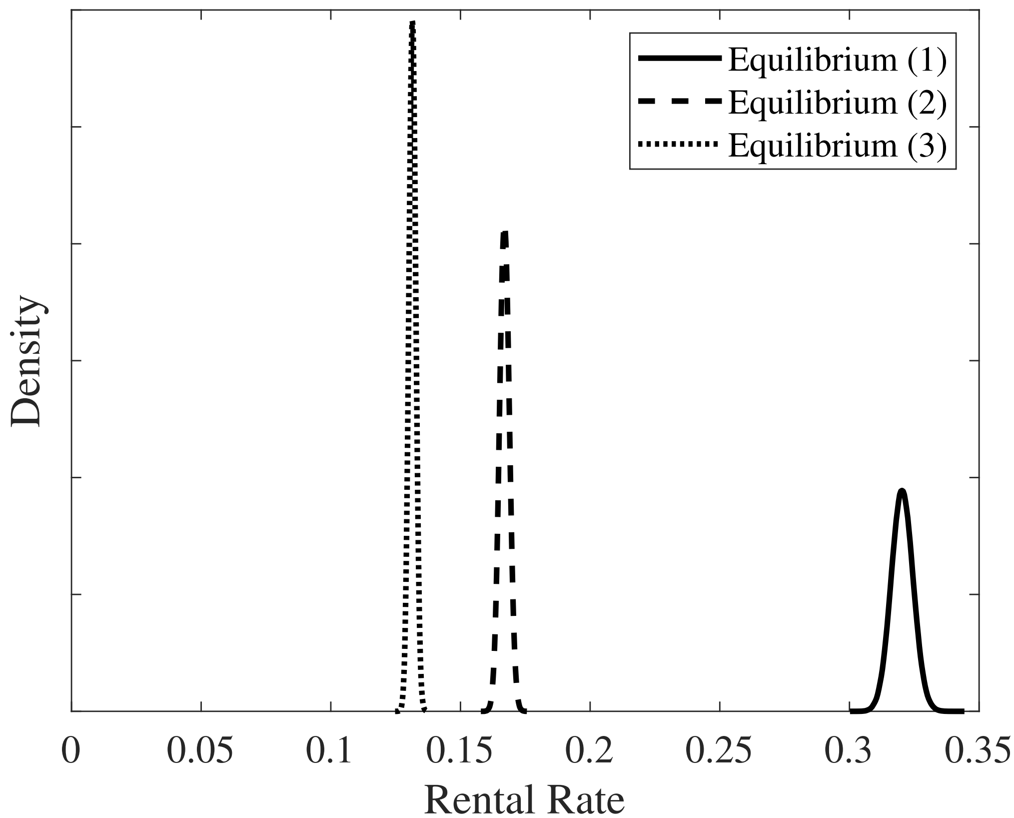

Figure 8: **Ergodic Rental Rate Densities for Three Krusell-Smith Equililibria**

Note: The figure plots the rental rate ergodic densities for the three Krusell-Smith equilibria (3.3), (3.4), and (3.5) based on the parameters from Figure 1 and aggregate productivity process given in the main text. To compute the densities, we apply kernel smoothing (ksdensity.m in MATLAB) after simulating the three laws of motion for one million periods.

three stationary Aiyagari (1994) equilibria at $r \approx 0.50$, $r \approx 0.62$ and $r \approx 2.25$.[13] Using the same aggregate productivity process and algorithm as before, we find two recursive competitive equilibria, corresponding to the highest and lowest rental rates:

$$\begin{aligned} \text{High } r: \log\left(K'\right) &= -0.9276 + 0.6879 \log\left(K\right) + 0.4039 \log\left(Z\right) \\ \text{Low } r: \log\left(K'\right) &= 0.0149 + 0.5002 \log\left(K\right) + 0.9993 \log\left(Z\right). \end{aligned}$$

Again, the functions fit nearly perfectly ($R^2 \approx 1$ and the maximum error is approximately 10^{-6}). The middle steady state $r \approx 0.62$ appears to be unstable; the economy, when simulated using aggregate productivity shocks, diverges to one of the other steady states. There also does not appear to be a transition path that converges to this steady state – iterating on the path does not converge to a fixed point – but there are transition paths away from it.

To see what is going on, we show the three stationary equilibria for this example in Figure 9. Note that the asset supply curve is very elastic in the neighborhood of the middle steady state. We conjecture that this "flatness" is what drives the local instability result: small deviations of r_t from r^* induce large movements in capital, and they are then attracted to a different steady state.

To explore this possibility, we perform an "MIT shock" exercise with respect to the middle equilibrium in the model without aggregate risk. Starting at this stationary equilibrium at time t, Z increases from 1 to $\bar{Z} = 1.0076289533$ at time $t+1$ and then drops back to 1 at $t+2$ (and stays at 1 thereafter). The one-time aggregate productivity shock induces a dynamic equilibrium path for the rental rate $r_{t+1}, r_{t+2}, \ldots$. We have already shown that beliefs about future rates can be self-fulfilling. To remove this effect, we now assume that agents are myopic in a particular sense. Specifically, at any time τ agents correctly forecast $r_{\tau+1}$ but assume (incorrectly) that it will persist forever. Let $k' = h\left(k, e, r, r'\right)$ be the decision rule of an agent currently facing rate r but believing the rate will be r' in all subsequent periods. Starting at $r_t = r^*$ and $\Omega_t = \Omega^*$ (the steady state), the dynamic MIT shock

[13] The biggest difference is that Açıkgöz (2018) uses $\alpha = 0.3$, but we were unable to construct multiple recursive equilibria with that value.

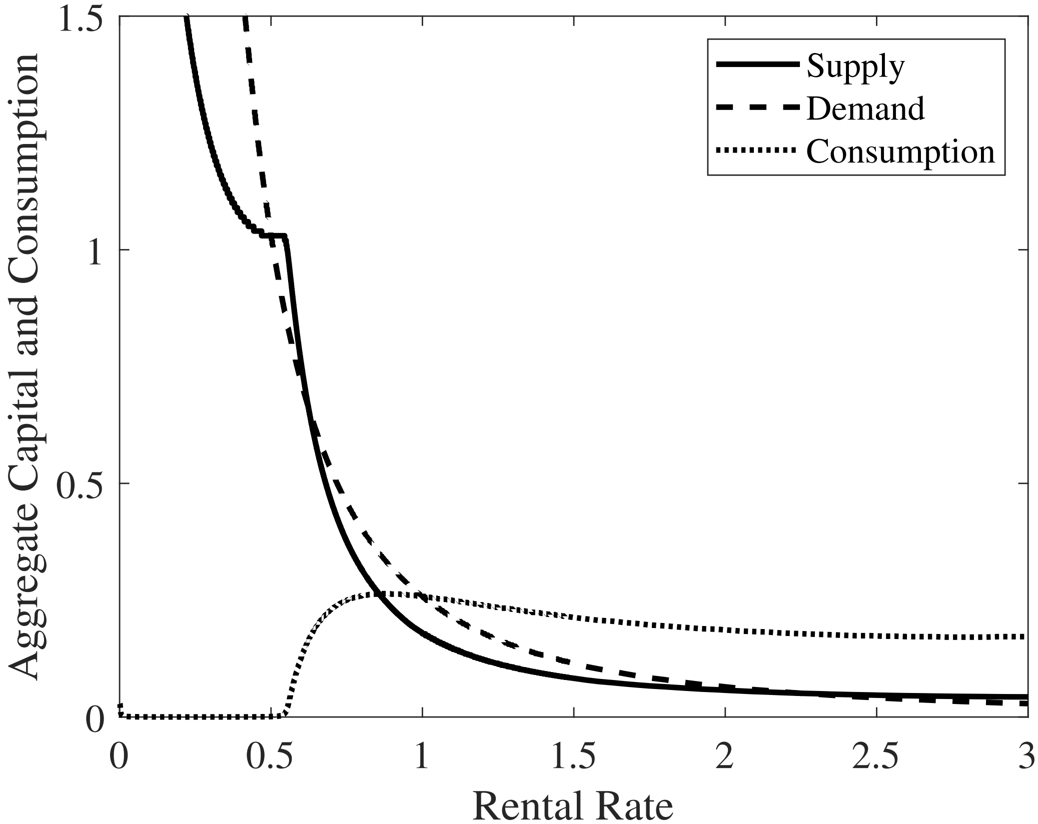

Figure 9: **Example with Unstable Middle Equilibrium**

Note: The figure plots capital supply and demand, $A(r)$ and $K(r)$, and aggregate consumption against the rental rate r. The parameters are similar to those in Açıkgöz (2018) except for $\alpha = 0.5$. The "low rate" equilibrium occurs at $r \approx \alpha\delta = 0.5$.

equilibrium rates solve

$$r_{t+1} = Z_{t+1}\alpha \left(\int h\left(k, e, r_t, r_{t+1}\right) d\Omega_t \right)^{\alpha-1} L^{1-\alpha},$$

where the path of Ω_t is induced by the decision rules (see the appendix for details). The result of the MIT shock is depicted in Figure 10. As conjectured, the middle equilibrium is unstable in the sense that the one-time shock causes the economy to transition to the lower rate equilibrium (vs. return to where it started).[14]

4 Conclusion

We have constructed many Aiyagari (1994) examples with three stationary equilibria, described which parameters appear most important for multiplicity, illustrated the possibility of transitions between equilibria, and constructed two Krusell and Smith (1998) economies with multiple recursive competitive equilibria. These stationary equilibria may be all locally stable or the middle one can be locally unstable.

Our results may seem esoteric at first glance. However, we will argue in this conclusion that they are relevant for many issues. The literature that attempts to estimate "r-star" – the "natural" or "neutral" rate of interest – finds that it moves around substantially over time.[15] Indeed, a downward trend in r-star since the 1980s is a popular explanation for falling real interest rates over the same period, as actual real rates are thought to be anchored to the natural rate in the long run through equilibrium forces. ECB Executive Board Member Isabel Schnabel's March 20, 2024 speech (Schnabel, 2024) discussed two competing theories for long-run declines in real rates, the role of r-star, and the importance of the debate for the conduct of monetary policy. The predominant view is the "savings-investment hypothesis," which says that demographic factors, slowing productivity growth, and

[14] We also conducted this experiment with a shock to Z_{t+1} that gradually decayed at rate $\rho = 0.95$. The economy transitioned to the high rate equilibrium, indicating that it is not stable with respect to this deviation either. We are unsure why the persistence of the deviation plays a role in determining which steady state attracts the economy.

[15] See, for example, Lubik and Matthes (2023).

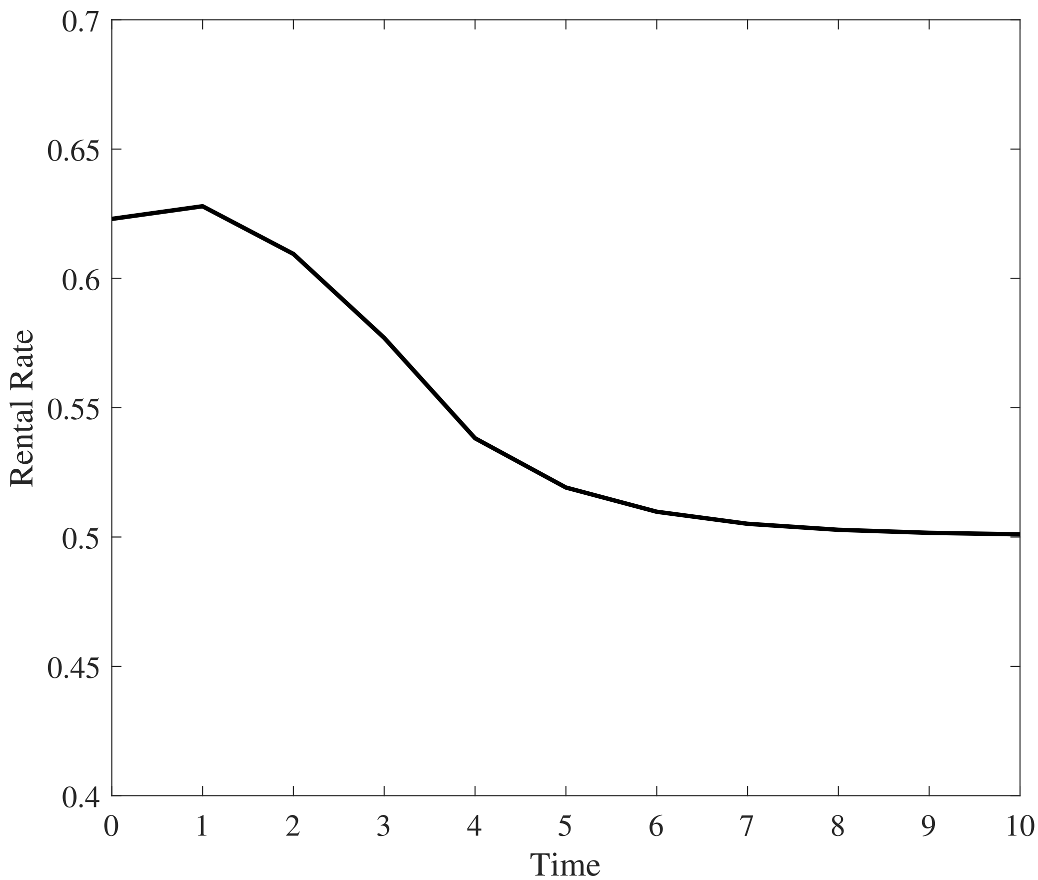

Figure 10: **Unstable Stationary Equilibrium**

Note: The figure shows the equilibrium path of rental rates following a small, one-time shock to aggregate productivity in an Aiyagari (1994) economy with parameters $\beta = 0.1$, $\alpha = 0.5$, $\gamma = 6.5$, $\delta = 1$, $\rho_e = 0.7$, $\sigma_e = 0.2$, and $\underline{e} = 10^{-6}$. At time 0, the economy begins at the middle steady state with $r^* \approx 0.62$ and $Z = 1$ (aggregate productivity). Z increases to ≈ 1.01 at time 1 and then returns to 1. Agents correctly forecast one-period-ahead rates but (incorrectly) assume they persist forever. The path converges to the lower steady state with $r \approx 0.50$.

a global savings glut have pushed down r-star (and hence real rates) over time. In this framework, central banks should cut rates to prevent passive contractionary policy, and unconventional monetary policy may be required if nominal rates and inflation are already low. A recent alternate theory is what Schnabel (2024) calls the "monetary policy hypothesis."[16] This perspective questions the empirical evidence for the savings-investment view and suggests that accommodative monetary policy itself has pushed down real rates (and potentially r-star). One possible channel is an informational one: low policy rates signal to consumers and firms a low r-star and a poor economy. The resulting pessimism of consumers and firms reduces consumption and growth, justifying past or future monetary easing and actually decreasing r-star. In this framework, overly-loose monetary policy creates a self-fulfilling prophecy of a poor economy without any initial change in fundamentals.

Our transition examples provide a possible microfoundation for the monetary policy hypothesis in the context of a heterogeneous agent model with capital. Starting in the high-rate equilibrium, we show that if agents believe real rates will fall, there is a rational, self-fulfilling equilibrium transition path to a new steady state with lower aggregate consumption. We suspect that in a HANK extension of our model with learning about r-star, a sufficiently accommodative monetary policy shock could trigger this transition. And once agents' beliefs and behavior shift, the central bank could have to accommodate the transition to avoid overly tight monetary policy relative to the new natural rate.[17]

To speculate further, another possibility is that monetary policy could be used to select between equilibria. While the transitions to lower rate equilibria reduce welfare for most agents in our model, the low-rate equilibria also exhibit less wealth inequality. So to the extent that agents learn about r-star from the actions of the central bank, monetary policy in a world with multiple stationary equilibria could be a powerful redistributive force, for better or worse.

Admittedly, even our leading example parameter set is not a standard calibration

[16]See, for example, Borio (2021), Borio et al. (2022), and Rungcharoenkitkul and Winkler (2023).

[17]See Sargent (1999) for examples of models in which central bank actions shift household expectations.

because of the high depreciation value, so we are not ready to propose our findings as a quantitative resolution in r-star discussions. But, on the other hand, our leading parameters do not strike us as particularly extreme, and we have considered only the simplest versions of the Aiyagari (1994) and Krusell and Smith (1998) models. So it seems plausible that more elaborate income processes, more assets, more general utility specifications, or more general production functions could generate multiplicity and rational equilibrium transitions in completely standard calibrations.[18]

More broadly, we hope that our numerical analysis will provide inspiration for future theoretical work on the Bewley class of models. The Bewley class is a cornerstone of both academic and policy-oriented macroeconomics, so it is essential that we understand its equilibrium properties. As Kehoe (1985) wrote, "Conditions that guarantee the uniqueness of equilibrium in models of economic competition are crucial to applications of these models in exercises of comparative statics."

[18]For example, multisector production models display indeterminacy at lower degrees of increasing returns to scale than one-sector models; see Benhabib and Farmer (1996).

Table 1: Aiyagari Result with Disaster State

A: Probit Regressions		
Dependent variable: indicator for multiplicity		
	β	-0.42
		(0.12)
	γ	0.04
		(0.02)
	ρ_e	4.29
		(0.28)
	σ_e	-1.45
		(0.25)
	δ	2.20
		(0.16)
	α	0.26
		(0.41)
	R^2	0.36
B: Parameter Ranges Conditional on Multiplicity		$\{\min, \max\}$
	β	$\{0.01, 0.98\}$
	γ	$\{1.49, 7.96\}$
	ρ_e	$\{0.47, 1.00\}$
	σ_e	$\{0.00, 0.50\}$
	δ	$\{0.19, 1.00\}$
	α	$\{0.20, 0.50\}$
C: Very Low Rate Equilibrium		Percentiles $\{1^{st}, 25^{th}, 50^{th}, 75^{th}, 99^{th}\}$
$\frac{\min\{r^*\}-\delta\alpha}{\delta\alpha}$ if multiplicity		$\{-0.17, 0.01, 0.01, 0.02, 0.21\}$
D: Other		
Minimum Pröhl number if multiplicity		1.25
Fraction Pröhl number > 1 if uniqueness		0.60
Fraction of runs with multiplicity		0.02 (250 examples)

Note: Drawing parameters from independent uniform distributions on $\beta \in [0, 1)$ (discount factor), $\gamma \in [0, 8]$ (risk aversion), $\rho_e \in [0, 1]$ (income persistence), $\sigma_e \in [0, 0.5]$ (income volatility), $\delta \in [0, 1]$ (depreciation), and $\alpha \in [0.2, 0.5]$, the table describes the relationship between the parameters and the incidence of stationary equilibrium multiplicity in the Aiyagari (1994) model with the lowest income realization set to $\underline{e} = 10^{-6}$. The table is based on 12,080 parameter sets. Panel A reports probit regressions with a constant (coefficient omitted). R^2 is the pseudo R-squared, and standard errors are in parentheses. Panel B reports the parameter $\{\min, \max\}$ across runs conditional on multiplicity. Panel C shows the distribution of the distance of the lowest rental rate from $\delta\alpha$ conditional on multiplicity. Panel D reports the minimum across runs of $\beta(1-\delta)^{1-\gamma}$ ("Pröhl number") conditional on multiplicity, the fraction of runs with $\beta(1-\delta)^{1-\gamma} > 1$ conditional on uniqueness, and the fraction of runs with multiplicity.

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A Computational Details

A.1 Aiyagari

Individual labor productivity processes are discretized to 7 states by Rouwenhorst's method. Individual agent problems are solved with a 160 point capital grid, and capital distributions are calculated on a $30,000$ point grid. Capital supply and demand are computed on a $1,001$ point grid for the rental rate, ranging from 0.001 to $1/\beta - 1 + \delta - 0.001$, since existence of a stationary equilibrium requires $r - \delta \in (-\delta, 1/\beta - 1)$.

Our codes were written in Fortran. We broke the $24,000$ sets of random parameters into 48 groups of 500 parameters each and calculated the $24,000$ supply and demand curves in parallel on 48 cores of the ETH Euler Cluster (which took around five days to finish). For a given set of parameters, we count the equilibria by finding all of the sign changes in excess capital supply.[19]

43 parameter sets generated non-number outputs, and 254 parameter sets resulted in stationary capital distributions hitting the maximum of the $30,000$ point grid for some rental rates. Of the remaining parameter sets, 73 yielded no stationary equilibrium. To be conservative, we dropped from our analysis the 327 runs with non-numbers, binding capital maximums, or no equilibrium, leaving the $23,673$ parameter sets considered in the main text.

We compute stationary equilibria using the following algorithm. Given r, the wage that is consistent with firm optimization is

$$w(r) = (1-\alpha)\left(\frac{r}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

[19]For the $24,000 \times 1,001$ excess supplies we computed, in exactly one instance excess supply was zero to machine precision.

The Euler equation for an individual household in state (k, e) is

$$
\begin{aligned}
&((1+r-\delta)\, k + we - k'\,(k,e;r))^{-\gamma} \geq \\
&\beta\,(1+r-\delta) \sum_{e'} \pi\,(e'|e)\,((1+r-\delta)\, k'\,(k,e;r) + w\,(r)\, e' - g\,(k'\,(k,e;r)\,, e';r))^{-\gamma}\,,
\end{aligned}
$$

where the π's are the Rouwenhorst probabilities. We use a nonlinear root finder to solve this equation given a guess for $g\,(k,e;r)$; if the root finder converges to $k'=0$ then the agent is borrowing constrained. We evaluate $g\,(k',e';r)$ using a Piecewise Cubic Hermite spline. We iterate until convergence. We then construct the invariant distribution $\Omega\,(k,e;r)$ using the method from Young (2010), and compute

$$
K\,(r) = \sum_{k} \sum_{e} k\Omega\,(k,e;r)\,.
$$

We then use Brent's method to adjust r until

$$
\alpha K\,(r)^{\alpha-1}\, L^{1-\alpha} = r.
$$

To compute the entire demand and supply curves for capital, we use a large number of different values of r and solve for $\Omega\,(k,e;r)$ for each of them, where

$$
\begin{aligned}
K^{d}\,(r) &= \left(\frac{r}{\alpha}\right)^{\frac{1}{\alpha-1}} L \\
K^{s}\,(r) &= \sum_{k} \sum_{e} k\Omega\,(k,e;r)\,.
\end{aligned}
$$

Using the grid to roughly identify the location of zeros, we can refine them using Brent's method as above.

For transition dynamics, we use the time-varying version of the Euler equation:

$$
\begin{aligned}
&((1+r_t-\delta)\, k + w_t e - k'_t\,(k,e))^{-\gamma} \geq \\
&\beta\,(1+r_{t+1}-\delta) \sum_{e'} \pi\,(e'|e)\,((1+r_{t+1}-\delta)\, k'_t\,(k,e) + w_{t+1} e' - g_{t+1}\,(k'_t\,(k,e)\,, e'))^{-\gamma}\,.
\end{aligned}
$$

We guess a sequence of rental rates $\{r_t\}_{t=1}^T$. We then solve the Euler equation backward from the terminal date $T = 800$, assuming the appropriate stationary decision rule governs the choice in period $T+1$. We then iterate the procedure from Young (2010) forward from an initial distribution, compute aggregate capital K_t in each time period, and update the sequence of rental rates by setting $r_t = \alpha K_t^{\alpha-1} L^{1-\alpha}$ until the sequence converges.

To compute welfare, we compare the value function in the initial steady state $v(k,e)$ with the value function in period 1 of the transition $v_1(k,e)$. Suppose that we augment consumption along the transition and in the new steady state by λ percent; using the homogeneity of the value function, the agent would value that allocation as

$$V(k,e;\lambda) = (1+\lambda)^{1-\gamma} v_1(k,e).$$

Setting this expression equal to the value of remaining in the initial steady state, we can solve for the welfare gain/loss:

$$\lambda(k,e) = \left(\frac{v(k,e)}{v_1(k,e)}\right)^{\frac{1}{1-\gamma}} - 1.$$

A.2 Krusell-Smith

To solve the version with aggregate shocks, we follow Krusell and Smith (1998) and suppose that households explicitly track only K. Given (K,Z), we can compute factor prices:

$$\begin{aligned} r(K,Z) &= \alpha Z K^{\alpha-1} L^{1-\alpha} \\ w(K,Z) &= (1-\alpha) Z K^{\alpha} L^{-\alpha}. \end{aligned}$$

We then guess $G(K,Z)$, the function that links the current state to the household's forecast of future aggregate capital:

$$G(K,Z) = \exp\left(b_0 + b_1 \log(K) + b_2 \log(Z)\right).$$

The household's Euler equation is

$$
\begin{aligned}
&\left(\left(1+r\left(K,Z\right)-\delta\right)k+w\left(K,Z\right)e-k'\left(k,e,K,Z\right)\right)^{-\gamma}\geq\\
&\beta\sum_{Z'}\pi\left(Z'|Z\right)\left(1+r\left(G\left(K,Z\right),Z'\right)-\delta\right)\times\\
&\sum_{e'}\pi\left(e'|e\right)\begin{pmatrix}(1+r\left(G\left(K,Z\right),Z'\right)-\delta)k'\left(k,e,K,Z\right)\\ +w\left(G\left(K,Z\right),Z'\right)\\ -g\left(k'\left(k,e,K,Z\right),e',K',Z'\right)\end{pmatrix}^{-\gamma};
\end{aligned}
$$

we use linear interpolation to compute g at values of K'. To simulate, we choose $T = 10,000$ random values of Z and iterate on the method from Young (2010). We then update the coefficients in $G(K,Z)$ using an OLS regression on the time series $\{K_t, Z_t\}_{t=1}^{T}$, where

$$K_t = \sum_k \sum_e k\Omega_t\left(k,e\right).$$

We initialize the simulation at the stationary distribution for each steady state rental rate r.

For the "MIT shock" experiment, we posit a sequence of productivity values $\{Z_t\}$, with $Z_2 = 1.0076$ and $Z_t = 1$ otherwise. Suppose we have computed the sequence up to time t. At time $t+1$, the agent will be solving the dynamic program

$$v_{t+1}\left(k,e;r_{t+1}\right) = \max_{k'\geq 0, c\geq 0}\left\{\frac{c^{1-\gamma}}{1-\gamma}+\beta\sum_{e'}\pi\left(e'|e\right)v_{t+1}\left(k',e';r_{t+1}\right)\right\}$$

$$c+k'\leq\left(1-\delta+r_{t+1}\right)k+w_{t+1}e$$

where

$$
\begin{aligned}
r_{t+1} &= \alpha Z_{t+1}K_{t+1}^{\alpha-1}L^{1-\alpha}\\
w_{t+1} &= \left(1-\alpha\right)Z_{t+1}K_{t+1}^{\alpha}L^{-\alpha}.
\end{aligned}
$$

That is, the households believe that whatever prices prevail tomorrow, they will remain constant from that point forward. The Euler inequality for all periods after

t is

$$\left((1-\delta+r_{t+1})\,k+w_{t+1}e-g^{*}_{t+1}(k,e)\right)^{-\gamma}\geq$$
$$\beta\,(1-\delta+r_{t+1})\sum_{e'}\pi\,(e'|e)\left((1-\delta+r_{t+1})\,g^{*}_{t+1}(k,e)+w_{t+1}e'-g^{*}_{t+1}\left(g^{*}_{t+1}(k,e)\,,e'\right)\right)^{-\gamma}.$$

Now we need to update the wealth distribution. At time t, the decision rule $k'=g_t(k,e)$ that households will follow solves

$$\left((1-\delta+r_t)\,k+w_te-k'\right)^{-\gamma}\geq$$
$$\beta\,(1-\delta+r_{t+1})\sum_{e'}\pi\,(e'|e)\left((1-\delta+r_{t+1})\,k'+w_{t+1}e'-g^{*}_{t+1}(k',e')\right)^{-\gamma}.$$

That is, they behave optimally today given (r_t, r_{t+1}), but mistakenly believe r_{t+1} will persist forever; that is, they assume their future savings will be governed by $g^{*}_{t+1}(k',e')$. The rule $g_t(k,e)$ is then used to update Ω_t to Ω_{t+1}.

Markets clear if

$$\begin{aligned}K_{t+1}&=\int g_t(k,e)\,d\Omega_t(k,e)\\ r_{t+1}&=\alpha Z_{t+1}K_{t+1}^{\alpha-1}L^{1-\alpha}\\ w_{t+1}&=(1-\alpha)\,Z_{t+1}K_{t+1}^{\alpha}L^{-\alpha}.\end{aligned}$$

Given the distribution and rental rate up to t, we use Brent's method to find r_{t+1} that clears the market at $t+1$ and then update the distribution using the decision rules evaluated at the market clearing value of r_{t+1}.