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Wage Adjustment in Efficient Long-Term Employment Relationships

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Abstract

We present a model in which efficient long-term employment relationships are sustained by wage adjustments prompted by shocks to idiosyncratic productivity and the arrival of outside job offers. In accordance with casual and formal evidence, these wage adjustments occur only sporadically, due to the presence of renegotiation costs. The model is amenable to analytical solution and yields new insights into a number of labor market phenomena, including: (1) key features of the empirical distributions of changes in pay among job stayers; (2) a property of near-"memorylessness" in wage dynamics that implies that initial hiring wages have only limited influence on later wages and allocation decisions; and (3) a crucial role for nonbase pay—specifically, recruitment and retention bonuses—in sustaining efficient employment relationships.

JEL codes: E24, E3, J3, J6.

Keywords: Sticky wages, business cycles.

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In a classic contribution, Becker (1962) articulated the central role of specific human capital in the durability of employment relationships. Since then, Becker's early insight has inspired a view of labor markets that acknowledges the empirical prevalence of long-term jobs (Hall 1982; Farber 1999), the role of idiosyncratic shocks in their creation and destruction (Mortensen and Pissarides 1994), and the scope for infrequent wage adjustment afforded by their rents (Barro 1977; Malcomson 1997; Hall 2005). Central to this view is the notion that *ex post* wage adjustments sustain *bilaterally efficient* long-term employment relationships that are subject to shocks.

In this paper, we formalize this view of labor markets with a novel framework that nests and generalizes many of the canonical models now in use. By so doing, we reveal which of those models' results survive relaxation of their more restrictive assumptions. In our framework, sporadic wage adjustment is prompted by two stimuli: costly bargaining threats, rendered credible by idiosyncratic shocks to match productivity; and outside offers, generated by on-the-job search. Echoing insights first articulated qualitatively by Malcomson (1997), we find that the model can capture the evolution of job separations by tenure, as well as key features of the distribution of wage changes among job stayers—a mass point at zero wage change, and a greater incidence of wage raises than wage cuts. The model further identifies a crucial role for nonbase pay—specifically, recruitment and retention bonuses—in sustaining efficient relationships; moreover, implied patterns of adjustment across these components of pay resemble their empirical counterparts. Finally, the model adds a note of caution regarding accounts of unemployment fluctuations based on hiring wage stickiness (as first proposed by Hall 2005). Because idiosyncratic shocks generate *ex post* wage adjustments that truncate the legacy of hiring wages, such accounts are unable to explain the magnitude of actual unemployment fluctuations.

We begin in Section 1 by describing our baseline model. Production is organized in worker-firm pairs, and match productivity is subject to persistent idiosyncratic shocks. Workers receive job offers both while unemployed and employed. We devise a novel protocol for wage adjustment that accommodates three channels. First, wages can be adjusted at any time, and at zero cost, by *mutual consent* of firm and worker. Second, each party can threaten to trigger a renegotiation of the wage *unilaterally*. Importantly, the presence of renegotiation costs renders such threats credible only intermittently. Together, these give rise to intermittent wage adjustments by mutual consent, sustained by unilateral threats of costly renegotiation. Third, upon arrival of an outside job offer, wages are renegotiated via an extension of the bargaining model of Dey and Flinn (2005), and Cahuc et al. (2006). Combined with the availability of costless wage adjustment by mutual consent, a corollary is that the protocol implies that all separations—both to unemployment and from job to job—are bilaterally efficient.

The result is a prototype model of efficient long-term employment relationships in which wages are adjusted only intermittently in response to idiosyncratic shocks to match productivity, and the realization of outside offers. The frequency of wage adjustment in the model is an equilibrium outcome: The greater are the costs of renegotiation, the less credible are the associated threats, and the less frequent is equilibrium wage adjustment.

An appealing feature of the model is its analytical tractability. Since the wage adjustment protocol preserves bilateral efficiency, the problem can be decoupled into subproblems for the match surplus and job durations, and wage adjustment. We show that the former admits analytical solutions for match surplus and job tenures, allowing a characterization of long-term employment relationships.

A key innovation of the model, however, is that it also admits an analytical characterization of wage adjustment. Wage dynamics follow a *drunken walk*, whereby wages adjust minimally (by mutual consent) such that neither firm nor worker has a unilateral incentive to trigger a (costly) renegotiation. In general, a solution to the associated adjustment thresholds is complicated: The decision of, say, the firm to initiate a renegotiation in the present depends on all the adjustment thresholds of both firm *and worker* in the *future*. A contribution of our framework is that the wage adjustment problem can be distilled down to a sequence of problems of two-sided instantaneous control, indexed by the current wage. This allows a novel analytical characterization of the division of total surplus between firm and worker, and thereby wage adjustment.

The model conveniently nests canonical approaches as special cases. As renegotiation costs are eliminated, continual wage bargaining arises, as in search and matching models. As the costs of renegotiation approach the total match surplus, wages are adjusted only when participation constraints bind. And, in the special case in which idiosyncratic shocks additionally are suspended, the environment further nests canonical sequential auctions approaches (Postel-Vinay and Robin 2002; Dey and Flinn 2005; Cahuc et al. 2006). Accordingly, our model extends the work of Postel-Vinay and Turon (2010), who study wage dynamics in a sequential auctions environment, but assumes for tractability that idiosyncratic shocks are *i.i.d.* As they note, accommodating persistent idiosyncratic shocks to match productivity has necessitated the use of numerical methods in prior work (Yamaguchi 2010; Lise et al. 2016). Our approach allows the option value of on-the-job search to be inferred analytically, facilitating a characterization of wage dynamics.

We show that the model can be inserted straightforwardly into canonical theories of equilibrium unemployment dynamics, permitting analysis of the allocative effects of wages. Interestingly, the latter yields new insights into the user cost of labor first proposed by

Kudlyak (2014), and revisited recently by Doniger (2021) and Bils et al. (2022). The presence of idiosyncratic shocks, on-the-job search, and the *ex post* wage adjustments they induce gives rise to novel components of user cost that moderate the allocative role of wages: Prospective changes in productivity, and realizations of turnover, or retention compensation, blunt the salience of current wages in the firm's flow cost of labor.

We then turn in Section 2 to exploring the model's quantitative implications. We first show that it is able to replicate salient empirical features of long-term employment relationships. Separation rates by tenure mirror the nonmonotonic empirical profile noted by Farber (1999). The model is thus able to capture the empirical durability of jobs.

The model also replicates key features of wage adjustment. It generates the hallmarks of the distribution of wage changes emphasized by the literature on empirical wage rigidity: a mass point of wage freezes and a relative scarcity of wage cuts. The latter asymmetry in wage changes emerges in the model even in the absence of an asymmetric friction (as anticipated by Malcomson 1997). Three forces give rise to this result. Accumulation of specific human capital induces an upward drift in wages on average. Disproportionate rates of separation among matches that otherwise would cut the wage naturally lower the incidence of wage cuts. And, the availability of bonus pay—which emerges endogenously in the model—provides a natural alternative to cuts in base pay.

The model's implications for its central outcomes—the durability of jobs and their wage and compensation dynamics—are thus very sensible. The remainder of the paper then studies three further applications of the model.

First, we study its implications for the allocative effects of wages. A key property of the model is that the marginal value of the wage to the firm is equal to its discounted *expected duration*: Bilateral efficiency requires wages to be adjusted *ex post* in response to idiosyncratic shocks to preserve profitable matches. Crucially, the latter imparts on wages a form of *memorylessness* whereby, upon adjustment, wages naturally are independent of the prior wage. The allocative legacy of any current wage is thus bounded by its duration.

This implication of the model provides an important point of contrast with the influential theory of hiring wage rigidity pioneered by Hall (2005). Hall's model can be viewed as a simple special case of our environment in which there are no idiosyncratic shocks to match productivity, no outside job offers, and neither firm nor worker can credibly enforce a renegotiation of the wage. In this case, the legacy of any initial entry wage is *indefinite*. Entry wages thus have profound effects on the valuation of a prospective match to a firm, and thereby on job creation, an implication used by Hall to propose a new channel for unemployment fluctuations. A celebrated feature of this channel is that it operates without invoking violations of bilateral efficiency.

Ironically, once Hall's model is generalized to allow for match-specific productivity shocks and outside offers, the *ex post* wage adjustments necessary to preserve bilaterally efficient matches unravel this result. Such wage adjustments truncate the legacy of the initial wage so that the model can no longer account for large unemployment fluctuations.

In a second application, we study how the interaction of on-the-job search and costly renegotiation naturally motivates a novel interpretation of base and nonbase pay. Upon realization of an outside offer, firms face an upper bound on the value they can credibly deliver to their workers solely through increases in the flow (base) wage: Further increases would be reversed almost immediately via subsequent renegotiations. Any remaining value must then be delivered as a lump-sum *bonus*, providing a novel theory of nonbase pay. Consistent with an empirical literature that emphasizes the role of nonbase pay in wage flexibility (Shin and Solon 2007), a quantitative assessment of the model reveals that this form of nonbase pay contributes substantially to flexibility in overall compensation. Mirroring recent results in Grigsby et al. (2021), compensation changes in the model exhibit considerably more flexibility than their base wage counterparts.

Crucially, we show that the availability of such bonuses is both allocative and intrinsically related to the pattern of base pay changes in the model. To illustrate, we explore the model's implications for the removal of the firm's ability to offer recruitment or retention bonuses. Doing so induces important adverse allocative consequences: The surplus of new matches declines by around 15 percent, and the rate of separation into unemployment rises by roughly 10 percent. Furthermore, base wages in this counterfactual exercise exhibit considerably more downward flexibility: The incidence of base pay cuts rises from around 5 percent to 15 percent when bonuses are unavailable. Intuitively, if firms must respond to outside offers solely via adjustments to base wages, the higher base wages that result are more likely to be subsequently renegotiated down.

The model thus provides a novel interpretation of recent empirical estimates of wage flexibility and its relation to the structure of compensation between base and nonbase pay, provided by Grigsby et al. (2021). A key lesson of the model is that, even in cases where base pay appears to be downward rigid and a minority of workers receive nonbase pay, the availability to firms of the *option* to deliver part of compensation as a recruitment or retention bonus in *marginal* matches is of considerable allocative value.

Finally, in Section 3, we show how the model can be extended to accommodate an inflationary environment and, thereby, speak to the adjustment of *nominal* wages. Following Malcomson (1997), we explore a view of contracts in which wages set in nominal terms are renegotiated only by mutual consent. A consequence is that real wages drift downward in the absence of adjustment. Though conceptually simple, this is accompanied by an analytical challenge whereby worker and firm valuations of a match are instead described by partial differential equations. We show, however, that the approach developed in the baseline model can be extended using a perturbation method to characterize an approximate solution around low inflation rates. Thus extended, the model has sensible implications for nominal wage adjustments, mirroring the empirical decay of the incidence of nominal wage freezes and cuts with inflation.

We conclude by offering thoughts on how to remedy the model's inability to account for unemployment fluctuations. Our analysis points to three avenues: First, one could explore violations of bilateral efficiency—e.g., due to asymmetric information (Hall and Lazear 1984), or due to costs associated with cutting wages (Bewley 1999; Bertheau et al. 2023; Davis and Krolikowski 2023). Second, and relatedly, one could break the memorylessness property by devising theories of history dependence in wages, such that the legacy of any (hiring) wage is longer than the time it takes for the wage to be adjusted. Finally, one could appeal to other sources of fluctuations that emphasize volatility in labor *demand*, echoing the conclusions of Bils et al. (2022). We hope the present paper will stimulate further research along these lines.

Related literature. In addition to the work already cited, our paper is related to the following strands of the literature.

First, our model echoes earlier work on long-term contracting. Most closely related is the model of MacLeod and Malcomson (1993), who emphasize the role of renegotiations by mutual consent in sustaining efficient long-term relationships. Holden (1994) develops related ideas in a union setting. Our contribution is to provide a quantitative model in which wage adjustments are prompted by both idiosyncratic shocks and on-the-job search, and the implied adjustment thresholds can be characterized analytically.

More distant to our approach are models of dynamic contracting that emphasize insurance motives, starting with the classic contribution of Thomas and Worrall (1988). The optimal contract in their setting takes the form of the drunken walk described above, but the motivation for it originates instead from risk-sharing. Rudanko (2009) embeds these insights into a model of equilibrium unemployment dynamics with aggregate shocks. Balke and Lamadon (2022) extend these ideas to a model of directed search on the job in which worker search effort is noncontractible. Although their model generates rich wage dynamics, it does not exhibit inaction in wage adjustments, a central object of our study.

A second strand of related literature builds on Hall's (2005) early insights into the potential allocative effects of stickiness in hiring wages. Gertler and Trigari (2009) embed these ideas into a model of time-dependent staggered wage adjustment and find that it gives rise to substantial unemployment fluctuations. By contrast, our focus is to study state-dependent wage adjustment prompted by plausible idiosyncratic shocks; we find instead that this does much to mute unemployment fluctuations.

In tandem, a growing empirical literature has sought to infer the extent of hiring wage stickiness from microdata (Pissarides 2009; Martins et al. 2012; Haefke et al. 2013; Gertler et al. 2020; Hazell and Taska 2020). A key message of our model, however, is that the degree of hiring wage stickiness may be less consequential for job creation, and thereby unemployment, in the presence of efficient *ex post* wage adjustments.

A third strand of related literature comprises recent work on nominal wage rigidity. Early empirical contributions by McLaughlin (1994), Card and Hyslop (1996), Kahn (1997), and Altonji and Devereux (2000) documented that distributions of year-to-year nominal wage changes for job stayers exhibit many instances of no wage change, more instances of wage increase, and a lower, though nontrivial, incidence of wage cuts. Since then, several researchers have sought to refine these survey-data-based measures by using more accurate administrative microdata (see the survey by Elsby and Solon 2019, as well as Jardim et al. 2019, Kurmann and McEntarfer 2019, and Grigsby et al. 2021). These latter measures are used to inform our quantitative analyses in what follows.

Our work is also related to a theoretical literature inspired by these facts. A common theme in these works is that they explore the allocative consequences of specific frictions that *impede* efficient wage adjustments. Elsby (2009) studies the wage-setting problem of a firm assumed to incur costs from wage cuts. Benigno and Ricci (2011) and Daly and Hobijn (2014) study New Keynesian models with spot labor markets in which wage cuts are assumed to be (stochastically) infeasible. Dupraz et al. (2022) extend these themes to a search and matching model with durable jobs. Fukui (2020) and Gottfries (2021) study the allocative role of rigid wages in models with on-the-job search in which firms are unable to respond to outside offers. Finally, like us, Blanco et al. (2022) study an environment with idiosyncratic shocks and long-term jobs, but their focus is instead on the role of constraints to wage adjustments in generating inefficient separations.

By contrast, the impetus behind our study is to evaluate a parsimonious model in which wages *can* be adjusted when necessary to sustain efficient relationships. We see the two approaches as complementary: One interpretation of our finding that the latter view cannot account for unemployment fluctuations is that it motivates further study of frictions that impede efficient wage adjustments. We further hope that progress to this end will, in turn, be aided by the analytical methods we develop in this paper.

Finally, our study is related to recent work that has begun to explore the macroeconomic implications of flexibility in components of pay. Broer et al. (2023) study labor contracts that specify a base wage, plus "marginal" wages that vary with *ex post* hours worked. They show that the availability of marginal wages moderates labor market fluctuations relative to a fully rigid base wage, but that rigidity of the contract with respect to shocks amplifies labor market fluctuations relative to a neoclassical benchmark. Gaur et al. (2022) study incentive pay motivated by the presence of worker moral hazard. Optimal incentive pay in their model offsets changes in workers' incentives to supply effort, and so incentive pay flexibility is *irrelevant* for labor fluctuations, by the envelope theorem. Along with our findings on the allocative role of recruitment and retention compensation, these works underscore the importance of the microeconomic origins of nonbase pay flexibility in shaping macroeconomic outcomes.

1. A model of wage adjustment

In this section, we devise a new model of wage determination that delivers intermittent wage adjustment as an equilibrium outcome. The model nests a class of canonical models that share the property that wages adjust to preserve bilateral efficiency. The model further admits an analytical characterization of an array of outcomes, in particular the endogenous thresholds for wage adjustment. The section closes by demonstrating how the wage protocol can be inserted straightforwardly into canonical models of aggregate labor market equilibrium, allowing an analysis of its implications for unemployment dynamics.

1.1 Environment

Time is continuous and the horizon infinite. The labor market is composed of workers and firms, with production organized in worker-firm matches.

Workers are risk neutral and occupy one of two labor force states: unemployment and employment. While unemployed, workers receive a flow payoff b, and realize job offers at rate λ. While employed, workers receive a flow wage w, and realize job offers at rate $s\lambda$; s thus indexes the relative efficiency of on-the-job search. For now we treat the arrival rate λ as a parameter but endogenize it in Section 1.5.

A new worker-firm match generates an initial flow product for the firm x_0. Thereafter, the idiosyncratic flow product x evolves according to a geometric Brownian motion,

$$\mathrm{d}x = \mu x \mathrm{d}t + \sigma x \mathrm{d}z, \tag{1}$$

where $\mathrm{d}z$ is the increment to a standard Brownian motion. Matches are destroyed exogenously at rate δ and endogenously subject to the free disposal of worker and firm.

We impose three further assumptions. First, to ensure the existence of a stationary distribution of productivity x across matches, we assume $\mu < \delta$. Second, to economize on notation, we confine ourselves in the main text to the case in which $\mu = \sigma^2/2$, so that $\ln x$ is a driftless Brownian motion. Results for general μ are reported in the Appendix. Finally, we assume zero inflation, so that payoffs can be read as both real and nominal. In Section 3, we show how to modify the baseline model to accommodate inflation.

Productivity x is realized at the beginning of each $\mathrm{d}t$ period, after which firm and worker may choose to separate. If the match continues, outside job offers are realized. Wage determination—the focus of our analysis—then proceeds as follows.

In the absence of an outside offer, wage adjustment may occur through two channels. First, the wage can be adjusted, at zero cost, subject to mutual consent of firm and worker. Second, either party may subsequently initiate a (re)negotiation of the wage. Importantly, the latter is not always costless: If the employee (firm) *unilaterally* initiates a renegotiation, there is an initial probability Δ_e (Δ_f) that the relationship breaks down and is destroyed.[1] By contrast, if both worker and firm initiate a negotiation, there is no risk of a breakdown. Absent a breakdown, wages are then determined according to a Nash bargain over the total match surplus, with worker bargaining power $\beta \in (0,1)$. If neither channel of wage adjustment is exercised, the wage remains unchanged.

In the presence of an outside offer, the worker chooses whether to exercise the offer. If the offer is not exercised, wage determination proceeds as above. If the offer is exercised, wages are determined according to a variation of the bargaining protocol proposed by Dey and Flinn (2005) and Cahuc et al. (2006). As in these papers, the worker can use the total surplus of the unsuccessful match as her outside option when bargaining. If the outside match is more productive, then both the outside firm and the worker initiate a renegotiation, and the worker captures an additional share β of the difference in surplus between the two matches. If the outside match is less productive, the bargain is initiated *unilaterally* by the worker, and the risk of a breakdown again applies. An implication is that the worker quits if and only if the outside firm has a greater match surplus.

Upon completion of wage setting and worker turnover decisions, production takes place and wages are paid in the relevant firm, and the period concludes.

We highlight several useful features of this environment. First, it nests canonical approaches to wage determination. Models that invoke *ex post* wage bargaining correspond to the case in which unilateral threats to renegotiate are fully credible, $\Delta_e = \Delta_f = 0$. Thus,

[1] This has the tractable implication that the match either ends in the event a breakdown occurs or continues with its surplus unimpaired. As will become clear, an implication is that separations are bilaterally efficient. Risk of a breakdown appeals to us as a tractable way of modeling renegotiation costs, but we also think of it as a proxy for other renegotiation costs, such as time, effort, and stress.

the search and matching models of Pissarides (1985) and Mortensen and Pissarides (1994) are accommodated, respectively, by the special cases $\sigma = s = \Delta_e = \Delta_f = 0$, and $\sigma > 0 = s = \Delta_e = \Delta_f$. Alternatively, models in which renegotiation of an existing agreement may occur only by mutual consent of both firm and worker correspond to the case in which unilateral threats to initiate a renegotiation are not credible, $\Delta_e = \Delta_f = 1$ (MacLeod and Malcomson 1993). Thus, Hall's (2005) model of entry wage rigidity is accommodated by the special case in which $\sigma = s = 0$ and $\Delta_e = \Delta_f = 1$. Finally, the environment nests canonical models in which renegotiation is prompted by the receipt of outside offers generated by on-the-job search. The sequential auctions model of Postel-Vinay and Robin (2002) corresponds to the special case $s > 0 = \sigma = \beta$ and $\Delta_f = 1$, whereas the model of Dey and Flinn (2005) and Cahuc et al. (2006) corresponds to $s > 0 = \sigma = \Delta_e$ and $\Delta_f = 1$.

A second useful feature of the environment is that it extends these canonical models to accommodate intermittent wage adjustment in the presence of idiosyncratic shocks. Central distinctions between canonical approaches are polar assumptions on the credibility of unilateral threats to renegotiate and the presence of idiosyncratic shocks. Our generalized environment allows the credibility of these threats to be varied flexibly via the renegotiation costs Δ_e and Δ_f. Combined with idiosyncratic shocks to productivity x, this gives rise to a first channel of sporadic equilibrium wage adjustment.

To see how, consider the case in which an outside offer is absent. Denoting the match surplus by S, the firm (respectively, worker) can guarantee an expected surplus equal to $(1-\beta)(1-\Delta_f)S$ (respectively, $\beta(1-\Delta_e)S$) by unilaterally triggering a renegotiation. The latter is costly to the match, however, as it risks a breakdown. Instead, (costly) off-equilibrium unilateral threats sustain (costless) equilibrium wage adjustment by mutual consent. Suppose that, upon realization of x, the worker's surplus under the existing agreement falls below $\beta(1-\Delta_e)S$. Rather than countenance the risk of a breakdown Δ_e, the firm will accede to a wage increase by mutual consent that restores the worker surplus to $\beta(1-\Delta_e)S$. Symmetrically, whenever the firm's surplus under the existing agreement falls below $(1-\beta)(1-\Delta_f)S$, the worker will agree to a wage cut by mutual consent that restores the firm's surplus to $(1-\beta)(1-\Delta_f)S$. Otherwise, both worker and firm surpluses lie above their respective thresholds, neither party has a credible unilateral threat to initiate a renegotiation, and the wage remains unchanged.

The case in which an outside offer is present is more straightforward. Since the employee always has the option not to exercise an outside offer, the only case in which the bargain is initiated unilaterally is when the employee does so. This will arise in the case in which the outside firm knows that the worker will not quit, and the current firm knows renegotiation will only raise the worker's compensation. In this case, the current firm accedes to an increase in compensation, by mutual consent, that renders the employee indifferent to unilateral initiation of a bargain. Per the bargaining protocol of Dey and Flinn (2005) and Cahuc et al. (2006), the worker receives $S(x_0) + \beta(1 - \Delta_e)[S(x) - S(x_0)]$. Otherwise, in the case in which the outside firm is more productive, both the outside firm and the worker initiate renegotiation, and the worker receives $S(x) + \beta[S(x_0) - S(x)]$.

The protocol thus links the frequency of wage adjustment to both the credibility of unilateral threats to renegotiate (the Δs) and the arrival of outside job offers (at rate $s\lambda$).

A final virtue of the model is that it is amenable to analytical solution. Key to tractability is the observation that the wage protocol preserves bilateral efficiency. The availability of costless wage adjustment by mutual consent ensures that the costs of unilaterally triggered renegotiation are never realized on the equilibrium path, and that matches separate only if the total match surplus S is extinguished. A consequence is that solution of the model can be decoupled into an optimal stopping problem that determines the durability of matches, and a novel wage determination problem, the solution of which is a key contribution of this paper. We now describe each of these in turn.

1.2 Surplus, separations, and tenure

Bilateral efficiency implies that total match surplus is independent of wage determination. We thus denote the total surplus of a match with current productivity x by $S(x)$. Separations then occur whenever the match surplus reaches zero.

Define a separation threshold x_l. Then, for all $x \in (x_l, \infty)$, $S(x)$ satisfies

$$(r + \delta)S(x) = x - rU + \beta s\lambda \max\{S(x_0) - S(x), 0\} + \mu x S_x + \frac{1}{2}\sigma^2 x^2 S_{xx}. \tag{2}$$

The flow surplus of the match is given by flow output x, less the annuitized value of unemployment to the worker rU. The match then faces capital gains from two sources. First, at rate $s\lambda$ the worker receives an offer from an outside firm of productivity x_0. Since, upon job-to-job transition, the worker captures the entirety of the surplus of her previous match, $S(x)$, plus a share β of the difference, $S(x_0) - S(x)$, on-the-job search is surplus enhancing. Second, there are idiosyncratic shocks to match productivity, which evolve according to the stochastic law of motion (1). Ito's lemma implies the form in (2).

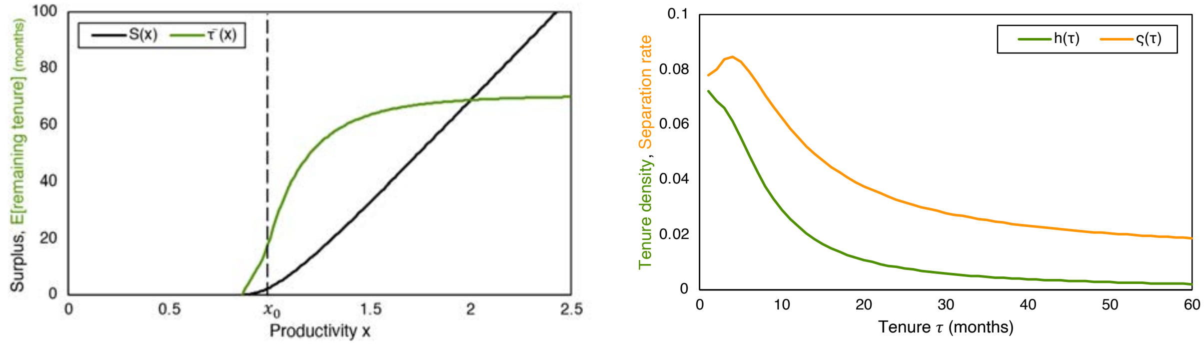
The definition of the separation threshold x_l, and optimality, imply the following value-matching and smooth-pasting conditions,

$$S(x_l) \equiv 0, \quad S'(x_l) = 0, \quad S(x_0^-) = S(x_0^+), \text{ and } S'(x_0^-) = S'(x_0^+). \tag{3}$$

Applying these boundary conditions to (2) yields a solution for the total surplus.

Figure 1. Match surplus, tenure, and separations

A. Surplus $S(x)$ and expected tenure $\bar{\tau}(x)$ B. Density $h(\tau)$ and separation rate $\varsigma(\tau)$



Notes. Parameter values are based on the model calibrated as described in Table 1.

Proposition 1 *(i) The total match surplus is given by*

$$S(x)=\begin{cases}\dfrac{x}{r+\delta-\mu+\beta s\lambda}-\dfrac{rU-\beta s\lambda S(x_0)}{r+\delta+\beta s\lambda}+\mathcal{S}_1 x^{-\sqrt{\frac{r+\delta+\beta s\lambda}{\mu}}}+\mathcal{S}_2 x^{\sqrt{\frac{r+\delta+\beta s\lambda}{\mu}}} & \text{if} \quad x<x_0,\\[2ex] \dfrac{x}{r+\delta-\mu}-\dfrac{rU}{r+\delta}+S_1 x^{-\sqrt{\frac{r+\delta}{\mu}}} & \text{if} \quad x\geq x_0,\end{cases} \tag{4}$$

for all $x\in(x_l,\infty)$. *The coefficients* $\mathcal{S}_1$, $\mathcal{S}_2$ *and* S_1, *and the separation boundary* x_l *can then be recovered from the boundary conditions (3).*

(ii) Expected remaining tenure at each productivity level can be recovered from

$$\bar{\tau}(x)=-S_{rU}(x)|_{S(x_0),r=0,\beta=1}. \tag{5}$$

The affine terms in (4) represent the value of the match in the absence of the option to separate, and absent the possibility of switching between states in which contacted workers quit ($x<x_0$), or are retained ($x\geq x_0$). The nonlinear terms capture the values of these future prospects. Figure 1A illustrates the implied shape of $S(x)$.

Proposition 1 further reveals that the solution for the match surplus $S(x)$ can be used to recover a solution for expected remaining tenure at each productivity level, $\bar{\tau}(x)$. Intuitively, holding fixed the surplus of a new match $S(x_0)$, a marginal decrease in the annuitized value of unemployment rU raises current match surplus $S(x)$ by the expected discounted value of a unit flow over the match's duration. When discounting is suspended ($r=0$), and when the arrival of an outside offer induces a full loss of value ($\beta=1$), this thought experiment recovers expected remaining tenure. Figure 1A superimposes the implied shape of $\bar{\tau}(x)$, which commences at zero at x_l, and then increases with x. A kink emerges at x_0 as contacted workers cease to quit, and $\bar{\tau}(x)$ then converges to $1/\delta$.

Instructive special cases. It is possible to refine and extend Proposition 1 to solve additionally for the distribution of tenure in instructive special cases. This, in turn, relates the results in Proposition 1 to classic results on Brownian motion. Lemma 1 summarizes.

Lemma 1 *The following results for special cases are available.*

(i) If $s = 0$, or $\beta = 0$, then

$$S(x) = \frac{x}{r+\delta-\mu} - \frac{rU}{r+\delta}\left[1 - \frac{\mu}{r+\delta+\mu}\left(\frac{x}{x_l}\right)^{-\sqrt{\frac{r+\delta}{\mu}}}\right], \text{ and } x_l = \frac{r+\delta-\mu}{r+\delta+\mu} rU. \tag{6}$$

(ii) If $s = 0$, the distribution function of completed tenure spells τ is given by

$$H(\tau) = 1 - \exp(-\delta\tau)\,[1 - H_0(\tau)]. \tag{7}$$

where $H_0(\tau)$ is Inverse Gaussian with associated density

$$h_0(\tau) = \frac{\ln(x_0/x_l)}{\sigma\tau^{3/2}}\phi\left(\frac{\ln(x_0/x_l)}{\sigma\tau^{1/2}}\right), \tag{8}$$

and where $\phi(\cdot)$ is the standard normal density.

(iii) If $x_l = 0$, the distribution function of completed tenure spells τ is given by

$$H(\tau) = 1 - \exp\left[-\left(\delta + \frac{s\lambda}{2}\right)\tau\right] I_0\left(\frac{s\lambda\tau}{2}\right), \tag{9}$$

where $I_{\cdot}(\cdot)$ is the modified Bessel function of the first kind.

Result (i) underscores the link between the solution for match surplus in our environment, and standard results on optimal stopping problems for the special case in which surplus is unaffected by job-to-job turnover ($\beta s = 0$). Here, the solution for $S(x)$ takes the simpler form in (6), and a closed-form solution for x_l is available, reiterating results reported in Dixit (1993), Moscarini (2005), Buhai and Teulings (2014), *inter alia.*

Results (ii) and (iii) then reveal that one can also infer the *distribution* of match tenure in two polar cases for the sources of endogenous separation.

Result (ii) suspends job-to-job separations ($s = 0$), and adapts classic results on first passage times originally applied to labor turnover by Whitmore (1979). Given initial productivity $x_0 > x_l$, an endogenous separation occurs in this case if, and only if, x first hits the lower boundary x_l, where the total surplus is zero. The match survival distribution is then the product of the exponential survival distribution of exogenous separations, and the Inverse Gaussian survival distribution of first passage times.

More novel, result (iii) suspends endogenous separations into unemployment ($x_l = 0$). Matches terminate in this case exogenously at rate δ, and endogenously at rate $s\lambda$

whenever $x < x_0$. The distribution of tenure thus depends on the distribution of the time spent by x below x_0. The latter is provided by a classic result on occupation times due to Lévy (1939): The share of time spent by x below x_0 has an arcsine distribution, which is U-shaped, and symmetric about its mean (and, thereby, median) of one half. The match survival distribution that emerges is the product of the exponential survival distribution that would arise if x spent half the time below x_0 with certainty, and an uncertainty adjustment that is formalized by a modified Bessel function of the first kind. The latter is increasing and convex in $s\lambda\tau/2 \geq 0$, and satisfies $I_0(0) = 1$.

Away from these special cases, it is straightforward to infer the tenure density $h(\tau)$ numerically. Separation rates by tenure can then be recovered from the hazard $\varsigma(\tau) = h(\tau)/[1 - H(\tau)]$. Figure 1B illustrates $h(\tau)$ and $\varsigma(\tau)$. Of particular note are the latter separation rates, which are initially hump-shaped, and thereafter declining. These features are consistent with empirical evidence on hazard rates of job ending documented by Farber (1999). The hump-shape has its origins in result (ii) of Lemma 1: New matches are created at $x_0 \gg x_l$, and it takes time for shocks to x to accumulate sufficiently to induce endogenous separations to unemployment. By contrast, the declining hazard emerges from both results (ii) and (iii) of Lemma 1. Both capture a form of dynamic selection whereby low-productivity matches—which are more likely to separate both to unemployment (result (ii)), and from job to job (result (iii))—are weeded out at lower tenures.

1.3 Wage adjustment

A key contribution of our analysis is to show that the wage protocol further admits a characterization of the surpluses of both firm and worker and, thereby, the path of wages.

To see how, consider the firm surplus, which we denote by $J(w, x)$. As the notation anticipates, this will depend on both the current wage w, and the productivity of the match x. The presence of two state variables raises a potential analytical challenge: Since the firm faces expected capital gains not only from future changes in productivity x, but also from future changes in the wage w, the firm surplus $J(w, x)$ in general will satisfy a *partial* differential equation, the analytical solution of which is complicated. A symmetric argument applies to the worker surplus, which we denote $V(w, x)$.

The wage protocol provides a considerable simplification, however. We show that it yields a solution for the wage policy characterized by three further thresholds for productivity for any current wage, which we shall denote by $x_e(w)$, $x_f(w)$, and $x_n(w)$. The boundaries $x_e(w)$ and $x_f(w)$ trace out loci along which a renegotiation is initiated by the *employee* and *firm* to raise and cut the wage, respectively. The boundary $x_n(w)$ traces out the locus along which an outside offer is rendered *noncompetitive*.

Importantly, wages are adjusted if, and only if, the boundaries $x_e(w)$ and $x_f(w)$ are attained, or a competitive outside offer arrives. Otherwise, wages remain unchanged and, for any current wage w, firms face capital gains solely from future changes in productivity x. In this case, the firm surplus $J(w,x)$ satisfies the *ordinary* differential equation

$$\begin{aligned}(r+\delta)J(w,x) &= x - w - s\lambda \mathbf{1}_{\{x<x_0\}} J(w,x) \\ &\quad + s\lambda \mathbf{1}_{\{x\geq x_0\}} \min\{[1-\beta(1-\Delta_e)][S(x)-S(x_0)] - J(w,x), 0\} \\ &\quad + \mu x J_x + \frac{1}{2}\sigma^2 x^2 J_{xx},\end{aligned} \tag{10}$$

and the worker surplus $V(w,x)$ satisfies the analogous ordinary differential equation

$$\begin{aligned}(r+\delta)V(w,x) &= w - rU + s\lambda \mathbf{1}_{\{x<x_0\}}\{S(x) + \beta[S(x_0)-S(x)] - V(w,x)\} \\ &\quad + s\lambda \mathbf{1}_{\{x\geq x_0\}} \max\{S(x_0) + \beta(1-\Delta_e)[S(x)-S(x_0)] - V(w,x), 0\} \\ &\quad + \mu x V_x + \frac{1}{2}\sigma^2 x^2 V_{xx}.\end{aligned} \tag{11}$$

Absent wage adjustment, the firm receives a flow surplus equal to output x less the current wage w, and the worker receives a flow surplus of w less the annuitized value of unemployment rU. The match then faces capital gains from two sources.

First, at rate $s\lambda$ the worker receives an offer from an outside firm of productivity x_0. If $x < x_0$, the worker quits, the firm realizes a capital loss equal to its value of the match $J(w,x)$, and the worker realizes a capital gain $S(x) + \beta[S(x_0) - S(x)] - V(w,x)$. If $x \geq x_0$, the worker is retained, and receives a capital gain of $S(x_0) + \beta(1-\Delta_e)[S(x) - S(x_0)] - V(w,x)$, or zero, whichever is greater. The firm receives the remainder of the surplus.

The second source of capital gains arises from idiosyncratic shocks to match productivity, which evolve according to the stochastic law of motion (1). Application of Ito's lemma yields the latter two terms in (10) and (11). Importantly, these capture not only the direct value to the firm of future changes in output x, but also the indirect value of future adjustments in wages induced by changes in x, or the arrival of outside offers. Formally, the latter are encoded in the boundary conditions for this problem, which follow from the bounds to firm and worker surpluses implied by the wage adjustment protocol.

Specifically, along the $x_f(w)$ boundary, wages are cut such that the firm is indifferent between continuing under the adjusted wage and unilaterally initiating a renegotiation. Optimal exercise of such threats implies that the firm's surplus must satisfy the value-matching and smooth-pasting conditions,

$$J\big(w, x_f(w)\big) \equiv (1-\beta)(1-\Delta_f)S\big(x_f(w)\big), \text{ and}$$
$$J_x\big(w, x_f(w)\big) = (1-\beta)(1-\Delta_f)S'\big(x_f(w)\big), \text{ for all } w. \tag{12}$$

Symmetrically, along the $x_e(w)$ boundary, wages are increased such that the employee is indifferent between continuing under the adjusted wage and unilaterally initiating a renegotiation. The worker's surplus must then satisfy

$$V\big(w, x_e(w)\big) \equiv \beta(1-\Delta_e)S\big(x_e(w)\big), \text{ and}$$
$$V_x\big(w, x_e(w)\big) = \beta(1-\Delta_e)S'\big(x_e(w)\big), \text{ for all } w. \tag{13}$$

Finally, along the $x_n(w)$ boundary, the worker receives a surplus that renders her indifferent to unilateral initiation of a bargain subject to breakdown risk Δ_e. Optimality further requires that $V(w,x)$ be continuously differentiable through the boundary,

$$V\big(w, x_n(w)\big) \equiv S(x_0) + \beta(1-\Delta_e)\big[S\big(x_n(w)\big) - S(x_0)\big], \text{ and}$$
$$V_x\big(w, x_n^-(w)\big) = V_x\big(w, x_n^+(w)\big), \text{ for all } w. \tag{14}$$

The boundary conditions reiterate the analytical tractability afforded by the decoupling of match surplus, and the surplus of each party, implied by bilateral efficiency. Given the solution for match surplus $S(x)$ in Proposition 1, the boundary conditions in (12), (13) and (14) can be evaluated, and the wage adjustment problem can be solved.

To do so, we define the *inaction sets* $\mathrm{I}_e(w)$ and $\mathrm{I}_f(w)$ that describe, for each current wage w, the set of productivities x such that employee and firm respectively cannot credibly issue a unilateral threat to renegotiate. It follows that $x_e(w) = \partial \mathrm{I}_e(w)$, and $x_f(w) = \partial \mathrm{I}_f(w)$, the boundaries of the respective inaction sets. Similarly, we define the *noncompetitive set* $\mathrm{N}(w)$ that describes, for each w, the set of x such that an outside offer is rendered noncompetitive. Accordingly, $x_n(w) = \partial \mathrm{N}(w)$. Finally, to accommodate the possibility that $\mathrm{N}(w)$, or its complement $\bar{\mathrm{N}}(w)$, is nonconvex, we partition each into subintervals indexed by i, denoted respectively by $\mathrm{N}_i(w)$ and $\bar{\mathrm{N}}_i(w)$. With this notation in hand, Proposition 2 summarizes the implied solutions for the firm and worker surpluses, and their implications for the allocative effects of wages.

Proposition 2 *(i) For any w, i, and $x \in N_i(w)$, the firm surplus has general solution*

$$J(w,x) = \frac{x}{r+\delta-\mu} - \frac{w}{r+\delta} + J_{i1}(w)x^{-\sqrt{\frac{r+\delta}{\mu}}} + J_{i2}(w)x^{\sqrt{\frac{r+\delta}{\mu}}}. \tag{15}$$

For any w, i, and $x \in \bar{N}_i(w)$, the firm surplus has general solution

$$J(w,x) = \frac{x}{r+\delta-\mu+s\lambda} - \frac{w}{r+\delta+s\lambda} + s\lambda\mathcal{P}(x) + \mathcal{J}_{i1}(w)x^{-\sqrt{\frac{r+\delta+s\lambda}{\mu}}} + \mathcal{J}_{i2}(w)x^{\sqrt{\frac{r+\delta+s\lambda}{\mu}}}, \tag{16}$$

where

$$\mathcal{P}(x) = \frac{1-\beta(1-\Delta_e)}{2(r+\delta+s\lambda)}\int_{x_0}^{\max\{x,x_0\}}\left[\left(\frac{x}{\tilde{x}}\right)^{\sqrt{\frac{r+\delta+s\lambda}{\mu}}} - \left(\frac{x}{\tilde{x}}\right)^{-\sqrt{\frac{r+\delta+s\lambda}{\mu}}}\right]\frac{S(\tilde{x})-S(x_0)}{\tilde{x}}\mathrm{d}\tilde{x}. \tag{17}$$

The coefficients $J_{i1}(w)$, $J_{i2}(w)$, $\mathcal{J}_{i1}(w)$, $\mathcal{J}_{i2}(w)$, *and boundaries* $x_e(w)$, $x_f(w)$, $x_n(w)$, *are implied by the boundary conditions (12), (13), and (14). The worker surplus* $V(w,x) = S(x) - J(w,x)$ *follows from (4).*

(ii) The expected duration until next wage adjustment can be recovered from

$$\bar{\tau}^w(w,x) = -J_w(w,x)|_{r=0} = V_w(w,x)|_{r=0}. \tag{18}$$

Proposition 2 reveals that, for each wage w, the solution for the firm (and worker) values can be recovered using an extension of the methods applied to the total surplus in Proposition 1. The affine terms in (15) capture the firm's surplus absent the option to adjust wages, and absent the possibility of switching between states in which outside offers are competitive ($x \in \overline{\mathrm{N}}(w)$), or noncompetitive ($x \in \mathrm{N}(w)$). The power functions in (15) respectively value the prospects of wage cuts in adverse future states, wage increases in favorable future states, and of switching in and out of $\mathrm{N}(w)$, for each current wage w.

A key analytical challenge resolved by Proposition 2 is the presence of additional capital gains associated with the prospect of an outside offer in (10) and (11). The insight is twofold. First, the solution for the total match surplus $S(x)$ in Proposition 1 delivers a solution for these additional capital gains. Second, the value of these capital gains to firm and worker can be inferred analytically in the form of the particular solution $\mathcal{P}(x)$ given in (17). This, in turn, implies a solution for the option value of on-the-job search.

Figure 2A illustrates the firm and worker values that emerge, reiterating the intuition that the intermittent ability of firm and worker to issue credible unilateral threats to renegotiate the wage places bounds on their valuations of the match.

By providing an analytical characterization of the firm and worker valuations of the match, Proposition 2 aids the solution for the boundaries $x_e(w)$, $x_f(w)$, and $x_n(w)$. These are illustrated in Figure 2B, which reveals that, in general, the wage adjustment boundaries take the form of correspondences. Most notably, the boundary $x_e(w)$ acts as both an upper and a lower bound for productivity at low current wages w. The upper bound reflects the fact that increases in productivity x raise the possible surplus that the employee can capture by issuing a unilateral threat to renegotiate. The lower bound reflects the fact that, as x declines, the surplus that the worker can extract in the event of an outside offer also declines. Both act as a potential stimulus to wage increases in the model. An analogous logic applies to the boundary along which outside offers become noncompetitive, and so an echo of these same forces can be seen in the shape of $x_n(w)$.

The boundaries in turn determine the path of wages for any given initial wage and realization of the path of match productivity x. As indicated by the arrows in Figure 2B, wages remain constant for periods of time, punctuated by adjustments from two sources.

First, productivity shocks trigger incremental wage adjustments at the upper and lower boundaries induced by unilateral bargaining threats. These adjustments regulate the joint path of (w, x) to remain within the boundaries $x_e(w)$ and $x_f(w)$.

Second, the arrival of outside offers may also trigger an adjustment to compensation. In matches with productivity $x < x_0$, contacted workers quit to accept the outside match. In matches of productivity $x \geq x_0$, however, contacted workers are retained. If, in addition, the outside offer is competitive, $x \in \overline{\mathrm{N}}(w)$, workers realize a retention package that raises their compensation just enough to render the outside offer noncompetitive.

Observe that these two stimuli to adjustment are fundamentally different. Adjustments at $x_e(w)$ and $x_f(w)$ are *incremental*, triggered by shocks to *flow* productivity x that are *persistent*. By contrast, adjustments induced by outside offers are *discrete*, triggered by shocks to the outside *value* available to the worker *temporarily*.

This distinction has natural implications for the structure of compensation between flow wages and lump-sum bonuses in each case. Changes to flow productivity x at the boundaries $x_e(w)$ and $x_f(w)$ are naturally resolved by adjustments to the flow wage w: Given the persistence of shocks to x, lump-sum bonuses would instead resolve the impetus to renegotiation only temporarily, requiring a period of *continual* bonuses.[2]

By contrast, temporary realizations of outside offers can be naturally resolved either by flow wage increases or by instantaneous bonus pay. Interestingly, a novel implication of the model is that it places an upper bound on the former: Firms can credibly commit to flow wage increases only up to $x_f^{-1}(x)$; a further increase would trigger a subsequent renegotiation, and be reversed, almost immediately. Any remaining value must then be delivered as a lump-sum bonus—lump-sum because the firm has no other means to pledge value credibly at the point of counteroffer. Thus, the model provides a novel theory of nonbase pay in the form of recruitment and retention bonuses.

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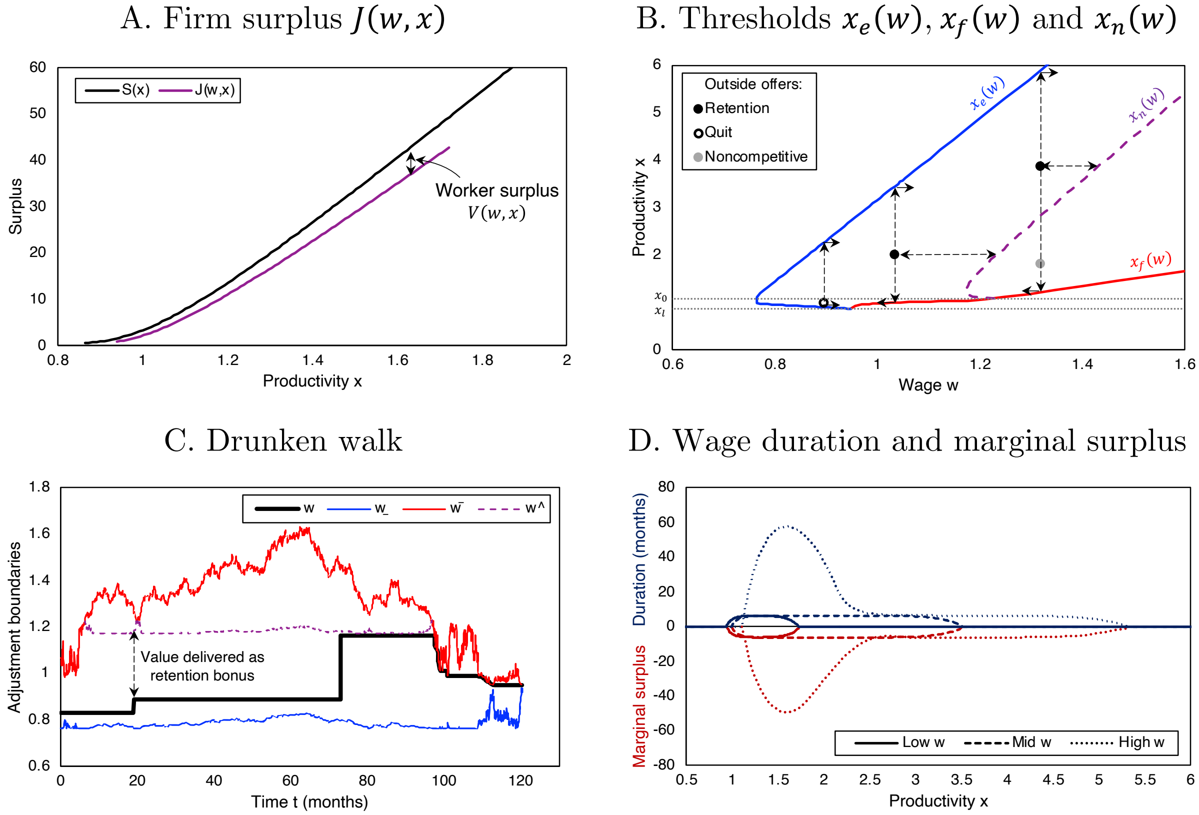
[2] Furthermore, at the wage cut boundary $x_f(w)$, this alternative would also require a period of continual lump-sum payments *from worker to firm*.

Figure 2. Baseline model outcomes



Notes. Parameter values are based on the model calibrated as described in Table 1.

Returning to Figure 2B, the upshot is that the arrival of outside offers induces adjustments to compensation that lie along the horizontal arrows. Note that a consequence is that, once a match enters the right-hand region in which the wage cut boundary $x_f(w)$ is operative, it will remain in that region until its termination.

Finally, Figure 2B anticipates a useful simplification of the form of the adjustment boundaries—namely that they appear to approach linear functions as the wage w rises.

Lemma 2 *For $\beta \in (0,1]$, and $\Delta_e, \Delta_f \in [0,1)$, the adjustment boundaries $x_e(w)$, $x_f(w)$ and $x_n(w)$ become linear as $w \to \infty$.*

The seemingly complex wage dynamics implied by Proposition 2 are thus quite simple for sufficiently productive matches. The model therefore does not demand excessive sophistication of many firms and workers who adhere to near-linear adjustment rules.

Figure 2C provides a parallel view of the implied wage dynamics. The adjustment thresholds can be inverted to define three wage thresholds: A lower wage threshold, $\underline{w}(x) \equiv x_e^{-1}(x)$, an upper threshold $\overline{w}(x) \equiv x_f^{-1}(x)$, and a threshold wage at which outside offers cease to be competitive, $\widehat{w}(x) \equiv x_n^{-1}(x)$. Given an initial wage w_0, and sample path of productivity x, wages evolve according to the *drunken walk* depicted in Figure 2C, remaining constant on the interior of the thresholds, adjusting minimally when a shock induces them to bind, and adjusting discretely upon arrival of a competitive outside offer.

Finally, result (ii) of Proposition 2 establishes a close relationship between firm and worker surpluses and the expected duration of wage spells. Specifically, expected wage durations are intimately related to the allocative effects of intermittent wage adjustment, as captured by the marginal values of the wage to firm and worker, $J_w(w,x)$ and $V_w(w,x)$.

Figure 2D illustrates. Because wage adjustment is two-sided—wage spells terminate when match productivity x first hits *either* the relevant lower *or* upper boundary—the expected duration until next wage adjustment is zero at these boundaries (and, by extension, at the separation boundary x_l). Away from these extremes, the expected time until adjustment is hump-shaped for productivities outside the noncompetitive set $\mathsf{N}(w)$. If productivity further traverses through $\mathsf{N}(w)$, a larger second hump emerges as outside offers cease to be competitive, and wages no longer need to adjust to their arrival.

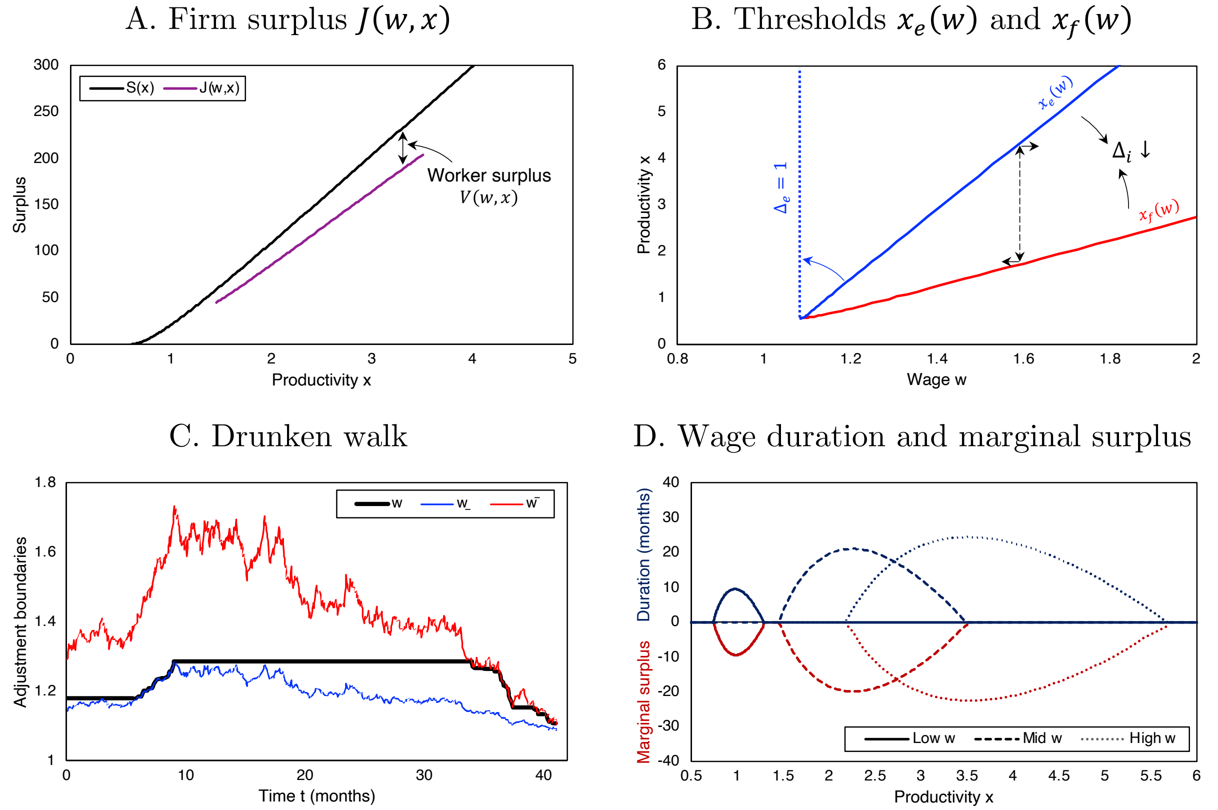
Figure 2D also reiterates that the marginal value of the wage is equal to (minus) the *discounted* expected wage duration. Intuitively, each party knows that any future adjustment of the wage will be made solely on the basis of contemporaneous productivity, and thus will be independent of the history of wages and productivity up to that point. Firm and worker therefore care simply about the duration of the current wage, appropriately discounted. We will see that this aspect of Proposition 2 has important implications for the allocative effects of intermittent wage adjustment on unemployment.

1.4 Instructive special cases

To build further intuition, we now consider two special cases that correspond to the two stimuli to wage adjustment in the general model and are contributions in themselves.

Costly renegotiation. First, we consider the case in which there is no on-the-job search ($s = 0$), and wage adjustment is prompted solely by intermittent unilateral threats to initiate a renegotiation. This case has the tractable implication that simple analytical solutions are available for match surplus $S(x)$, the separation boundary x_l, and the distribution of completed tenure spells $h(\tau)$, as summarized in results (i) and (ii) of Lemma 1. The solution for $S(x)$ can in turn be used directly to evaluate the boundary conditions in (12) and (13), permitting a solution to the wage adjustment problem. A tractable feature of this case is that the wage adjustment boundaries $x_e(w)$ and $x_f(w)$ can be distilled into a pair of nonlinear equations, which we provide in the Appendix.

Figure 3. Model outcomes: Costly renegotiation case ($s = 0$)



Notes. Parameter values based on a recalibration that omits the target moment for s in Table 1.

Proposition 3 *In the special case $s = 0$, the general solution for the firm surplus is*

$$J(w,x) = \frac{x}{r+\delta-\mu} - \frac{w}{r+\delta} + J_1(w)x^{-\sqrt{\frac{r+\delta}{\mu}}} + J_2(w)x^{\sqrt{\frac{r+\delta}{\mu}}}, \tag{19}$$

for all $w \geq \beta x_l + (1-\beta)rU$, and $x \in (x_f(w), x_e(w))$. The coefficients $J_1(w)$, $J_2(w)$ can be recovered from the boundaries $x_e(w)$ and $x_f(w)$, which in turn solve a known pair of equations (provided in the Appendix).

Figure 3 illustrates model outcomes in this case as a point of comparison to Figure 2. The boundaries simplify to an upper bound provided by the employee's unilateral renegotiation threshold $x_e(w)$, and a lower bound provided by the firm threshold $x_f(w)$, as in Figure 3B. This reinforces the role of the credibility of unilateral threats to initiate a renegotiation, indexed by the Δs, in the incidence of equilibrium wage adjustment. Smaller Δs raise the credibility of such threats, narrowing the adjustment thresholds, and raising the frequency of wage adjustment. Larger Δs have the opposite effect. When both Δs are zero, threats to initiate a unilateral renegotiation are always credible, the adjustment boundaries degenerate, and the model recovers the continual Nash bargaining applied extensively in the search and matching literature (Mortensen and Pissarides 1994).

Notice, however, a fundamental asymmetry in this special case: Firms receive shocks to their flow payoff x; workers *do not*. Even in the face of a complete breakdown risk ($\Delta_f = 1$), there remain states in which the firm can credibly threaten to walk away from the match at the current wage, and thereby enforce a wage cut. By contrast, the worker can *never* credibly enforce a wage increase in the face of a complete breakdown risk ($\Delta_e = 1$), as illustrated in Figure 3B. This asymmetry is resolved in the general model by the presence of shocks to the worker's payoff, driven by the arrival of outside job offers.

Sequential auctions. The second special case we explore turns to the opposite polar case in which wage adjustment is prompted solely by the availability of outside offers generated by on-the-job search ($s > 0$). To isolate its signature implications for wage dynamics, we suspend the forces of the previous subsection: Unilateral bargaining threats are no longer credible, $\Delta_f = 1$, and workers no longer have bargaining power, $\beta = 0$.

This special case corresponds to the canonical sequential auctions model of Postel-Vinay and Robin (2002), extended to incorporate persistent idiosyncratic shocks to match productivity x, and drunken walk wage dynamics. Postel-Vinay and Turon (2010) study a similar environment, but with i.i.d. productivity shocks. As they note, the present case with persistent shocks poses an analytical challenge that has necessitated the use of numerical methods in prior work (Yamaguchi 2010; Lise et al. 2016). We show how analytical progress can be made for this canonical special case.

To begin, note that total match surplus takes the same form as in the preceding case: Although the presence of on-the-job search induces additional turnover, match surplus remains unimpaired since, upon job-to-job transition, the worker receives the entire surplus of her previous match. A direct implication is that the solutions in Lemma 1 for the match surplus $S(x)$ and the threshold for separation into unemployment x_l in (6) apply equally to this environment with sequential auctions. This again facilitates evaluation of the boundary conditions in (12), (13), and (14), which again reduces the solution for the boundaries $x_e(w)$, $x_f(w)$, and $x_n(w)$ to a set of nonlinear equations.

Proposition 4 *In the special case in which $\beta = 0$ and $\Delta_f = 1$, the general solution for the firm surplus is*

$$J(w,x) = \begin{cases} \dfrac{x}{r+\delta-\mu+s\lambda} - \dfrac{w}{r+\delta+s\lambda} + s\lambda\mathcal{P}(x) + \mathcal{J}_1(w)x^{-\sqrt{\frac{r+\delta+s\lambda}{\mu}}} + \mathcal{J}_2(w)x^{\sqrt{\frac{r+\delta+s\lambda}{\mu}}} & \text{if} \quad x < x_n(w), \\ \dfrac{x}{r+\delta-\mu} - \dfrac{w}{r+\delta} + J_1(w)x^{-\sqrt{\frac{r+\delta}{\mu}}} & \text{if} \quad x \geq x_n(w), \end{cases} \tag{20}$$

for all w*, and* $x \in \left(max\{x_e(w), x_f(w)\}, \infty\right)$*, where* $\mathcal{P}(x)$ *is given in (17). The coefficients* $\mathcal{J}_1(w)$*,* $\mathcal{J}_2(w)$*, and* $J_1(w)$*, and boundaries* $x_e(w)$*,* $x_f(w)$*, and* $x_n(w)$ *are known (implicit) functions of the parameters of the problem (provided in the Appendix).*

Figure 4 illustrates model outcomes for this case, revealing several points of contrast relative to Figures 2 and 3 above. Figure 4A reveals that the worker's share of the surplus is highest (and the firm's share lowest) in the neighborhood of $x_0 = 1$, since the worker's outside option is bounded above by $S(x_0)$. And, because the latter is small compared to $S(x)$ for $x \gg x_0$, the worker's surplus share decays as x rises.

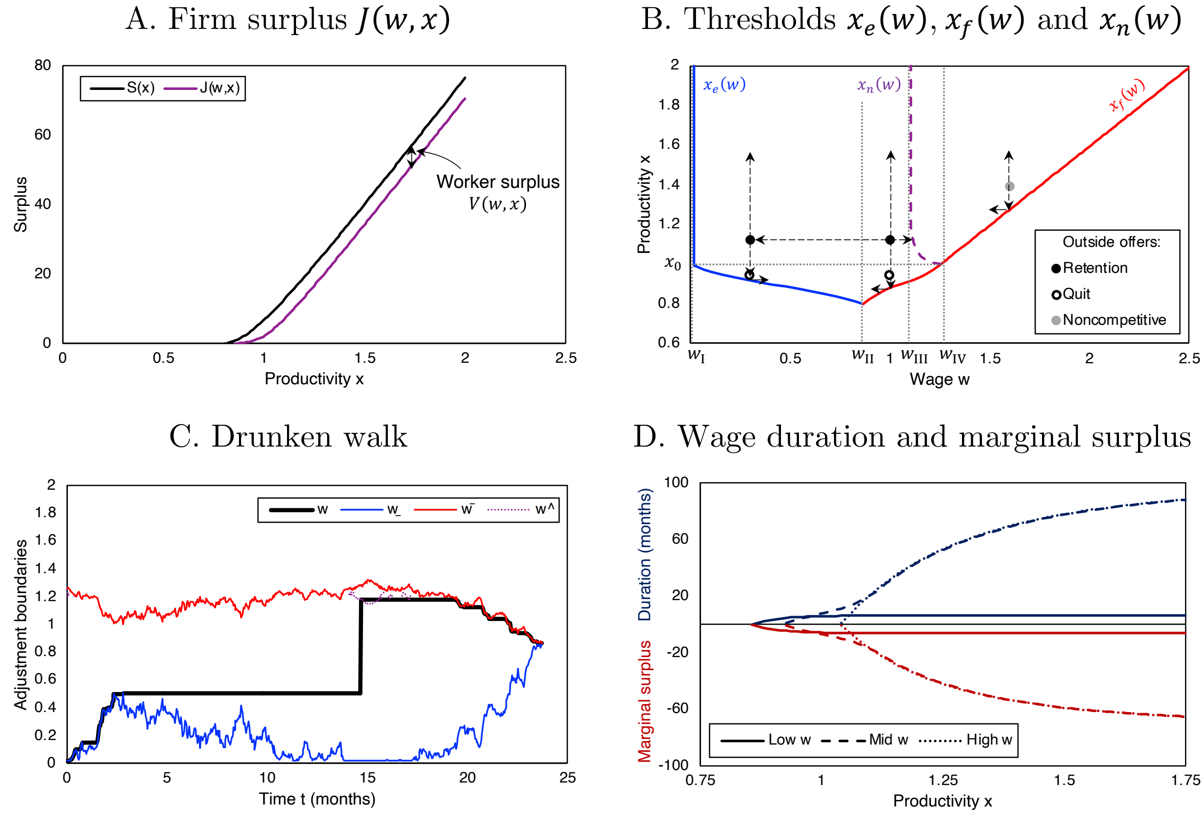
Figure 4B then illustrates the implied wage adjustment thresholds. This reveals that the downward-sloping sections in Figure 2B have their origins in the presence of on-the-job search: Both $x_e(w)$ and $x_n(w)$ slope downward in the sequential auctions environment. Intuitively, since the worker has no bargaining power, these trace out indifference loci over wages and productivity for the worker, $V\left(w, x_e(w)\right) \equiv 0$ and $V\left(w, x_n(w)\right) \equiv S(x_0)$. Since increases in x raise recruitment compensation paid in the event of an outside offer, $S(x)$, workers in matches with productivity $x < x_0$ value both higher wages and higher productivity, and $x_e(w)$ slopes downward for such matches. And since increases in x reduce the likelihood of future wage cuts, $x_n(w)$ also slopes downward.

Relatedly, Figure 4B reveals that wage adjustment at the thresholds is *one-sided* in this case. Fixing the current wage w and successively raising match productivity x no longer induce wage increases, due to the absence of worker bargaining power. A perhaps surprising corollary is that wage increases induced at the $x_e(w)$ threshold must arise from *reductions* in match productivity. Intuitively, reductions in x lower the worker's recruitment compensation in the event of an outside job offer, lowering her value of the match, and necessitating a raise to obviate a quit to unemployment.

The remaining panels of Figure 4 highlight two corollaries of the one-sided nature of wage adjustment. First, the drunken walk in Figure 4C reveals that the bounds for wages are *inversely* correlated in the sequential auctions case. Second, in Figure 4D the expected duration of the wage rises monotonically with productivity x, converging to $1/(\delta + s\lambda)$ when outside offers are competitive, and to $1/\delta$ when they are not. Reiterating the link between the duration of wages and their allocative effects, the marginal value of the wage behaves (near-)symmetrically, converging to $1/(r + \delta + s\lambda)$, and $1/(r + \delta)$, respectively.

Like the previous costly renegotiation example, the sequential auctions case also has extreme implications: The absence of worker bargaining power implies that workers can capture surplus only through receipt of outside offers, no matter how productive the match. Juxtaposing Figures 2, 3, and 4 suggests that the two forces of wage adjustment

Figure 4. Model outcomes: Sequential auctions case ($\Delta_f = 1$, $\beta = 0$)



Notes. Parameter values as described in Table 1, but with $\Delta_f = 1$ and $\beta = 0$.

encapsulated in these two special cases complement one another, capturing adjustments prompted by rent sharing on the one hand and external competition on the other. These results further underscores the value of the general model characterized in Proposition 2.

1.5 Labor market equilibrium

We now show how this model of wage determination can be embedded into aggregate labor market equilibrium, facilitating an analysis of its implications for unemployment.

To underscore the novel features of the theory, we consider its implications in the context of standard models in the Diamond-Mortensen-Pissarides search and matching tradition, extended to accommodate on-the-job search. The flow of new contacts arising from u unemployed searchers, $1-u$ employed searchers, and v vacancies is determined by a constant-returns-to-scale matching function $m(u+s(1-u),v)$. Labor market tightness, $\theta \equiv v/[u+s(1-u)]$, is thus a sufficient statistic for contact rates: Vacancies contact a searcher at rate $q(\theta) = m/v = m(1/\theta, 1)$; unemployed and employed searchers respectively contact a vacant job at rates $\lambda(\theta) = m/[u+s(1-u)] = m(1,\theta)$, and $s\lambda(\theta)$.

Four further conditions then complete aggregate labor market equilibrium.

First, the value of unemployment to a worker U satisfies the Bellman equation,

$$rU(w_0, x_0; \theta) = b + \lambda(\theta)V(w_0, x_0; \theta). \tag{21}$$

While unemployed, a worker receives a flow payoff b. At rate $\lambda(\theta)$ she finds a new job with initial wage w_0 and initial productivity x_0, yielding a surplus to the worker of $V(w_0, x_0; \theta)$. Observe that the solution for $V(w_0, x_0; \theta)$ is provided by Proposition 2 above.

Second, the model is amenable to different protocols for the determination of the entry wage of a new hire from unemployment w_0. Consistent with our assumption that bargaining is costless when both parties wish to initiate a negotiation, our benchmark will be that w_0 is determined by Nash bargaining. A newly hired worker from unemployment thus receives a value $\beta S(x_0)$. But the model is also amenable to alternatives, such as Hall's (2005) proposed fixity of w_0, and variations thereof, a point to which we will return.

Third, equilibrium labor market tightness is determined by free entry into vacancy creation. This requires the expected firm surplus from meeting a searcher to be equal to the expected cost of meeting a searcher. Denoting the distribution function of employees over productivity by $G(x; \theta)$, and the flow cost of a vacancy by c_v, this implies

$$\frac{u}{u + s(1-u)} J(w_0, x_0; \theta) + \frac{s(1-u)}{u + s(1-u)} (1-\beta) \int_{x_l(\theta)}^{x_0} [S(x_0; \theta) - S(x; \theta)] \mathrm{d}G(x; \theta) = \frac{c_v}{q(\theta)}. \tag{22}$$

In the event of hiring an unemployed worker, the firm realizes a surplus of $J(w_0, x_0; \theta)$. Recruitment of a worker previously employed in a match of productivity $x < x_0$ realizes a surplus of $(1-\beta)[S(x_0; \theta) - S(x; \theta)]$. Weighting the latter by their respective shares of the population of searchers delivers the expected firm surplus in (22). Observe that solutions for the total surplus $S(x; \theta)$, and the firm's valuation of a new job $J(w_0, x_0; \theta)$, are provided respectively by Propositions 1 and 2.

Finally, denoting the aggregate separation rate into unemployment by $\varsigma(\theta)$, the unemployment rate evolves according to the law of motion,

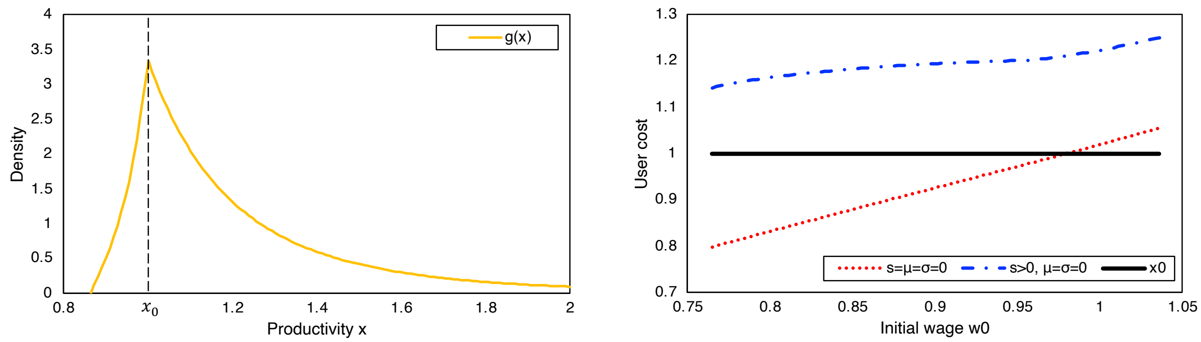
$$\frac{\mathrm{d}u}{\mathrm{d}t} = \varsigma(\theta)(1-u) - \lambda(\theta)u. \tag{23}$$

To close the model, it remains to solve for the stationary density of workers over productivity, $g(x; \theta)$. This takes a standard "double-Pareto" form illustrated in Figure 5 (and derived in the Online Appendix). Job creation "pours" workers into jobs at x_0, yielding a peak at that point. Endogenous job destruction "pours" workers out of jobs at $x_l(\theta)$ such that there is no density at this point.

Figure 5. Stationary worker density and user cost of labor

A. Stationary worker density

B. User cost of labor



Notes. Parameter values are based on the model calibrated as described in Table 1.

1.6 User cost of labor

The preceding analysis characterizes outcomes in terms of *values*—most notably the value of a prospective match to a firm, $J(w_0, x_0; \theta)$. We now explore a parallel interpretation in terms of *flows*. This is aided by an approach identified by Kudlyak (2014), who notes that the shadow flow price of labor in long-term employment relationships has a *user cost* interpretation, mirroring the analogous concept in capital theory (Jorgenson 1963).

Kudlyak's user cost of labor concept can be extended to our environment using the out-of-steady-state analogue of the Bellman equation for the firm surplus in (10). Specifically, evaluating the latter at initial productivity x_0, a straightforward extension of the approach to user cost devised by Abel and Eberly (1996) in a model of investment under uncertainty, implies that the user cost of labor for a new match takes the form

$$\begin{aligned} \omega(w_0, x_0; \theta) = w_0 + [r + \delta + s\lambda(\theta)]J(w_0, x_0; \theta) - \mu x_0 J_x(w_0, x_0; \theta) \\ - \frac{1}{2}\sigma^2 x_0^2 J_{xx}(w_0, x_0; \theta) - \frac{\partial}{\partial t} J(w_0, x_0; \theta). \end{aligned} \tag{24}$$

A useful point of contrast is provided by the analogous expression for the canonical Diamond-Mortensen-Pissarides model considered by Kudlyak, in which $s = \mu = \sigma = 0$:

$$\omega(w_0, x_0; \theta)|_{s=\mu=\sigma=0} = w_0 + (r + \delta)J(w_0, x_0; \theta)|_{s=\mu=\sigma=0} - \frac{\partial}{\partial t} J(w_0, x_0; \theta)\Big|_{s=\mu=\sigma=0}. \tag{25}$$

Comparing the latter expressions provides a useful perspective on the economic forces in the model. Absent idiosyncratic shocks that induce *ex post* adjustments of the wage—arising either from on-the-job search ($s = 0$), or variations in productivity ($\mu = \sigma = 0$)—user cost in (25) is simply composed of the flow wage w_0, and the flow-equivalent costs associated with discounting and exogenous job destruction $(r+\delta)J(w_0, x_0; \theta)$, set against the flow value of any future changes in entry wages w_0, productivity x_0, and tightness θ.

The presence of on-the-job search ($s > 0$), productivity shocks ($\mu \neq 0 \neq \sigma$), and the *ex post* wage adjustments that they induce further gives rise to two additional components of user cost in (24), with important economic implications.

First, on-the-job search generates additional costs to the firm associated with job-to-job quits, and the retention compensation for contacted workers. Among new matches of productivity x_0, the latter coincide, with a user cost contribution of $s\lambda(\theta)J(w_0, x_0; \theta)$. Although this is reminiscent of the analogous contribution of exogenous job destruction δ, a fundamental economic difference is the *endogeneity* of the contact rate $s\lambda(\theta)$, which will naturally vary, for example, with the economic cycle.

Second, drift and variance in productivity give rise to the user cost contributions $-\mu x_0 J_x(w_0, x_0; \theta) - (\sigma^2/2)x_0^2 J_{xx}(w_0, x_0; \theta)$, which capture the flow value of future changes in productivity. Notice, however, that these terms encompass a variety of economic forces: both the direct effect of changes in productivity, and the indirect effects associated with endogenous wage adjustments, and endogenous job destruction at x_l, induced by them.

To illustrate, note first from the Bellman equation for firm surplus (10) that the sum of all components of the user cost in (24) by definition must equal the productivity of a new match x_0. Figure 5 then traces out the remaining subcomponents of the steady-state user cost of labor as a function of the initial wage w_0.

The first captures the user cost contribution $w_0 + (r+\delta)J(w_0, x_0; \theta)$. Dominated by the initial flow wage, this subcomponent is increasing in w_0 and near-linear. Adding in the turnover costs, $s\lambda(\theta)J(w_0, x_0; \theta)$, then has two effects. First, it substantially raises the user cost contribution, indicating that turnover costs are nontrivial to the firm. Second, the slope of the user cost contribution in the initial wage w_0 is reduced. Intuitively, the prospect that the worker may quit, or require retention compensation, in the future blunts the salience of the initial wage in the firm's effective flow cost of labor.

What remains to satisfy the identity that user cost equals initial productivity x_0 is the effect of shocks to idiosyncratic productivity. The message of Figure 5 is that these act as a source of moderation to the user cost of labor, and one that further diminishes the importance of the initial wage w_0. Intuitively, mirroring an earlier lesson of the model—that the allocative effects of wages are tied to their expected duration—the prospect that future changes in productivity will induce future changes in wages limits the effects of the initial wage w_0 on the user cost of labor.

2. Quantitative exploration

We now provide a quantitative illustration of the model's ability to capture relevant empirical evidence on the prevalence of long-term jobs and the sporadic nature of wage adjustment, and their associated implications for the allocative effects of wages.

Table 1 summarizes an illustrative calibration of the model. We focus on a parsimonious case in which $\mu = \sigma^2/2$, so that log match productivity is driftless, and $\Delta_e = \Delta_f$, so that the credibility of unilateral threats is symmetric across employees and firms.

Given this, we begin by normalizing the initial productivity of a match, $x_0 \equiv 1$, and setting the monthly discount rate r to replicate an annual real interest rate of 5 percent. The remaining parameters of the model are then calibrated as follows.

First, we choose the annuitized value of unemployment rU, the intensity of on-the-job search s, and the rate of exogenous separations δ to replicate three moments related to the empirical durability of employment relationships. Specifically, we target a steady-state unemployment rate of 6 percent; a monthly job-to-job (E-to-E) transition rate of 2.5 percent (Fujita et al. 2021); and a limiting monthly separation rate among high-tenure jobs of 1 percent (Farber 1999, Figure 5). Intuitively, the annuitized value of unemployment rU shapes the separation rate into unemployment via the reservation productivity x_l; the intensity of on-the-job search s determines the arrival rate of outside offers and, thereby, the rate of job-to-job transitions; and the exogenous separation rate δ is the limit of the total separation rate as tenure rises.

Second, we choose the volatility of idiosyncratic productivity shocks σ and the breakdown probabilities associated with unilaterally triggered renegotiations $\Delta_e = \Delta_f$ to replicate two moments related to compensation adjustment. To do so, we must specify how recruitment and retention compensation is apportioned between base and bonus pay in the model. Note that, per (12), the maximum share of the match surplus that the firm can credibly deliver in the form of base pay is $1 - (1 - \beta)(1 - \Delta_f)$. Furthermore, were the base wage raised to satisfy this bound, subsequently it would almost surely be adjusted. We adopt a simple middle ground whereby, in the event of an outside offer, the recruitment or retention package delivers a value up to a share β of its surplus by raising the worker's base wage, and delivers any remaining value as a lump-sum bonus.[3]

[3] This rule has the virtue of simplicity. It also implies a form of equal treatment, whereby base wages in hiring firms deliver the same surplus to newly hired workers, regardless of whether they are hired from unemployment or poached from another firm. Clearly, one could generalize this simple rule to accommodate further moments of wage adjustment. In the interest of parsimony, though, we start here.

Table 1. Parameters and targeted moments (monthly frequency)

Parameter		Value	Reason / Moment	Target
A. Externally calibrated				
x_0	Initial productivity	1	Normalization	—
r	Discount rate	0.004	Annual real interest rate	0.05
B. Long-term employment relationships				
rU	Annuitized value of unemployment	1.114	Unemployment rate	0.060
s	Employed search intensity	0.586	E-to-E rate	0.025
δ	Exogenous separation rate	0.010	Limiting high-tenure sep. rate	0.010
C. Wage adjustment				
σ	Standard deviation of x	0.034	Std. dev. annual log wage change	0.053
$\Delta_{e,f}$	Breakdown probability	0.382	Incidence of compensation freezes	0.170
β	Worker bargaining power	0.200	See text	—
D. Aggregate parameterization				
α	Matching elasticity	0.5	Standard	0.5
m_0	Matching efficiency	0.25	Tightness θ	1
c_v	Flow vacancy cost	0.088	Job-finding rate λ	0.25
b	Flow unemployment payoff	0.979	Annuitized value rU	1.114

Notes. The rationale and source for each targeted moment are explained in detail in the main text.

Given this, we target estimates of the incidence and size of changes in the base wage and compensation among job stayers documented by Grigsby et al. (2021, Section 8). Recalling our initial focus on a zero-inflation environment, a convenient aspect of Grigsby et al.'s data on nominal pay changes is that they pertain to a period of very low inflation. Accordingly, we choose σ to target a standard deviation of changes in annual log base wage among job stayers of 0.053, and $\Delta_e = \Delta_f$ to replicate an incidence of compensation (base plus bonus pay) freezes of 17 percent. Intuitively, the volatility of shocks σ naturally passes through into dispersion in wage changes, while the credibility of unilateral threats, indexed by $\Delta_e = \Delta_f$, shapes the incidence of pay changes. We return to the model's implications for nominal wage adjustments in an inflationary environment in Section 3.

It remains to specify worker bargaining power β. We find that setting β equal to 0.2 gives rise to several reasonable outcomes in the calibrated model. First, it yields a wage passthrough—the change in wages induced by a unit rise in match productivity—equal to 0.37, which is in the range of recent estimates reported by Kline et al. (2019, Table VIII).

Second, the magnitude of hiring costs (net of recruitment bonuses) in the calibrated model corresponds to 1.4 months of wages. This is consistent with the early results of Oi (1962), which have been borne out in more recent work (Manning 2011; Gavazza et al. 2018). Third, the calibrated model implies that the base pay share of compensation is 94 percent at the median; Grigsby et al. (2021, Table 1) report an analogous statistic of 97 percent. Fourth, the flow payoff to unemployment b corresponds in the model to approximately half of average match productivity. This implies an average flow surplus to employment relationships that is in the range of values considered plausible in the related literature (Shimer 2005; Hall and Milgrom 2008; Mortensen and Nagypal 2007).

The final ingredients are related to aggregate labor market equilibrium. Since our goal is to draw out the novel implications of the theory, we adopt a conventional approach. The matching function is assumed to be Cobb-Douglas,

$$m(u + s(1-u), v) = m_0 \cdot [u + s(1-u)]^{\alpha} v^{1-\alpha}. \tag{26}$$

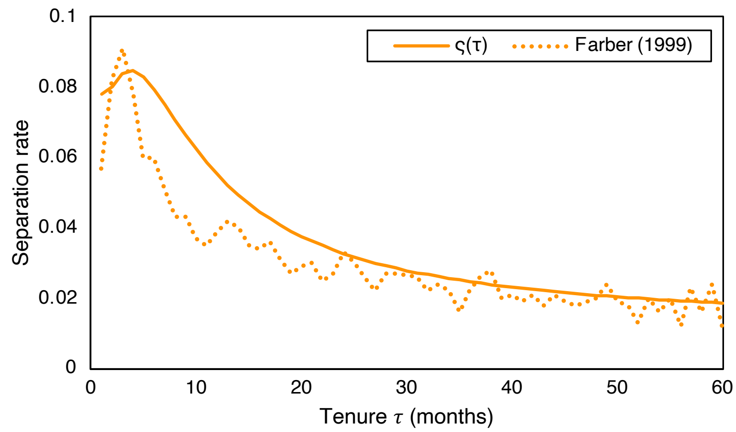
We set the matching elasticity α equal to 0.5. Matching efficiency m_0 and the flow vacancy cost c_v are then chosen to deliver a (normalized) equilibrium labor market tightness θ of 1, and a monthly job-finding rate $\lambda(\theta)$ of 0.25, consistent with the unemployment-to-employment transition rate in the Current Population Survey's gross flows data. Finally, the flow payoff from unemployment b is set to generate the equilibrium annuitized value of unemployment rU pinned down earlier.

In the subsections that follow, we explore the model's implications for the allocative effects of wages and bonus pay. Before we do so, we first confront the model with further moments of the stylized facts that informed it: the prevalence of long-term employment relationships, and the intermittent adjustment of wages, and compensation more broadly.

2.1 Durability of employment relationships

Figure 6 reiterates the model-generated pattern of separation rates by tenure from Figure 1, and juxtaposes it against analogous estimates reported by Farber (1999). In addition to replicating the (targeted) limiting separation rate, the calibrated model also generates a (nontargeted) pattern of separation rates by tenure that resembles Farber's estimates: Similar to the data, separation rates in the model are hump-shaped among low-tenure matches, with a peak of just over 0.08 at around 3 to 5 months of tenure, and declining thereafter. The model understates the hump and overstates separation rates at middling levels of tenure; but there is a clear broad similarity between the model and the data.

Figure 6. Separation rates by tenure: Model vs. data



Notes. Parameter values are based on the model calibrated as described in Table 1. Data are transcribed from the lower-left panel of Figure 5 in Farber (1999).

2.2 Wage and compensation adjustment

A natural implication of Figure 2C is that the distribution of wage and compensation changes takes a form reminiscent of their analogues in microdata (Malcomson 1997). Intermittent adjustment yields a mass point at zero change; idiosyncratic shocks and outside offers generate tails of wage cuts and increases. The calibration targets a subset of these outcomes: the dispersion of changes in the base wage, and the frequency of changes in compensation. We now assess model implications for these outcomes more generally.

Figure 7 depicts the distributions of changes in annual base wage (panel A) and compensation (i.e., base wage plus bonus pay, panel B) among job stayers implied by the model. These display several key properties that resemble their empirical analogues.

Consider first Figure 7A. In addition to the paucity of base wage cuts, the model-implied distribution features a large incidence of base wage freezes: Over 30 percent of job stayers in the model realize no change in their base pay from year to year. Furthermore, the distribution is prominently asymmetric, with many more realizations of base pay increases than cuts. Both are key features of empirical distributions of base pay changes, as documented by Altonji and Devereux (2000) using data for a large corporation, and by Grigsby et al. (2021) using microdata from ADP, a payroll processing company.

An interesting corollary is that the model is able to capture the asymmetry of changes in the base wage *despite* the presence of a symmetric friction—recall that the parameters underlying Figure 7 are such that $\Delta_e = \Delta_f$. This observation, anticipated qualitatively by Malcomson (1997), emerges in the model from three natural forces. First, and most directly, positive drift in match productivity, $\mu > 0$, generates more wage increases than wage cuts. Second, separations are concentrated in matches that otherwise would have cut the wage. A third reason for the asymmetry of changes in the base wage is related to the structure of compensation. Figure 7B depicts the analogous distribution of changes in compensation (i.e., base wage plus bonus pay) in the model. In addition to the much lower (targeted) incidence of pay freezes, this further reveals that changes in compensation are also much more symmetric than changes in the base wage. Intuitively, when a portion of pay is delivered as a bonus—in this case, for recruitment and retention purposes—two additional forces come into play. First, there is a greater mass of increases in overall compensation driven by the realization of these bonuses. Second, the flipside is that there is a greater mass of cuts in overall compensation driven by workers who have received bonuses in the past, but not in the present.

This implication of the model in turn dovetails with two dimensions of related empirical evidence. First, recent studies of annual changes in overall hourly earnings report a substantially greater incidence of cuts than the picture for base wages in Figure 7A (Kurmann and McEntarfer 2019; Jardim et al. 2019). Second, studies that differentiate components of pay find that nonbase pay is responsible for much of the flexibility in overall compensation. Shin and Solon (2007) note that a large part of the procyclicality of real wages can be traced to variation in nonbase pay. And Grigsby et al. (2021) report that the distribution of annual changes in base plus bonus pay exhibits considerably greater flexibility, and symmetry, compared to their analogous results for base wages.[4]

The remaining panels of Figure 7 depict analogous results for special cases. Outcomes are very similar in the costly renegotiation case (with on-the-job search suspended, $s = 0$) in Figure 7C. However, the sequential auctions case ($\Delta_f = 1$, $\beta = 0$) in Figure 7D is considerably different, greatly overstating the incidence of pay freezes.[5] Echoing Postel-Vinay and Turon (2010), the arrival rate of job offers $s\lambda$ consistent with the empirical job-to-job transition rate is too low to account for the empirical incidence of wage change.

By contrast, the theory of wage determination developed in the preceding sections is able to capture many of the salient features of both the durability of employment relationships, and the evolution of wages over their course. That the model can faithfully replicate these stylized facts lends credence to its use in studying the allocative effects of wages and bonus pay, topics that we now take up in the subsections that follow.

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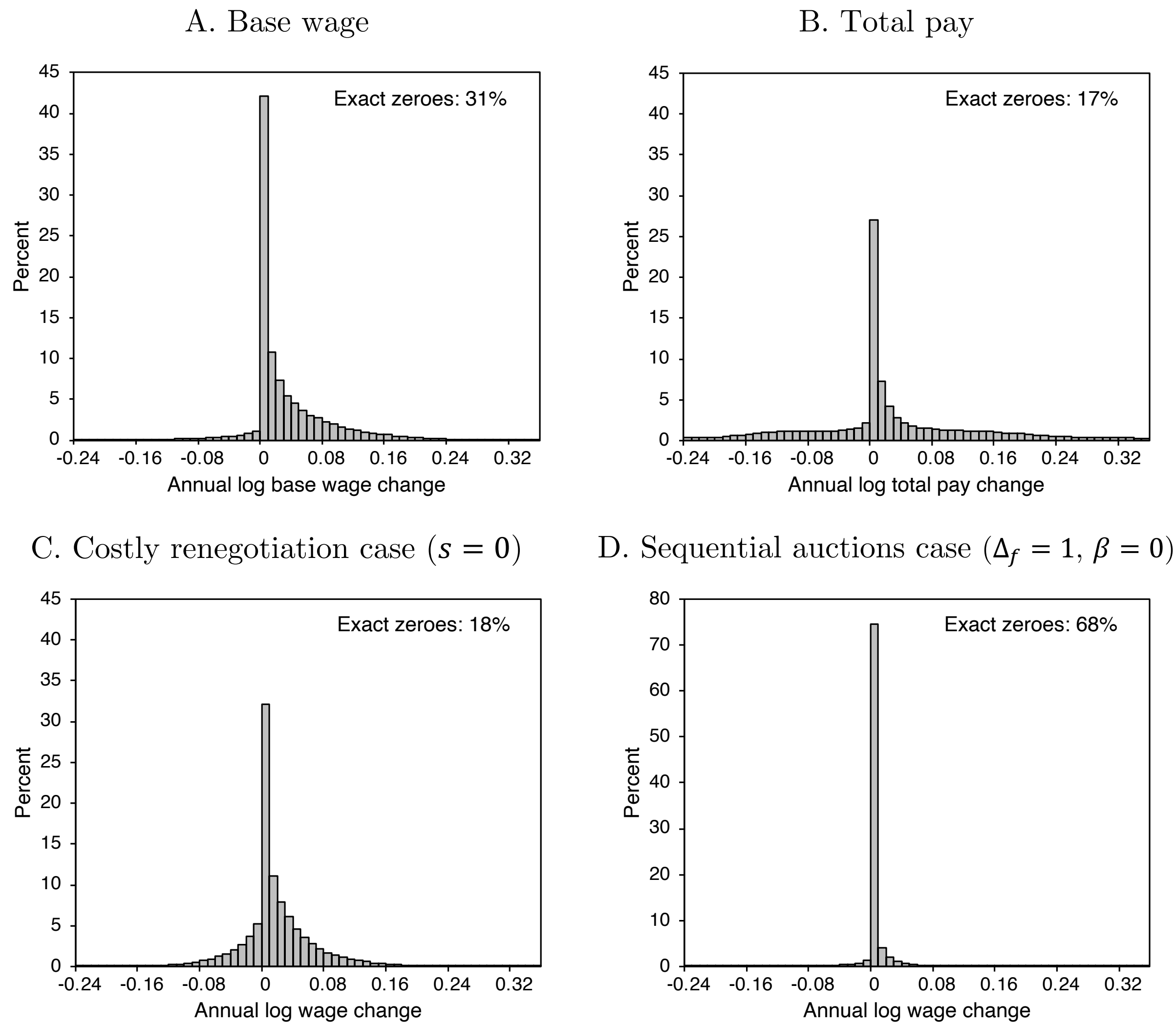
[4] Lebow et al. (2003) report similar results using job-level microdata from the Employment Cost Index.

[5] That the sequential auctions case is not recalibrated is in fact conservative in this regard: Offer matching in this case yields extreme wage changes that *overstate* the dispersion of wage changes relative to Table 1.

Figure 7. Model-implied distributions of wage changes among job stayers



Notes. Panels A and B: Parameter values for the baseline model are calibrated as described in Table 1. Panel C: Parameter values based on a recalibration that omits the target moment for s in Table 1. Panel D: Parameter values as described in Table 1, but with $\Delta_f = 1$ and $\beta = 0$.

2.3 The allocative effects of entry wages

We begin by revisiting an influential theory of the allocative effects of rigidity in entry wages developed by Hall (2005). Hall's model can be viewed as a simple special case of our environment in which there is no on-the-job search ($s = 0$), there are no idiosyncratic shocks to match productivity ($\mu = \sigma = 0$), and neither firm nor worker can credibly issue unilateral threats to renegotiate the wage ($\Delta_e = \Delta_f = 1$).

The drunken walk illustrated in Figure 2C is particularly simple in this case, as any initial wage w_0 need *never* be adjusted *ex post*. This feature of the model yields a theory of unemployment fluctuations whereby rigidity in entry wages squeezes firm surplus in a recession, retarding job creation and raising unemployment. A celebrated feature is that the theory generates unemployment fluctuations without violating bilateral efficiency.

In this subsection, we show that the presence of on-the-job search and idiosyncratic shocks to match productivity has an important bearing on the role of entry-wage rigidity in unemployment fluctuations. To do so, we modify the match product to equal px, and consider a perturbation to aggregate labor productivity p. We allow the entry wage of a new hire from unemployment w_0 to vary flexibly with this perturbation, with elasticity $\epsilon_{w_0,p}$. Thus, as in Hall (2005), the response of w_0 to p in general will deviate from that implied by Nash surplus sharing. And, as in Hall (2005), this will imply no violation of the wage adjustment bounds, given a small perturbation to p.

Given the conventional search and matching structure, a summary statistic for the implied volatility of job creation is the elasticity of labor market tightness θ with respect to p, denoted $\epsilon_{\theta,p}$. Rewrite the job creation condition (22) as $J(w_0, p; \theta) = [c_v/q(\theta)] \cdot R(p, \theta)$, where R is the scaling of the effective hiring cost due to on-the-job search. Totally differentiating allows $\epsilon_{\theta,p}$ to be decomposed into four components,

$$\epsilon_{\theta,p} = \frac{\epsilon^m_{J,p} + \epsilon^m_{J,w_0} \cdot \epsilon_{w_0,p} + \epsilon^M_{J,p} + \epsilon^M_{R,p}}{\alpha}, \tag{27}$$

where a superscript m denotes a *micro* (or partial equilibrium) elasticity, and a superscript M denotes a *macro* (or general equilibrium) elasticity.

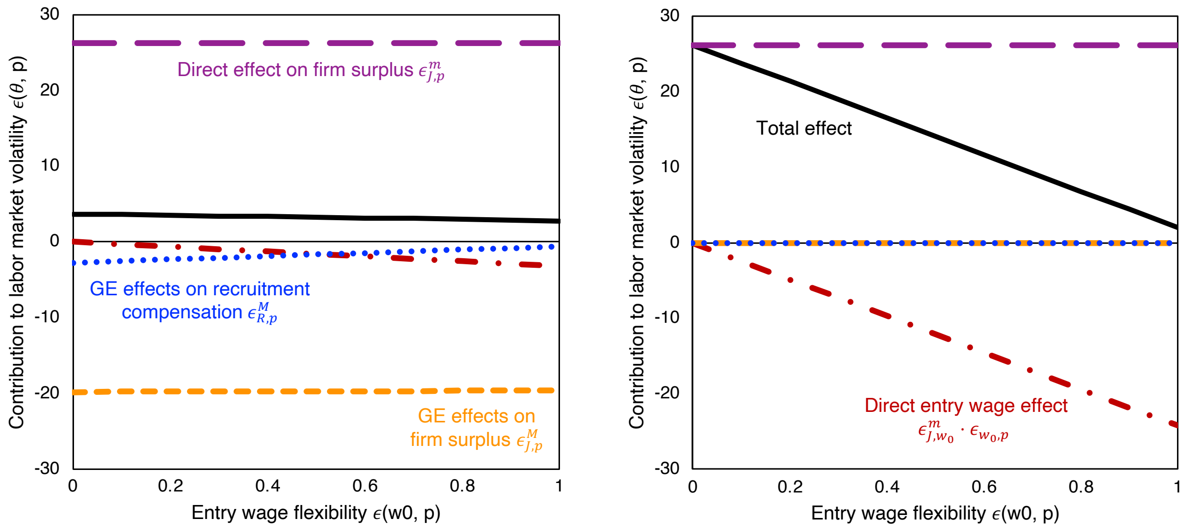
The micro elasticities capture changes in a firm's job creation incentives for a fixed labor market equilibrium. $\epsilon^m_{J,p} > 0$ summarizes the direct effect of an increase in aggregate productivity on a firm's surplus. $\epsilon^m_{J,w_0} \cdot \epsilon_{w_0,p} \leq 0$ is the partial equilibrium effect of entry wages. To the extent that the entry wage is flexible, $\epsilon_{w_0,p} \geq 0$, and the entry wage affects a firm's surplus, $\epsilon^m_{J,w_0} \leq 0$, this will moderate the rise in a firm's incentives to create jobs.

The macro elasticities capture changes in job creation incentives due to changes in labor market equilibrium. $\epsilon^M_{J,p} \leq 0$ summarizes the total effect of equilibrium changes on firm surplus (combining the effects of increases in the offer arrival rate for employed workers $s\lambda$, and rises in the annuitized value of search for unemployed workers rU). Similarly, $\epsilon^M_{R,p} \leq 0$ is the total effect of equilibrium changes on recruitment compensation. Intuitively, firms anticipate that their workers will quit—or require retention compensation—with greater frequency, and that wages will have to be raised more in the future when outside options improve. Both moderate job creation incentives.

Finally, the implications of all four effects for labor market volatility are shaped by the denominator in (27), the elasticity of the matching function, $\alpha = 0.5$.

Figure 8. The allocative effects of entry wages

A. Baseline model $(s, \sigma > 0)$ B. No *ex post* adjustments $(s = \sigma = 0)$



Notes. Panel A: Parameter values are based on the model calibrated as described in Table 1. Panel B: Idiosyncratic shocks and on-the-job search are eliminated $s = \mu = \sigma = 0$; unilateral threats to renegotiate are not credible, $\Delta_e = \Delta_f = 1$, so bargaining power β is redundant; w_0 is set to replicate the direct effect on firm surplus $\epsilon^m_{J,p}$. The latter, in turn, is similar to the empirical magnitudes emphasized by Shimer (2005).

This decomposition provides an important point of contrast with Hall (2005). There, the absence of on-the-job search and idiosyncratic productivity shocks implies that wages need never be adjusted. The legacy of any initial entry wage is thus indefinite, and the initial wage has profound effects on the valuation of a prospective match to a firm, and thereby on job creation. Formally, result (ii) of Proposition 2 implies that the marginal valuation of the wage to the firm, and thereby the allocative effects of entry wages, are *maximal* in this special case, $J_w(w, x)|_{s=\mu=\sigma=0} = -1/(r + \delta)$.

By contrast, in the presence of on-the-job search and idiosyncratic productivity shocks, the duration of the hiring wage is truncated by endogenous *ex post* wage adjustments, and the allocative implications of entry wages are moderated. Bilateral efficiency plays a key role in this logic. Intuitively, wages must change *ex post* to preserve profitable matches. In turn, adjusted wages are naturally independent of the initial wage. This imparts on wages a form of *memorylessness*: Return to the drunken walk in Figure 2C, and consider a perturbation of the initial wage. Conditional on adjustment, the subsequent sample path of wages is unaltered by the initial perturbation. Upon adjustment, the entry wage is "forgotten," and its allocative effects end.

This implication of the model dovetails well with the substantial empirical literature initiated by Beaudry and DiNardo (1991). Most of that literature finds that, once one conditions on the history of economic conditions since initiation of the employment relationship, economic conditions at the outset of the relationship have no discernible explanatory power for the current wage.

Returning to the special case studied by Hall (2005), note that the absence of both on-the-job search and idiosyncratic shocks to match productivity implies that no *ex post* wage adjustments are required by bilateral efficiency in this case. The magnitude of the allocative effects of entry wages rests on this.

Figure 8 provides a quantitative illustration of this point. It applies the decomposition of labor market volatility in (27) to the baseline model parameterized as in Table 1, and to a comparable model without on-the-job search ($s = 0$), or idiosyncratic shocks ($\mu = \sigma = 0$). To underscore the role of entry-wage flexibility, the decomposition is presented as a function of the flexibility of wages for hires from unemployment, summarized by $\epsilon_{w_0,p}$.

This highlights two implications of *ex post* wage adjustments. First, as foreshadowed above, the entry-wage effect, $\epsilon_{J,w_0}^{m} \cdot \epsilon_{w_0,p}$, is much less consequential. Second, general equilibrium effects on firm surplus, $\epsilon_{J,p}^{M}$, provide a considerable source of moderation of labor market volatility. Put simply, firms anticipate that wages will have to be raised more frequently in the future following expansionary aggregate shocks, limiting the increased incentive to create jobs.

The upshot is that *ex post* wage adjustments required by the combination of on-the-job search, idiosyncratic shocks, and bilateral efficiency dampen labor market volatility, and render it much less responsive to wages at the point of hiring.

2.4 The allocative effects of recruitment and retention bonuses

A further key implication of the model is that it provides a novel interpretation of base and nonbase pay. Specifically, the flexibility afforded by nonbase pay has important *allocative* consequences that take a novel form. Absent an ability to respond fully to outside offers via recruitment and retention bonuses, job-to-job separations will be bilaterally *inefficient*, and matches will face additional costs associated with turnover. Anticipating this *ex ante*, valuations of a match will be retarded.

We now explore these novel implications of the model. Since bonuses arise from firms responding to workers' outside offers, we explore their allocative effects using an extension of the approach to incomplete offer matching developed in Elsby and Gottfries (2022). Specifically, we consider a simple extension of the bargaining model of Cahuc et al. (2006) invoked thus far. There, upon realization of an outside offer, the two firms bid for the worker in an initial stage, and the worker then uses the less productive firm as an outside

Table 2. The allocative effects of recruitment and retention bonuses

Outcome	**Baseline**	**Without bonuses**	
		Retention	Retention and recruitment
Incidence of base pay cuts	0.069	0.149	0.135
Loss of surplus in new match	0%	14.1%	15.1%
Rise in E-to-U rate	0%	8.73%	9.50%

Notes. Parameter values are based on the model calibrated as described in Table 1.

option for bargaining with the more productive firm in a second stage. Our approach here is to introduce a probability ζ that the bid of the less productive firm remains available after a breakdown in negotiation with the more productive match. (With complementary probability $1-\zeta$, the worker becomes unemployed following such a breakdown.) This limits the role of outside offers in the determination of overall compensation and, thereby, the necessity of recruitment and retention bonuses.

Accordingly, the recruitment surplus delivered to a contacted worker currently in a match of productivity $x \leq x_0$ will be $\zeta S(x) + \beta[S(x_0) - \zeta S(x)]$. And the retention surplus delivered to a contacted worker currently in a match of productivity $x > x_0$ will be $\zeta S(x_0) + \beta(1-\Delta_e)[S(x) - \zeta S(x_0)]$. These in turn have some convenient implications. First, note that the special case in which $\zeta = \Delta_e = \Delta_f = 0$ gives rise to continual linear surplus sharing, as in models of continual *ex post* renegotiation without offer matching (see Pissarides 1994, and the microfoundation provided by Gottfries 2021). Second, coupled with the maximum value that can be credibly delivered to a worker via increases in the flow wage, $\left[1-(1-\beta)\left(1-\Delta_f\right)\right]S(x)$, one can find values of ζ that eliminate retention and recruitment bonuses, facilitating a study of their allocative consequences.[6]

Table 2 reports the implications of these two counterfactuals in the model, and compares them with the baseline model. A first message is that eliminating firms' ability to respond to outside offers with bonus pay considerably alters the incidence of cuts in base pay, which approximately doubles. Thus, the patterns of changes in base and overall compensation in the baseline model studied earlier are intrinsically related.

A second message of Table 2 is that removing firms' ability to deliver bonuses has detrimental allocative consequences. First, Table 2 reports the percentage reduction in total match surplus among new matches, $S(x_0)$, implied by the elimination of bonus pay.

[6] Specifically, setting ζ equal to $[\beta\Delta_e + (1-\beta)\Delta_f]/[1-\beta(1-\Delta_e)]$ eliminates the payment of retention bonuses. Further lowering ζ to equal Δ_f additionally eliminates the payment of recruitment bonuses.

This is substantial, around 15 percent of total surplus in the baseline model. Second, Table 2 reports the rise in the separation rate into unemployment. This too is nontrivial, rising by around 10 percent relative to the baseline model.

3. Nominal wage adjustment and inflation

Recall that the foregoing analyses can be interpreted equally as a characterization of *real* wage adjustments or of *nominal* wage adjustments in a zero-inflation environment. In this section, we address this distinction by extending the model to accommodate inflation. We suppose instead that, absent adjustment, it is the *nominal* wage that is held fixed; equivalently, the real wage w drifts downward at the rate of inflation π, $\mathrm{d}w/\mathrm{d}t = -\pi w$.

We study this case for two reasons. First, it formalizes a view of nominal wage adjustments proposed by Malcomson (1997) whereby wage contracts set in nominal terms are renegotiated only by mutual consent. Second, as noted by Malcomson, it provides a simple interpretation of the wealth of microdata-based evidence on nominal wage changes.

In contrast to the baseline model, the firm and worker now must anticipate not only future capital gains associated with changes in match productivity x, but also in the real wage w. Formally, absent adjustment, the Bellman equation for the firm's surplus now satisfies a *partial* differential equation,

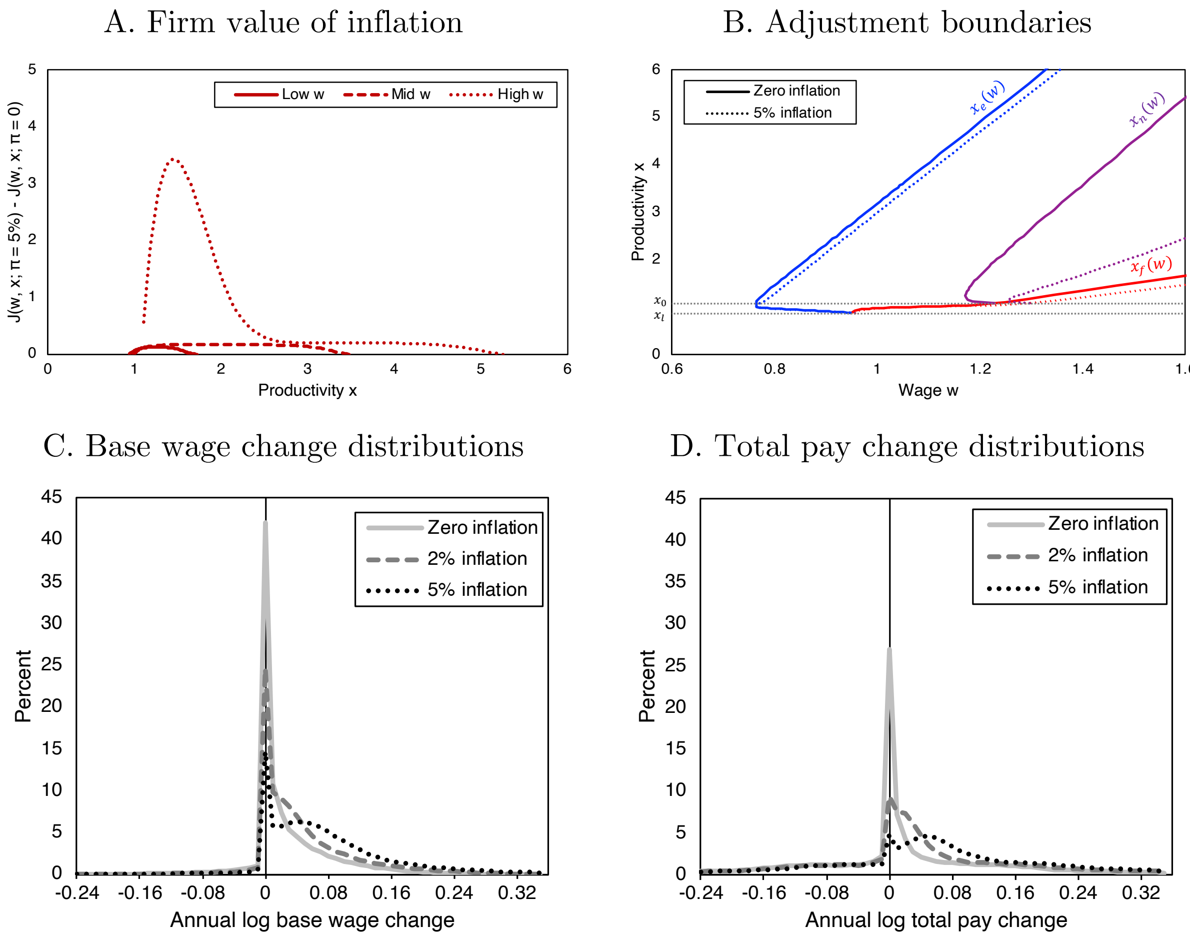
$$
\begin{aligned}
(r+\delta)J(w,x;\pi) = x - w - s\lambda\mathbf{1}_{\{x<x_0\}}J(w,x;\pi) \\
+ s\lambda\mathbf{1}_{\{x\geq x_0\}}\min\{[1-\beta(1-\Delta_e)][S(x)-S(x_0)] - J(w,x;\pi), 0\} \\
-\pi w J_w + \mu x J_x + \frac{1}{2}\sigma^2 x^2 J_{xx}
\end{aligned}
\tag{28}
$$

Relative to its analogue in the baseline model (10), downward drift in real wages is valued by the firm in proportion to the marginal value of the wage, yielding the capital gain $-\pi w J_w$. The presence of this additional capital gain complicates analytical solution.

It is possible to make progress, however, by seeking an approximate solution. Specifically, we derive a Taylor series expansion of $J(w,x;\pi)$ that decouples the partial differential equation (28) into two ordinary differential equations (a method due to Fleming 1971).[7] These in turn can be solved by an extension of our baseline approach.

[7] Thanks to Ben Moll for alerting us to Fleming's early work. The method has also recently been applied in consumption-savings problems by Kasa and Lei (2018), and Fagereng et al. (2021).

Figure 9. Nominal wage adjustment and inflation



Notes. Parameter values are based on the model calibrated as described in Table 1.

Proposition 5 *For any* w*,* i*, and* $x \in N_i(w;\pi)$*, the firm surplus has the general solution*

$$J(w,x;\pi) = J(w,x;0) + \pi \mathbb{P}_{\mathrm{N}}(w,x) + \pi j_{i1}(w) x^{-\sqrt{\frac{r+\delta}{\mu}}} + \pi j_{i2}(w) x^{\sqrt{\frac{r+\delta}{\mu}}} + O(\pi^2), \tag{29}$$

where $J(w,x;0)$ *is given in (15), and*

$$\mathbb{P}_{\mathrm{N}}(w,x) = -\frac{w}{2(r+\delta)} \int^{x} \left[\left(\frac{x}{\tilde{x}}\right)^{\sqrt{\frac{r+\delta}{\mu}}} - \left(\frac{x}{\tilde{x}}\right)^{-\sqrt{\frac{r+\delta}{\mu}}} \right] \frac{J_w(w,\tilde{x};0)}{\tilde{x}} \mathrm{d}\tilde{x}. \tag{30}$$

For any w*,* i*, and* $x \in \overline{N}_i(w;\pi)$*, the firm surplus has the general solution*

$$J(w,x;\pi) = J(w,x;0) + \pi \mathbb{P}_{\overline{\mathrm{N}}}(w,x) + \pi \bar{\jmath}_{i1}(w) x^{-\sqrt{\frac{r+\delta+s\lambda}{\mu}}} + \pi \bar{\jmath}_{i2}(w) x^{\sqrt{\frac{r+\delta+s\lambda}{\mu}}} + O(\pi^2), \tag{31}$$

where $J(w,x;0)$ *is given in (16), and*

$$\mathbb{P}_{\bar{\mathrm{N}}}(w,x) = -\frac{w}{2(r+\delta+s\lambda)}\int^{x}\left[\left(\frac{x}{\tilde{x}}\right)^{\sqrt{\frac{r+\delta+s\lambda}{\mu}}} - \left(\frac{x}{\tilde{x}}\right)^{-\sqrt{\frac{r+\delta+s\lambda}{\mu}}}\right]\frac{J_w(w,\tilde{x};0)}{\tilde{x}}\mathrm{d}\tilde{x}. \tag{32}$$

The coefficients $j_{i1}(w)$, $j_{i2}(w)$, $\jmath_{i1}(w)$, $\jmath_{i2}(w)$, *and boundaries* $x_e(w;\pi)$, $x_f(w;\pi)$, $x_n(w;\pi)$, *are implied by the boundary conditions (12), (13), and (14). The worker surplus* $V(w,x;\pi) = S(x) - J(w,x;\pi)$ *follows from (4).*

Two insights are key to the solution in Proposition 5. First, the marginal value of the wage in the absence of inflation, $J_w(w,x;0)$, is provided by the baseline solution, allowing approximation of the capital gain due to inflation, $-\pi w J_w$. Second, the value of this capital gain can be inferred analytically in the form of the particular solutions $\mathbb{P}_{\mathrm{N}}(w,x)$ and $\mathbb{P}_{\bar{\mathrm{N}}}(w,x)$ in (30) and (32), implying a solution for the value of inflation to firm and worker.

Inflation induces downward drift in real wages in the absence of renegotiation, providing an additional source of value to the firm (and loss to the worker). We illustrate this in Figure 9A. A consequence is that positive inflation lowers the adjustment boundaries. For a given productivity, firms are less likely to demand (nominal) wage cuts, and workers are more likely to demand (nominal) wage increases, as in Figure 9B.

The final panels of Figure 9 demonstrate that this simple extension of the baseline model has sensible implications for how nominal wage adjustments vary with inflation. They plot implied distributions of the growth in the nominal base wage and compensation. These exhibit a spike at zero change that decays with inflation, a familiar feature of the empirical literature on nominal wage adjustment. Again, model outcomes are very sensible.

The upshot is that a simple interpretation of the baseline model extended to an inflationary environment can accommodate further salient empirical facts emphasized by the large empirical literature on nominal wage adjustments and their relation to inflation. More generally, the solution in Proposition 5 accommodates additional interpretations. Partial indexation would manifest as a downward drift in real wages, but at a slower rate. Contracts that link wages to expected productivity growth would instead manifest as an *upward* drift in real wages. Proposition 5 can also be extended to *state-dependent* contracts that link wage drift to match productivity x. We leave these extensions for future work.

4. Summary and discussion

In this paper, we have explored an interpretation of sporadic wage adjustment in which long-term employment relationships are held to be bilaterally efficient, a view that has its origins in the influential early work of Becker (1962), and which pervades important subsequent insights from Barro (1977), Malcomson (1997), and Hall (2005).

We offer two contributions. First, we propose a model in which efficient long-term employment relationships are sustained by wage adjustments prompted by idiosyncratic shocks and outside job offers. A useful feature of the model is that it nests canonical approaches to wage determination (Mortensen and Pissarides 1994; Postel-Vinay and Robin 2002; Cahuc et al. 2006).

Our second contribution is to draw out insights into the durability of employment relationships, wage adjustments, the structure of pay, and unemployment dynamics. Echoing the qualitative insights of Malcomson (1997), our quantitative exploration of the model reveals that it is able to account for a range of empirical facts. Separation hazards are hump-shaped in tenure, mirroring the empirical results of Farber (1999). Distributions of wage changes for job stayers exhibit spikes at zero and a relative lack of wage cuts, mirroring the data (e.g., Elsby and Solon 2019).

The model naturally yields a novel theory of base and nonbase pay. The implied dynamics of the components of pay further resemble recent empirical findings (Grigsby et al. 2021). Interestingly, the model also implies that this form of nonbase pay is allocative: Absent the ability to use nonbase pay, firm value is reduced, and separations are inefficiently elevated.

Despite these successes, an important failing of the model is that it is unable to account for unemployment fluctuations. This emerges from a novel perspective on Hall's (2005) influential theory of rigid hiring wages. The presence of idiosyncratic shocks and outside job offers mutes this source of unemployment fluctuations due to a *memorylessness* property: *Ex post* adjustments required to preserve bilaterally efficient employment relationships render future wages independent of the hiring wage.

This insight suggests fruitful directions for work on the sources of unemployment fluctuations. First, one could entertain possible violations of bilateral efficiency. Second, one could pursue theories of history dependence in wages, such that the legacy of any (hiring) wage is longer than the time it takes for it to be adjusted. Finally, one could appeal to sources of unemployment fluctuations that emphasize instead volatility in labor *demand*. We hope the present paper will stimulate further research along these lines.

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Appendix

A. Proofs of main results

Proof of Proposition 1. (i) See online appendix.

(ii) Expected remaining tenure at productivity x satisfies

$$\left(\delta + s\lambda \mathbf{1}_{\{x<x_0\}}\right)\bar{\tau}(x) = 1 + \mu x \bar{\tau}'(x) + \frac{1}{2}\sigma^2 x^2 \bar{\tau}''(x). \tag{33}$$

The boundary conditions are $\bar{\tau}(x_l) = 0$, $\bar{\tau}(x_0^-) = \bar{\tau}(x_0^+)$, and $\lim_{x\to\infty} \bar{\tau}(x) = 1/\delta$.

The marginal response of the total surplus to rU satisfies

$$\left(r + \delta + \beta s\lambda \mathbf{1}_{\{x<x_0\}}\right)S_{rU}(x) = -1 + \mu x S_{rUx} + \frac{1}{2}\sigma^2 x^2 S_{rUxx}. \tag{34}$$

with boundary conditions are $S_{rU}(x_l) = 0$, $S_{rU}(x_0^-) = S_{rU}(x_0^+)$, and $\lim_{x\to\infty} S_{rU}(x) = 1/(r+\delta)$. Symmetry of (33) and (34), and their boundary conditions yields the result.

Proof of Lemma 1. (i) and (ii) See online appendix.

(iii) If $x_l = 0$, matches almost surely never reach x_l. Instead, they separate at exogenous rate δ, and additionally at rate $s\lambda$ whenever $x < x_0$. If $\mu = \sigma^2/2$, then $\ln x_t$ is a driftless Brownian motion started at $\ln x_0$. Defining $\Gamma_+(\tau) \equiv \int_0^\tau \mathbf{1}_{\{\ln x_s > \ln x_0\}} \mathrm{d}s$, Lévy's Arcsine Law for the occupation time of Brownian motion states that

$$\Pr(\Gamma_+(\tau) \le \theta) = \frac{2}{\pi} \cdot \arcsin\sqrt{\theta/\tau}. \tag{35}$$

See Karatzas and Shreve (1998, page 273). By symmetry, $\Gamma_-(\tau) \equiv \int_0^\tau \mathbf{1}_{\{\ln x_s < \ln x_0\}} \mathrm{d}s$ has the same distribution. It follows that the distribution of completed tenure spells is

$$\begin{aligned} H(\tau) &= 1 - \int_0^\tau \exp(-\delta\tau - s\lambda\theta) \frac{\partial}{\partial\theta}\left(\frac{2}{\pi}\arcsin\sqrt{\theta/\tau}\right) d\theta \\ &= 1 - \exp(-\delta\tau)\frac{1}{\pi}\int_0^\tau \frac{\exp(-s\lambda\theta)}{\sqrt{\theta(\tau-\theta)}} d\theta = 1 - \exp\left[-\left(\delta + \frac{s\lambda}{2}\right)\tau\right] I_0\left(\frac{s\lambda\tau}{2}\right), \end{aligned} \tag{36}$$

as stated. Akahori (1995) generalizes Lévy's Arcsine Law to Brownian motion with drift.

Proof of Proposition 2. The firm surplus satisfies the Bellman equation in (10), where the $S(x)$ is given by Proposition 1. Note from the boundary conditions (12), (13), and (14) that $x \in \mathrm{N}_i(w)$ implies $x \ge x_0$. Thus, for $x \in \mathrm{N}_i(w)$, the general solution to (10) is

$$J(w,x) = \frac{x}{r+\delta-\mu} - \frac{w}{r+\delta} + J_{i1}(w)x^{\gamma_1} + J_{i2}(w)x^{\gamma_2}, \tag{37}$$

where $\gamma_1 < 0$ and $\gamma_2 > 1$ are the roots of $\rho(\gamma) = 0$, and

$$\rho(\gamma) \equiv -\frac{1}{2}\sigma^2\gamma^2 - \left(\mu - \frac{1}{2}\sigma^2\right)\gamma + r + \delta = 0. \tag{38}$$

For $x \in \bar{\mathrm{N}}_i(w)$, the general solution to (10) is

$$J(w,x) = \frac{x}{r+\delta-\mu+s\lambda} - \frac{w}{r+\delta+s\lambda} + s\lambda\mathcal{P}(x) + \mathcal{J}_{i1}(w)x^{\psi_1} + \mathcal{J}_{i2}(w)x^{\psi_2} \tag{39}$$

where $\psi_1 < 0$ and $\psi_2 > 1$ are the roots of $\rho(\psi) + s\lambda = 0$, and $\mathcal{P}(x)$ is a particular solution determined by the method of variation of parameters,

$$\mathcal{P}(x) = \frac{1-\beta(1-\Delta_e)}{(\sigma^2/2)(\psi_2-\psi_1)} \int_{x_0}^{\max\{x_0,x\}} \left[\left(\frac{x}{\tilde{x}}\right)^{\psi_2} - \left(\frac{x}{\tilde{x}}\right)^{\psi_1}\right] \frac{S(\tilde{x}) - S(x_0)}{\tilde{x}} \mathrm{d}\tilde{x}. \tag{40}$$

Setting $\mu = \sigma^2/2$ implies $\psi_2 = \sqrt{(r+\delta+s\lambda)/\mu} = -\psi_1$, $\gamma_2 = \sqrt{(r+\delta)/\mu} = -\gamma_1$ and thereby the stated solution.

(ii) The expected duration until next wage adjustment satisfies

$$\left(\delta + s\lambda\mathbf{1}_{\{x\in\bar{\mathrm{N}}_i(w)\}}\right)\bar{\tau}^w(w,x) = 1 + \mu x\bar{\tau}_x^w + \frac{1}{2}\sigma^2x^2\bar{\tau}_{xx}^w. \tag{41}$$

The boundary conditions are $\bar{\tau}^w\left(w, x_f(w)\right) = 0 = \bar{\tau}^w\left(w, x_e(w)\right)$, and $\bar{\tau}^w\left(w, x_n^-(w)\right) = \bar{\tau}^w\left(w, x_n^+(w)\right)$. The marginal value of the wage to the firm satisfies

$$\left(r + \delta + s\lambda\mathbf{1}_{\{x\in\bar{\mathrm{N}}_i(w)\}}\right)J_w(x) = -1 + \mu xJ_{wx} + \frac{1}{2}\sigma^2x^2J_{wxx}, \tag{42}$$

subject to $J_w\left(w, x_f(w)\right) = 0 = J_w\left(w, x_e(w)\right)$, $J_w\left(w, x_n^-(w)\right) = J_w\left(w, x_n^+(w)\right)$. The result follows from the symmetry of (41) and (42), and their boundary conditions.

Proof of Lemma 2. See online appendix.

Proof of Proposition 3. The general solution to (10), evaluated at $s = 0$, is

$$J(w,x) = \frac{x}{\rho(1)} - \frac{w}{\rho(0)} + J_1(w)x^{\gamma_1} + J_2(w)x^{\gamma_2}, \tag{43}$$

where $\gamma_1 < 0$ and $\gamma_2 > 1$ are the roots of $\rho(\gamma) = 0$ in (38). $J_1(w)$ and $J_2(w)$, and the boundaries $x_e(w)$ and $x_f(w)$, that satisfy the boundary conditions (13) and (14) are inferred from an extension of the method of Abel and Eberly (1996) in the online appendix.

Proof of Proposition 4. We propose and verify a solution characterized by the four regions in Figure 4, with cutoffs $w_{\mathrm{I}} < w_{\mathrm{II}} < w_{\mathrm{III}} < w_{\mathrm{IV}}$. w_{IV} satisfies $x_f(w_{\mathrm{IV}}) \equiv x_n(w_{\mathrm{IV}}) = x_0$. For $w \geq w_{\mathrm{IV}}$, the general solution for the firm's surplus is as in (37). The boundary conditions are $J\left(w, x_f(w)\right) \equiv 0$ and $J_x\left(w, x_f(w)\right) = 0$. Since $\gamma_2 > 1$, it must be that $J_2(w) = 0$. $J_1(w)$, the boundary $x_f(w)$, and the wage cutoff w_{IV} are then

$$J_1(w) = -\frac{1}{\gamma_1}\frac{\left[x_f(w)\right]^{1-\gamma_1}}{\rho(1)},\ x_f(w) = -\frac{\gamma_1}{1-\gamma_1}\frac{\rho(1)}{\rho(0)}w,\ \text{and,}\ \ w_{\mathrm{IV}} = -\frac{1-\gamma_1}{\gamma_1}\frac{\rho(0)}{\rho(1)}x_0. \tag{44}$$

w_{III} satisfies $\lim_{w\to w_{\mathrm{III}}} x_n(w) = \infty$. In this limit, the prospect of any future wage adjustment becomes negligible, $\lim_{w\to w_{\mathrm{III}}} V\big(w, x_n(w)\big) = (w_{\mathrm{III}} - rU)/(r+\delta) = S(x_0)$, and

$$w_{\mathrm{III}} = rU + (r+\delta)S(x_0) = \frac{\rho(0)}{\rho(1)}x_0\left[1 - \frac{1}{\gamma_1}\left(\frac{x_0}{x_l}\right)^{\gamma_1-1}\right] < w_{\mathrm{IV}}, \tag{45}$$

where the second equality follows from the solutions for $S(x)$ and x_l in Lemma 1 (i).

w_{I} satisfies $\lim_{w\to w_{\mathrm{I}}} x_e(w) = \infty$. In this limit, the prospect of future wage adjustment again is negligible, so $\lim_{w\to w_{\mathrm{I}}} V\big(w, x_e(w)\big) = [w_{\mathrm{I}} - rU + s\lambda S(x_0)]/(r+\delta) = 0$, and $w_{\mathrm{I}} = rU - s\lambda S(x_0) < w_{\mathrm{III}}$.

Finally, w_{II} satisfies $x_f(w_{\mathrm{II}}) \equiv x_e(w_{\mathrm{II}}) = x_l$. To establish that $w_{\mathrm{II}} < w_{\mathrm{III}}$ note that, since $V(w_{\mathrm{II}}, x_l) = V_x(w_{\mathrm{II}}, x_l) = 0$, it must be that $V_{xx}(w_{\mathrm{II}}, x_l) \geq 0$, and so (11) implies $w_{\mathrm{II}} = rU - (1/2)\sigma^2 V_{xx}(w_{\mathrm{II}}, x_l) \leq rU < w_{\mathrm{III}}$. To establish that $w_{\mathrm{II}} > w_{\mathrm{I}}$, consider two matches, match A currently at (w_{II}, x_l), and match B currently at (w_{I}, x_0). Note that $V^A \equiv V(w_{\mathrm{II}}, x_l) = 0 = V^B \equiv V(w_{\mathrm{I}}, x_0)$. Fix a sequence of innovations to x, arrivals of job offers, and job destruction shocks that are the same for both matches, and denote by T the first time one of the following events is realized: (i) x falls below x_l for match A, such that it is endogenously destroyed; (ii) an outside offer arrives; and (iii) the job is exogenously destroyed. Then, one can write $V^B - V^A = \mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}(w_{\mathrm{I}} - w_{\mathrm{II}})\mathrm{d}t + \mathrm{e}^{-rT}\big(\tilde{V}_T^B - \tilde{V}_T^A\big)\right] = 0$, where, with a slight abuse of notation, $\tilde{V}_T^j$ is the terminal surplus at T of the worker currently employed in match $j \in \{A, B\}$. It must be that $\tilde{V}_T^B > 0 = \tilde{V}_T^A$: If event (i) occurs at T, $\tilde{V}_T^B \geq 0 = \tilde{V}_T^A$ since $x_0 > x_l$, and the sequence of innovations to x is the same for both matches; if event (ii) occurs at T, $\tilde{V}_T^B \geq \tilde{V}_T^A$, with strict inequality if productivity in match A is below at x_0 at T; if event (iii) occurs at T, $\tilde{V}_T^B = 0 = \tilde{V}_T^A$. It follows that $w_{\mathrm{I}} < w_{\mathrm{II}}$.

Given the ordering of the wage cutoffs, the coefficients of the firm surplus and the boundaries are solved for region by region in the online appendix.

Proof of Proposition 5. For $x \in \overline{\mathrm{N}}_i(w;\pi)$, we can rewrite (28) as

$$\begin{aligned}&(r+\delta+s\lambda)J(w,x;\pi)\\&= x - w + s\lambda\mathbf{1}_{\{x\geq x_0\}}[1-\beta(1-\Delta_e)][S(x)-S(x_0)] - \pi w J_w + \mu x J_x + \frac{1}{2}\sigma^2 x^2 J_{xx}.\end{aligned} \tag{46}$$

A Taylor series approximation to the firm surplus is $J(w,x;\pi) = J(w,x;0) + J_\pi(w,x;0)\pi + O(\pi^2)$. Recall that the general solution for $J(w,x;0)$ takes the form in (39). Differentiating the Bellman equation, the marginal value of inflation satisfies

$$(r+\delta+s\lambda)J_\pi(w,x;0)=\left[-wJ_w+\mu xJ_{\pi x}+\frac{1}{2}\sigma^2x^2J_{\pi xx}\right]_{\pi=0}. \tag{47}$$

The general solution to the latter takes the form

$$J_\pi(w,x;0)=\mathbb{P}_{\bar{\mathrm{N}}}(w,x)+\jmath_{i1}(w)x^{\psi_1}+\jmath_{i2}(w)x^{\psi_2}, \tag{48}$$

for some coefficients $\jmath_{i1}(w)$ and $\jmath_{i2}(w)$, where $\mathbb{P}_{\bar{\mathrm{N}}}(w,x)$ is a particular solution to (47). It follows that the general solution to the Bellman equation takes the form

$$J(w,x;\pi)=J(w,x;0)+\pi\mathbb{P}_{\bar{\mathrm{N}}}(w,x)+\pi\jmath_{i1}(w)x^{\psi_1}+\pi\jmath_{i2}(w)x^{\psi_2}+O(\pi^2). \tag{49}$$

Observing that $J_w(w,x;0)$ in (47) is provided by Proposition 2, the particular solution $\mathbb{P}_{\bar{\mathrm{N}}}(w,x)$ then follows from application of the method of variation of parameters,

$$\mathbb{P}_{\bar{\mathrm{N}}}(w,x)=-\frac{w}{\sigma^2/2}\frac{1}{\psi_2-\psi_1}\int^x\left[\left(\frac{x}{\tilde{x}}\right)^{\psi_2}-\left(\frac{x}{\tilde{x}}\right)^{\psi_1}\right]\frac{J_w(w,\tilde{x};0)}{\tilde{x}}\,\mathrm{d}\tilde{x}. \tag{50}$$

Now consider the case in which $x\in\mathrm{N}_i(w;\pi)$. Then we can rewrite (28) as

$$(r+\delta)J(w,x;\pi)=x-w-\pi wJ_w+\mu xJ_x+\frac{1}{2}\sigma^2x^2J_{xx}. \tag{51}$$

Recall that the general solution for $J(w,x;0)$ takes the form in (37). Applying the same steps as above, the general solution satisfies

$$J(w,x;\pi)=J(w,x;0)+\pi\mathbb{P}_{\mathrm{N}}(w,x)+\pi j_{i1}(w)x^{\gamma_1}+\pi j_{i2}(w)x^{\gamma_2}+O(\pi^2), \tag{52}$$

for some coefficients $j_{i1}(w)$ and $j_{i2}(w)$, where $\mathbb{P}_{\mathrm{N}}(w,x)$ is the particular solution

$$\mathbb{P}_{\mathrm{N}}(w,x)=-\frac{w}{\sigma^2/2}\frac{1}{\gamma_2-\gamma_1}\int^x\left[\left(\frac{x}{\tilde{x}}\right)^{\gamma_2}-\left(\frac{x}{\tilde{x}}\right)^{\gamma_1}\right]\frac{J_w(w,\tilde{x};0)}{\tilde{x}}\,\mathrm{d}\tilde{x}. \tag{53}$$

Online appendix

B. Additional proofs and derivations

Proof of Proposition 1, (i). We first establish that the surplus $S(x)$ is monotonically increasing in productivity x. Fix a separation boundary x_l at which $S(x_l) = 0$, and the surplus in new jobs $S(x_0) = \bar{S} > 0$, and conjecture that the implied $S(x)$ is monotonically increasing in x. Consider two matches with different initial productivities $x' > x$. Fix, for both matches, a given sample path for changes in idiosyncratic productivity, arrivals of outside job offers, and job destruction shocks. Denote by T the first time one of the following events occurs for match x: (i) the job is destroyed endogenously; (ii) an outside offer arrives; and (iii) the job is destroyed exogenously. Further denote by S_T the continuation value thereafter for match x, and S'_T the continuation value for match x'. Since we have fixed the sample path of shocks, and the arrivals of outside job offers, and job destruction shocks, it follows $S'_T \geq S_T$: If event (i) or (ii) is realized at T, $S'_T > S_T$; if event (iii) $S'_T = S_T = 0$. Since the match surplus is based on expectations over sample paths, $S(x') = \mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}(x'_t - rU)\mathrm{d}t + \mathrm{e}^{-rT} S'_T | x'\right] > \mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}(x_t - rU)\mathrm{d}t + \mathrm{e}^{-rT} S_T | x\right] = S(x)$. This confirms the conjecture. Continuity of the coefficients of the differential equation for $S(x)$ in (2) further implies that there exists a unique solution for given $S(x_0)$ and x_l, by the Picard-Lindelöf theorem. Therefore, for any $\bar{S}$ and x_l, there is a unique, monotonically increasing solution for $S(x)$.

The general solution to (2) is then given by

$$S(x) = \begin{cases} \dfrac{x}{r + \delta - \mu + \beta s\lambda} - \dfrac{rU - \beta s\lambda S(x_0)}{r + \delta + \beta s\lambda} + \mathcal{S}_1 x^{\tilde{\gamma}_1} + \mathcal{S}_2 x^{\tilde{\gamma}_2} & \text{if} \quad x < x_0, \\ \dfrac{x}{r + \delta - \mu} - \dfrac{rU}{r + \delta} + S_1 x^{\gamma_1} + S_2 x^{\gamma_2} & \text{if} \quad x \geq x_0, \end{cases} \tag{54}$$

where $\tilde{\gamma}_1 < \gamma_1 < 0$, and $\tilde{\gamma}_2 > \gamma_2 > 1$ are the roots of $\rho(\tilde{\gamma}) + \beta s\lambda = 0$ and $\rho(\gamma) = 0$, where

$$\rho(\gamma) \equiv -\frac{1}{2}\sigma^2\gamma^2 - \left(\mu - \frac{1}{2}\sigma^2\right)\gamma + r + \delta = 0. \tag{55}$$

The boundary conditions are given in (3). Since $\gamma_2 > 1$, the solution will explode as $x \to \infty$ unless $S_2 = 0$. The separation boundary x_l, and the remaining coefficients $\mathcal{S}_1$, $\mathcal{S}_2$, and S_1, can then be recovered from the boundary conditions. Setting $\mu = \sigma^2/2$ implies $\tilde{\gamma} = \pm\sqrt{(r + \delta + \beta s\lambda)/\mu}$ and $\gamma = \pm\sqrt{(r + \delta)/\mu}$, and thereby the stated solution.

Proof of Lemma 1, (i) and (ii). (i) If $\beta s = 0$, the general solution for the surplus is

$$S(x) = \frac{x}{r+\delta-\mu} - \frac{rU}{r+\delta} + S_1 x^{\gamma_1}. \tag{56}$$

Imposing the value-matching and smooth-pasting conditions $S(x_l) \equiv 0$ and $S'(x_l) = 0$,

$$S(x) = \frac{x}{r+\delta-\mu} - \frac{rU}{r+\delta}\left[1 - \frac{(x/x_l)^{\gamma_1}}{1-\gamma_1}\right], \qquad x_l = -\frac{\gamma_1}{1-\gamma_1}\frac{r+\delta-\mu}{r+\delta} rU. \tag{57}$$

Setting $\mu = \sigma^2/2$ implies $\gamma_1 = -\sqrt{(r+\delta)/\mu}$, and thereby the stated solution.

(ii) A standard result on first passage times implies

$$h_0(\tau) = \frac{\ln(x_0/x_l)}{\sigma\tau^{3/2}}\phi\left(\frac{\ln(x_0/x_l) + [\mu - (\sigma^2/2)]\tau}{\sigma\tau^{1/2}}\right). \tag{58}$$

See, for example, Buhai and Teulings (2014). Setting $\mu = \sigma^2/2$ simplifies the solution as stated. Equation (7) follows because exogenous separations are independent.

Proof of Lemma 2. For $\beta \in (0,1]$, and $\Delta_e, \Delta_f \in [0,1)$, the boundary conditions (12), (13), and (14), imply that the boundaries $x_e(w) \to \infty$, $x_f(w) \to \infty$, and $x_n(w) \to \infty$, as $w \to \infty$. Since $\gamma_1 < 0$, it follows from Proposition 1 that $S(x) \to [x/(r+\delta-\mu)] - [w_r/(r+\delta)]$ as $w \to \infty$. Furthermore, the definitions of $x_e(w)$ and $x_n(w)$ in (13) and (14) imply that $x_n(w) \to x_e(w)$ as $w \to \infty$. Combining these observations, it follows that

$$J(w,x) \to \frac{x}{r+\delta-\mu} - \frac{w}{r+\delta}, \text{ and, } V(w,x) \to \frac{w-rU}{r+\delta} \tag{59}$$

as $w \to \infty$. Recalling the definitions of the boundaries $x_e(w)$, $x_f(w)$, and $x_n(w)$ in (12), (13), and (14), it follows that these become affine as $w \to \infty$.

Proof of Proposition 3: Further detail. Since the arguments that follow hold for all levels of the current wage w, we treat w as parametric and, where necessary to avoid clutter, suppress notation for dependence on w.

The coefficients J_1 and J_2, and the boundaries x_e and x_f, that satisfy the boundary conditions (13) and (14) can be inferred from an extension of the solution method of Abel and Eberly (1996) as follows. Define the functions

$$\begin{aligned}\vartheta_1(\mathcal{G},a,b) &\equiv \frac{[1-(r+\delta-\mu)a]\mathcal{G}^{\gamma_2} - [1-(r+\delta-\mu)b]\mathcal{G}}{\mathcal{G}^{\gamma_2} - \mathcal{G}^{\gamma_1}},\\ \vartheta_2(\mathcal{G},a,b) &\equiv \frac{[1-(r+\delta-\mu)b]\mathcal{G} - [1-(r+\delta-\mu)a]\mathcal{G}^{\gamma_1}}{\mathcal{G}^{\gamma_2} - \mathcal{G}^{\gamma_1}}.\end{aligned} \tag{60}$$

Note that $\vartheta_1(\mathcal{G}^{-1}, a, b) = \mathcal{G}^{\gamma_1 - 1}\vartheta_1(\mathcal{G}, b, a)$, and $\vartheta_2(\mathcal{G}^{-1}, a, b) = \mathcal{G}^{\gamma_2 - 1}\vartheta_2(\mathcal{G}, b, a)$. Define the minimum surplus shares of the firm $\mathcal{B}_f \equiv (1-\beta)(1-\Delta_f)$, and worker $\mathcal{B}_e \equiv \beta(1-\Delta_e)$. Then the boundary conditions imply the following nonlinear equations in the coefficients,

$$\begin{aligned} J_1 &= -\frac{1}{\gamma_1}\frac{x_f^{1-\gamma_1}}{\rho(1)}\vartheta_1\left(\frac{x_e}{x_f}, \mathcal{B}_f S'(x_f), (1-\mathcal{B}_e)S'(x_e)\right), \text{ and,} \\ J_2 &= -\frac{1}{\gamma_2}\frac{x_f^{1-\gamma_2}}{\rho(1)}\vartheta_2\left(\frac{x_e}{x_f}, \mathcal{B}_f S'(x_f), (1-\mathcal{B}_e)S'(x_e)\right), \end{aligned} \tag{61}$$

and the boundaries

$$\begin{aligned} \frac{x_f}{\rho(1)}\Bigg[1 - \frac{1}{\gamma_1}\vartheta_1\left(\frac{x_e}{x_f}, \mathcal{B}_f S'(x_f), (1-\mathcal{B}_e)S'(x_e)\right) \\ - \frac{1}{\gamma_2}\vartheta_2\left(\frac{x_e}{x_f}, \mathcal{B}_f S'(x_f), (1-\mathcal{B}_e)S'(x_e)\right)\Bigg] = \frac{w}{\rho(0)} + \mathcal{B}_f S(x_f), \end{aligned} \tag{62}$$

and,

$$\begin{aligned} \frac{x_e}{\rho(1)}\Bigg[1 - \frac{1}{\gamma_1}\vartheta_1\left(\frac{x_f}{x_e}, (1-\mathcal{B}_e)S'(x_e), \mathcal{B}_f S'(x_f)\right) \\ - \frac{1}{\gamma_2}\vartheta_2\left(\frac{x_f}{x_e}, (1-\mathcal{B}_e)S'(x_e), \mathcal{B}_f S'(x_f)\right)\Bigg] = \frac{w}{\rho(0)} + (1-\mathcal{B}_e)S(x_e). \end{aligned} \tag{63}$$

Noting that a solution for the total surplus $S(x)$ is provided in Lemma 1, and that the preceding steps hold for any current wage w, completes the solution.

Proof of Proposition 4: Further detail. Since the arguments that follow hold for all levels of the current wage w, we treat w as parametric and, where necessary to avoid clutter, suppress notation for dependence on w.

The total match surplus $S(x)$ is as in Lemma 1, and the firm surplus satisfies the Bellman equation in (10), evaluated at $\beta = 0$. The solution involves repeated use of the quadratic $\rho(\gamma)$ in (38), and its cousin, $\varrho(\psi) \equiv \rho(\psi) + s\lambda$. The latter has roots $\psi_1 < \gamma_1 < 0$ and $\psi_2 > \gamma_2 > 1$. Note that each can be written as $\rho(\gamma) = -(\sigma^2/2)(\gamma - \gamma_1)(\gamma - \gamma_2)$, and $\varrho(\psi) = -(\sigma^2/2)(\psi - \psi_1)(\psi - \psi_2)$.

Region I: $w \in (w_{\mathrm{I}}, w_{\mathrm{II}})$. The general solution for the firm's surplus takes the form

$$J(w,x) = \frac{x}{\varrho(1)} - \frac{w}{\varrho(0)} + s\lambda\mathcal{P}(x) + \mathcal{J}_1(w)x^{\psi_1} + \mathcal{J}_2(w)x^{\psi_2}, \tag{64}$$

where, applying the method of variation of parameters,

$$\mathcal{P}(x) = \frac{1}{\sigma^2/2}\int_{x_0}^{\max\{x,x_0\}} \frac{(x/\tilde{x})^{\psi_1} - (x/\tilde{x})^{\psi_2}}{\psi_2 - \psi_1}\frac{S(\tilde{x}) - S(x_0)}{\tilde{x}}\,\mathrm{d}\tilde{x}. \tag{65}$$

The boundary conditions are

$$J\big(w, x_e(w)\big) \equiv S\big(x_e(w)\big), \text{ and, } \quad J_x\big(w, x_e(w)\big) = S'\big(x_e(w)\big). \tag{66}$$

Furthermore, since $\psi_2 > 1$, it must be that all terms in x^{ψ_2} in the general solution cancel. Using the solution for $S(x)$ in Lemma 1, the definitions of the roots, and expanding $\mathcal{P}(x)$, this implies (after some tedious algebra)

$$\mathcal{J}_2 = \frac{s\lambda}{\sigma^2/2}\frac{1}{\psi_2}\frac{1}{\psi_2 - \psi_1}\frac{x_0^{1-\psi_2}}{\rho(1)}\left[\frac{1}{\psi_2 - 1} - \frac{1}{\psi_2 - \gamma_1}\left(\frac{x_0}{x_l}\right)^{\gamma_1 - 1}\right]. \tag{67}$$

Applying the smooth-pasting condition, and observing that, under the proposed solution, $\mathcal{P}(x_e) = \mathcal{P}'(x_e) = 0$, yields the remaining coefficient,

$$\mathcal{J}_1 = -\frac{x_e^{1-\psi_1}}{\psi_1}\left[\frac{1}{\varrho(1)} + \mathcal{J}_2\psi_2 x_e^{\psi_2 - 1} - S'(x_e)\right]. \tag{68}$$

Further imposing the value-matching condition yields a nonlinear equation in the boundary $x_e(w)$,

$$\frac{x_e}{\varrho(1)}\left(1 - \frac{1}{\psi_1}\right) + \mathcal{J}_2 x_e^{\psi_2}\left(1 - \frac{\psi_2}{\psi_1}\right) - S(x_e) + \frac{1}{\psi_1}x_e S'(x_e) - \frac{w}{\varrho(0)} = 0. \tag{69}$$

Region II: $w \in (w_{\mathrm{II}}, w_{\mathrm{III}})$. The general solution for the firm's surplus takes the same form as in Region I. The boundary conditions are

$$J\left(w, x_f(w)\right) \equiv 0, \text{ and, } \quad J_x\left(w, x_f(w)\right) = 0. \tag{70}$$

As in Region I, since $\psi_2 > 1$, all terms in x^{ψ_2} in the general solution must cancel, and so the solution for $\mathcal{J}_2$ in (67) again holds. Applying the smooth-pasting condition, and noting that $x_f(w) \le x_0$ for all w in Region II implies $\mathcal{P}\big(x_f\big) = \mathcal{P}'\big(x_f\big) = 0$,

$$\mathcal{J}_1 = -\frac{x_f^{1-\psi_1}}{\psi_1}\left[\frac{1}{\varrho(1)} + \mathcal{J}_2\psi_2 x_f^{\psi_2 - 1}\right]. \tag{71}$$

Imposing the value-matching condition,

$$\frac{x_f}{\varrho(1)}\left(1 - \frac{1}{\psi_1}\right) + \mathcal{J}_2 x_f^{\psi_2}\left(1 - \frac{\psi_2}{\psi_1}\right) - \frac{w}{\varrho(0)} = 0. \tag{72}$$

Region III: $w \in (w_{\mathrm{III}}, w_{\mathrm{IV}})$. We divide Region III into two sub-regions:

Region III(a): $x \in \left(x_f(w), x_n(w)\right)$. The general solution in this case takes the same form as in Regions I and II. The value-matching conditions are

$$J\left(w, x_f(w)\right) \equiv 0, \text{ and, } \quad J\big(w, x_n(w)\big) \equiv S\big(x_n(w)\big) - S(x_0), \tag{73}$$

and the smooth-pasting conditions are

$$J_x\left(w, x_f(w)\right) = 0, \text{ and, } \quad J_x(w, x_n(w)^-) = J_x(w, x_n(w)^+) \equiv \kappa. \tag{74}$$

Mirroring the approach taken in (60), it will be useful to define the functions

$$\begin{aligned} \Theta_1(\mathcal{G}, a, b) &\equiv \frac{[1-(r+\delta+s\lambda-\mu)a]\mathcal{G}^{\psi_2} - [1-(r+\delta+s\lambda-\mu)b]\mathcal{G}}{\mathcal{G}^{\psi_2} - \mathcal{G}^{\psi_1}}, \\ \Theta_2(\mathcal{G}, a, b) &\equiv \frac{[1-(r+\delta+s\lambda-\mu)b]\mathcal{G} - [1-(r+\delta+s\lambda-\mu)a]\mathcal{G}^{\psi_1}}{\mathcal{G}^{\psi_2} - \mathcal{G}^{\psi_1}}. \end{aligned} \tag{75}$$

As before, we have $\Theta_1(\mathcal{G}^{-1}, a, b) = \mathcal{G}^{\psi_1 - 1}\Theta_1(\mathcal{G}, b, a)$, and $\Theta_2(\mathcal{G}^{-1}, a, b) = \mathcal{G}^{\psi_2 - 1}\Theta_2(\mathcal{G}, b, a)$. Observing that $x_f(w) \le x_0$ for all w in Region III(a) implies $\mathcal{P}\big(x_f(w)\big) = \mathcal{P}'\big(x_f(w)\big) = 0$, yields the coefficients

$$\begin{aligned} \mathcal{J}_1 &= -\frac{1}{\psi_1}\frac{x_f^{1-\psi_1}}{\varrho(1)}\Theta_1\left(\frac{x_n}{x_f}, 0, \kappa - s\lambda\mathcal{P}'(x_n)\right), \text{ and} \\ \mathcal{J}_2 &= -\frac{1}{\psi_2}\frac{x_f^{1-\psi_2}}{\varrho(1)}\Theta_2\left(\frac{x_n}{x_f}, 0, \kappa - s\lambda\mathcal{P}'(x_n)\right). \end{aligned} \tag{76}$$

and the boundaries

$$\frac{x_f}{\varrho(1)}\left[1 - \frac{1}{\psi_1}\Theta_1\left(\frac{x_n}{x_f}, 0, \kappa - s\lambda\mathcal{P}'(x_n)\right) - \frac{1}{\psi_2}\Theta_2\left(\frac{x_n}{x_f}, 0, \kappa - s\lambda\mathcal{P}'(x_n)\right)\right] = \frac{w}{\varrho(0)}, \tag{77}$$

and

$$\begin{aligned} \frac{x_n}{\varrho(1)}\left[1 - \frac{1}{\psi_1}\Theta_1\left(\frac{x_f}{x_n}, \kappa - s\lambda\mathcal{P}'(x_n), 0\right) - \frac{1}{\psi_2}\Theta_2\left(\frac{x_f}{x_n}, \kappa - s\lambda\mathcal{P}'(x_n), 0\right)\right] \\ = \frac{w}{\varrho(0)} + S(x_n) - S(x_0) - s\lambda\mathcal{P}(x_n). \end{aligned} \tag{78}$$

Region III(b): $x > x_n(w)$. The general solution in this case takes the form

$$J(w, x) = \frac{x}{\rho(1)} - \frac{w}{\rho(0)} + J_1(w)x^{\gamma_1} + J_2(w)x^{\gamma_2}. \tag{79}$$

The boundary conditions are

$$J\big(w, x_n(w)\big) \equiv S\big(x_n(w)\big) - S(x_0), \text{ and, } J_x(w, x_n(w)^-) = J_x(w, x_n(w)^+) \equiv \kappa. \tag{80}$$

Since $\gamma_2 > 1$, it must be that $J_2(w) = 0$. Applying the smooth-pasting condition yields the remaining coefficient,

$$J_1 = -\frac{x_n^{1-\gamma_1}}{\gamma_1}\left[\frac{1}{\rho(1)} - \kappa\right]. \tag{81}$$

Applying the value-matching condition yields the boundary,

$$\frac{x_n}{\rho(1)}\left(1 - \frac{1}{\gamma_1}\right) = \frac{w}{\rho(0)} + S(x_n) - S(x_0) - \frac{1}{\gamma_1}\kappa x_n. \tag{82}$$

The latter provides a solution for $\mathcal{J}_1$, $\mathcal{J}_2$, and J_1, and the boundaries x_f and x_n, for a given κ. It remains to pin down $\kappa(w) \equiv J_x\big(w, x_n(w)\big)$. Equating x_n across Regions III(a) and III(b) yields the following nonlinear equation x_f, x_n, and κ,

$$\begin{aligned}\frac{x_n}{\varrho(1)}\left[1 - \frac{\varrho(1)}{\rho(1)}\left(1 - \frac{1}{\gamma_1}\right) - \frac{1}{\psi_1}\Theta_1\left(\frac{x_f}{x_n}, \kappa - s\lambda\mathcal{P}'(x_n), 0\right) - \frac{1}{\psi_2}\Theta_2\left(\frac{x_f}{x_n}, \kappa - s\lambda\mathcal{P}'(x_n), 0\right)\right] \\ = \frac{w}{\varrho(0)}\left[1 - \frac{\varrho(0)}{\rho(0)}\right] - s\lambda\mathcal{P}(x_n) + \frac{1}{\gamma_1}\kappa x_n.\end{aligned} \tag{83}$$

Region IV: $w \in (w_{\mathrm{IV}}, \infty)$. The solution is given in (44) in the main appendix.

Derivation of the worker density $g(x;\theta)$. For notational simplicity, we suppress notation for dependence on tightness θ. The Fokker-Planck (Kolmogorov Forward) equation for the evolution of the worker density takes the form

$$\begin{aligned}\frac{\partial g(x)}{\partial t} = -\mu\frac{\partial}{\partial x}[xg(x)] + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}[x^2 g(x)] - \big(\delta + s\lambda\mathbf{1}_{\{x<x_0\}}\big)g(x) \\ + \left[\lambda\frac{u}{1-u} + s\lambda G(x_0)\right]\mathbf{1}_{\{x=x_0\}}.\end{aligned} \tag{84}$$

The final term captures the inflow of workers due to the creation of new matches. An inflow λu is hired from unemployment into new matches with initial productivity x_0. This inflow is scaled by $1-u$ to conform to the definition of $g(x)$ as the density of *employees* over productivity. A further inflow $s\lambda G(x_0)$ is hired from employment in matches with productivity $x < x_0$.

Noting that, in steady state, $\partial g(x)/\partial t = 0$ and $u/(1-u) = \varsigma/\lambda$, and integrating once,

$$\big(\delta + s\lambda\mathbf{1}_{\{x<x_0\}}\big)G(x) + (\mu - \sigma^2)xg(x) = \frac{1}{2}\sigma^2 x^2 g'(x) + \varsigma\mathbf{1}_{\{x\geq x_0\}} + C_1, \tag{85}$$

where C_1 is a constant of integration. The general solution is

$$G(x) = \begin{cases} G_1^- x^{\tilde{\xi}_1} + G_2^- x^{\tilde{\xi}_2} + G_0^- & \text{if} \quad x < x_0, \\ G_1^+ x^{\xi_1} + G_2^+ x^{\xi_2} + G_0^+ & \text{if} \quad x \geq x_0, \end{cases} \tag{86}$$

where C_2 is a further constant, and $\tilde{\xi}_1 < \xi_1 < 0$, $\tilde{\xi}_2 > \xi_2 > 1$ are the roots of

$$-\frac{1}{2}\sigma^2\tilde{\xi}^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\tilde{\xi} + \delta + s\lambda = 0, \text{ and, } -\frac{1}{2}\sigma^2\xi^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\xi + \delta = 0. \tag{87}$$

Since $\xi_2 > 1$, the solution will explode as $x \to \infty$ unless $G_2^+ = 0$. Furthermore, since x_l is an absorbing barrier, it follows that $g(x_l) = 0$. The remaining boundary conditions are $G(x_l) = 0$, $g(x_0^-) = g(x_0^+)$, $G(x_0^-) = G(x_0^+)$, and $\lim_{x\to\infty} G(x) = 1$. Imposing these, solving for the remaining coefficients, and noting that $\tilde{\xi}_1\tilde{\xi}_2 = -(\delta + s\lambda)/(\sigma^2/2)$ and $\xi_1\xi_2 = -\delta/(\sigma^2/2)$, yields the following solution for the worker distribution

$$G(x) = \begin{cases} g_0\left[\frac{1}{\tilde{\xi}_2}\left(\frac{x}{x_l}\right)^{\tilde{\xi}_2} - \frac{1}{\tilde{\xi}_1}\left(\frac{x}{x_l}\right)^{\tilde{\xi}_1}\right] + G_0^- & \text{if} \quad x < x_0, \\ g_0\left[\left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_2} - \left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_1}\right]\frac{1}{\xi_1}\left(\frac{x}{x_0}\right)^{\xi_1} + 1 & \text{if} \quad x \geq x_0, \end{cases} \tag{88}$$

where

$$\begin{aligned} g_0 &= \left[\left(\frac{1}{\tilde{\xi}_2} - \frac{1}{\xi_1}\right)\left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_2} - \left(\frac{1}{\tilde{\xi}_1} - \frac{1}{\xi_1}\right)\left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_1} - \frac{(\sigma^2/2)(\tilde{\xi}_2 - \tilde{\xi}_1)}{(\delta + s\lambda)}\right]^{-1}, \\ G_0^- &= -\left\{\frac{(\delta + s\lambda)}{(\sigma^2/2)(\tilde{\xi}_2 - \tilde{\xi}_1)}\left[\left(\frac{1}{\tilde{\xi}_2} - \frac{1}{\xi_1}\right)\left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_2} - \left(\frac{1}{\tilde{\xi}_1} - \frac{1}{\xi_1}\right)\left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_1}\right] - 1\right\}^{-1}. \end{aligned} \tag{89}$$

Differentiating yields the following solution for the density $g(x)$.

$$g(x) = \begin{cases} g_0 \cdot \frac{1}{x}\left[\left(\frac{x}{x_l}\right)^{\tilde{\xi}_2} - \left(\frac{x}{x_l}\right)^{\tilde{\xi}_1}\right] & \text{if} \quad x < x_0, \\ g_0 \cdot \frac{1}{x}\left[\left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_2} - \left(\frac{x_0}{x_l}\right)^{\tilde{\xi}_1}\right]\left(\frac{x}{x_0}\right)^{\xi_1} & \text{if} \quad x \geq x_0, \end{cases} \tag{90}$$

The solution for the separation rate into unemployment is then given by $\varsigma = \delta + (\sigma^2/2)x_l^2 g'(x_l)$—see, for example, Moscarini (2005, Section 6.5).