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Random Walk Forecasts of Stationary Processes Have Low Bias

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Abstract

We study the use of a zero mean first difference model to forecast the level of a scalar time series that is stationary in levels. Let bias be the average value of a series of forecast errors. Then the bias of forecasts from a misspecified ARMA model for the first difference of the series will tend to be smaller in magnitude than the bias of forecasts from a correctly specified model for the level of the series. Formally, let P be the number of forecasts. Then the bias from the first difference model has expectation zero and a variance that is $O(1/P^2)$, while the variance of the bias from the levels model is generally $O(1/P)$. With a driftless random walk as our first difference model, we confirm this theoretical result with simulations and empirical work: random walk bias is generally one-tenth to one-half that of an appropriately specified model fit to levels.

Keywords: ARMA models, overdifferenced, prediction, macroeconomic time series, simulation.
JEL Codes: C22, C53, E37, E47

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1 Introduction

One common empirical measure of forecast performance is bias, constructed as the average value of a time series of forecast errors. The measure is featured prominently in analyses of survey forecasts (e.g., Croushore (2010)). While subsidiary to root mean square prediction error, it also appears prominently in some comparisons of forecasts of econometric models. A recent example is Bennett and Owyang (2022).

In this paper, we document a surprising feature of models to forecast the level of a scalar time series that is stationary in levels: forecasts relying on a misspecified ARMA model for the difference of the series tend to have lower bias than forecasts relying on a model for the level of the series. That the model for the difference of the series is "misspecified" requires that it not be just an overdifferenced version of an invertible ARMA model for the level of the series. Lower bias for the misspecified differenced model holds even if the model for the levels is correctly specified.

We establish this result theoretically, confirm it with simulations, and illustrate it with forecasts of some aggregate US series.

This paper is motivated by Lunsford and West (2023). In that research, we used a set of models, including a driftless random walk, to forecast each of a collection of macroeconomic time series. The time series included some that show low persistence. We found that modeling the low-persistence series via a random walk resulted in forecasts with low bias – a finding we did not anticipate but wanted to understand. Hence the present paper. To be clear, we are not necessarily arguing for the use of differenced models to forecast series that are stationary in levels. Rather, we are documenting an interesting effect of doing so.

A simple example will illustrate why our result holds. Suppose that one is making one-step-ahead forecasts for a stationary univariate time series x_t. The data run up to $t = P + 1$. One predicts first x_2, then x_3, ... and finally x_{P+1}–P predictions in all. For a driftless random walk model, the prediction of x_2 is x_1, the prediction of x_3 is x_2, ..., the prediction of x_{P+1} is x_P . The time series of forecast errors is thus $x_2 - x_1$, $x_3 - x_2$, ... $x_{P+1} - x_P$. The average value of the forecast error (i.e., bias)[1] is

$$b = \frac{(x_2 - x_1) + (x_3 - x_2) + ... + (x_{P+1} - x_P)}{P}$$

$$= \frac{x_{P+1} - x_1}{P}.$$

Let "var" denote "variance" and "cov" denote "covariance." Clearly

$$Eb = 0, \text{var}(b) = \frac{2\text{var}(x_t) - 2\text{cov}(x_t, x_{t+P})}{P^2} \leq \frac{c}{P^2}, c = 4\text{var}(x_t).$$

That is, bias b has expectation zero, and variance that is $O(1/P^2)$.

In Section 2, we generalize this theoretical result to: allow $h>1$, to consider asymptotics both with h fixed (as was implicit in the above analysis) and with h growing with P at a suitable rate, and to allow forecasts not just from a random walk but from an ARMA model in the first difference of x_t. The result just given continues to hold, subject to a qualification spelled out below when h grows with P. Our theory is corroborated by simulation results reported in Section 3 and empirical results reported in Section 4. Section 5 concludes. The appendix has the proof of our result. An online appendix has some simulation and empirical results omitted from the paper to save space.

[1]Here and throughout, we use bias to refer to a sample average.

2 Theoretical results

We are interested in forecasting the average value of a scalar stationary time series x_t over the next h periods. For a forecast $E_t(.)$, the period $t+h$ forecast error is thus

$$\frac{x_{t+1}+x_{t+2}+...+x_{t+h}}{h} - E_t\frac{x_{t+1}+x_{t+2}+...+x_{t+h}}{h}. \tag{2.1}$$

(Our results continue to hold if one is forecasting the point in time value x_{t+h} rather than the average value.) Let us make the innocuous assumption that $E_t x_t = x_t$. Then with a little algebra, the forecast error (2.1) can be written

$$\frac{h\Delta x_{t+1}+(h-1)\Delta x_{t+2}+...+\Delta x_{t+h}}{h} - E_t\frac{h\Delta x_{t+1}+(h-1)\Delta x_{t+2}+...+\Delta x_{t+h}}{h} \tag{2.2}$$

We assume that a forecast from an invertible ARMA model for Δx_t is substituted for E_t in (2.2). Specifically, the forecast satisfies

$$\text{period } t \text{ forecast of } \Delta x_{t+j} = \psi_j\Delta x_t + \psi_{j+1}\Delta x_{t-1} + \psi_{j+2}\Delta x_{t-2} + ... + \psi_{t+j-1}\Delta x_1, \tag{2.3}$$

$$|\psi_j| \le c\rho^j \text{ for some } c>0 \text{ and } 0\le\rho<1.$$

Equation (2.3) is consistent with use of an ARMA model in Δx_t that assumes that Δx_t has mean zero with ARMA parameters that satisfy the usual stationarity and invertibility conditions. The invertibility condition implies misspecification: $\rho<1$ rules out a unit moving average root. Because x_t is stationary (per our assumptions below), a correctly specified process for Δx_t will have a unit moving average root. In other words, (2.3) is not just an overdifferenced version of an invertible ARMA model for the level of x_t

The random walk model used in the previous section fits into (2.3) with $\psi_i = 0$ for all i. The ψ_j's will be nonzero for all j if one uses an ARMA model for Δx_t with a moving average component. For a final example, suppose forecasts are made assuming the AR(1) model $\Delta x_t = \varphi\Delta x_{t-1}+$ unforecastable disturbance, with $|\varphi|<1$. (Since Δx_t is overdifferenced, it is wrong to assume that the "unforecastable disturbance" really is unforecastable.) Then in (2.3) we have $\psi_j = \varphi^j$; $\psi_{j+1} = \psi_{j+2} = ... = \psi_{t+j-1} = 0$; $c=1$ and $\rho = |\varphi|$. In practice φ would presumably be an estimate of $\text{cov}(\Delta x_t, \Delta x_{t-1})/\text{var}(\Delta x_t)$.

The investigator uses the formula (2.3) to compute the $E_t(.)$ term in (2.2). Bias is

$$b = \frac{\sum_{t=1}^{P}[\frac{h\Delta x_{t+1}+(h-1)\Delta x_{t+2}+...+\Delta x_{t+h}}{h} - \text{ period } t \text{ forecast of } \frac{h\Delta x_{t+1}+(h-1)\Delta x_{t+2}+...+\Delta x_{t+h}}{h}]}{P} \tag{2.4}$$

Our theoretical result assumes:

$$x_t \text{ is covariance stationary} \tag{2.5}$$

and either

$$(\text{a1})\sum_{j=0}^{\infty}|\text{cov}(x_t, x_{t-j})|<\infty \text{ and } (\text{a2})h/P = \text{O}(1) \tag{2.6}$$

or

$$(b)h \text{ is fixed as } P\to\infty. \tag{2.7}$$

In (2.6), absolute summability of x_t's autocovariances is weaker than (though consistent with) x_t following an ARMA process. Equation (2.6) allows the horizon h to grow with the sample size P. This is consistent with the environment of Richardson and Stock (1989) or Müller and Watson (2016), who make an equivalent assumption. Under (2.7), we do not require absolute summability of those autocovariances of x_t. This allows, for example, stationary fractionally integrated processes.

Our basic theoretical result is:

Theorem 2.1 Define b as in (2.4). Under (2.3), (2.5) and (2.6), $Eb = 0$ and $\text{var}(b) = \text{O}(h/P^2)$. Under (2.3), (2.5) and (2.7), $Eb = 0$ and $\text{var}(b) = \text{O}(1/P^2)$.

The proof is in the Appendix. Comments on Theorem 2.1:

1. Theorem 2.1 continues to hold if one is forecasting point-in-time x_{t+h} rather than the average over the next h periods.

2. Suppose that $h = cP$ for some $c > 0$, as in Müller and Watson (2016) . Then $h/P^2 = c/P$ and $\text{var}(b) = \text{O}(1/P)$, as is the case for the variance of bias from a stationary model. According to this metric, then, bias from the differenced model is no less variable than is bias from a stationary model applied to the level. However, if h/P is small, one can still expect less noise from a unit root forecast. Indeed, our simulations and empirical work indicate that a random walk forecast tends to have lower bias than stationary processes even when h/P is as large as 0.6, although, consistent with our theory, the advantages of a random walk forecast diminish when h/P is large.

3. We have been assuming a zero mean model in first differences. One may wonder about the properties of a forecast based on a zero mean model in higher-order integer differences. Let us consider forecasts from the (misspecified) I(2) process $\Delta^2 x_t = u_t \tilde{}$ iid. Then when h is fixed, as in (2.7), we again obtain $\text{var}(b) = \text{O}(1/P^2)$. On the other hand, for this I(2) process, if h grows with P, $\text{var}(b)$ does not go to zero rapidly, and in fact diverges when $h = cP$ as in our previous point. This suggests that superior performance will require relatively small values of h/P. Hence our focus on a first difference model.

4. One can ask, which zero mean first difference ARMA process yields the smallest $\text{var}(b)$? The answer depends on the particulars of the x_t process. In our simulations and empirical work, we simply use a driftless random walk.

3 Simulation results

To study the small sample properties, we use simulations when the DGP follows an AR(2) process. We have also completed simulations replacing the AR(2) process with an iid process and with a stationary fractionally integrated $I(d)$ process. To save space, these are reported in our online appendix; they deliver results similar to those for the AR(2), except as noted below.

In our AR(2) DGP, $x_t = 0.5x_{t-1} - 0.1x_{t-2} + u_t$,with $u_t \sim$ iid $N(0, 1)$. The first-order autocorrelation of x_t is about 0.45, which is representative of the first-order autocorrelations of the data studied in the next section. The modulus of both autoregressive roots is about 0.32.

We structure the simulations to parallel the quarterly empirical results in the next section, and the two sections together supply complementary details on our setup. The total sample size is $T + 1 = 156$ (39 years of quarterly data). In each simulation sample, we draw initial values from the unconditional distribution of x_t, i.e., from a bivariate $N(0, V)$ distribution in which the diagonals of V are the variance of x_t and the off-diagonals are the first-order autocovariance of x_t.

We use a sample size of $R = 48$ to compute the initial forecasts. That leaves 108=156-48 observations for forecasting and for extending the sample used to make forecasts. We compute forecasts for the average value of x over horizons h = 1, 4, 8, 12, 20, and 40. For a given forecast horizon, there are $P = 108 - (h\text{-}1)$ forecasts available for computing the bias. The end dates of the samples used to construct forecasts are $\tau = R, ..., T + 1 - h$. The forecasts use both recursive and rolling estimation samples. (See West (2006), for example, for definitions and illustrations of these two schemes.)

In a sample that ends at date τ, the forecast from the driftless random walk ("RW") model is x_τ for both the recursive and rolling samples, and for all horizons. For the AR(2) model, ordinary least squares is used to estimate parameters using $\{x_1, ..., x_\tau\}$ (recursive sample) or $\{x_{\tau-R+1}, ..., x_\tau\}$ (rolling sample). The estimated parameters are then used to compute forecasts using the chain rule of forecasting. We also report results from the infeasible AR(2) forecast that relies on population parameters.

We run 1000 simulations. In each simulation, we compute forecast bias and root mean squared prediction error (RMSPE) for both models. We then compute (1) the absolute value of relative forecast biases $|b_j^{RW}/b_j^{AR}|$, where the subscript j denotes simulation j, and (2) relative RMSPEs, $RMSPE_j^{RW}/RMSPE_j^{AR}$. We report the median across our 1000 simulations of relative bias and relative RMSPE, as well as the percentage of simulations in which each ratio is less than 1. To keep the presentation smooth, we will generally omit "absolute value" when we reference relative bias: "smaller bias" should be understood to mean "smaller bias in absolute value," for example.

We note that our paper makes no new predictions about relative RMSPEs. We report these to reassure the reader that our setup delivers the expected result that in a stationary environment, the RMSPE of a forecast from a well specified stationary model is less than that of a RW forecast.[2]

Table 3.1 summarizes results. Each column reports results for a given horizon. For convenience, lines (5) and (6) give the number of predictions P and the ratio h/P.

Line (2) reports the variance of bias from the RW model, normalized by P^2/h. Our theory suggests that this normalization should generate a number that is approximately constant across horizons, at least for large P. For our values of P, this does seem to be the case.

Lines (3a)-(3c) support our central theoretical prediction. The RW model delivered smaller bias in 73 percent - 95 percent of the 1000 samples, with the percentage tending to be higher for shorter horizons. Relatively good performance for shorter horizons is consistent with one interpretation of our asymptotic results (see the comments under Theorem 2.1). Note that smaller bias results even when the RW forecast is compared to the infeasible forecast that relies on population parameters (line (3c)). The value of 0.2 for h=1, population (line 3c), for example, means that in 500 of our 1000 simulation samples, RW bias was at most two-tenths of AR(2) bias.

Lines (4a)-(4c) present results for relative RMSPE. Since the DGP's autoregressive roots are comfortably below one, we expect the AR(2) model to have lower RMSPE. And that is indeed the case. The "0%" figures in the table are exact and not just rounded down to zero: in none of the 1000 samples was the RMSPE lower for the RW than for the AR(2) model.

As noted above, we repeated these calculations with a DGP that was iid ($x_t = u_t$) and one that was fractionally integrated ($(1-L)^d x_t = u_t$) with d=0.31. The value d=0.31 was chosen because it implies a first-order autocorrelation for x_t of 0.45. For the $I(d)$ DGP, to our eyes there seemed to be a tendency for var$(b^{RW}) \times P^2/h$ to grow, rather than stay approximately constant, as the horizon h increased. We experimented with a sample size of 400 (instead of 156), and the tendency to grow with h persisted. In this respect, our theory works less well for I(d) data than for I(0) data. Apart from this discrepancy from Table 3.1, results for both the iid and $I(d)$ DGPs were very similar to those just discussed. In particular, RW bias was typically a small fraction of the bias from the stationary model (either iid or I(d)), with superior performance of the RW model most notable for shorter horizons.

We conclude that the simulations are consistent with the theory proposed in the previous section.

[2]Of course, if the DGP were a stationary model with a near unit autoregressive root, the RW model might tend to have lower RMSPE than an estimated stationary model.

4 Empirical results

This section presents some results from pseudo-out-of-sample forecasting. We compare the bias of a driftless random walk and either an AR(2) process or an AR process with lag length chosen by BIC. We report results for AR(2) here, with the very similar results for BIC-chosen lag length reported in our online appendix.

We use four quarterly US series. The series were chosen to encompass a range of persistence while focusing on series in which estimates of first-order autocorrelation coefficients are consistent with stationarity. We wished to encompass a range of persistence because the closer the first-order autocorrelation is to 1 perhaps the less surprising it might be to have a RW forecast yield low bias. With growth rates computed with log differences, the four series are:

- per capita real GDP growth (US BEA (2023); estimated first-order autocorrelation coefficient $\rho_1 = 0.39$);
- CPI-U inflation (US BLS (2023a); $\rho_1 = 0.33$). Quarterly price levels computed from averages of monthly data;
- labor productivity growth for nonfarm business (US BLS (2023b); $\rho_1 = 0.08$); and
- M3 growth (OECD (2023); $\rho_1 = 0.58$). Quarterly level set to last month of quarter.

As in the simulations, the horizons are h=1, 4, 8, 12, 20 and 40 quarters. Our baseline sample ran 1984:1-2022:4, with data prior to 1996 used only for estimates of the parameters of the AR models. The base for the first prediction thus is 1995:4. The realization for the last prediction was 108 quarters later in 2022:4. For h=4, for example, the first value predicted was the average value over the four quarters 1996:1-1996:4; the second value predicted the average value for the four quarters 1996:2-1997:1, and so on, ending with a final prediction for the average over the four quarters 2022:1-2022:4. This allowed 105 h=4 quarter ahead predictions in all. More generally, for each horizon h the number of predictions P satisfies P=108-(h-1). The various values of h, P and the implied values of h/P are given in lines (5) and (6) of Table 3.1.

An alternative sample used no data past 2019:4, to make sure that the steep fall and rapid recovery of the COVID recession was not skewing results. When we stop the comparison in 2019:4 instead of 2022:4, P of course falls by 12 for each horizon.

In estimation of the AR(2) model, we used both rolling and recursive sampling schemes as we moved the estimation end point forward. We checked for stationarity of the estimates and substituted a driftless RW forecast if, for a given sample, the estimates did not satisfy stationarity conditions. (In practice, this substitution was never necessary.)

Table 4.1 gives detailed results for GDP growth, with our online appendix presenting details for the other three variables. Panel A gives results when the sample ends in 2022:4, panel B when the sample ends in 2019:4. Line 2 gives the bias in the RW forecast (b^{RW}) , over the various horizons. The magnitude of the values might be gauged by a comparison to the mean value of GDP growth, which is 1.7 percent. In both panels of Table 4.1, and for all horizons but h=40, the absolute value of b^{RW} is a small fraction–less than 1/8–of the mean of 1.7 percent. In our experience in forecasting, that is a good result.

Rows (3a) and (3b) are the test of whether our paper's theoretical results are reflected in the data. In these rows, a value less than 1 means that bias was smaller in the RW forecast. Upon examining the 24 values in these two rows, we see that 22, or 92 percent, of the figures are below 1. The superior performance of the RW model is not because it is better according to a mean squared error criterion. Of the 24 values in lines (4a) and (4b), only 1, or 4 percent, is less than 1.

Table 4.2 presents summary results for all four variables. Columns (2)-(4) present in-sample summary statistics, computed 1984:1-2019:4. Columns (5)-(9) present forecasting results, with columns (6) and (7) the ones that are central to our paper. In columns (6) and (7), medians are computed across 24 comparisons and percentages are computed from a fraction of 24 comparisons. The 24 comparisons are those for 6 horizons × 2 sampling schemes (rolling and recursive) × 2 sample endpoints (2022:4 and 2019:4). For example, the "92%" in row (1), column (6) of Table 4.2 reflects the fact that b^{RW} was lower than b^{AR} in 22 of 24, or 92 percent, of the entries in rows (3a) and (3b) of Table 4.1.

Column (5) in Table 4.2 indicates that the (absolute) value of bias generally was usually very small–by a factor of 20 or more–than the mean given in column (1). Columns (6) and (7) present our central results. For three of the four series, b^{RW} was almost always smaller than b^{AR}, and typically by a substantial magnitude: in column (7), the median value (across 24 comparisons per series) of $|b^{RW}/b^{AR}|$ was 0.4, 0.1 and 0.2 for per capita GDP growth, CPI inflation and M3 growth. Labor productivity growth was an exception: the ratio was less than one in just slightly over half of the 24 comparisons (line (3), column (6)). We see in columns (8) and (9) that by a mean squared error criterion, the RW model fares poorly compared to the AR model, for all four data series.

Lines (5) to (7) in Table 4.2 give some statistics aggregated over all four series. Line (5) states that in 84 percent of the 96 (=24×4) comparisons, b^{RW} was smaller than b^{AR}, with the median value of $|b^{RW}/b^{AR}|$ being 0.3. Lines (6) and (7) in columns (6) and (7) indicate that there is a modest tendency for the benefits of RW to decline with horizon. This is consistent with asymptotics in which h increases along with P (see the discussion in Section 2).

On comparing these results to those in the simulations, one difference is that the excess RMSPE of the RW model (column (9)) is smaller here than in the simulations. There are many possible reasons for this, one of which is that the AR(2) model actually generated the simulation data. But even though the AR(2) model is only an approximating model for the actual data, the reduction in bias (columns (6) and (7)) is comparable to what is reported for the simulations in Table 3.1.

In sum, the empirical results are strongly supportive of our theoretical and simulation results. In forecasting a stationary process, bias will generally be more concentrated around 0 for a forecast from a RW than from a stationary model.

5 Conclusion

Using asymptotic theory, simulations, and empirical estimates, we have shown that if the data are stationary, a zero mean ARMA model in the first difference of the data will tend to have forecast bias more concentrated around zero than will a stationary model. This applies even for the infeasible forecast that relies on population parameters from the stationary process that generated the data. Our empirical work indicates that a substantial reduction in bias can occur in practice; for some aggregate US data, bias from a random walk forecast was generally one-tenth to one-half of that of a plausibly specified stationary model.

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7 Appendix

In this Appendix, we prove Theorem 2.1. Throughout, c is a generic constant that varies from statement to statement. Define

$$\alpha_t \equiv h\psi_t + (h-1)\psi_{t+1} + ... + \psi_{t+h-1}. \tag{7.1}$$

From (2.3), we have

$$\text{period } t \text{ forecast of } [h\Delta x_{t+1} + (h-1)\Delta x_{t+2} + ... + \Delta x_{t+h}] = \alpha_1 \Delta x_t + \alpha_2 \Delta x_{t-1} + ... + \alpha_t \Delta x_1. \tag{7.2}$$

Using (7.2) in (2.4) , we have

$$\begin{aligned} b = (\frac{1}{Ph}) \times \{ & [h\Delta x_2 + (h-1)\Delta x_3 + ... + \Delta x_{h+1}] - [\alpha_1 \Delta x_1] \\ & + [h\Delta x_3 + (h-1)\Delta x_4 + ... + \Delta x_{h+2}] - [\alpha_1 \Delta x_2 + \alpha_2 \Delta x_1] \\ & + [h\Delta x_4 + (h-1)\Delta x_5 + ... + \Delta x_{h+3}] - [\alpha_1 \Delta x_3 + \alpha_2 \Delta x_2 + \alpha_3 \Delta x_1] \\ & + ... \\ & + [h\Delta x_{P+1} + (h-1)\Delta x_P + ... + \Delta x_{P+h}] - [\alpha_1 \Delta x_P + \alpha_2 \Delta x_{P-1} + ... + \alpha_P \Delta x_1] \}. \end{aligned} \tag{7.3}$$

In (7.3), sum the terms multiplied by h (yielding $h(x_{P+1} - x_1)$), sum the terms multiplied by $h-1$, ..., sum the terms multiplied by α_1, ..., sum the terms multiplied by α_P. We get

$$\begin{aligned} b = (\frac{1}{Ph}) \times \{ & [h(x_{P+1} - x_1) + (h-1)(x_{P+2} - x_2) + ... + (x_{P+h} - x_h)] \\ & - [\alpha_1(x_P - x_0) + \alpha_2(x_{P-1} - x_0) + ... + \alpha_P(x_1 - x_0)] \} \\ \equiv (\frac{1}{Ph}) & \times (A_1 - A_2 - B_1 + B_2), \\ & A_1 \equiv hx_{P+1} + (h-1)x_{P+2} + ... + x_{P+h}, \\ & A_2 \equiv hx_1 + (h-1)x_2 + ... + x_h, \\ & B_1 \equiv \alpha_1 x_P + \alpha_2 x_{P-1} + ... + \alpha_P x_1, \\ & B_2 \equiv (\alpha_1 + \alpha_2 + ... + +\alpha_P)x_0. \end{aligned} \tag{7.4}$$

For h fixed, it suffices to show that the variance of each of the four terms is O(1). This is immediate for A_1 and A_2. For B_1 and B_2: it is easily shown that $|\alpha_t| \leq c\rho^t$ for a constant c that does not depend on P. Then B_2 =O(1) is immediate. For B_1: Let

$$\gamma_j = \text{cov}(x_t.x_{t-j}).$$

We have $\text{var}(B_1) = \sum\sum_{s,t=1}^{P} \alpha_s \alpha_t \gamma_{|s-t|}$. Also, $|\alpha_t| \leq c\rho^t \Rightarrow \sum\sum_{s,t=1}^{\infty} \alpha_s \alpha_t \leq 2c^2/[(1-\rho^2)(1-\rho)]$ $\equiv c_\alpha \Rightarrow \text{var}(B_1) \leq c_\alpha \gamma_0$.

When h/P =O(1), it suffices to show that the variance of each of the four terms is O(h^3). Clearly var(B_2) is bounded, even if h grows. I will write out the arguments for $A_1$and B_1. The argument for A_2 is similar.

We have

$$
\begin{aligned}
var(A_1) = &[h^2 + (h-1)^2 + ... + 1]\gamma_0 \\
&+ 2[h(h-1) + (h-1)(h-2) + ... + 2]\gamma_1 \\
&+ 2[h(h-2) + (h-1)(h-3) + ... + 3]\gamma_2 \\
&+ ... \\
&+ 2h\gamma_{h-1} \\
&\leq 2[h^2 + (h-1)^2 + ... + 1][|\gamma_0| + |\gamma_1| + |\gamma_2| + ... + |\gamma_{h-1}|] \\
&\leq 2(\sum_{j=0}^{\infty} |\gamma_j|)[h^2 + (h-1)^2 + ... + 1] \\
&= O(h^3),
\end{aligned} \tag{7.5}
$$

because $\sum_{j=0}^{\infty} |\gamma_j|$ is finite and $h^2 + (h-1)^2 + ... + 1$=O(h^3). For var(B_1), note first that

$$
\begin{aligned}
|\alpha_t| &\leq h|\psi_t| + (h-1)|\psi_{t+1}| + ... + |\psi_{t+h-1}| \\
&\leq c\rho^t[h + (h-1)\rho + ... + \rho^{h-1}] \\
&\leq c\rho^t h(1 + \rho + ...\rho^{h-1}) \\
&\leq c_2 h\rho^t, c_2 \equiv c/(1-\rho).
\end{aligned} \tag{7.6}
$$

Then logic similar to that used in analyzing var(A_1) yields var $(B_1) \leq 2(\sum_{j=0}^{\infty} |\gamma_j|)(\alpha_1^2 + \alpha_2^2 + ... + \alpha_P^2)$ and it suffices to show $\alpha_1^2 + \alpha_2^2 + ... + \alpha_P^2$ =O(h^3). Using (7.6), we have

$$
\begin{aligned}
\alpha_1^2 + \alpha_2^2 + ... + \alpha_P^2 &\leq c_2^2 h^2[\rho^2 + ... + \rho^{2P}] \\
&\leq c_2^2 h^2 \frac{\rho^2}{1-\rho^4} \\
&= O(h^2).
\end{aligned}
$$

Table 3.1

Simulation results on bias and RMSPE, random walk relative to AR(2) forecasts

	(1)		(2)		(3)		(4)		(5)		(6)	
(1)	h=1		h=4		h=8		h=12		h=20		h=40	
(2) var(b^{RW})×P^2/h	2.63		1.92		1.85		1.85		1.72		1.75	
	med	%<1	med	%<1	med	%<1	med	%<1	med	%<1	med	%<1
(3) $\lvert b^{RW} / b^{AR}\rvert$												
(3a) Rolling	0.3	90%	0.3	90%	0.4	86%	0.5	82%	0.6	79%	0.7	73%
(3b) Recursive	0.2	95%	0.2	94%	0.2	91%	0.3	90%	0.4	88%	0.6	80%
(3c) Population	0.2	95%	0.2	95%	0.3	94%	0.3	91%	0.4	88%	0.7	79%
(4) RMSPE^{RW} / RMSPE^{AR}												
(4a) Rolling	1.1	0%	1.6	0%	2.0	0%	2.3	0%	2.8	0%	3.5	0%
(4b) Recursive	1.2	0%	1.6	0%	2.1	0%	2.4	0%	3.0	0%	3.9	0%
(4c) Population	1.2	0%	1.7	0%	2.2	0%	2.6	0%	3.3	0%	5.0	0%
(5) P	108		105		101		97		89		69	
(6) h/P	0.01		0.04		0.08		0.13		0.22		0.58	

Notes:

1. 1000 simulation samples of size T+1=156 were generated from a stationary AR(2) model. The DGP was: $x_t = 0.5x_{t-1}-0.1x_{t-2}+u_t$, $u_t \sim$ iid N(0,1). Predictions were made from a driftless random walk model (superscript RW) and an AR(2) model (superscript AR). In lines labeled "rolling" and "recursive," AR(2) predictions were made using least squares estimates; in lines labeled "population," population parameters were used.

2. For horizons given in row (1), the number of predictions in a given sample is given in row (5). In a given sample, P predictions (row (5)) were used to compute bias b (row (3)) and root mean squared prediction error (RMSPE, row (4)). See text for additional details.

3. In row (2), var(b^{RW}) is the variance of random walk bias across the 1000 samples. Here and throughout, bias is the sample average of the difference between realization and prediction.

4. In rows (3a)-(3c), the "med" column reports the median value, across the 1000 samples, of the absolute value of the ratio of the bias of the random walk model to the bias of the estimated AR(2) model. A value less than 1 indicates that the RW forecast had lower bias (in absolute value). The "%<1" column reports the percentage of the 1000 samples in which this ratio was less than 1.

5. Rows (4a)-(4c) are analogous to (3a)-(3b), with RMSPE replacing bias. Here and throughout, RMSPE is the square root of the average of squared forecast errors. A median value greater than 1 indicates that the RW forecast had a higher RMSPE.

Table 4.1

Bias and RMSPE, random walk relative to AR forecasts, per capita GDP growth

		A: Last forecast 2022:4						B: Last forecast 2019:4					
(1)		h=1	h=4	h=8	h=12	h=20	h=40	h=1	h=4	h=8	h=12	h=20	h=40
(2)	b^{RW}	0.0	0.1	0.1	-0.2	-0.2	-0.3	0.0	0.0	0.0	-0.1	-0.2	-0.4
(3)	$\|b^{RW} / b^{AR}\|$												
(3a)	Rolling	0.1	0.2	0.4	1.3	0.8	0.5	0.1	0.4	0.8	0.5	0.6	0.5
(3b)	Recursive	0.1	4.5	0.7	0.4	0.4	0.4	0.0	0.0	0.1	0.2	0.4	0.5
(4)	RMSPE^{RW} / RMSPE^{AR}												
(4a)	Rolling	0.9	1.1	1.5	1.7	2.1	2.4	1.2	1.3	1.6	1.7	2.1	2.4
(4b)	Recursive	1.1	1.7	2.4	1.9	2.3	2.6	1.2	1.4	1.7	1.9	2.4	2.6

Notes:

1. b^{RW} is bias in predictions of the average value of GDP growth over the horizons indicated in row 1, from a model that assumes that GDP growth follows a driftless random walk. All forecasting exercises are pseudo-out-of-sample.

2. For a given horizon h, the first forecast is for average GDP growth from 1996:1 through the next h-1 quarters. In panel A, the last forecasted observation is for average GDP growth over h quarters ending in 2022:4; the number of forecasts and forecast errors P is 108-(h-1) (see line (5) in Table 3.1). In panel B, the date of the first forecast is the same but the ending quarter is 2019:4; P is smaller by 12, for each horizon.

3. b^{AR} is bias in forecasts computed from a series of estimates of an AR(2) model. In the regression relying on the first sample used to estimate the AR model, the left-hand-side variable spans the 48 quarters from 1984:1 to 1995:4. For h=1, the left-hand-side in the last estimation sample in panel A runs from 2009:4 to 2022:3 (rolling) or 1984:1 to 2022:3 (recursive); the dates in panel B are 2007:4-2019:3 and 1984:1-2019:3. End dates for h>1 are shifted back to accommodate the longer horizon (e.g., for h=4, the final recursive sample in panel A runs 1984:1-2021:4).

4. Rows (3a) and (3b) give the absolute value of the ratio of the bias from the random walk forecast to the bias from the AR forecast.

5. Rows (4a) and (4b) give the ratio of root mean squared prediction errors (RMSPEs).

Table 4.2

Bias and RMSPE, random walk relative to AR forecasts: summary statistics

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
				median	– $\lvert b^{RW}/b^{AR}\rvert$ –		RMSPE^{RW} / RMSPE^{AR}	
	mean	s.d.	ρ_1	$\lvert b^{RW}\rvert$	%<1	median	%<1	median
(1) Per capita GDP growth	1.7	2.3	0.4	0.10	92%	0.4	4%	1.7
(2) CPI inflation	2.6	1.9	0.3	0.05	96%	0.1	0%	2.0
(3) Labor prod growth	1.9	2.5	0.1	0.05	54%	0.9	0%	2.1
(4) M3 growth	5.5	2.8	0.6	0.05	96%	0.2	0%	1.7
(5) All 4 series, all horizons				0.06	84%	0.3	1%	1.9
(6) All 4 series, $h \leq 12$				0.05	86%	0.2	2%	1.7
(7) All 4 series, h>12				0.23	81%	0.4	0%	2.4

Notes:

1. Columns (2)-(4) give summary statistics for the series indicated in column (1), computed using quarterly data 1984:1-2019:4. In column (4), ρ_1 is the first-order autocorrelation coefficient.

2. For per capita GDP growth (row 1), columns (5)-(9) are computed from the values in Table 4.1.

   a. Column (5): "median b^{RW}" is median bias for the random walk model, computed from the 12 values in row (2) of Table 4.1.

   b. Column (6): "$\lvert b^{RW}/b^{AR}\rvert$ %<1" is the percentage of forecast comparisons in which bias for the random walk model is less than that for the AR(2) model, computed from the 24 values in rows (3a) and (3b) of Table 4.1

   c. Column (7): "$\lvert b^{RW}/b^{AR}\rvert$ % median" is the median value of the absolute value of the ratio of the bias of random walk model to the bias of the AR model, again computed from the 24 values of rows (3a) and (3b) of Table 4.1.

   d. Columns (8) and (9) are comparable values for root mean squared prediction error rather than bias, computed from the 24 values in rows (4a) and (4b) in Table 4.1.

3. The values in rows (2)-(7) are computed analogously to those for GDP growth. The underlying values comparable to those in Table 4.1 for GDP growth are given in the online appendix.

Random Walk Forecasts of Stationary Processes Have Low Bias

Online Appendix

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This online appendix presents simulation and empirical results omitted from the paper to save space.

1. Tables A3.1a and A3.1b present simulation results analogous to those presented in Table 3.1, but with a different DGP and a different stationary forecasting model.

    a. Table A3.1a: the DGP is $x_t = u_t \sim$ iid N(0,1); the stationary model for comparison with the random walk model is iid possibly around a nonzero mean. That is, the stationary forecast is $x_{t+h}=\bar{x}$ for all h, where $\bar{x}$ is the sample mean from the relevant rolling or recursive sample. The population forecast is $x_{t+h}=0$ for all h.

    b. Table A3.1b: the DGP is $(1-L)^d x_t = u_t \sim$ iid N(0,1), with d=0.31.

        i. As in Chung and Baillie (1993), data in a given simulation sample were drawn from an N(0,V) DGP where V is the variance-covariance matrix of a (156×1) vector of x's. For example, the diagonals of V were set to the variance of x_t (i.e., to $\Gamma(1-2d)/\Gamma^2(1-d)$, where Γ is the gamma function and, again, d=0.31). Off-diagonals were set in accordance with the autocovariances of x_t. See Brockwell and Davis (1993, p 522) for formulas.

        ii. The technique of Geweke and Porter-Hudak (1983) was used to estimate d, setting the estimate to -0.49 or 0.49 if the estimate fell outside the interval (-0.49,0.49). Forecasts were made using the autoregressive representation of x_t. The autoregression was truncated at lag 48 (rolling samples) or at the first observation in the sample (recursive samples). As in our AR(2) forecasts, h-period-ahead-forecasts were made using the chain rule of forecasting.

        For example, for rolling samples, forecasts were based on the AR(48)

        $$x_t-\bar{x} = \hat{\varphi}_1(x_{t-1}-\bar{x}) + \hat{\varphi}_2(x_{t-2}-\bar{x}) + \ldots + \hat{\varphi}_{48}(x_{t-48}-\bar{x}).$$

        Here $\bar{x}$ is the sample mean, $\hat{\varphi}_1=\hat{d}$, $\hat{\varphi}_2=-\hat{d}(\hat{d}-1)/2!$, ... and $\hat{d}$ is the estimate of d. Population forecasts were made from this AR representation using population values, i.e., setting $\hat{d}$=0.31 and $\bar{x}$=0 (=population mean of x).

    c. Table A3.1c: the same as Table A3.1b, except that T+1=400 instead of T+1=156.

2. Tables A4.1a, A4.1b, and A4.1c present results for CPI inflation, labor productivity growth, and M3

growth analogous to the results presented for GDP growth in Table 4.1.

3. Table A4.2 presents results analogous to Table 4.2, except that lag length was chosen by BIC with a maximum lag length of 4. (Table 4.2 relied on an AR(2) model for its results.)

Additional references:

Brockwell, Peter J. and Richard A. Davis, 1993, *Time Series: Theory and Methods*, Springer-Verlag: New York. https://www.doi.org/10.1007/978-1-4419-0320-4

Chung, Ching-fan and Richard T. Baillie, 1993, "Small Sample Bias in Conditional Sum-of-Squares Estimators of Fractionally Integrated ARMA Models," *Empirical Economics* 18, 791-806. https://doi.org/10.1007/BF01205422

Geweke, John and Susan Porter-Hudak, 1983, "The Estimation and Application of Long Memory Time Series Models," *Journal of Time Series Analysis* 4(4), 221-238. https://doi.org/10.1111/j.1467-9892.1983.tb00371.x

Table A3.1a

Simulation results on bias and RMSPE, random walk relative to iid forecasts

	(1)		(2)		(3)		(4)		(5)		(6)	
(1)	h=1		h=4		h=8		h=12		h=20		h=40	
(2) var(b^{RW})×P^2/h	2.08		0.90		0.75		0.73		0.65		0.70	
	med	%<1	med	%<1	med	%<1	med	%<1	med	%<1	med	%<1
(3) $\|b^{RW} / b^{iid}\|$												
(3a) Rolling	0.3	91%	0.3	89%	0.4	86%	0.5	83%	0.6	80%	0.7	74%
(3b) Recursive	0.1	95%	0.2	94%	0.2	92%	0.3	90%	0.3	88%	0.6	80%
(3c) Population	0.1	96%	0.2	94%	0.3	93%	0.3	90%	0.4	87%	0.7	78%
(4) RMSPE^{RW} / RMSPE^{iid}												
(4a) Rolling	1.4	0%	2.2	0%	2.8	0%	3.3	0%	4.1	0%	5.3	0%
(4b) Recursive	1.4	0%	2.2	0%	2.9	0%	3.5	0%	4.4	0%	5.8	0%
(4c) Population	1.4	0%	2.3	0%	3.1	0%	3.7	0%	4.9	0%	7.4	0%
(5) P	108		105		101		97		89		69	
(6) h/P	0.01		0.04		0.08		0.13		0.22		0.58	

Notes: See notes to Table 3.1.

Table A3.1b

Simulation results on bias and RMSPE, random walk relative to I(d) forecasts

	(1)		(2)		(3)		(4)		(5)		(6)	
(1)	h=1		h=4		h=8		h=12		h=20		h=40	
(2) var(b^{RW})×P^2/h	2.53		1.99		2.31		2.68		3.14		3.93	
	med	%<1	med	%<1	med	%<1	med	%<1	med	%<1	med	%<1
(3) $\|b^{RW} / b^{I(d)}\|$												
(3a) Rolling	0.1	97%	0.3	91%	0.4	90%	0.4	87%	0.5	84%	0.7	76%
(3b) Recursive	0.1	96%	0.2	95%	0.2	92%	0.3	92%	0.4	89%	0.6	78%
(3c) Population	0.1	95%	0.2	93%	0.2	92%	0.3	90%	0.4	87%	0.6	79%
(4) RMSPE^{RW} / RMSPE$^{I(d)}$												
(4a) Rolling	1.2	3%	1.5	0%	1.7	0%	1.8	0%	2.0	0%	2.3	0%
(4b) Recursive	1.2	6%	1.5	0%	1.7	0%	1.9	0%	2.1	0%	2.4	0%
(4c) Population	1.2	0%	1.6	0%	1.8	0%	2.0	0%	2.3	0%	2.8	0%
(5) P	108		105		101		97		89		69	
(6) h/P	0.01		0.04		0.08		0.13		0.22		0.58	

Notes: See notes to Table 3.1.

Table A3.1c

Simulation results on bias and RMSPE, random walk relative to I(d) forecasts, T+1=400

	(1)		(2)		(3)		(4)		(5)		(6)	
(1)	h=1		h=4		h=8		h=12		h=20		h=40	
(2) $\text{var}(b^{RW})\times P^2/h$	2.44		2.20		2.62		3.04		3.91		5.40	
	med	%<1	med	%<1	med	%<1	med	%<1	med	%<1	med	%<1
(3) $\lvert b^{RW} / b^{I(d)}\rvert$												
(3a) Rolling	0.1	98%	0.2	94%	0.3	92%	0.4	90%	0.5	86%	0.6	82%
(3b) Recursive	0.1	99%	0.1	97%	0.1	97%	0.2	95%	0.2	95%	0.3	92%
(3c) Population	0.1	97%	0.1	96%	0.1	96%	0.1	94%	0.2	93%	0.3	90%
(4) RMSPE^{RW} / $\text{RMSPE}^{I(d)}$												
(4a) Rolling	1.1	0%	1.4	0%	1.7	0%	1.8	0%	2.0	0%	2.2	0%
(4b) Recursive	1.2	1%	1.5	0%	1.7	0%	1.8	0%	2.0	0%	2.4	0%
(4c) Population	1.2	0%	1.6	0%	1.8	0%	1.9	0%	2.2	0%	2.6	0%
(5) P	353		350		346		342		334		314	
(6) h/P	0.003		0.01		0.02		0.04		0.06		0.13	

Note: This table differs from Table A3.1b in that T+1=400 instead of T+1=156.

Table A4.1a

Bias and RMSPE, random walk relative to AR forecasts, CPI inflation

		A: Last forecast 2022:4						B: Last forecast 2019:4					
(1)		*h*=1	*h*=4	*h*=8	*h*=12	*h*=20	*h*=40	*h*=1	*h*=4	*h*=8	*h*=12	*h*=20	*h*=40
(2)	b^{RW}	0.1	0.1	0.2	0.0	0.0	-0.3	0.0	0.0	0.0	0.0	-0.1	-0.3
(3)	$\|b^{RW} / b^{AR}\|$												
(3a)	Rolling	0.4	1.7	0.8	0.1	0.0	0.4	0.0	0.1	0.1	0.1	0.2	0.4
(3b)	Recursive	0.2	0.3	0.3	0.1	0.0	0.3	0.0	0.0	0.0	0.0	0.2	0.3
(4)	$RMSPE^{RW} / RMSPE^{AR}$												
(4a)	Rolling	1.1	1.4	1.7	2.1	2.6	3.3	1.2	1.9	2.2	2.6	2.9	3.4
(4b)	Recursive	1.2	1.4	1.7	1.9	2.1	2.3	1.2	1.7	1.9	2.0	2.1	2.4

See notes to Table 4.1.

Table A4.1b

Bias and RMSPE, random walk relative to AR forecasts, labor productivity growth

		A: Last forecast 2022:4						B: Last forecast 2019:4					
(1)		*h*=1	*h*=4	*h*=8	*h*=12	*h*=20	*h*=40	*h*=1	*h*=4	*h*=8	*h*=12	*h*=20	*h*=40
(2)	b^{RW}	0.0	0.0	0.0	0.1	-0.1	-0.5	0.0	0.0	0.0	-0.1	-0.2	-0.8
(3)	$\|b^{RW} / b^{AR}\|$												
(3a)	Rolling	0.2	0.9	0.9	0.5	0.3	1.2	0.2	0.1	0.4	0.6	0.9	2.6
(3b)	Recursive	0.1	0.3	12.9	4.0	1.0	2.4	0.7	1.1	2.9	3.2	3.3	9.7
(4)	$RMSPE^{RW} / RMSPE^{AR}$												
(4a)	Rolling	1.4	2.1	2.4	2.1	2.2	2.3	1.3	1.8	2.0	2.1	2.1	2.5
(4b)	Recursive	1.3	1.9	2.3	2.2	2.6	3.1	1.4	1.9	2.1	2.2	2.5	3.3

See notes to Table 4.1.

Table A4.1c

Bias and RMSPE, random walk relative to AR forecasts, M3 growth

		A: Last forecast 2022:4						B: Last forecast 2019:4					
(1)		h=1	h=4	h=8	h=12	h=20	h=40	h=1	h=4	h=8	h=12	h=20	h=40
(2)	b^{RW}	0.1	0.1	0.3	0.9	0.4	0.2	0.0	0.0	0.0	0.0	0.0	0.0
(3)	$\|b^{RW} / b^{AR}\|$												
(3a)	Rolling	1.5	0.3	0.4	0.8	0.5	0.2	0.3	0.2	0.1	0.1	0.0	0.0
(3b)	Recursive	0.2	0.2	0.3	0.6	0.4	0.1	0.2	0.1	0.0	0.0	0.0	0.0
(4)	$\text{RMSPE}^{RW} / \text{RMSPE}^{AR}$												
(4a)	Rolling	1.0	1.2	1.3	1.5	1.7	2.0	1.1	1.4	1.7	1.8	1.9	1.9
(4b)	Recursive	1.1	1.3	1.6	1.5	1.7	2.0	1.1	1.4	1.8	2.0	2.2	2.4

See notes to Table 4.1.

Table A4.2

Bias and RMSPE, random walk relative to AR forecasts: summary statistics
Lag length chosen by BIC

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
				median	– $\|b^{RW}/b^{AR}\|$ –		RMSPE^{RW} / RMSPE^{AR}	
	mean	s.d.	ρ_1	$\|b^{RW}\|$	%<1	median	%<1	median
(1) Per capita GDP growth	1.7	2.3	0.4	0.10	92%	0.4	0%	1.8
(2) CPI inflation	2.6	1.9	0.3	0.05	96%	0.2	0%	1.9
(3) Labor prod growth	1.9	2.5	0.1	0.05	50%	1.0	0%	2.1
(4) M3 growth	5.5	2.8	0.6	0.05	100%	0.2	0%	1.7
(5) All 4 series, all horizons				0.06	84%	0.3	0%	1.9
(6) All 4 series, $h \leq 12$				0.05	84%	0.2	0%	1.7
(7) All 4 series, h>12				0.23	84%	0.4	0%	2.4

See notes to Table 4.2. Columns (1)-(5) are identical to columns (1)-(5) in Table 4.2.