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Federal Reserve Balance-Sheet Policy in an Ample Reserves Framework: An Inventory Approach

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Abstract

I apply techniques from stochastic inventory theory to calibrate the optimal balance-sheet buffer needed to implement monetary policy in an ample reserves regime. I quantify the size of the buffer to be about $60 billion. This is small relative to the reserves needed for an ample reserves regime, even though the FOMC appears to act as if the cost of too few reserves is over 20 times as high as the cost of too many.

Keywords: Reserves, Monetary Policy
JEL codes: E58, D25

1 Introduction

In May 2022, the Federal Open Market Committee (FOMC) announced plans to reduce its holding of securities, and thereby its balance sheet, stating that "the Committee intends to maintain securities holdings in amounts needed to implement monetary policy efficiently and effectively in its ample reserves regime" (FOMC 2022). Implementing this plan requires holding enough securities so that shocks to demand and supply do not push reserves below the ample level. Using techniques from stochastic inventory theory calibrated to the Federal Reserve's balance sheet and the federal funds rate, this paper shows that the level of securities needed may be relatively small, despite the FOMC acting as if the costs of reserves falling below "ample" far exceed the costs of having a large balance sheet.

Reserves are ample when the FOMC can control its target, the federal funds rate, by changing the administered rates it controls, rather than increasing or decreasing reserves via open market operations. In the words of the 2020 Monetary Policy Report:

> "In such a system, active management of reserves through frequent open market operations is not required, and control over the level of the federal funds rate and other short-term interest rates is exercised primarily through the setting of the Federal Reserve's administered rates."

*Federal Reserve Bank of Cleveland, PO Box 6387, Cleveland, OH 44101-1387, 216 579 2802 jhaubrich@clev.frb.org. The views expressed here are solely those of the author and not necessarily those of the Federal Reserve Bank of Cleveland or the Board of Governors of the Federal Reserve System. This paper is my attempt to think through some issues of an ample reserves regime. In memory of Marvin Goodfriend, who both in person and through his work taught me a lot about monetary regimes. Thanks to Matt Sobel for help on inventories, Tom Phelan for useful discussions, and Rachel Widra and Chris Healy for excellent research assistance.

In textbook accounts, the FOMC increases bank reserves, a liability on its balance sheet, by purchasing securities, which are recorded as assets. Having an ample level of reserves implies having a sufficiently large balance sheet: questions about the appropriate level of reserves become questions about the appropriate size of the Fed's balance sheet. In reality, however, there is not a one-to-one correspondence between security holdings and reserves, as there are other liabilities, particularly currency in circulation, deposits of the US Treasury, and overnight reverse repos (ONRRP). Shifts in these factors constitute a supply shock to the level of reserves. Figure 1 plots the ratio to GDP of bank reserves and the Fed's security holdings, known as the System Open Market Account (SOMA).

The level of reserves that qualifies as "ample" is to some degree uncertain, as is the size of the buffer needed to defend against shocks to demand and supply. On one side, the FOMC in its 2014 *Policy Normalization Principles and Plans* stated that the Federal Reserve will "hold no more securities than necessary to implement monetary policy efficiently and effectively." Conversely, a current and former Fed official argued that

> "The minimum level of reserves is conceptually murky, impossible to estimate, and likely to vary over time. The best approach is to steer well clear of it, especially since maintaining a higher level of reserves as a buffer has no meaningful cost" (Gagnon and Sack, 2019).

If the balance sheet is too small, then demand and supply shocks will move reserves below the ample level and the federal funds rate will fluctuate. A larger balance sheet makes this less likely, but has costs of its own, such as interest paid on reserves and a larger footprint in financial markets. This is the central trade-off in an ample reserves regime–the cost of fed funds rate variability against the cost of a large balance sheet. The repo market volatility of September 2019 indicated that reserves at the time were not sufficiently ample, and the Fed intervened, increasing the quantity of reserves. The market disruptions of March 2020 led to even further increases in the balance sheet. As of this writing (October 2023), the FOMC is continuing with the plan announced in May 2022 to reduce the balance sheet, and how long the quantitative tightening will last depends on judgments about the appropriate size of the balance sheet, about how large the target level of reserves should be, and about how much variability in the fed funds market is acceptable.

Choosing the optimal level of the balance sheet has similarities to the news vendor problem in operations research, where a retailer must choose inventory to balance storage costs against lost sales. Applying the techniques of stochastic inventory theory (Porteus 1990, 2002) I find that even though the FOMC appears to behave as if fed funds rate variability is much more costly than a large balance sheet, the optimal buffer stock is small compared to the minimum level of reserves needed for an ample reserves regime.

This paper contributes to the recent but growing literature on implementing monetary policy with a large central bank balance sheet. Ihrig, Senyuz, and Weinbach (2020) provide a detailed description of the ample reserves approach, while Craig and Millington (2017) document changes in the federal funds market stemming from a large balance sheet. Afonso et al. (2020) provide a sophisticated theoretical and empirical justification for such a regime (that partly motivates this paper). Relative to them, I develop a simpler model of the reserves market but use techniques from inventory theory to quantify the optimal buffer more explicitly. Early explanations of using administered rates as a tool of monetary policy include Goodfriend (2002) and Kiester, Martin, and McAndrews (2008), which build on the early work on reserves markets of Poole (1968) and Frost (1971). Afonso et al. (2022b) estimate the demand for reserves and what constitutes an ample level, an issue also explored in Copeland, Duffie, and Yang (2021) and Afonso et al.(2022a). Reserve demand and interest rate control are discussed by Lopez-Salido and Vissing-Jorgensen (2023).

2 A Simple Model

Understanding how demand and supply shocks interact with the size of the balance sheet and produce interest rate variability requires a more explicit model of the reserves market. This section develops a simple model of an ample reserves regime based on the important work of Afonso et al. (2020). Their approach incorporates the equilibrium of supply and demand by expressing the federal funds rate as a spread above a floor, here assumed to be interest on reserve balances (IORB), where the spread depends positively on the demand and negatively on the supply of reserves.[1]

$$rate = IORB + spread(reservesupply, reservedemand). \tag{1}$$

Equation (1) expresses the ability of the central bank to move the target rate by changing the administered rate $IORB$, and how shocks to demand and supply can cause fluctuations around that rate.

I further specialize the form of the spread to the following inverse demand function

$$spread = Max[D + \delta - a(T + s), 0] \tag{2}$$

where D is the y-intercept of the inverse demand curve, T is the (target) level of reserves, δ is the demand shock, modeled as a parallel shift in the demand curve shifting the intercept by the amount δ, s is the supply shock, and a is the slope of the inverse demand curve. See Figure 2. This captures, in a simple way, the main features of the current reserves market: a demand for reserves that slopes downward until it reaches a floor at the rate of interest on reserves, at which point any amount supplied will be willingly held. It can be re-written to put the demand and supply shocks on the same footing:

$$spread = Max[D - a(T + s - \frac{\delta}{a}), 0]. \tag{3}$$

This formulation, following Goodfriend (2002), puts a kink point in the demand function at what we will also call the "saturation point" of reserves: adding more reserves beyond this point does not lower the spread: demand is saturated. An ample reserves framework puts reserves above this saturation point, and policy moves the fed funds rate by changing the IORB.

This formulation makes several modeling choices, which have several pros and cons. Like Afonso et al. (2020) and Lopez-Salido and Vissing-Jorgensen (2023), the relevant price is not the federal funds rate per se but the spread, so that an increase in the IORB shifts up the demand curve. This captures an essential element of how such floor systems are supposed to work. It also assumes that the IORB, the interest on reserve balances, functions as the floor, which ignores the often complicated relationships between the IORB and other administered rates such as the overnight reverse repo (ONRRP) rate. (Lopez-Salido and Vissing-Jorgensen explore this issue in detail.) Lately, the ONRRP rate and the IORB have been 5 and 15 basis points above the bottom of the target fed funds rate range established by the FOMC. Likewise, this formulation treats the other portions of the Fed's balance sheet as exogenous factors: as of July 5, 2023, while the system held

[1] Richer equilibrium approaches to the federal funds market can be found in Hamilton (2020), Afonso, Armenter, and Lester (2019), Bianchi and Bigio (2022), and Lagos and Navarro (2023).

\$3.1 trillion of reserves, currency in circulation was \$2.3 trillion and reverse repurchase agreements were \$2.3 trillion. In monetary policy discussions it is tempting to equate the Fed's balance sheet with the level of reserves, but they are by no means identical. The demand curve is piecewise linear, making it a simplification of Afonso et al. (2020), and although less realistic, allows for a cleaner calibration in the inventory context.[2] The formulation also assumes that the aggregate level of reserves is what matters for the fed funds rate, whereas there is some evidence that the distribution of reserves among banks matters (Copeland, Duffie, and Yang, 2021). See Ihrig, Senyuz, and Weinback (2020) for a more detailed description of the ample reserves regime.

The question of rate control is then about how to respond to the shocks that move the spread, and how much tolerance the Committee has for deviations from the target. Again, this takes as given a specific target for the federal funds rate. The monetary policy question of the appropriate level for the target fed funds rate is separate from the level of reserves needed for efficient and effective implementation of that policy.

3 Inventory

The above model of the reserve market illustrates the interaction between reserve demand, reserve supply via the Federal Reserve's balance sheet, and the level of interest rates. Shocks to demand and supply create interest rate variability when reserves (and the balance sheet) are low. The optimal balance sheet trades off the costs of interest rate variability against the costs of a larger balance sheet, and this maps quite naturally into a stochastic inventory problem, where a vendor balances the costs of excessive inventory against the costs of running out.[3]

Consider first the single-period version of the problem. The central bank aims to have a balance sheet no larger than necessary for control of the fed funds rate. I interpret this as, without shocks, the balance sheet should be at the smallest level that puts the FFR at the IORB floor (in the Afonso et al. 2020 model, that would be a spread of zero). Without loss of generality, I label the combined demand and supply shocks $(s - \frac{\delta}{a})$ as a supply shock that adds s to the balance sheet, with a cumulative distribution $F(s)$ and density $f(s)$ with mean μ. If the balance sheet drops below the minimum level and the spread rises above zero, there is a per dollar penalty cost c_p of the deficiency (which might be reputational, of course). With a linear demand for reserves, as in Section 2, this is equivalent to assigning a penalty to spreads above zero but is also compatible with a non-linear demand for reserves, as long as the cost of reserve shortfalls is linear in quantities. Holding a large balance sheet has a per dollar cost c_B for assets on the balance sheet. An alternative model might follow the inventory approach more slavishly and only assign costs to balance sheets above the minimum level of ample reserves, but the results are quite similar.[4] There is a per dollar cost c of adding to the balance sheet, reflecting the time and processing costs involved in conducting open market operations.

[2] For example, Afonso et al. (2022b) use the functional form

$$spread = p^* + (\arctan(\frac{\theta_1 - q + q*}{\theta_2}) + \frac{\pi}{2})\theta_3.$$

[3] This tradition has an impressive pedigree. Arrow (1958) traces the origin of stochastic inventory theory to work by Edgeworth (1888), which examines the balance sheet of the Bank of England.

[4] For a discussion of the costs of having a large balance sheet, related to political economy and credit allocation, see Copeland, Duffie, and Yang (2021).

3.1 One-period case

Let T (for target) denote the amount of reserves after the open market operation. Let A denote the minimum level of the reserves needed for an ample balance sheet, the minimum level where the fed funds rate equals the IORB floor and the spread is zero. (In the model of Section 2, $A = \frac{D}{a}$.) Then, following Porteus (1990), the one-period holding and shortage cost is

$$L(T) = \int_{-\infty}^{\infty} c_B(T+s)f(s)ds + \int_{-\infty}^{A-T} c_p(A-T-s)f(s)ds. \tag{4}$$

Given reserves of size T, the rate rises above the floor if the shock is negative enough to drive reserves below A, that is, if $T+s < A$ or equivalently $s < A-T$, making the penalty cost $c_p(A-T-s)$. Given a shock s, the total balance sheet is $T+s$, resulting in a cost of $c_B(T+s)$.[5] The overall objective function for the size of the balance sheet is

$$g(T) = cT + L(T) \tag{5}$$

where

$$g(T) = cT + \int_{-\infty}^{\infty} c_B(T+s)f(s)ds + \int_{-\infty}^{A-T} c_p(A-T-s)f(s)ds. \tag{6}$$

Re-writing (6) as

$$(c+c_B)T + c_B\mu + c_p \int_{-\infty}^{A-T} (A-T-s)f(s)ds \tag{7}$$

and using Leibniz's rule,[6] we can find the first-order condition

$$\frac{dg}{dT} = (c+c_B) + c_p[-\int_{-\infty}^{A-T} f(s)ds] = 0. \tag{8}$$

Noting that $\int_{-\infty}^{A-T} f(s)ds = F(A-T)$ the optimal balance-sheet buffer T^* is

$$F(A-T^*) = \frac{c+c_B}{c_p} \tag{9}$$

where $\frac{c+c_B}{c_p}$ is the *critical fractile* and gives the optimal probability of letting the fed funds rate rise above the floor. Conversely, the optimal level is

$$A - T^* = F^{-1}(\frac{c+c_B}{c_p}) \tag{10}$$

[5] This embeds the not entirely satisfactory assumption that for extreme negative values of s, the balance sheet is negative and the balance-sheet holding cost becomes negative. This has an aspect of realism, as under a corridor system there is a borrowing as well as a lending rate. But in any case, I judge the probability to be small and empirically irrelevant. Section 6.2 in the Appendix works out the case of a lower limit.

[6] Recall Leibniz's rule (Boas, 1966):

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt.$$

The reverse of this, $B = T^* - A = -F^{-1}(\frac{c+c_B}{c_p})$, gives the optimal "buffer" level of reserves above the point of transition to reserve scarcity.

Intuitively, $F(A - T^*)$ has the properties we expect. As c gets larger, $F(A - T^*)$ gets larger, meaning the probability of reserves falling below ample gets larger, implying a T^* smaller in absolute value; intuitively, as the cost of supplying reserves increases, the target level of reserves should decrease. As c_B, the cost of holding a larger balance sheet, increases, again $F(A - T^*)$ increases and T^* falls. Provided $c_B \geq 0$, an increase in the penalty cost c_p decreases $F(A-T^*)$ and increases T^*; as the cost of letting the fed funds rate rise above its target increases, the balance sheet increases to reduce that probability. Figure 3a illustrates these comparative statics. It may be more intuitive to consider the buffer, B, and Figure 3b illustrates the comparative statics from that perspective.

3.2 Initial stock and set-up costs

The single-period case has an unrealistic element in that it posits that a per dollar cost of adding to the balance sheet must be applied to the entire balance sheet, whereas intuition suggests it should be closer to an adjustment cost, responding to a shock that is small relative to the overall size of the balance sheet. This can be easily accommodated by defining the initial level of reserves as S, before the FOMC adjusts reserves. Then the objective function becomes

$$G(T, S) = c(T - S) + L(T) = g(T) - cS, \tag{11}$$

where $g(T)$ is defined by equation (6). If no action is taken, the cost is $G(S, S)$ and so the cost savings of moving to T is $G(S, S) - G(T, S)$ which should be positive if it is worth moving from S to T. Note that $G(S, S) - G(T, S) = g(S) - g(T)$. But if we take T to be the optimal level T^* it follows from the linearity of marginal costs and the convexity of $g()$ that it is optimal to move from S to T^*.[7] Of course, fixed costs of changing the portfolio will affect this result, leading to a zone of inaction (Porteus, 2002, chapter 9): if there is a fixed set-up cost c_T of adjusting the balance sheet, the cost savings, $g(T) - g(S)$ must be larger than the fixed cost c_T or no adjustment takes place.

This can be generalized in a straightforward manner to a discrete time dynamic model. The critical fractile becomes

$$F(A - T^*) = \frac{c(1-\beta) + c_B}{c_p}. \tag{12}$$

where β is the per period discount factor. Appendix 6.1 provides the details.

4 Application: Calibrating relative costs and optimal buffer

The theory in Section 3 interpreted the point at which reserves become scarce and the effective fed funds rate rises above the floor as similar to the inventory concept of stock-out, where inventory hit zero. The optimal level of inventory, determined by the critical fractile, is then the optimal buffer stock for keeping reserves in the ample regime. This section calculates that level using data on bank reserves, the Fed's balance sheet, and the fed funds market. The critical fractile, equation (9), is calibrated as the fraction of time that the fed funds rate rises above the floor. That also provides an estimate of the relative costs of rate volatility and balance-sheet size. Given the value of the critical fractile, an estimate for the distribution of supply shocks implies a value for the optimal

[7] It follows from the convexity of $g(b)$, and since $g'(y) = (c + c_B) - c_pF(y)$ from (8), and $g''(y) = c_pf(y) \geq 0$.

buffer. As a robustness check, I use several defintions of what it means for the fed funds rate to be off the floor, and several estimates of the distribution of shocks, leading to a range of buffer values.

4.1 Data

There are several ways to determine when the fed funds rate is off the floor and reserves are no longer ample. The one that best matches the model of Section 2 compares the effective fed funds rate with the interest on reserves. Using this definiton, the floor is the interest rate on excess reserves, IOER, from October 15, 2008, until it is replaced by the interest on reserve balances on July 29, 2021, which I use until July 5, 2023. Both series are taken from the Federal Reserve's Data Download site.[8] The effective fed funds rate is calculated by the Federal Reserve Bank of New York "as a volume-weighted median of overnight federal funds transactions reported in the FR 2420 Report of Selected Money Market Rates" (FRB NY). Since March 1, 2016, the Federal Reserve Bank of New York has also reported additional quantiles besides the median, providing an alternative view of when rates get off the floor. A more conservative approach (in the sense of less time above the floor) would consider when the EFFR rises above the target range established by the FOMC. The target range for the fed funds rate starts December 16, 2008, and so starts later than the payment of interest on reserves. This is taken from the Federal Reserve via FRED: DFF Federal Funds Effective Rate, Percent, Daily, Not Seasonally Adjusted, DFEDTARU Federal Funds Target Range - Upper Limit, Percent, Daily, Not Seasonally Adjusted. One downside of using different definitions of off-the floor is that the data are available for different periods. Interest on reserves starts in October 2008, but percentiles of the fed funds rate start on March 1, 2016, while the ample reserves regime starts in December 2015.

There is a notable change in the relationship between the fed funds rate and the target rate starting with interest on reserves. The standard deviation of the difference between the EFFR and the target FFR dropped from a pre-great financial crisis level of 0.141 (January 1997–December 2006) to a level of 0.041 between the start of the target range on December 16, 2008, and the start of the ample regime on December 23, 2015, rising to 0.044 in the ample regime up to July 5, 2023.

The second step in calibrating the buffer involves estimating $F(s)$, the distribution of supply shocks to the Fed's balance sheet. For this we look at changes in non-reserve liabilities over the ample reserves regime, which, following Ihrig, Senyuz, and Weinbach (2020), starts in December 2015 and continues to the end of my data in July 2023. Given a size of the balance sheet, variation in non-reserve liabilities can affect the level of reserves. The FOMC has little control over these factors. For example, depositors may demand cash from their bank, in which case reserves would fall and currency would increase. The demand for currency represents a large portion of the Fed's liabilities ($2.3 trillion), but it is not particularly variable (though it has been in the past, as in 1933). A more important source of variability is the Treasury General Account (TGA) ($ 0.4 trillion), whose level is controlled by the Deparment of the Treasury; if the Treasury spends money, drawing down the TGA, reserves increase. Another large source of variability is the overnight

[8] `https://www.federalreserve.gov/datadownload/Choose.aspx?rel=PRates` The Federal Reserve has paid interest on reserve balances since October 6, 2008. Initially there were two rates: one for required reserves (interest on required reserves, IORR), and one for excess reserves (interest on excess reserves, IOER). Reserve requirements were reduced to zero effective March 24, 2020. Not needing to distinguish the rates, the Board of Governors established the interest rate on reserve balances effective July 29, 2021. Since its inception, the IORB has been set at 15 basis points above the lower limit of the federal funds target range. The IORR stood at 25 bp above the bottom of the range from the end of 2008 to June of 2018, when it decreased to 20 bp, after which it fluctuated slightly, reaching a low of 5 bp in 2019 before rising to 15 again in the summer of 2021, just before it was replaced by the IORB.

reverse repo facility ($2.3 trillion), in which the Fed sells securities to eligible counterparties and repurchases them the next day.[9] The eligible counterparties include primary dealers, large banks, government sponsored enterprises (GSEs) and money market mutual funds (MMFs). The money market dynamics can get complicated, but conceptually, the impact on reserves can be illustrated by a simple case where a SOMA purchase of Treasury securities initially increases reserves, which banks then reduce by participating in the ONRRP facility. The non-reserve liabilities are often called autonomous factors and constitute supply shocks to the reserves market. Writers such as Lopez-Salido and Vissing-Jorgensen do not include reverse repos as part of autonomous factors, as the split between reserves and RRPs is endogenous to the financial sector. For that reason it makes sense to consider the distribution shock with and without ONRRPs.

Again refer to Figure 1. The two measures of non-reserve liabilities are total assets less deposits (Assets: Total Assets: Total Assets (Less Eliminations from Consolidation): Wednesday Level, Millions of U.S. Dollars, Weekly, Not Seasonally Adjusted,WALCL) (Liabilities and Capital: Liabilities: Deposits: Other Deposits Held by Depository Institutions: Wednesday Level, Millions of U.S. Dollars, Weekly, Not Seasonally Adjusted WLODLL). For the other measure I also subtract reverse repo holdings (WLRRAL Liabilities and Capital: Liabilities: Reverse Repurchase Agreements: Wednesday Level, Millions of U.S. Dollars, Weekly, Not Seasonally Adjusted). Data are taken from the Federal Reserve H.4.1 release via FRED.

4.2 The critical fractile and relative costs

As explained above, the critical fractile is the fraction of time that the federal funds rate rises above the floor. The EFFR is usually below the IORB, indicating ample reserves. Figure 4 shows the EFFR and the IORB since the Federal Reserve Banks began paying interest on reserve balances, in October 2008.[10] However, 4.6 percent of the time, the EFFR exceeds the IORB. This pins down the critical fractile of stocking out $F(A - T*) = \frac{c+c_B}{c_p}$ from equation (9) to 4.6 percent.

Since the critical fractile is determined by a ratio of costs, it also estimates their relative size, by a revealed preference argument. If we temporarily assume the direct cost of adding reserves (c) is negligible, then $\frac{c_B}{c_p} = 0.046$ or $c_P = 21.7c_B$. This is consistent with the FOMC acting as if a dollar of reserve deficiency is more than 20 times as costly as maintaining an extra dollar on the balance sheet.

We can take this one step further. There are few, if any, estimates of the direct costs of a large balance sheet, but Lucas (2022) takes a fiscal approach and notes that between 2008 and 2019, IORB exceeded the rate on 3-month Treasury bills by 15 basis points. (As of August 9, 2023, the IORB rate was 5.40 percent and the Treasury Constant Maturity 3-month T-bill rate was 5.28 percent, per the H.15 report.) That represents a cost of $1.5 million per $1 billion of excess balance sheet, which, using the 21.7 ratio calculated above, implies a cost of reserves dropping below the ample level by $1 billion to be $32.6 million.

[9] Readers familiar with money markets may note that this is a description of a repurchase agreement, or repo: the Federal Reserve denotes this from the standpoint of the Fed's counterparties, who buy and then sell, which is a reverse repo.

[10] See `https://www.federalreserve.gov/monetarypolicy/reserve-balances.htm`.

4.3 Buffer size

To use the critical fractile, equation (9), to estimate the optimal buffer requires estimating the distribution of balance-sheet shocks, $F(s)$. This does not require assuming a zero value for c. As mentioned above, I use two approaches to these so called autonomous factors: total assets less reserves (WALCL-WLODLL) and total assets less reserves and reverse repo balances (WALCL - WLODLL-WLRRAL). Following Ihrig, Senyuz, and Weinbach (2020), I look at the ample reserves regime starting in December 2015 and continuing to July 2023. The mean of weekly changes is $7.6 billion and the standard deviation is $80.0 billion. Figure 5 plots the weekly changes for the ample reserves regime of 2015 to 2023.

Although the quantity data are weekly, the reserves market operates according to what Hamilton (1996) calls "the daily market for federal funds." The interest rate on reserve balances is set daily, though usually only changed pursuant to FOMC meetings.[11] In accordance with Regulation D of the Federal Reserve System, IORB is calculated on daily reserve balances.[12] The auction for the overnight reverse repo facility occurs daily between 12:45 pm and 1:15 pm.[13] Adjusting to daily data, assuming a five-day work week, the distribution changes to a mean of $1.52 billion with a standard deviation of $35.76 billion. This makes the inverse of the critical fractile -$58.6 billion.

How does the estimated buffer size compare with the amount of reserves needed to be ample (an estimate of A above)? Waller (2022) noted that"financial markets worked well" when bank reserves were about 8 percent of GDP, which as of 2023 Q2 equals about $3.06 trillion, equivalent to a SOMA size of about $7.51 trillion at the July 2023 SOMA to reserves ratio (July 6, 2023 H.4.1). This makes the optimal buffer about 1.9 percent of reserves or 0.8 percent of the SOMA portfolio. Afonso et al. (2022a) estimate the transition between scarce and ample reserves happens at about 8 percent of bank assets, which, as of February 2023, stood at $22,895 billion (Federal Reserve H.8), suggesting reserves of $1,832 billion or a SOMA level of $4,487 billion. Either number suggests the buffer size is relatively small compared to the minimal level required for ample reserves. It is also smaller than the suggestion of Lopez-Salido and Vissing-Jorgensen (2023, p.25), who remark "A buffer of several hundred billion dollars does not seem unresonable given recent TGA volatility."

A normal distribution is the obvious place to start (Porteus, 2002, p.12) but it may not be the best choice. To judge between distributions, Figure 6 shows a percentile comparison or QQ plot (Wilk and Gnanadesikan 1968), plotting 1 percent quantiles from a normal and a logit against the empirical quantiles of the changes in non-reserve liabilities. It may be a matter of taste, but the logit appears more linear and thus preferred. Applying the same adjustments as the normal for daily data, the critical fractile of 0.046 corresponds to a buffer of $58.0 billion, slightly less than the $58.6 billion buffer required if the distribution were normal.[14]

The above calculations estimate the standard deviation over the entire ample reserves regime, but a glance at Figure 5 shows that variability has changed over time, and so the full-sample standard deviation may understate the current variability of the autonomous factors. Figure 7 plots the rolling trailing 52-week standard deviation of non-reserve liabilities for the December 2016 to July 2023 period (having dropped the first 52 weeks). The maximum standard deviation is $123.53 billion. Adjusting for daily data, with a normal distribution and a critical fractile of 0.046, the optimal buffer is $91.3 billion.

[11] `https://www.federalreserve.gov/monetarypolicy/reserve-balances.htm`

[12] 12 CFR 204.10(b)(1) `https://www.federalregister.gov/documents/2021/06/04/2021-11758/regulation-d-reserve-requirements-of-depository-institutions`

[13] `https://www.newyorkfed.org/markets/rrp_faq`

[14] In their theoretical example, Afonso et al. (2020) use a uniform distribution.

The above results all assume that the penalty cost starts when the effective fed funds rate moves above the IORB rate. A more conservative approach might judge the fed funds rate as excessive when the effective fed funds rate breaks above the upper limit of the target range. This has happened only once in the 3699 business days between the start of the target range on December 16, 2008, and July 5, 2023. This implies a critical fractile of 0.027 percent, in turn implying a ratio of penalty costs to balance-sheet costs of 3704, and an optimal buffer size of $121 billion under a normal distribution.

The reported effective fed funds rate is just one snapshot of a decentralized market, and other measures might better capture the idea of the fed funds rate not being effectively controlled. The 75th percentile of the fed funds rate transactions exceeded the IORB on 271 of the 1892 business days between the start of the data on March 1, 2016, and July 5, 2023, or 14.3 percent of the time. This would correspond to a buffer under the normal distribution (N($1.52, $35.76)) of $36.6 billion.

The first panel of Table 1 collects the calibrated buffers according to the various criteria.

Table 1: Optimal Buffer Levels Under different Assumptions

Criterion	Distribution	Sample	Critical Fractile	Buffer
Assets Less Reserves				
EFFR >IORB	Normal	2008-2023	4.6%	58.6
	Logistic			58.0
	Max rolled s.d.			91.3
EFFR> Upper Target	Normal		0.03%	122.2
75^{th} EFFR>IORB	Normal	2016-2023	14.3%	36.6
Assets Less Reserves and Reverse Repo				
EFFR >IORB	Normal	2008-2023	4.6%	49.2
	Logistic			48.8
	Max rolled s.d.			75.6
EFFR> Upper Target	Normal		0.03%	102.0
75^{th} EFFR>IORB	Normal	2016-2023	14.3%	31.0

Buffers in billion dollars. Source: Author's calculations

For a robustness check, it is important to exclude Fed holdings of reverse repos from autonomous factors because of the substitutability between reserves and ONRRPs, instead defining autonomous factors as $WAALCL - WLODLL - WLRRAL$ in FRED mnemonics. The second panel of Table 1 reports the results for this definition of autonomous factors. The results are qualitatively and quantitatively quite similar to the results using only assets less reserves, though generally smaller.

5 Buffers to limit FFR variability

The inventory approach provided one way to calculate an optimal balance-sheet buffer. Afonso et al. (2020) provide some qualitative results on the optimal buffer when the object is either to keep the fed funds rate in a given range or to minimize the probability of moving outside that range. Answering these questions using the simple reserves market model of Section 2 provides an additional perspective and can serve as a qualitative robustness check on the stochastic inventory approach. The logic here follows Section 4.4 of Afonso et al. (2020) quite closely, but considers a more general probability distribution that eliminates the need to consider a maximum demand or supply shock.

5.1 Buffer size

Consider the buffer needed to keep the rate in range. If, as defined above, T is the target level of reserves, denote ϵ to be the tolerance for deviation from the rate target, that is, the maximum deviation of the spread that will be allowed. Assume that the target T is set so that in the absence of supply and demand shocks the spread (and thus the rate) is at the desired level. Then the buffer B must be large enough that

$$(D - a(T + B + s - \frac{\delta}{a})) - (D - aT) \leq \epsilon. \tag{13}$$

or

$$B \geq \frac{\delta - \epsilon}{a} - s. \tag{14}$$

This concept of the buffer assumes that the target spread is zero and T is on the flat part of the demand curve, or at least close. This is in some sense the definition of an ample reserves regime. A positive supply shock, adding reserves, has no impact on the spread and thus on rates, which remain at the IORB. This would not be true in a scarce reserves regime, where more reserves would push the spread below the target.

A related question is how far reserves could be allowed to fall below the target. This depends on the interaction between the demand shock δ and the supply shock s. Then the minimum allowed level of reserves, call it M, follows from

$$(D - a(M + s - \frac{\delta}{a})) - (D - aT) \leq \epsilon. \tag{15}$$

or

$$M \geq T + s + \frac{\epsilon - \delta}{a}. \tag{16}$$

In other words, the buffer makes sure the supply shock won't drive the level of reserves below the minimum allowable level, or equivalently, allow the spread to exceed its maximum permitted level.

5.2 Minimizing intervention probability

The previous section calculated a buffer size, given a tolerance for deviations from the target fed funds rate. The tolerance parameter was given exogenously, and the natural next step is to derive the tolerance from an optimizing model, which can then compute the optimal level of reserves. Of course, that depends on the loss function for the economy. A plausible class of loss functions, discussed by Afonso et al. (2020), has the social planner minimizing the probability of intervening in the market (preventing the spread from exceeding the ϵ tolerance) plus some cost of having a large balance sheet.

The simple case is where T is high enough that we only have to worry about injecting reserves, the case where increases in supply or decreases in demand, which lower the spread, will only take it to zero and thus stay within the tolerance band. Where is this? We know the kink point in the demand curve is at the point $D - ax = 0$ or $x = \frac{D}{a}$, which implies the target level of reserves must be at $D - aT < \epsilon$ or

$$T \geq \frac{D}{a} - \frac{\epsilon}{a}. \tag{17}$$

or, equivalently, T can be at most $\frac{\epsilon}{a}$ below the kink point $\frac{D}{a}$ of the demand function. Figure 8 illustrates this point.

So if condition (17) is satisfied, we only have to worry about supply shocks that decrease reserves and demand shocks that increase demand.

$$Pr[D - a(T + s - \frac{\delta}{a}) > \epsilon]. \tag{18}$$

or

$$Pr[(\frac{D}{a} - T) - \frac{\epsilon}{a} > s - \frac{\delta}{a}]. \tag{19}$$

This has a natural interpretation: the first term $(\frac{D}{a} - T)$ is the difference between the kink point of the demand curve and the target rate, and from that is subtracted the reserve equivalent amount of the spread tolerance $\frac{\epsilon}{a}$. This is compared with the reserves equivalent of the supply and demand shocks, which, if negative enough, will force the spread (and thus the interest rate) above the tolerance range.

Given the intervention probability in equation (19), the loss function used by Afonso et al. (2020), which trades off the probability of intervening against the size of the balance sheet, takes the following form:

$$min\ L = Pr[(\frac{D}{a} - T) - \frac{\epsilon}{a} > s - \frac{\delta}{a}] + kT. \tag{20}$$

This form assumes that the Fed doesn't care if the rate fluctuates *within* the tolerance range, and that there is a fixed cost of intervening, unrelated to the required size of the intervention. Letting F be the cumulative distribution function (f the density)

$$\frac{\partial L}{\partial T} = \frac{\partial}{\partial T}\{F[(\frac{D}{a} - T) - \frac{\epsilon}{a} > s - \frac{\delta}{a}] + kT\}. \tag{21}$$

$$= -f() + k = 0.$$

More compactly,

$$f() = k. \tag{22}$$

The second-order conditions are satisfied when $f' > 0$.

The Committee should increase the target level of reserves until the increased probability of intervening just equals the increased cost of a larger balance sheet. The cost of intervening here is normalized to 1, so k, the cost of a larger balance sheet, is in terms of the intervention cost.

A simple comparative static result is also illuminating. Implicitly differentiating (22), we have

$$\frac{dT}{da} = -\frac{Tf'()}{af'()} = -\frac{T}{a}. \tag{23}$$

This indicates that a steeper demand curve implies that the target balance sheet should be larger.

For completeness, we should state the results for Case II, where the Committee is working a scarce reserves regime and has to worry about breaching the boundary on both the upside and the downside. In that case the probability is

$$Pr[D - aT - \epsilon < D - a(T + s - \frac{\delta}{a}) < D - aT + \epsilon]. \tag{24}$$

but that simplifies to

$$Pr[-\epsilon < -as + \delta < \epsilon]. \tag{25}$$

In other words, far enough into the decreasing section of the demand curve, moving the target level of reserves does not change the probability of breaching the barrier. Obviously, this is a consequence of linearity.

This loss function is not fully satisfactory, however. The tolerance range, ϵ, is given exogenously, and while in principle it might be determined by the legislature or otherwise specified in advance, ideally it would arise from the costs of trading off interest rate variability, intervention costs, and balance-sheet size. Otherwise, we can have both a small balance sheet and no intervention, provided we put up with a variable funds rate.[15] One advantage of the inventory approach is that it explicitly considers the trade-off.

6 Conclusion

The problem of keeping reserves at a level no larger than needed for effective and efficienct interest rate control maps naturally into a question of inventory policy. A revealed preference approach using the resulting critical fractile indicates that the FOMC appears to act as if the cost of interest rate variability is much higher than the costs associated with a larger balance sheet. Even so, given the estimated distribution of shocks, the size of the optimal buffer is small relative to the balance sheet required to maintain an ample regime, that is, to keep the funds rate at the interest rate floor. This has implications for the optimal size of the Fed's balance sheet and therefore the allowable level of quantitative tightening (QT).

[15] Ghironi and Ozhan (2020) discuss using the tolerance range as a policy tool itself.

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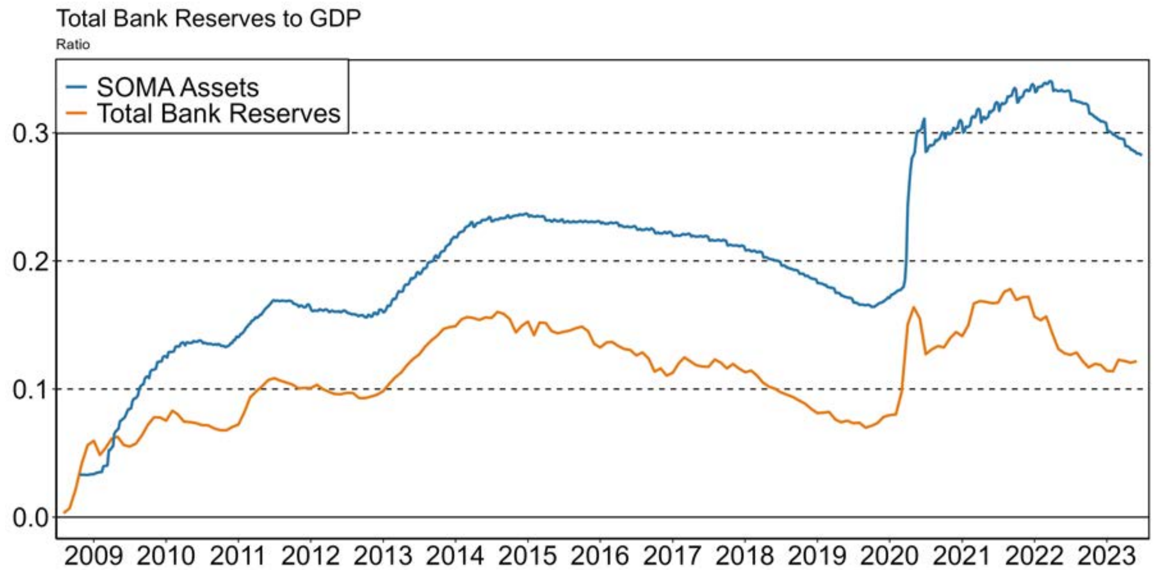


Figure 1: **Ratio of Total Bank Reserves and SOMA to GDP** Ratio of total bank reserves and security holdings of the Federal Reserve to nominal GDP. Source: Federal Reserve H.4.1 release.

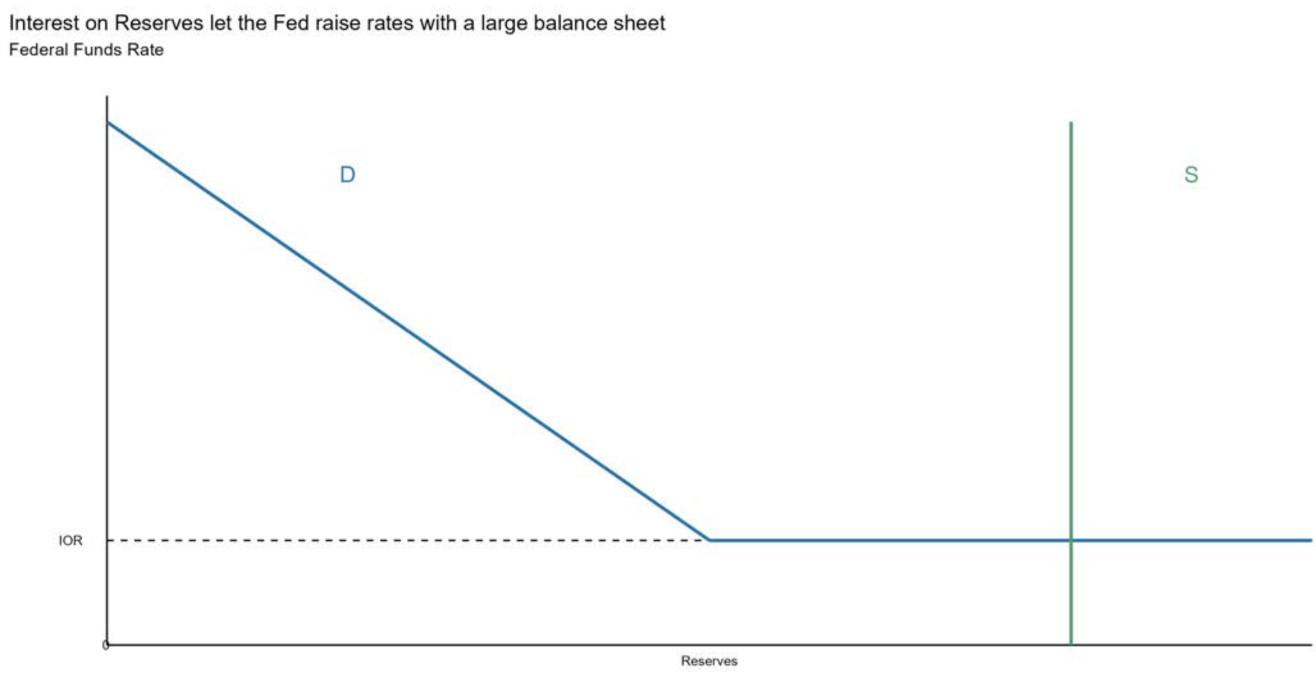
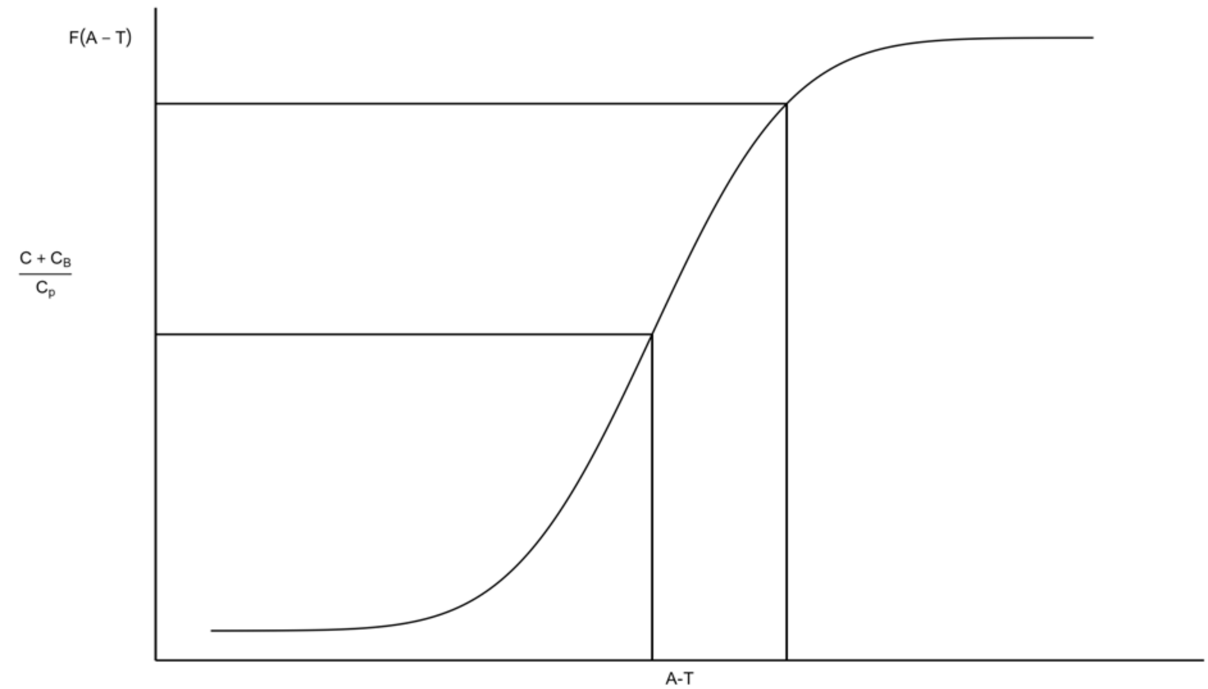
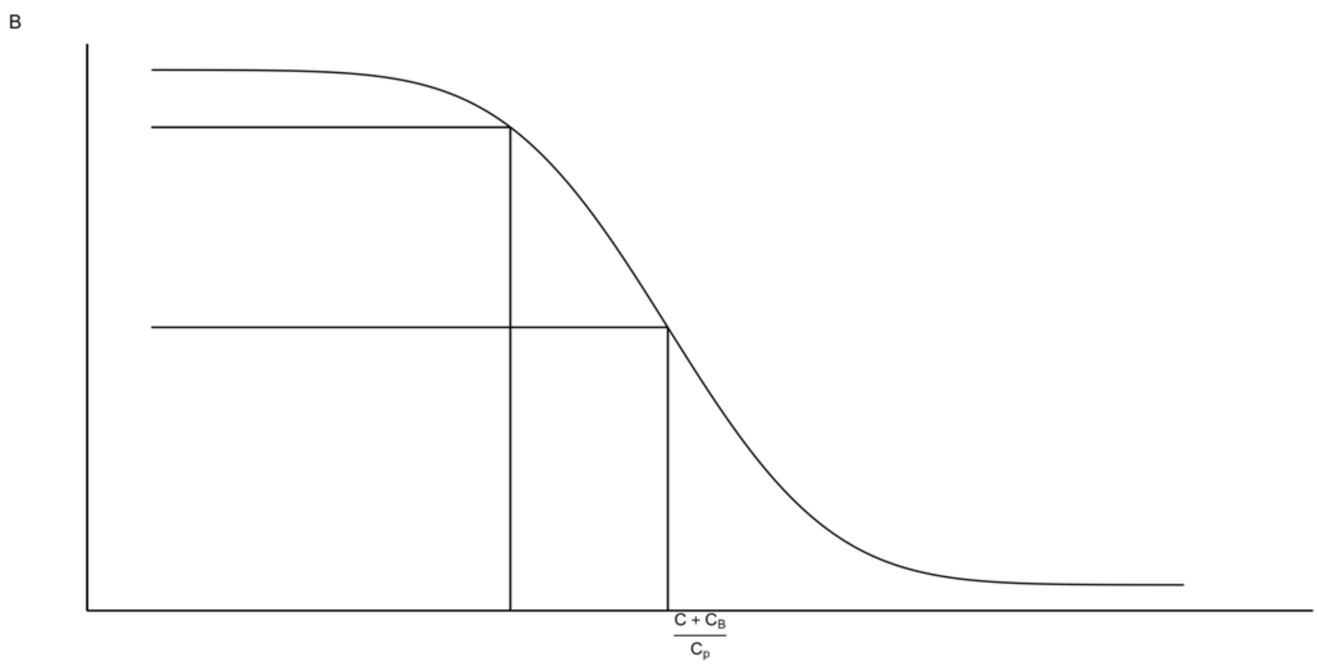


Figure 2: **The reserves market with ample reserves** This illustrates the piecewise linear demand curve of equation (3).



(a) **Comparative Statics** This illustrates how changes in the critical fractile change the optimal buffer by inverting the distribution function F.



(b) Comparative Statics for the Buffer

Figure 3: Comparative Statics for the Balance Sheet.

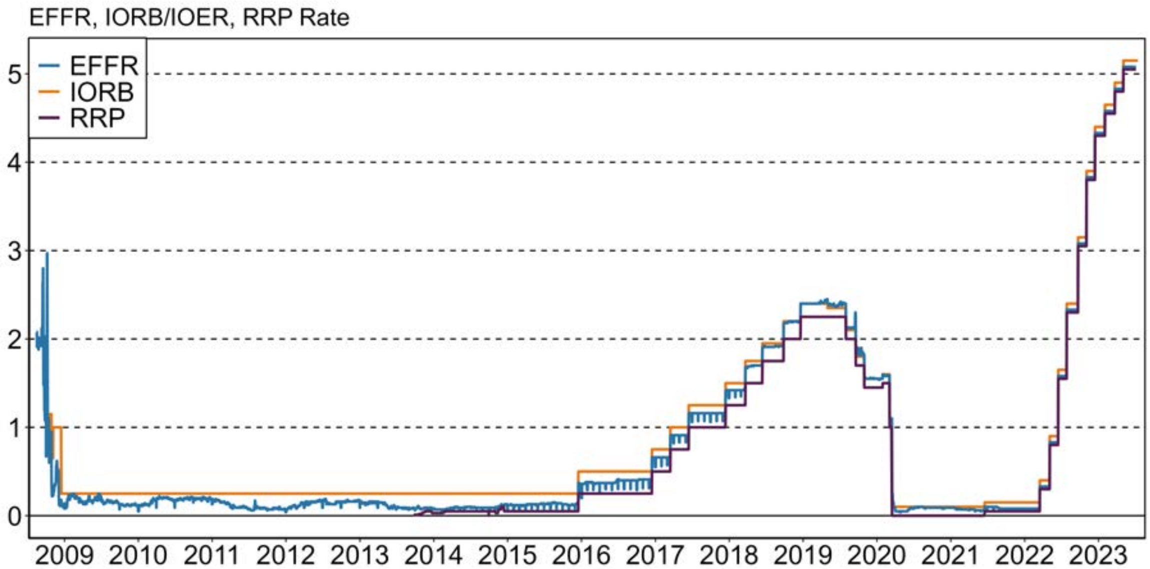


Figure 4: **EFF, ONRRP, IORB rates** This plots the effective fed funds rate, the overnight reverse repo rate, and the interest on reserve balances from the start of paying interest on reserves until July 2023. Source: Federal Reserve H.15 via FRED.

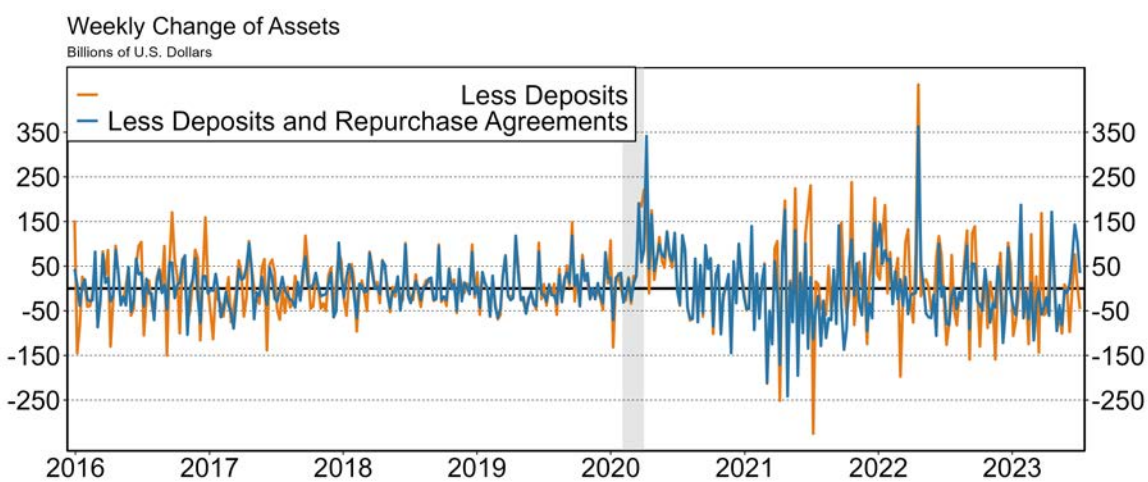


Figure 5: **Weekly change, autonomous factors** Weekly changes in autonomous factors, defined as Federal Reserve total assets less reserves, and as total assets less reserves and reverse repo. Source: Federal Reserve H.4.1 via FRED.

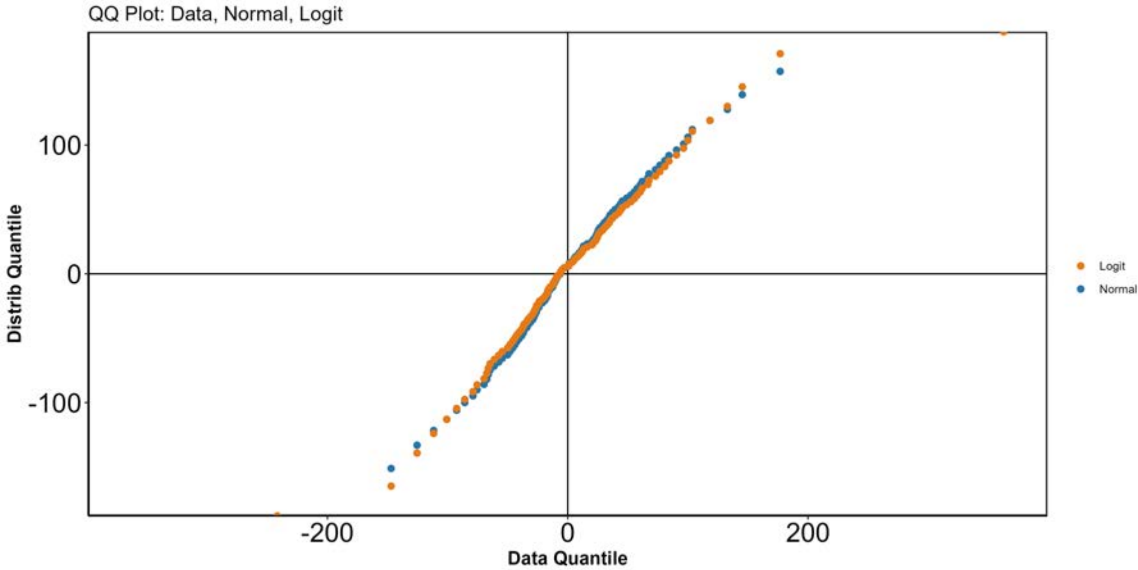


Figure 6: **QQ Plot** A comparison of quantiles of weekly changes of autonomous factors (total assets less reserves and RRP) at one percentile intervals of a fitted normal and fitted logistic distribution against quantiles of the data. Source: author's calculations and H.4.1 via FRED.

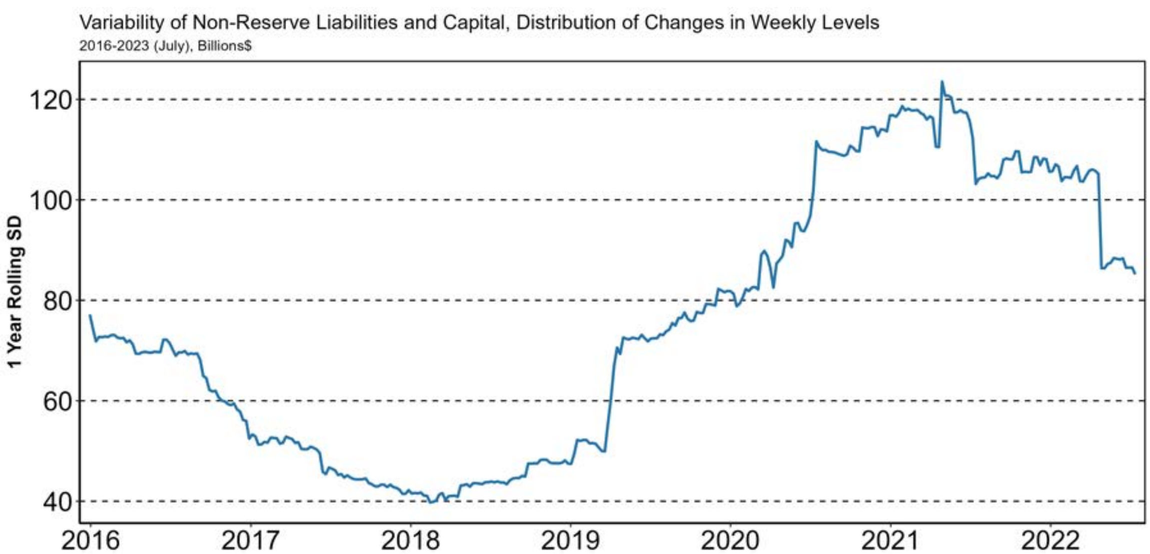


Figure 7: **Rolling Standard Deviation of Supply Shocks** Trailing 52-week standard deviation of weekly change in autonomous factors. Source: author's calculations and Figure 6.

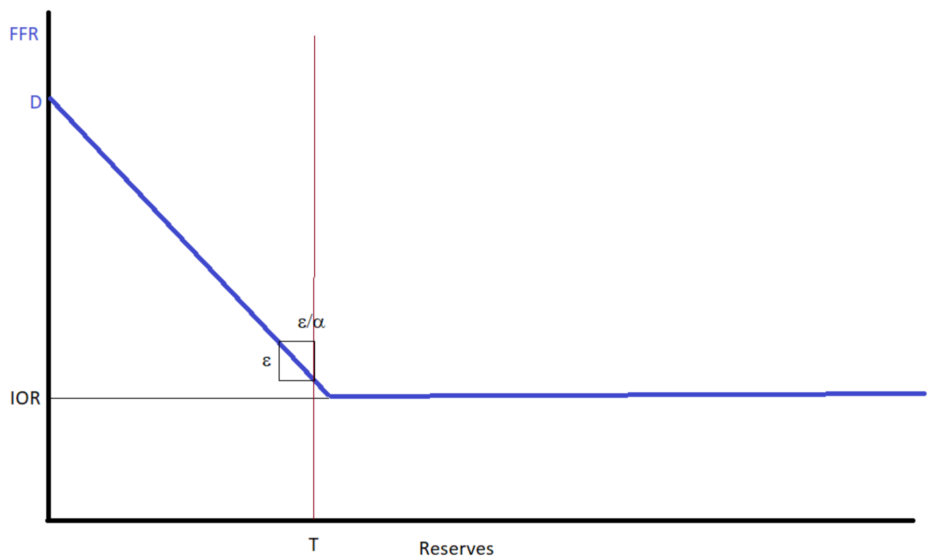


Figure 8: **Slope and Tolerance** Illustrates the relationship between slope of reserve demand α and the tolerance parameter ϵ.

Appendix:

6.1 A discrete time dynamic inventory approach

Consider a dynamic version of the inventory question, where the balance sheet is carried over to the next period. To move to a more dynamic problem (effectively Porteus 2002 chapter 4) requires a few additional assumptions. As before, if the reserves are below the ample level, there is a per unit cost c_p and reserves above the ample level bear a per unit cost c_B, the cost of holding inventory. There is a per unit cost of acquiring assets (respectively, inventory) of c. The state variable is x_t, the level of reserves before the central bank intervention, and y_t is the level of reserves after the intervention (but before the shock), so that $x_{t+1} = y_t + s$. There is a terminal value function, where the final reserves of size x are valued at $v_{\mathbb{T}}(x) = -c(x - A)$. The balance sheet must be brought up to the ample level, but anything above that is sold at cost c. This is similar to the assumption of back-orders in the inventory literature. The one-period discount factor is β and shocks are a random variable s with density f and distribution F.

This problem has a recursive formulation and can be solved via backward induction. Let $f_t(x)$ denote the minimum expected cost starting in period t with reserves at level x. The optimality equations become

$$f_t = \min_y \{c(y - x) + L(y) + \beta \int_{-\infty}^{\infty} f_{t+1} + (y - s)f(s)ds\}. \tag{26}$$

Letting

$$G_t(y) := cy + L(y) + j\beta \int_{-\infty}^{\infty} f_{t+1} + (y - s)f(s)ds \tag{27}$$

then the optimality conditions can be rewritten as

$$f_t = \min_y \{G_t(y) - cx\}. \tag{28}$$

Note that if f_{t+1} is convex, $G_t(y)$ is convex, since it is the sum of three convex functions. Hence a y that minimizes $G_t(y)$ gives an optimal level of reserves. Furthermore, f_t is convex, following from convexity preservation under minimization, theorem A.4 of Porteus (2002).

The optimal level can be found more explicity. As a preliminary step, consider the problem of the last period, assuming reserves are at zero. The problem is to minimize

$$cy + L(y) - \beta \int_{-\infty}^{\infty} c(y - s)f(s)ds = c(1 - \beta)y + L(y) + \beta c\mu. \tag{29}$$

As above, $L(y) = \int_{-\infty}^{\infty} c_B(y + s)f(s)ds + \int_{-\infty}^{A-y} c_p(A - y - s)f(s)ds$. Then letting $g(b) := c(1 - \beta)y + L(y)$ (29) can be written as

$$g(y) + \beta c\mu. \tag{30}$$

Thus, the optimal balance-sheet size S solves

$$g'(S) = 0. \tag{31}$$

Using Leibniz's rule to show that $L'(y) = -c_P + (c_B + c_P)F(y)$ the optimal balance sheet level is defined implicitly as

$$F(A - S) = \frac{c_B + (1 - \beta)c}{c_P}. \tag{32}$$

For this to be finite, the fractile must fall between zero and one, and for that it is sufficient if the cost of a balance sheet below the ample level c_P and the cost of holding excess balances, c_B, are both greater than zero, and that the penalty for a low balance sheet is greater than the discounted cost of acquiring and holding assets (otherwise it is optimal to do nothing) or $c_p > (1-\beta)c + c_B$.

So far, equation (32) is only the solution for the last period of the problem. However, we can show that the optimal value functions f_t also have the same slope as the terminal value function, $v_{\mathbb{T}} = -c$, and so (12) will be optimal in each period.

$$f_N(x) = G_N(S) - cx \tag{33}$$

Hence,

$$f'_N(x) = G'_N(S) - c. \tag{34}$$

So $f_N(b)$ has slope $-c$. This argument of course extends back to previous time periods and establishes the recursion.

6.2 A non-negative balance sheet

For the one-period problem, impose the condition that the balance sheet must be non-negative or, equivalently, that the balance sheet holding cost only applies to a non-negative balance sheet (for example, if a major cost is the interest paid on reserves). The analogue of the basic equation (6) becomes

$$g(T) = cT + \int_{-T}^{\infty} c_B(T+s)f(s)ds + \int_{-\infty}^{A-T} c_p(A-T-s)f(s)ds. \tag{35}$$

Differentiating via Leibniz's rule yields

$$c + c_B[1 - F(-T)] - c_p F(A-T) = 0. \tag{36}$$

This involves two fractiles, so is not as easily interpretable as the critical fractile (12) but the comparative statics are not difficult to compute. Differentiating implicitly,

$$\frac{dx}{dc_B} = \frac{-[c_B f(-x) + c_p f(A-x)]}{1 - F(-x)}. \tag{37}$$

$$\frac{dx}{dc_p} = \frac{[c_B f(-x) + c_p f(A-x)]}{F(A-x)}. \tag{38}$$

More explicit solutions can be obtained by assuming a functional form for the distribution. For example, letting $F \sim \text{Uniform}[-K, K]$ results in

$$x = \frac{K}{C_p - c_B}[c_p - c_B - 2c] + \frac{Ac_p}{c_p - c_B}. \tag{39}$$