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Accounting for Risk in a Linearized Solution: How to Approximate the Risky Steady State and Perturb Around It

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Abstract

We propose a novel approximation of the risky steady state and construct first-order perturbations around it for a general class of dynamic equilibrium models with time-varying and non-Gaussian risk. We offer analytical formulas and conditions for their local existence and uniqueness. We apply this approximation technique to models featuring Campbell-Cochrane habits, recursive preferences, and time-varying disaster risk, and show how the proposed approximation represents the implications of the model similarly to global solution methods. We show that our approximation of the risky steady state cannot be generically replicated by higher-order perturbations around the deterministic steady state, which cannot account well for the effects of risk in our applications even up to third order. Finally, we argue that our perturbation can be viewed as a generalized version of the heuristic loglinear-lognormal approximations commonly used in the macro-finance literature.

JEL classification: C63; G12; E32; E44.

Keywords: Perturbation methods, Risky steady state, Macroeconomic uncertainty, Solving dynamic equilibrium models, Time-varying risk premia

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"The central fact of life that makes financial economics interesting is that risk premia are not at all second order." — Cochrane (2008)

1. Introduction

Risk plays a crucial role in modern dynamic equilibrium models in capturing asset pricing facts and studying the effects of uncertainty in the economy. For example, over the last two decades researchers have explored models with time-varying risk aversion, risk-sensitive preferences, stochastic volatility, variable disaster risk, and risky returns in small open economies.[2] But risk presents a challenge for extant solution techniques. Projection methods are accurate but computationally intensive and offer limited analytical insight. Perturbations around the *deterministic steady state* (DSS) are certainty equivalent at first order, while at higher orders they have disadvantages similar to those of projection methods and remain accurate only locally, especially when non-analytic functions are involved. Finally, the DSS can be an invalid expansion point.

In this context, a literature initiated by Coeurdacier, Rey, and Winant (2011), Juillard (2010) and, in a special application, Devereux and Sutherland (2011) started to explore approximations around a *risky steady state* (RSS). Specifically, the RSS can be defined as the limit point of the deterministic model in which all shocks are zero, but in which agents expect shocks to be realized according to their true distribution and form expectations consistent with the *exact* solution of the stochastic model. Since approximations around the RSS do not perturb the amount of uncertainty in the economy, they have a better chance of being close to the exact solution of the model than perturbations around the DSS. This definition, however, is typically impractical as it presupposes that a nonlinear accurate solution of the model is already available to evaluate expectations. Therefore, the literature has focused instead on a specific approximation of the RSS, namely, Coeurdacier et al. (2011) solve for the RSS in which agents form expectations consistent with a linear approximate solution.

We likewise focus on this definition, which we call the *first-order risky steady state* (FRSS) to avoid confusion with the RSS. More precisely, we define the FRSS as the limit point of the deterministic model in which all shocks are zero, but in which agents expect shocks to be realized according to their true distribution and form expectations consistent with a *first-order approximation* of the solution around the FRSS. Since such perturbations search

[2]For example, among many others, Campbell and Cochrane (1999); Bansal and Yaron (2004); Fernández-Villaverde et al. (2011); Binsbergen et al. (2012); Gourio (2012); Rudebusch and Swanson (2012); Wachter (2013); Lopez et al. (2015); Kehoe et al. (2022).

jointly for the expansion point and for the linear approximation coefficients, the problem has proved challenging. Indeed, analytical formulas to compute the approximation, a discussion of the uniqueness of the FRSS, and conditions for the stability and uniqueness of the local dynamics of the model have so far remained unknown. Furthermore, the setup sketched by Coeurdacier et al. (2011) does not accommodate heteroskedastic shocks and, hence, *time-varying* risk premia.

This paper aims to fill these gaps in four ways. First, we obtain simple analytical formulas for the FRSS and for the coefficients of the linear approximation around it. These formulas facilitate an analytical understanding of the implications of risk on equilibrium prices and quantities and are conducive to fast filtering techniques by the linearity of the approximate solution, which, as we show, is accurate in several applications.

Second, we characterize the existence and uniqueness of the local dynamics around the FRSS of the approximate solution by generalizing the Blanchard and Kahn (1980) saddle-path conditions. This result, as we show, implies that we can root these approximations in formal ground in perturbation theory.

Third, we show how, once the equations are written in the appropriate form, heteroskedastic and non-Gaussian shocks, and hence time-varying risk premia, are easily accommodated. We do so by using relative entropy, rather than variance, as the measure of dispersion, which can be characterized by its connection with the cumulant generating function of shocks. Therefore, while only a conventional approximation of at least third order can generate time-varying risk premia, our perturbation around the FRSS captures risk premia variation already at the first order and is therefore appropriate for models that speak to the initial quote by Cochrane (2008).

Fourth, we note that, whenever forward-looking difference equations are present, the system can be rewritten in multiple ways that lead to different approximations, a point so far not recognized by the literature. In fact, writing a forward-looking difference equation in recursive form or as a summation affects which variables are approximated as linear in the states when evaluating the expectations. We discuss how to pin down the approximation as the one that minimizes Euler equation errors.

We then discuss how our proposed perturbation noticeably differs from conventional perturbations. Generically, we show that first-order perturbations around the FRSS are not nested in perturbations around the DSS of arbitrary order, and hence cannot generically be replicated with output of conventional higher-order perturbations. In fact, as pointed out by Coeurdacier et al. (2011) and Devereux and Sutherland (2011), the DSS may not even

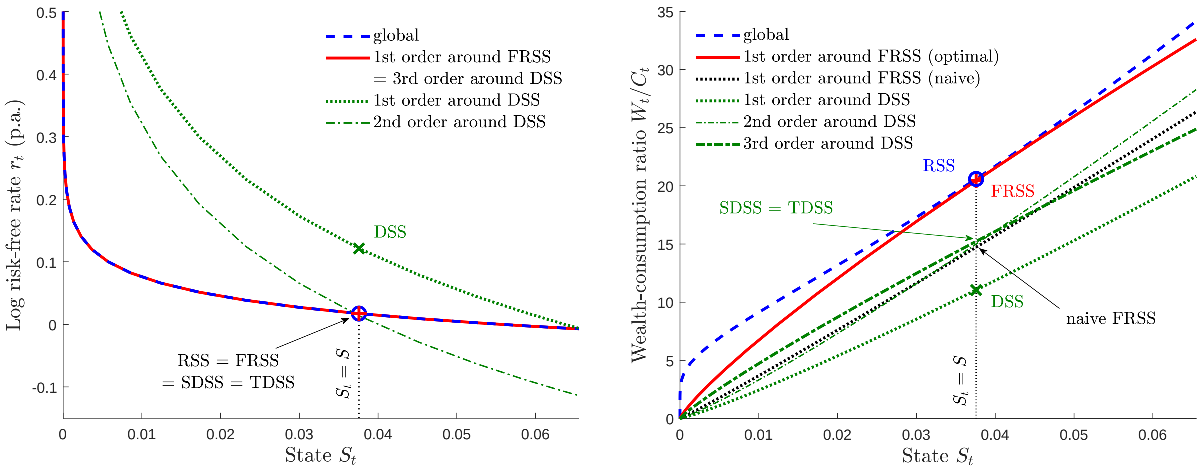


Figure 1: Equilibrium risk-free rate and price-dividend ratio of the consumption portfolio as a function of the state of the economy in Wachter (2006). Markers and arrows denote steady-state values under the different solution methods: global solution (cubic splines collocated over 200 Chebyshev nodes over the interval $(10^{-130}, 10^{-1})$ for S_t and shocks integrated by 20-point Gauss-Hermite quadrature), linear perturbation around the FRSS, perturbations around the DSS. RSS: risky steady state; FRSS: first-order risky steady state; DSS: deterministic steady state; SDSS: second-order DSS; TDSS: third-order DSS. Naive and optimal FRSS use a summation specification with 1 and 1500 terms, respectively, for the wealth-consumption ratio.

be well-defined in examples in which the FRSS is. In this context, note that the constant terms of conventional higher-order perturbations around the DSS are sometimes referred to as proxies for the RSS. To distinguish them from the RSS and FRSS, we will refer to such proxies for the RSS based on second- and third-order perturbations around the DSS as *second-* and *third-order deterministic steady states* (or SDSS and TDSS for brevity). Besides being problematic when the DSS is not well-defined, these alternative approximations of the RSS have the disadvantage of being inaccurate in important applications.

Indeed, Figure 1 illustrates the different approximations in a simple example: the pricing of a risk-free bond and of the wealth-consumption ratio in the Campbell-Cochrane habit model of Wachter (2006). (Section 4 elaborates this example.) First- and second-order perturbations around the DSS are severely inaccurate. Conventional third-order perturbations recover the global solution for the risk-free rate but remain inaccurate when characterizing equilibrium wealth. As is apparent, the constant terms of second- or third-order perturbations around the DSS, or SDSS and TDSS, are poor approximations of the RSS. In contrast, our approximation around the FRSS (labeled 'optimal' in the Figure) is much closer to the global solution; importantly, the FRSS and the RSS are nearly identical. Even though our strategy approximates other versions of the model less accurately, as shown in Section 4, the fact that our proposed FRSS is nearly identical to the RSS is all the more remarkable given that the Campbell and Cochrane (1999) model is notoriously highly nonlinear and requires projections on fine grids (e.g., Wachter, 2005).

Note also how our approximation differs from previous implementations of the FRSS. First, a setup that accommodates heteroskedastic shocks is necessary to solve models with the nonlinear habits of Campbell and Cochrane (1999), in which time-varying risk premia play a big role. Second, Figure 1 shows that a naive implementation of the FRSS that uses the standard recursive pricing equation of the wealth-consumption ratio performs poorly, similar to second- or third-order perturbations around the DSS. In contrast, our approach recognizes the multiplicity of the FRSS and exploits it to minimize Euler equation errors.

We first test the performance of our approximation in endowment economies. Besides the habit formation models, we study the solution to the disaster risk model of Wachter (2013). Projection methods are required to find the global solution under nonlinear habits, while rare disasters are main examples of non-Gaussian exogenous shocks that produce time-varying risk premia. Our approximation is accurate in solving for risk premia and volatilities of equities and bonds at both short and long durations.

We then turn to a production economy. The real business cycle model of Jermann (1998) with Campbell-Cochrane habits explored by Chen (2017) is appropriate for testing the accuracy of our solution in an environment where consumption risk is endogenous, while habits and capital adjustment costs generate volatile stock prices. In this application the full nonlinear solution is computationally expensive, while the FRSS approximation yields a fast and tractable solution with good accuracy.

Finally, we provide a user-friendly computer code for application to most DSGE models.

1.1. Relationship to the literature

Two strands of the literature have dealt separately with risk-adjusted linearizations. First, the macro-finance literature has used affine risk adjustments based on lognormality at least since Campbell (1993), with Bansal and Yaron (2004) being a prominent example and Malkhozov (2014) offering the most recent treatment.[3] However, these ad hoc approximations remain limited in scope and lack a formal justification based on perturbation theory.

Second, Coeurdacier et al. (2011), Juillard (2010), and Devereux and Sutherland (2011) were the first to study perturbations around (approximations of) the RSS, with de Groot

[3]Examples of applications of loglinear-lognormal methods include Jermann (1998); Lettau and Uhlig (2000); Uhlig (2007); Bekaert, Cho, and Moreno (2010); Kaltenbrunner and Lochstoer (2010); Dew-Becker (2014); Backus, Ferriere, and Zin (2015); Schorfheide, Song, and Yaron (2018); Itskhoki and Mukhin (2021) among many others.

(2013), Meyer-Gohde (2016), and Kliem and Uhlig (2016) offering recent applications, while Lopez et al. (2015) and Kehoe et al. (2022) are recent applications of the approximation we propose here. But, as discussed above, a characterization of the exact solution, uniqueness, and local stability properties of these approximations has so far been missing. In fact, extant treatments often rely on additional ad hoc approximations to the equations and on output from perturbations around the DSS, and ignore the dangers of constructing what we labeled 'naive' FRSS approximations.

Furthermore, the relationship of these perturbations with loglinear-lognormal risk-adjusted linearizations has not been clarified. In this context, we argue that first-order perturbations around the FRSS can be viewed as loglinear-lognormal approximations, suitably generalized. The extant affine methods subsumed by Malkhozov (2014) proceed in two steps: first, they linearize the equations around the DSS and, second, they adjust the solution by a risk correction. This correction can be time-varying when risk comes from shocks to exogenous variables but is constant when it comes from shocks to endogenous variables, and hence fails to capture sources of time variation in risk premia that are central in our applications.[4] Therefore, besides the extension to non-Gaussian shocks, we generalize extant loglinear-lognormal methods by evaluating the approximate functions around the FRSS rather than the DSS and by treating consistently innovations to the state vector. Still, our approximation can be derived heuristically in a way that is similar in spirit to how affine approximations are derived, namely, by first splitting the expectational equations into a certainty equivalent and a dispersion term and then evaluating them with a conjectured linear solution. In this precise sense we reconcile the two strands of the literature in macroeconomics and finance.

2. Approximation method: Heuristic algorithm

2.1. General framework

We aim to characterize the solution for jump variables $y_t \in \mathbb{R}^{n_y}$ and states $z_t \in \mathbb{R}^{n_z}$ of the dynamic system of equilibrium conditions with generic form:

$$
\begin{aligned}
0 &= \ln E_t \exp[f(y_t, z_t, y_{t+1}, z_{t+1})], \quad f(y_t, z_t, y_{t+1}, z_{t+1}) \equiv h(y_t, z_t) + f_3 \; y_{t+1} + f_4 \; z_{t+1} \\
z_{t+1} &= g(y_t, z_t) + \lambda(z_t)(y_{t+1} - E_t y_{t+1}) + q\sigma(z_t)\varepsilon_{t+1}
\end{aligned} \tag{1}
$$

[4]Examples include production economies with habits and intertemporal choice under uncertainty in small open economies, where affine approximations yield the determinacy emphasized by Coeurdacier et al. (2011) only when generalized to coincide with perturbations around the FRSS.

where $\lambda(z_t)(y_{t+1} - E_t y_{t+1})$ describes heteroskedastic endogenous risk that depends on innovations in jump variables and $q\sigma(z_t)\varepsilon_{t+1}$ is exogenous risk, where scalar $q \in [0,1]$ denotes the amount of risk in the economy. We adopt the convention that $q = 1$ corresponds to the model of interest. Operator $\ln E_t e^{[\cdot]}$ is applied elementwise to a vector-valued map, with E_t the expectations operator conditioned on the history up to time-t of state variables. Functions $f : \mathbb{R}^{2n_y+2n_z} \to \mathbb{R}^{n_y}$, $h : \mathbb{R}^{n_y+n_z} \to \mathbb{R}^{n_y}$, $g : \mathbb{R}^{n_y+n_z} \to \mathbb{R}^{n_z}$, $\lambda : \mathbb{R}^{n_z} \to \mathbb{R}^{n_z \times n_y}$ and $\sigma : \mathbb{R}^{n_z} \to \mathbb{R}^{n_z \times n_\varepsilon}$ are differentiable. We denote by $f_i, g_i, ...$ the derivatives of $f, g, ...$ with respect to the ith argument. Our framework requires function f to be linear in y_{t+1} and z_{t+1}, an unrestrictive technical assumption we will elaborate on later. The equilibrium conditions of most DSGE models can be cast into this framework after suitable redefinition of variables.

Exogenous shocks $\varepsilon_t \in \mathbb{R}^{n_\varepsilon}$ have a conditional mean of zero and distribution described by the differentiable, conditional cumulant generating function (ccgf):

$$\kappa[\alpha(z_t); z_t] \equiv \ln E_t e^{\alpha(z_t)'\varepsilon_{t+1}}, \quad \text{for any differentiable map } \alpha : \mathbb{R}^{n_z} \to \mathbb{R}^{n_\varepsilon}$$

For example, if $\varepsilon_t \sim Niid(0, I)$, one has $\kappa[\alpha(z_t); z_t] = .5diag[\alpha(z_t)\alpha(z_t)']$.

2.2. Linearization around the FRSS

The solution of the model with expectations formed according to that solution consists of the policy functions $y_t = y(z_t, q)$ and $z_t = z(z_{t-1}, q, \varepsilon_t)$. We define the FRSS of variables y_t and z_t and a linearized solution around it as the point $y = y(z, q)$ and $z = z(z, q, 0)$ and the linear approximate solution $y_t = y + \Psi(z_t - z)$ that solve system (1) with expectations formed consistently with the linear approximation. (Section 3 discusses this qualification.)

To solve for unknown coefficients $[y, z, \Psi]$, we rewrite the forward-looking equations as:

$$0 = h(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + L_t(f_3 y_{t+1} + f_4 z_{t+1}) \tag{2}$$

where $L_t(x_{t+1}) \equiv \ln E_t e^{x_{t+1}} - E_t x_{t+1}$ is a relative entropy measure—a nonnegative measure of dispersion that generalizes variance. The presence of entropy takes risk into account and breaks certainty equivalence. We then use the conjectured linear solution to rewrite (2) as

$$0 = h(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + L_t\big((f_3\Psi + f_4) z_{t+1}\big) \tag{3}$$

Since, if $I_{n_z} - \lambda(z_t)\Psi$ is invertible, innovations to the state have the approximate form

$$z_{t+1} - E_t z_{t+1} = \lambda(z_t)\Psi(z_{t+1} - E_t z_{t+1}) + q\sigma(z_t)\varepsilon_{t+1} = (I_{n_z} - \lambda(z_t)\Psi)^{-1} q\sigma(z_t)\varepsilon_{t+1}$$

it follows that rational expectations consistent with the linear solution imply the existence of a nonnegative function $\tilde{L} : \mathbb{R}^{n_z} \times [0,1] \to \mathbb{R}^{n_y}_+$ of the state vector:

$$\tilde{L}(z_t, q) \equiv L_t\big((f_3\Psi + f_4)z_{t+1}\big) = \kappa[(f_3\Psi + f_4)(I_{n_z} - \lambda(z_t)\Psi)^{-1}q\sigma(z_t); z_t] \tag{4}$$

where the connection with the ccgf follows from the definition of entropy.

We can therefore plug equation (4) into equation (3) and linearize it around the point $[y_t; z_t] = [y; z]$ as:

$$\begin{aligned} 0 &= h(y,z) + f_1(y_t - y) + f_2(z_t - z) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \tilde{L}(z,q) + \tilde{L}_1(z,q)(z_t - z) \\ E_t z_{t+1} &= g(y,z) + g_1(y_t - y) + g_2(z_t - z) \end{aligned}$$

with the notation $f_i \equiv f_i(y,z,y,z)$ and $g_i \equiv g_i(y,z)$. The conjectured linear solution

$$\begin{aligned} y_t &= y + \Psi(z_t - z) \\ z_{t+1} &= z + g_1(y_t - y) + g_2(z_t - z) + (I_{n_z} - \lambda(z_t)\Psi)^{-1}q\sigma(z_t)\varepsilon_{t+1} \end{aligned} \tag{5}$$

can be identified by matching coefficients. Namely, the unknowns $[y, z, \Psi]$ solve the system:

$$\begin{aligned} 0 &= g(y,z) - z \\ 0 &= h(y,z) + f_3 y + f_4 z + \tilde{L}(z,q) \\ 0 &= f_1\Psi + f_2 + (f_3\Psi + f_4)(g_1\Psi + g_2) + \tilde{L}_1(z,q) \end{aligned} \tag{6}$$

Here the entropy terms $\tilde{L}(z,q)$ and $\tilde{L}_1(z,q)$ capture both constant and dynamic risk corrections to an otherwise standard linearization. (In fact, when $q = 0$ we have $\tilde{L}(z,0) = 0$ and $\tilde{L}_1(z,0) = 0$ and recover the DSS and the linear perturbation around it.) For example, these terms will capture constant and time-varying risk premia, respectively.

To summarize, our approximation can be constructed as follows:

Algorithm 1. *With system (1) as a starting point, proceed stepwise:*

Step 1. *Write expectations as the sum of a certainty-equivalent and an entropy term [(2)].*
Step 2. *Conjecture a solution linear in the states and use it to characterize entropy [(4)].*
Step 3. *Identify the linear solution (5) by solving matrix equation (6).*

2.3. Discussion

There are two key implicit assumptions in representation (1). First, forward-looking arguments of the expectations operator must be strictly positive—a necessary property for a connection with entropy. This assumption is not without loss of generality, but in practice, most problems can be rewritten appropriately by splitting the argument into strictly positive components.

Second, function f must be linear in y_{t+1} and z_{t+1}. Note that this assumption is not restrictive—one can always define new variables to fit the linear structure. For example, difference equation $e^{a_t} = E_t e^{a_{t+1}} + e^{b_t}$ with an exogenous process b_t can be equivalently written in notation (1) as $0 = \ln E_t e^{f(y_t, z_t, y_{t+1}, z_{t+1})}$ with $f(y_t, z_t, y_{t+1}, z_{t+1}) = a_{t+1} - \ln(e^{a_t} - e^{b_t})$, with $y_t = a_t$ and $z_t = b_t$. In fact, it is quite the opposite; there are infinite ways to represent a model in this form whenever a forward-looking *difference* equation is present. The difference equation in the previous example can be equivalently written as $e^{a_t} = E_t e^{a_{t+N}} + \sum_{n=1}^{N} E_t e^{b_{t+n}}$ for each $N \in \mathbb{N}$ and can accordingly be written in form (1) by expanding the y_t vector to include the expectations $E_t e^{a_{t+N}}$ and $E_t e^{b_{t+n}}$.[5]

Generically, a degree of freedom N in the approximate solution appears for each forward-looking difference equation present. Although the value of N we choose to write the model in form (1) has no consequences for conventional riskless perturbations, the choice matters for approximations around the FRSS because it determines which variables are approximated as linear in the states, and that in turn affects entropy calculations—a point so far unrecognized by the literature.

How should we select the best specification, and hence remove the degree of freedom by pinning down N? Intuitively, one should pick N, and hence the definition of variables y_t and z_t, so that the exact (unknown) solution of vector $f_3 y_t + f_4 z_t$ is as close to linear in the state as possible. In practice, one should select the one whose associated approximation minimizes the difference equation's Euler equation error. While in our applications the optimal approximation is achieved for large N, we conjecture that the optimality of a summation specification with large N over the recursive specification holds more generally. In practice, our advice is to increase the number of strips progressively until the FRSS solution changes less than some tolerance level. We illustrate these points in the examples of Section 4.

[5] Namely, we can define $P_{nt} = E_t e^{b_{t+n}}$ and express it recursively as $P_{nt} = E_t(E_{t+1} e^{b_{t+n}}) = E_t P_{n-1,t+1}$ with $P_{0t} = e^{b_t}$. Similarly, we define $R_{nt} = E_t e^{a_{t+n}}$ and express it recursively as $R_{nt} = E_t(E_{t+1} e^{a_{t+n}}) = E_t R_{n-1,t+1}$ with $R_{0t} = e^{a_t}$. Letting $p_{nt} = \ln(P_{nt})$ and $r_{nt} = \ln(R_{nt})$, we therefore write $0 = \ln E_t e^{p_{n-1,t+1} - p_{nt}}$ and $0 = \ln E_t e^{r_{n-1,t+1} - r_{nt}}$. The difference equation can therefore be written as $0 = \ln E_t e^{f(y_t, z_t, y_{t+1}, z_{t+1})}$ with

$$f(y_t, z_t, y_{t+1}, z_{t+1}) = \begin{bmatrix} e^{r_{nt}} + \sum_{n=1}^{N} e^{p_{nt}} - e^{a_t} \\ p_{n-1,t+1} - p_{nt}, \ n = 1, ..., N \\ r_{n-1,t+1} - r_{nt}, \ n = 1, ..., N \end{bmatrix}, \qquad y_t = \begin{bmatrix} a_t \\ p_{nt}, \ n = 1, ..., N \\ r_{nt}, \ n = 1, ..., N \end{bmatrix}, \quad z_t = b_t$$

There are also two minor assumptions in the representation of innovations that can be relaxed easily. First, we can generalize representation (2) to handle a dependence also on jump variables y_t of functions λ and σ—that would be replaced by functions $\tilde{\lambda}(y_t, z_t)$ and $\tilde{\sigma}(y_t, z_t)$. In that case define $\lambda(z_t) = \tilde{\lambda}(y + \Psi(z_t - z), z_t)$ and $\sigma(z_t) = \tilde{\sigma}(y + \Psi(z_t - z), z_t)$ and proceed as before. Second, when describing the dynamics of the state vector in (5), one may also choose to approximate the volatility of innovations $(I_{n_z} - \lambda(z_t)\Psi)^{-1}\sigma(z_t)$ around $z_t = z$. This approximation would not affect the approximation coefficient, Ψ—it would only affect simulations from the model—but it offers no practical advantage.

Finally, the solution of matrix equation (6) in Step 3 of Algorithm 1 deserves some comment. Constant terms $[y, z]$ and dynamic coefficient Ψ are necessarily identified jointly at the end of the algorithm, as the expansion point depends on expectations, which are formed according to the solution. Expression (6) includes nonlinear matrix equations in the unknown coefficients that are amenable to straightforward Newton-type numerical solution methods. Appendix C discusses simple numerical algorithms to solve matrix equation (6) as well as a simplified two-step approach that can be used in the interest of speed when the iterative procedure is considered to be unnecessarily slow. However, these matrix equations are sufficiently nonlinear to allow for multiple solutions and to complicate the characterization of the local uniqueness of the constant terms and of the determinacy of the approximate solution's dynamics.

In this context, Proposition 1 characterizes the saddle-point stability of any solution of the nonlinear equations (6), thereby adapting Blanchard and Kahn (1980) conditions to our context. These conditions can be readily checked to assess the legitimacy of a solution.

Proposition 1. *A solution of system* (6) *has unique and bounded dynamics if and only if matrices*

$$\Gamma \equiv \begin{bmatrix} f_4 & f_3 \\ I_{n_z} & 0 \end{bmatrix} \qquad \textit{and} \qquad \Upsilon \equiv \begin{bmatrix} -f_2(y,z) - \tilde{L}_1(z,q) & -f_1(y,z) \\ g_2(y,z) & g_1(y,z) \end{bmatrix}$$

have n_z generalized eigenvalues $\alpha(\Gamma, \Upsilon) \equiv \{\alpha \in \mathbb{C} : \det(\Gamma\alpha - \Upsilon) = 0\}$ inside the unit circle and n_y outside the unit circle.

Appendix A provides a proof of Proposition 1. Relative to conventional linearizations, the determinacy of equilibrium dynamics is affected by the evaluation of derivatives at a different expansion point *and* by the presence of a dynamic entropy component that affects eigenvalues.

2.4. Relationship with Coeurdacier et al. (2011)

Although we follow Coeurdacier et al. (2011) in solving for the FRSS and a linear perturbation around it, their setup and solution strategy need to be extended in two important ways. First, Coeurdacier et al. rely on an additional approximation to solve for the perturbation coefficients. Their procedure starts by a second-order approximation of the original function f around $y_{t+1} = E_t y_{t+1}$ and $z_{t+1} = E_t z_{t+1}$. When applied to our equation (1), their strategy ends up with equation:

$$0 \approx h(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \frac{1}{2} var_t \left(f_3 y_{t+1} + f_4 z_{t+1} \right)$$

This procedure boils down in our context to replacing entropy with half the variance, and hence is correct under normal shocks but is not under any other distributions whose entropy does not coincide with half the variance.

Second, the framework sketched by Coeurdacier et al. (2011) assumes homoskedastic shocks, and hence does not accommodate our applications with Campbell and Cochrane (1999) habits or Epstein-Zin preferences with time-varying risk, including the two most prominent examples of variable disaster risk (illustrated in our applications of Section 4) and stochastic volatility.

In this context, we showed how to draw a connection with the cumulant generating function of shocks and easily accommodate both non-Gaussian shocks, and perhaps more importantly, heteroskedasticity and time-varying risk premia. The key to achieving our simple formulas was to write the problem in the exponential form (1) to then exploit the properties of the conditional cumulant generating function.

Relatedly, de Groot (2013) further approximates the calculations of Coeurdacier et al. (2011) using output from second-order approximations around the DSS. Again, these additional approximations are unnecessary, and they are by Proposition 3 below generically inappropriate. For example, in the simple example behind Figure 1 that contains no endogenous states, the RSS approximated by the method proposed by de Groot would reduce to the SDSS, which as shown in Figure 1 offers a bad approximation for both the FRSS and the RSS.

3. Approximation method: Formal statement

The reader interested in applications can skip this section on first reading.

Proposition 2 provides the mathematical foundation for the approximation treated heuristically in Section 2 by showing that it can indeed be justified based on perturbation theory, i.e., on the implicit function and Taylor theorems. We then discuss the relationship of first-order perturbations around the FRSS with perturbations around the DSS and with loglinear-lognormal affine methods.

3.1. Formal derivation

To set the ground for perturbations, we consider the parameterized family of system (1):

$$
\begin{aligned}
0 &= E_t x_{t+1} + \tau L_t(x_{t+1}) + (1-\tau)\tilde{L}(z_t, q) \\
z_{t+1} &= g[y(z_t, q, \tau), z_t] + \lambda(z_t)(E_{t+1} - E_t) y[z(z_t, q, \varepsilon_{t+1}, \tau), q, \tau] + \sigma(z_t) q \varepsilon_{t+1} \\
x_{t+1} &\equiv h[y(z_t, q, \tau), z_t] + f_3 y[z(z_t, q, \varepsilon_{t+1}, \tau), q, \tau] + f_4 z(z_t, q, \varepsilon_{t+1}, \tau)
\end{aligned}
\tag{7}
$$

where $\tilde{L}(z, q) \equiv \kappa[(f_3\Psi + f_4)(I - \lambda(z)\Psi)^{-1}\sigma(z)q; z]$ is a differentiable function for all $q \in [0, 1]$ with $\tilde{L}(z, 0) = 0$. We are looking for solutions for jump and state variables $y_t = y(z_t, q, \tau)$ and $z_{t+1} = z(z_t, q, \varepsilon_{t+1}, \tau)$, where scalar q denotes the amount of risk in the economy and scalar τ indicates whether entropy is evaluated using the true policy function or using the linear function $y_t = \tilde{y} + \tilde{\Psi}(z_t - \tilde{z})$ for coefficients $\tilde{y}$, $\tilde{z}$, and $\tilde{\Psi}$. Under $q = \tau = 1$ the dynamics coincide with the original model (1). Entropy $w(z_t, q, \tau) \equiv L_t(x_{t+1})$ is assumed to be differentiable in z_t for all $q \in [0, 1]$.

It is useful to rewrite the solution of system (7) as the root of the functional

$$
F([y, z], \varepsilon, q, \tau) \equiv \left\{ \begin{bmatrix} h(y_t, z_t) + f_3 E_t y_{t+1} + f_4 E_t z_{t+1} + \tau w(z_t, q, \tau) + (1-\tau)\tilde{L}(z_t, q) \\ z_{t+1} - g(y_t, z_t) - \lambda(z_t)(y_{t+1} - E_t y_{t+1}) - q\sigma(z_t)\varepsilon_{t+1} \end{bmatrix} \right\}_{t=0}^{\infty}
$$

whose tth coordinate maps $q \in [0, 1]$, $\tau \in [0, 1]$, $\varepsilon_{t+1} \in \mathbb{R}^{n_\varepsilon}$ and essentially bounded functions $[y_t; z_t; y_{t+1}; z_{t+1}]$ of the history of shocks $\{\varepsilon_s\}_{s \le t+1}$ into the Banach space of essentially bounded functions of the history of shocks $\{\varepsilon_s\}_{s \le t}$.

Definition. A *risky steady state* (RSS) of system (7) is a point $z_t = \tilde{z}$ and $\tau = 1$ such that $F([y(\tilde{z}, q, 1), \tilde{z}], 0, q, 1) = 0$. In words, the limit point of the deterministic system in which all shocks are zero but in which agents *i)* expect shocks to be realized according to their true distribution and *ii)* form expectations consistent with the exact solution.

The RSS is a more relevant expansion point than the DSS $z = \bar{z}$ such that $F([y(\bar{z}, 0, 0), \bar{z}], 0, 0, 0) = 0$ because it does not restrict q to 0. The RSS is a point around which the nonlinear system fluctuates after a long sequence of small shocks, while nothing guarantees that the DSS is.

But note that point ii of the definition of RSS relies on a specific description of how people form expectations over future states of nature. As previously discussed, whenever the exact solution is unknown to the modeler, the calculation of the RSS is unfeasible; so a more useful definition in this context is that of a first-order risky steady state, which coincides with the definition given in Section 2.

Definition. A *first-order risky steady state* (FRSS) of system (7) is a point $z_t = \tilde{z}$ and $\tau = 0$ such that $F([\tilde{y}, \tilde{z}], 0, q, 0) = 0$ with $\tilde{y} = y(\tilde{z}, q, 0)$ and $\tilde{\Psi} = y_1(\tilde{z}, q, 0)$. In words, the limit point of the deterministic system in which all shocks are zero but in which *i)* agents expect shocks to be realized according to their true distribution and *ii)* agents' approximate decision rules are computed using a first-order approximation around the FRSS.

Proposition 2. *Suppose that an FRSS $(z_t, \tau) = (\tilde{z}, 0)$ is such that the associated matrices Γ and Υ defined in Proposition 1 have n_z generalized eigenvalues inside the unit circle and n_y outside the unit circle. Then,* i) *implicit functions $y_t = y(z_t, q, \tau)$ and $z_{t+1} = z(z_t, q, \varepsilon_{t+1}, \tau)$ are unique and differentiable in a neighborhood of the FRSS; and* ii) *the coefficients (y, z, Ψ) of the approximate solution of system (6) are indeed the coefficients from a linear perturbation around the FRSS of system (7), i.e., $y = \tilde{y}$, $z = \tilde{z}$, $\Psi = y_1(\tilde{z}, q, 0)$.*

It follows that an FRSS with the property in the premise of Proposition 2 is a *saddle point*.

The first part of Proposition 2 follows from the implicit function theorem. To be able to invoke the theorem, we must show the invertibility of the derivative operator of map F evaluated at the expansion point $(z_t, \tau) = (\tilde{z}, 0)$. Such an operator has tth coordinate

$$D_{F,t}[\hat{y}; \hat{z}] = \Gamma \begin{bmatrix} E_t \hat{z}_{t+1} \\ E_t \hat{y}_{t+1} \end{bmatrix} - \Upsilon \begin{bmatrix} \hat{z}_t \\ \hat{y}_t \end{bmatrix}$$

and it maps an a.s.-bounded sequence of perturbed arguments $\{\hat{y}_t; \hat{z}_t\}_{t=0}^{\infty}$ into a unique a.s.-bounded process $u = \{u_t\}_{t=0}^{\infty}$ that is a measurable function of the history of shocks.[6] Invertibility of the derivative operator then means that an inverse exists that maps an a.s.-bounded process $\{u_t\}_{t=0}^{\infty}$ into unique a.s.-bounded processes $\{\hat{y}_t; \hat{z}_t\}_{t=0}^{\infty}$. It turns out

[6]Note how the derivative operator is well-defined because the ccgf of exogenous shocks exists and is differentiable. Also, the existence of the ccgf of exogenous shocks is not a local property and yet is a necessary regularity condition, as the moment $E_t e^{\alpha(z_t) q \varepsilon_{t+1}}$ for a real map α need not exist otherwise, even for arbitrarily small $q > 0$. Jin and Judd (2002) and Kim et al. (2008) make a similar point about the existence of moments of shocks.

that the saddle-point condition of the FRSS is the same one that guarantees invertibility. After having proved the first part by the implicit function theorem, we are then able to invoke the Taylor theorem to prove the second part of Proposition 2, as the uniqueness and differentiability of the implicit functions imply that we can now approximate the local solution around the FRSS $(z_t, \tau) = (\tilde{z}, 0)$. Appendix B provides the full proof that fleshes out these steps.

3.2. Relationship with conventional perturbations

First-order perturbations around the FRSS are not nested in conventional perturbations around the DSS $(z_t, q, \tau) = (\bar{z}, 0, 0)$ (for example, as in Schmitt-Grohé and Uribe, 2004), and hence cannot be replicated with conventional perturbations.[7] For example, in our applications conventional perturbations of third order are still far from both the FRSS and the RSS. Proposition 3 provides sufficient conditions under which nesting does not occur.

Proposition 3. *If $y(z, 1, 0) \neq y(\bar{z}, 1, 0)$ or $y_1(z, 1, 0) \neq y_1(\bar{z}, 1, 0)$, then perturbations around the FRSS are not nested in perturbations around the DSS of arbitrary order ℓ.*

The proof follows by recognizing that one can at most reconstruct the implicit functions $y(\bar{z}, q, 0)$ and $y_1(\bar{z}, q, 0)$ using output from ℓth-order perturbations around the DSS $(z_t, q, \tau) = (\bar{z}, 0, 0)$ as:

$$y(\bar{z}, q, 0) = \lim_{\ell \to \infty} \sum_{i=1}^{\ell} \frac{1}{i!} \frac{\partial^i y(\bar{z}, q, 0)}{\partial q^i}\Big|_{q=0} q^i, \qquad y_1(\bar{z}, q, 0) = \lim_{\ell \to \infty} \sum_{i=0}^{\ell} \frac{1}{i!} \frac{\partial^i y_1(\bar{z}, q, 0)}{\partial q^i}\Big|_{q=0} q^i$$

as long as the implicit functions $y(\bar{z}, q, 0)$ and $y_1(\bar{z}, q, 0)$ have convergent Taylor series at $q = 0$ with a sufficiently large radius of convergence. In this context, the implicit functions of interest are $y(z, 1, 0)$ and $y_1(z, 1, 0)$; so a necessary (but not sufficient) condition for nesting is that the radius of convergence of the Taylor series be larger than one.

It follows that the constant term of a k-order approximation around the DSS does not generically coincide with the FRSS. Indeed, in our applications, including in Figure 1, we can immediately see that both SDSS and TSS are still quite far from the FRSS (as well as from the RSS), and provide a relatively bad approximation when risk matters.

[7] The online appendix discusses this further by comparing analytically the first-order perturbation around the FRSS and a third-order perturbation around the DSS.

3.3. Relationship with loglinear-lognormal affine methods

The macro-finance literature has used affine loglinear-lognormal approximation methods at least since Campbell (1993). We focus on the exposition of the affine method in Malkhozov (2014), which is the most general formulation of extant loglinear-lognormal methods. He shows how affine approximations are nested in second-order perturbations around the DSS. Since Proposition 3 established that approximations around the FRSS cannot in general be nested in second-order perturbations around the DSS, it follows that his affine method does not coincide with an approximation around the FRSS.

More precisely, Malkhozov's strategy is to split expectational equations into a certainty equivalent and variance terms and then linearize the system around the DSS. In our notation, his approximated system is:

$$0 = \bar{f}_1(y_t - \bar{y}) + \bar{f}_2(z_t - \bar{z}) + f_3 E_t(y_{t+1} - \bar{y}) + f_4 E_t(z_{t+1} - \bar{z}) + \frac{1}{2} var_t \left(f_3 y_{t+1} + f_4 z_{t+1}\right)$$
$$z_{t+1} = \bar{f}_1(y_t - \bar{y}) + \bar{g}_2(z_t - \bar{z}) + \lambda(\bar{z})(y_{t+1} - E_t y_{t+1}) + \sigma(z_t)\varepsilon_{t+1}$$

where barred variables denote their values at the DSS, and $\bar{f}_i$ and $\bar{g}_i$ denote the derivatives of functions f and g with respect to their ith element evaluated at the DSS. It follows that affine approximations coincide with our approximation in the special case of Gaussian shocks (when entropy and half the variance coincide), linear functions h and g (so $\bar{f}_1 = f_1$, $\bar{f}_2 = f_2$, $\bar{g}_2 = g_2$), and a constant function λ. The last qualification in particular implies that his method corrects differently for exogenous risk variation, which manifests through $\sigma(z_t)$, than for endogenous risk variation, which manifests through $\lambda(z_t)$. (The online appendix elaborates with a simple example.)

More deeply, however, the heuristic construction of the perturbation around the FRSS in Section 2 mimics in fundamental ways the construction of loglinear-lognormal approximations. In particular, the decomposition of expectational equations (1) into a certainty equivalent and a dispersion term in expression (2) is precisely the hallmark of affine solution methods. In this precise sense, risky steady state approximation methods can be viewed as affine methods.

3.4. Discussion

Our definition of FRSS is precisely the definition in Coeurdacier et al. (2011), who conjecture a linear solution and plug it into the nonlinear equation. Intuitively, the FRSS is the fixed point of a problem that searches jointly for an approximate solution and for expectations evaluated using that same solution. The FRSS can differ from the RSS even though it remains closer to it than the DSS, as seen, for example, in Figures 1 and 2. Generically, the RSS can be computed only by a global solution technique as the point where the nonlinear system converges after a long simulation of zero shocks.

The FRSS remains in our applications a preferable expansion point over the DSS to study the risk implications of a model. Here note that the sufficient condition for the absence of nesting of FRSS perturbations in DSS perturbations listed in Proposition 3 holds in relevant examples. The production economy in Section 4 and the small open economy described in Coeurdacier et al. (2011), and rederived in the online appendix, are key examples where nesting does not occur. Moreover, even when nesting is possible, the speed of convergence as the order of approximation ℓ increases can be impractically large. For example, in the models of Campbell and Cochrane (1999) and Wachter (2006), Figures 1 and 2 show how third-order perturbations fall short of providing a sufficiently accurate approximation of the solution. In those basic examples, the risk implications are severely biased around the DSS, even up to the popular third order.

4. Applications

We illustrate our approximation in the context of three models, and compare it with conventional perturbations. Models with Campbell-Cochrane habits are particularly suited to test our approximation as they display strong heteroskedasticity; the state of the economy is driven by consumption innovations, which are endogenous objects outside an endowment economy. Models with risk-sensitive preferences and time-varying disaster risk similarly produce variation in risk premia, while non-Gaussianities make loglinear-lognormal methods inapplicable.[8]

We start by pricing a risk-free bond and wealth in the Gaussian endowment economies with the habit formation of Campbell and Cochrane (1999) and Wachter (2006), which we illustrated in Figure 1. When pricing wealth we will illustrate the multiplicity problem discussed in Section 2.3 and how to choose which specification to retain. We then show how to handle a non-Gaussian disaster component by pricing a risk-free bond and wealth in the model of Wachter (2013). Finally, we apply our approximation to a more challenging model—the production economy of Jermann (1998) extended as in Chen (2017) to incorporate Campbell-Cochrane habits, and hence a larger role for risk.

[8]Alternatively, models with risk-sensitive preferences and stochastic volatility offer another prominent example of heteroskedasticity that would fit our framework nicely. We select here the example with variable disaster risk because it illustrates both how to deal with non-Gaussianities and heteroskedasticity.

To compare the different solutions, we report policy functions over the state space—an obvious metric—and risk pricing at different horizons by comparing the term structures of zero-coupon claims and, in production economies, multiperiod Euler equation errors. This second exercise decomposes the quality of the approximation at different time horizons, and for claims that are the basis for pricing other, more complex assets and characterizing the welfare costs of fluctuations and the investors' marginal utility. We define errors in the n-period Euler equation from a solution for consumption $c^{(0)}(z_t)$ as:

$$EEE^{(n)}(z_t) \equiv \log_{10} \left| 1 - e^{c^{(n)}(z_t) - c^{(0)}(z_t)} \right|$$

where $c_t^{(n)}(z_t)$ solves equation $0 = \ln E_t e^{m_{t+1}[c_{t+1}^{(n-1)}(z_t), c_t^{(n)}(z_t)] + r_t}$, for points z_t that cover a high-probability region of the state space, and a stochastic discount factor m_{t+1} that is a function of consumption. Intuitively, an n-period Euler equation error of $-\varepsilon$ implies that the consumer is making a one dollar mistake in how much she decides to save over an n-period horizon for every 10^ε dollars spent. Since errors accumulate as the horizon increases, multiperiod Euler equation errors provide an indication of how good the approximation is for long-term valuations.

(In what follows lower-case letters and hat variables will denote, respectively, logarithms and log deviations from the expansion point.)

4.1. Habit formation

A representative consumer with Campbell and Cochrane (1999) preferences

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1 - \gamma}$$

lives in an endowment economy that describes the equilibrium risk-free rate (r) and the log wealth-consumption ratio ($w - c$) as a function of two state variables—the consumption process (C), which follows a random walk $c_{t+1} = \mu + c_t + \sigma\varepsilon_{t+1}$, with $\varepsilon_t \sim Niid(0, 1)$, and a process for surplus consumption ($S \equiv 1 - X/C$) relative to an external habit level (X) with law of motion

$$s_{t+1} = \rho_s s_t + \Lambda(s_t)(c_{t+1} - E_t c_{t+1})$$

for some nonlinear function $\Lambda(s_t)$. Parameter β is the rate of time preference and $1/\gamma$ is the elasticity of intertemporal substitution.

This endowment economy is described by the pricing equation for the risk-free rate

$$0 = \ln E_t e^{m_{t+1}+r_t} \tag{8}$$

where $m_{t+1} = \ln(\beta) - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1}$ is the log stochastic discount factor, and by the pricing equation for the wealth portfolio $W_t = C_t + E_t M_{t+1} W_{t+1}$:

$$e^{w_t - c_t} = 1 + E_t e^{m_{t+1}+\Delta c_{t+1}+w_{t+1}-c_{t+1}} \tag{9}$$

$$= E_t e^{m_{t,t+N}+c_{t+N}-c_t+w_{t+N}-c_{t+N}} + \sum_{n=0}^{N-1} E_t e^{m_{t,t+n}+c_{t+n}-c_t} \tag{10}$$

where $m_{t,t+n} = \sum_{j=1}^{n} m_{t+j}$ is the n-period log stochastic discount factor. Because it is a forward-looking difference equation, equation (9) can be written as (10), so there are infinite ways to cast it into form (2). Namely, for a given $N > 0$, we must solve the system of $2N$ equations:

$$e^{w_t - c_t} = e^{rc_t^{(N)}} + \sum_{n=0}^{N-1} e^{pc_t^{(n)}}, \quad pc_t^{(n)} = \ln E_t e^{m_{t+1}+\Delta c_{t+1}+pc_{t+1}^{(n-1)}}, \quad rc_t^{(n)} = \ln E_t e^{m_{t+1}+\Delta c_{t+1}+rc_{t+1}^{(n-1)}} \tag{11}$$

with boundary conditions $pc_t^{(0)} = 0$ and $rc_t^{(0)} = w_t - c_t$, where $pc_t^{(n)}$ describes the log price-consumption ratio of the nth consumption strip, i.e., a claim to n-periods-ahead consumption, and $rc_t^{(n)}$ is the log value of a claim to n-periods-ahead wealth as a fraction of consumption.

Including difference specification (9)—or (10) with $N = 1$—among the equilibrium conditions implies approximating the price of the sum of strips as conditionally lognormal. Including specification (10) for $N \to \infty$ implies approximating each strip price as conditionally lognormal. Since the sum of lognormals is generically not a lognormal, and in contrast with conventional riskless perturbations, it follows that it matters which specification we choose to approximate, as discussed in Section 2.3.

4.1.1. Perturbation around the FRSS

We use the algorithm in Section 2 to approximate equations (8) and (10).

Step 1. Write expectational equations in terms of a certainty equivalent and entropy:

$$0 = \ln(\beta) - \gamma E_t \Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + r_t + L_t \left(-\gamma \Delta c_{t+1} - \gamma \Delta s_{t+1}\right)$$
$$0 = \ln(\beta) + (1-\gamma) E_t \Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)} + L_t \left((1-\gamma)\Delta c_{t+1} - \gamma \Delta s_{t+1} + pc_{t+1}^{(n-1)}\right)$$

for $n = 1, ..., N$ and with boundary condition $pc_t^{(0)} = 0$.

Step 2. Conjecture linear solutions $r_t = r + \psi_r s_t$ and $pc_t^{(n)} = pc^{(n)} + \psi^{(n)} s_t$ and use the Gaussian ccgf to characterize the entropy terms as:

$$L_t \left(-\gamma \Delta c_{t+1} - \gamma \Delta s_{t+1}\right) = \gamma^2 [1 + \Lambda(s_t)]^2 \frac{\sigma^2}{2}$$
$$L_t \left((1-\gamma)\Delta c_{t+1} - \gamma \Delta s_{t+1} + pc_{t+1}^{(n-1)}\right) = \left(1 - \gamma[1 + \Lambda(s_t)] + \psi^{(n-1)} \Lambda(s_t)\right)^2 \frac{\sigma^2}{2}$$

Step 3. Identify the solution by solving matrix equation (6) or, equivalently, linearize:

$$0 = \ln(\beta e^{-\gamma\mu}) + \gamma(1-\rho_s)s_t + r + \psi_r s_t + \gamma^2[1 + \Lambda(s_t)]^2 \frac{\sigma^2}{2}$$
$$\approx r + \ln(\beta e^{-\gamma\mu}) + \gamma^2 [1 + \Lambda(0)]^2 \frac{\sigma^2}{2} + \left[\psi_r + \gamma(1-\rho_s) + \gamma^2[1 + \Lambda(0)]\Lambda_1(0)\sigma^2\right] s_t$$
$$0 = \ln(\beta) + (1-\gamma)E_t \Delta c_{t+1} - \gamma E_t \Delta s_{t+1} + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)} + \left(1 - \gamma[1 + \Lambda(s_t)] + \psi^{(n)} \Lambda(s_t)\right)^2 \frac{\sigma^2}{2}$$
$$\approx pc^{(n-1)} - pc^{(n)} + \ln(\beta e^{(1-\gamma)\mu}) + \left(1 - \gamma[1 + \Lambda(0)] + \psi^{(n-1)}\Lambda(0)\right)^2 \frac{\sigma^2}{2}$$
$$+ \left[\psi^{(n-1)}\rho_s - \psi^{(n)} + \gamma(1-\rho_s) + \left(1 - \gamma[1 + \Lambda(0)] + \psi^{(n-1)}\Lambda(0)\right)(\psi^{(n-1)} - \gamma)\Lambda_1(0)\sigma^2\right] s_t$$

for $n = 1, ..., N$, and match coefficients to identify the unknown vector $[r; \psi_r; pc^{(n)}; \psi^{(n)}]$ as:

$$r = -\ln(\beta e^{-\gamma\mu}) - \gamma^2 [1 + \Lambda(0)]^2 \frac{\sigma^2}{2} \qquad (12)$$
$$\psi_r = -\gamma(1-\rho_s) - \gamma^2 [1 + \Lambda(0)]\Lambda_1(0)\sigma^2$$
$$pc^{(n)} = pc^{(n-1)} + \ln(\beta e^{(1-\gamma)\mu}) + \left(1 - \gamma[1 + \Lambda(0)] + \psi^{(n-1)}\Lambda(0)\right)^2 \frac{\sigma^2}{2} \qquad (13)$$
$$\psi^{(n)} = \psi^{(n-1)}\rho_s + \gamma(1-\rho_s) + \left(1 - \gamma[1 + \Lambda(0)] + \psi^{(n-1)}\Lambda(0)\right)(\psi^{(n-1)} - \gamma)\Lambda_1(0)\sigma^2$$

with boundary condition $pc^{(0)} = \psi^{(0)} = 0$.

Analogously, and relevant for $N < \infty$, $rc_t^{(n)} = rc^{(n)} + \varphi^{(n)} s_t$ where

$$\begin{aligned} rc^{(n)} &= rc^{(n-1)} + \ln(\beta e^{(1-\gamma)\mu}) + \left(1 - \gamma[1+\Lambda(0)] + \varphi^{(n-1)}\Lambda(0)\right)^2 \frac{\sigma^2}{2} \\ \varphi^{(n)} &= \varphi^{(n-1)}\rho_s + \gamma(1-\rho_s) + \left(1 - \gamma[1+\Lambda(0)] + \varphi^{(n-1)}\Lambda(0)\right)(\varphi^{(n-1)} - \gamma)\Lambda_1(0)\sigma^2 \end{aligned}$$

with $rc^{(0)} = \ln\left(e^{rc^{(N)}} + \sum_{n=0}^{N-1} e^{pc^{(n)}}\right)$ and $\varphi^{(0)} = (e^{rc^{(N)}}\varphi^{(N)} + \sum_{n=0}^{N-1} e^{pc^{(n)}}\psi^{(n)})/(e^{rc^{(N)}} + \sum_{n=0}^{N-1} e^{pc^{(n)}})$.

We can then write the approximate solution for the wealth-consumption ratio as:

$$e^{w_t - c_t} = e^{rc^{(N)} + \varphi^{(N)} s_t} + \sum_{n=0}^{N-1} e^{pc^{(n)} + \psi^{(n)} s_t} \to \sum_{n=0}^{\infty} e^{pc^{(n)} + \psi^{(n)} s_t} \tag{14}$$

with the associated the FRSS of the log wealth-consumption ratio

$$wc_{\text{FRSS}} = \ln\left(\sum_{n=0}^{\infty} e^{pc^{(n)}}\right) \tag{15}$$

4.1.2. Perturbations around the DSS

Compare our approximation with a conventional third-order approximation of the solution for the risk-free rate, the price-dividend ratio of the consumption portfolio, and the price-dividend ratio of consumption strips. Even though, as discussed, the choice of N does not matter for conventional perturbations, for ease of comparison with our approximation we derive the coefficients for the specification of the model with $N \to \infty$. Namely,

$$r_t = -\ln(\beta e^{-\gamma\mu}) - \gamma(1-\rho_s)s_t - \underbrace{\gamma^2[1+\Lambda(0)]^2\frac{\sigma^2}{2}}_{\text{2nd order term}} - \underbrace{\gamma^2[1+\Lambda(0)]\Lambda_1(0)\sigma^2 s_t}_{\text{3rd order term}} \tag{16}$$

$$w_t - c_t = -\ln\left(1 - \beta e^{(1-\gamma)\mu}\right) + (1 - \beta e^{(1-\gamma)\mu}) \sum_{n=0}^{\infty} \beta^n e^{(1-\gamma)n\mu} \left[pc_t^{(n)} - n\ln(\beta e^{(1-\gamma)\mu})\right] \tag{17}$$

$$pc_t^{(n)} = n\ln(\beta e^{(1-\gamma)\mu}) + \bar{\psi}_1^{(n)} s_t + \underbrace{\bar{\psi}_2^{(n)}\sigma^2}_{\text{2nd order term}} + \underbrace{\bar{\psi}_3^{(n)}\sigma^2 s_t}_{\text{3rd order term}} \tag{18}$$

where we highlighted the terms that are progressively captured as the order of approximation increases, and where

$$\begin{aligned}
\bar{\psi}_1^{(n)} &= \bar{\psi}_1^{(n-1)}\rho_s + \gamma(1-\rho_s) \\
\bar{\psi}_2^{(n)} &= \bar{\psi}_2^{(n-1)} + \frac{1}{2}\Big(1-\gamma[1+\Lambda(0)] + \bar{\psi}_1^{(n-1)}\Lambda(0)\Big)^2 \\
\bar{\psi}_3^{(n)} &= \bar{\psi}_3^{(n-1)}\rho_s + \Big(1-\gamma[1+\Lambda(0)] + \bar{\psi}_1^{(n-1)}\Lambda(0)\Big)\,(\bar{\psi}_1^{(n-1)} - \gamma)\Lambda_1(0)
\end{aligned} \tag{19}$$

with $\bar{\psi}_1^{(0)} = \bar{\psi}_2^{(0)} = \bar{\psi}_3^{(0)} = 0$.

Accordingly, we can recover the SDSS and TDSS, which coincide, in this example, for the risk-free rate and for the log wealth-consumption ratio by setting $s_t = 0$ in (16) and (17). Here note that the SDSS and TDSS values of the risk-free rate coincide with the FRSS value, but the FRSS wealth-consumption ratio differs from the SDSS and TDSS values. Figure 1 illustrates these differences for two benchmark parameterizations discussed below.

The reason for these differences can be understood by comparing expressions (13) and (19). Namely, to evaluate the value of the nth consumption strip, and in particular the risk premium it commands, it is necessary to evaluate how the price of the $(n-1)$th strip varies with the state of the economy. Our approximation solves jointly for the price and for the elasticity of the strips, while conventional perturbations proceed iteratively by first solving for the elasticity of the $(n-1)$th strip in the absence of risk premia and then using that elasticity to evaluate the risk premium commanded by the nth strip. This iterative approach results in a loss of accuracy in our example.

4.1.3. Numerical example

We specify sensitivity function $\Lambda(s_t) = S^{-1}\sqrt{1-2s_t} - 1$ and calibrate the model using the values in Campbell and Cochrane (1999) and Wachter (2006) reported in Table 1. Figures 1 and 2 compare the exact solution to our approximation by plotting the map from the value of the state variable (surplus consumption) into the price-dividend ratio of the consumption portfolio.

As discussed in Section 2.3, N should be picked to minimize Euler equation errors of the original forward-looking difference equation (9). In this example, our approximation with $N \to \infty$ (we use $N = 1500$ in practice) offers the most accurate approximation around the expansion point. Indeed, Figure 2a shows the approximation for two extreme values of N, $N = 1$ (labeled 'naive' in Figure 1) plotted in black and $N = 1500$ (labeled 'optimal' in Figure 1) in red. The approximation around the FRSS in the specification with $N = 1$ is

Parameter	Habit formation		Disaster risk
	Camp-Coch	Wachter	Wachter
Frequency	monthly	quarterly	quarterly
Subjective discount factor, β	$.89^{1/12}$	.9843	$\exp(-.012/4)$
Utility curvature parameter γ	2	2	3
Utility curvature parameter ρ			$\{1/3, 1, 3\}$
Habit persistence, ρ_s	$.87^{1/12}$	$.89^{1/4}$	
Steady-state surplus consumption ratio, S	.057	.038	
Mean growth rate (in %), μ	1.89/12	2.20/4	2.52/4
Standard deviation of consumption innovations (in %), σ	$1.50/\sqrt{12}$	$.86/\sqrt{4}$	$2.00/\sqrt{4}$
Average number of disasters per period (in %), p			3.55/4
Mean reversion, ρ_p			$.92^{1/4}$
Volatility of disaster intensity, $\phi\sigma$			.067/4
Impact of disaster, θ			$-.26$
Volatility of disaster impact, $\theta\delta$			$-.10$

Table 1: Deep parameters and their calibration.

noticeably worse than the one associated with $N = 1500$. This dramatic difference illustrates how the FRSS is not uniquely defined, as previously discussed, and how to pin down the best approximation of the RSS.

Furthermore, in this application it is interesting to notice that we can actually reconstruct very accurately the global solution using the output of the first-order approximation around the FRSS. Figure 2b plots a solution that adds the approximate prices of strips as $\sum_{n=0}^{\infty} e^{pc_t^{(n)}}$ under the different approximations of the consumption strip prices $pc_t^{(n)}$, using the fact that the wealth-consumption ratio equals that sum. It is apparent that such a sum of strip prices under the linearization around the FRSS reconstructs almost exactly the global solution. This property follows from the fact that the log price-consumption ratios of consumption strips are nearly linear in the state vector under the calibration in Wachter (2006). In fact, this property holds quite generally in this class of models, with the exception, illustrated in Figure 2c, of the knife-edge parameterization of Campbell and Cochrane (1999) that generates exactly constant risk-free rates at all maturities, which is notoriously more nonlinear and harder to approximate. In these examples with knife-edge parameterizations, our approximation performs similarly only to global solution methods with relatively coarse grids. In this context, recall in fact that Wachter (2005) warned us of a difference in level that is apparent when using an insufficiently coarse grid for the original Campbell and Cochrane (1999) parameterization.

In these examples, conventional third-order perturbations deliver a much less accurate approximation as they capture to a lower extent the nonlinear effect of risk on prices.

Figures 5 and 6 complement Figures 1 and 2 by comparing the global solution for several risk premia and return volatilities with our proposed solution. The figures report the term structures of equilibrium risk premia and realized return volatilities of zero-coupon equities and bonds. Relative to the global solution, a first-order approximation around the FRSS manages to capture the level, amplitude, and, to a varying degree of accuracy, even the shape of the term structures.

4.2. Disaster risk

Consider a discrete-time version of the endowment economy in Wachter (2013). Investors have Epstein-Zin recursive preferences:

$$v_t = c_t + \frac{1}{1-\rho} \ln\left(1 - \beta + \beta e^{(1-\rho)(x_t - c_t)}\right), \quad x_t = \frac{1}{1-\gamma} \ln E_t e^{(1-\gamma)v_{t+1}} \tag{20}$$

where β is the rate of time preference, $1/\rho$ the elasticity of intertemporal substitution, and γ the risk aversion coefficient. The log stochastic discount factor is $m_{t+1} = \ln(\beta) - \rho \Delta c_{t+1} - (\gamma - \rho)(v_{t+1} - x_t)$ and x represents a certainty equivalent. When $\rho \neq 1$ we can rewrite these preferences as:

$$\begin{aligned} v_t - c_t &= \frac{1}{1-\rho} \ln\left(1 - \beta + \beta e^{(1-\rho)(x_t - c_t)}\right) \\ w_t - c_t &= -\ln(1-\beta) + (1-\rho)(v_t - c_t) \end{aligned} \tag{21}$$

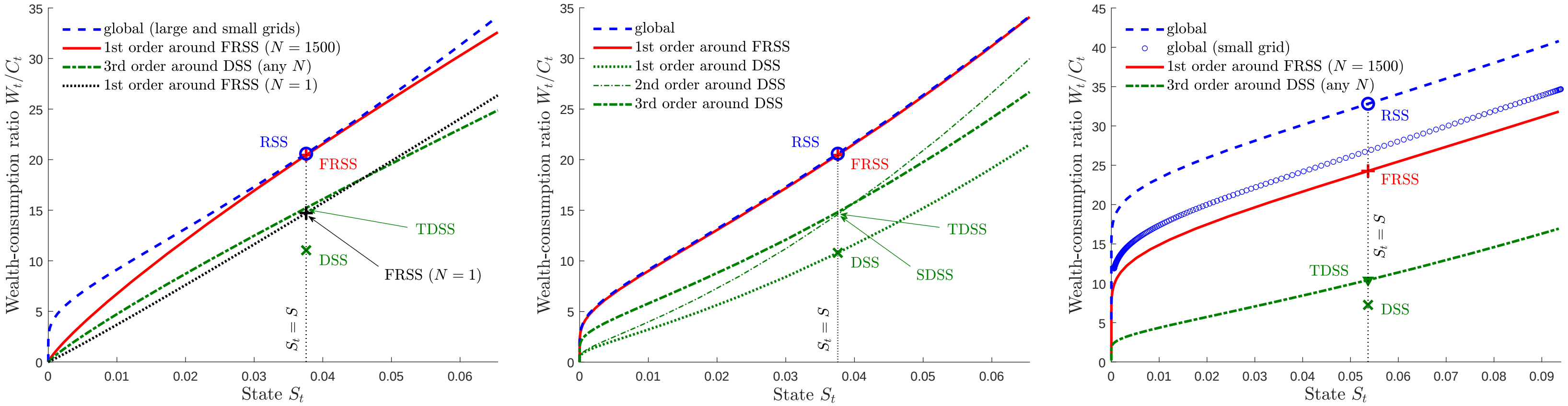
along with equation (11) with boundary conditions $pc_t^{(0)} = 0$ and $rc_t^{(0)} = w_t - c_t$. (These expressions are derived in the online appendix.) When $\rho = 1$, log utility is $v_t = (1-\beta)c_t + \beta(1-\gamma)^{-1} \ln E_t e^{(1-\gamma)v_{t+1}}$.

Investors live in an endowment economy in which log consumption growth has a normal component ε^c as well as a disaster component ξ modeled as a Poisson mixture of normals:

$$c_{t+1} = \mu + c_t + \sigma \varepsilon_{t+1}^c - \theta \xi_{t+1}$$

where $\varepsilon_t^c \sim Niid(0,1)$ and $\xi_t | j_t \sim N(j_t, j_t \delta^2)$, with the number of jumps $j_{t+1} \sim Poisson(p_t)$. We assume that ε_{t+1}^c and $\varepsilon_{t+1}^\xi \equiv \xi_{t+1} - E_t \xi_{t+1}$ are independent, where $E_t \xi_{t+1} = E_t j_{t+1} = p_t$. Disaster intensity p_t evolves according to the discrete-time square-root process:

$$p_{t+1} = (1-\rho_p)p + \rho_p p_t + \sqrt{p_t}\phi\sigma\varepsilon_{t+1}^p$$



(a) *Wachter (2006)*. (b) *Wachter (2006)*. (c) *Campbell and Cochrane (1999)*.

Figure 2: Comparison of solutions for the wealth-consumption ratio in models with Campbell and Cochrane (1999) habits. Markers and arrows denote steady-state values under the different solution methods: global solution (cubic splines collocated over 200 Chebyshev nodes over the interval $(10^{-130}, 10^{-1})$ for S_t and shocks integrated by 20-point Gauss-Hermite quadrature), linear perturbation around the FRSS, perturbations around the DSS. RSS: risky steady state; FRSS: first-order risky steady state; DSS: deterministic steady state; SDSS: second-order DSS; TDSS: third-order DSS. The small-grid global solution uses the same cubic spline approach but over an interval $(10^{-2}, 10^{-1})$ for S_t.

with $\varepsilon_t^p \sim Niid(0,1)$ and independent of ε_t^c and ε_t^ξ. Thus, the shock $\varepsilon_t = [\varepsilon_t^c; \sqrt{p_{t-1}}\varepsilon_t^p; \varepsilon_t^\xi]$ has ccgf:

$$\kappa([\alpha_c; \alpha_p; \alpha_\xi]; p_t) = \frac{1}{2}\alpha_c^2 + \left[\frac{1}{2}\alpha_p^2 + (e^{\alpha_\xi + \frac{1}{2}\alpha_\xi^2\delta^2} - 1) - \alpha_\xi\right] p_t$$

As in the previous example, for each $N > 0$ there is a way to write model (21) in form (1). Finally, the Euler equation that characterizes the risk-free rate r_t is

$$0 = \ln E_t e^{\ln(\beta) - \rho\Delta c_{t+1} + (\rho-\gamma)(v_{t+1} - x_t) + r_t}$$

4.2.1. Perturbation around the FRSS

Step 1. Write expectational equations in terms of a certainty equivalent and entropy:

$$\begin{aligned} 0 &= \ln(\beta) + (1-\rho)E_t\Delta c_{t+1} + (\rho-\gamma)(E_t v_{t+1} - x_t) + E_t pc_{t+1}^{(n-1)} - pc_t^{(n)} \\ &\quad + L_t\left((1-\rho)\Delta c_{t+1} + (\rho-\gamma)v_{t+1} + pc_{t+1}^{(n-1)}\right) \end{aligned} \tag{22}$$

$$0 = \ln(\beta) - \rho E_t\Delta c_{t+1} + (\rho-\gamma)(E_t v_{t+1} - x_t) + L_t\left(-\rho\Delta c_{t+1} + (\rho-\gamma)v_{t+1}\right) + r_t \tag{23}$$

Step 2. Conjecture a linear solution for stationary variables $pc_t^{(n)} = \alpha_0^{(n)} + \alpha_1^{(n)}\hat{p}_t$, $w_t - c_t = wc + \psi_w\hat{p}_t$, $v_t - c_t = vc + \psi_v\hat{p}_t$, $x_t - c_t = xc + \psi_x\hat{p}_t$, and $r_t = r + \psi_r\hat{p}_t$ and combine it with the ccgf to characterize entropy:

$$\begin{aligned} L_t\left((1-\rho)\Delta c_{t+1} + (\rho-\gamma)v_{t+1} + pc_{t+1}^{(n-1)}\right) &= \frac{(1-\gamma)^2\sigma^2}{2} \\ &\quad + \left[\frac{[(\rho-\gamma)\psi_v + \alpha_1^{(n-1)}]^2\phi^2\sigma^2}{2} + e^{(\gamma-1)\theta + \frac{(\gamma-1)^2\theta^2\delta^2}{2}} - 1 + (1-\gamma)\theta\right] p_t \\ L_t\left(-\rho\Delta c_{t+1} + (\rho-\gamma)v_{t+1}\right) &= \frac{\gamma^2\sigma^2}{2} + \left(\frac{(\rho-\gamma)^2\psi_v^2\phi^2\sigma^2}{2} + e^{\gamma\theta + \frac{\gamma^2\theta^2\delta^2}{2}} - 1 - \gamma\theta\right) p_t \end{aligned}$$

Step 3. Identify the linear solution by solving matrix equation (6) or, equivalently, linearize (21):

$$\begin{aligned} 0 &= \frac{1}{1-\rho}\ln\left(1-\beta+\beta e^{(1-\rho)xc}\right) + \frac{\beta e^{(1-\rho)xc}\psi_x}{1-\beta+\beta e^{(1-\rho)xc}}\hat{p}_t - vc - \psi_v\hat{p}_t \\ 0 &= \ln(1-\beta) + wc + \psi_w\hat{p}_t - (1-\rho)(vc + \psi_v\hat{p}_t) \\ 0 &= \ln\left(\sum_{n=0}^{\infty} e^{\alpha_0^{(n)}}\right) + \frac{\sum_{n=0}^{\infty} e^{\alpha_0^{(n)}}\alpha_1^{(n)}}{\sum_{n=0}^{\infty} e^{\alpha_0^{(n)}}}\hat{p}_t - wc - \psi_w\hat{p}_t \end{aligned}$$

plug the entropy term in equations (22) and (23) as

$$0 = \ln(\beta e^{(1-\gamma)\mu}) + (\rho-\gamma)[vc - xc + (\psi_v \rho_p - \psi_x)\hat{p}_t] + \alpha_0^{(n-1)} - \alpha_0^{(n)} + (\alpha_1^{(n-1)}\rho_p - \alpha_1^{(n)})\hat{p}_t$$
$$+ \frac{(1-\gamma)^2\sigma^2}{2} + \left[\frac{[(\rho-\gamma)\psi_v + \alpha_1^{(n-1)}]^2\phi^2\sigma^2}{2} + e^{(\gamma-1)\theta + \frac{(\gamma-1)^2\theta^2\delta^2}{2}} - 1\right] p_t$$
$$0 = \ln(\beta e^{-\gamma\mu}) + (\rho-\gamma)[vc - xc + (\psi_v\rho_p - \psi_x)\hat{p}_t] + \frac{\gamma^2\sigma^2}{2} + \left[\frac{(\rho-\gamma)^2\psi_v^2\phi^2\sigma^2}{2} + e^{\gamma\theta + \frac{\gamma^2\theta^2\delta^2}{2}} - 1\right] p_t + r + \psi_r \hat{p}_t$$

and match coefficients to identify the unknown vector $[\alpha_0^{(n)}; \alpha_1^{(n)}; wc; \psi_w; vc; \psi_v; xc; \psi_x; r; \psi_r]$.

4.2.2. A special case: $\rho = 1$

Note that we can derive an exact solution in the limit as $\rho \to 1$: it is easy to verify that a linear solution for $v_t - c_t$ in p_t solves the problem $v_t = (1-\beta)c_t + \beta(1-\gamma)^{-1}\ln E_t e^{(1-\gamma)v_{t+1}}$.[9] Therefore, our approximation recovers naturally the exact solution, as the identification conditions in step 3 are identical; namely, they reduce to $wc = -\ln(1-\beta)$ and $\psi_w = 0$, which imply $\alpha_0^{(n)} = n\ln(\beta)$ and $\alpha_1^{(n)} = 0$, and to $vc = \beta xc$ and $\psi_v = \beta\psi_x$. Likewise, we recover also the exact risk-free rate

$$r_t = -\ln(\beta e^{-\mu}) - \frac{\gamma^2\sigma^2}{2} + \frac{(\gamma-1)^2\sigma^2}{2} - \left(e^{\gamma\theta + \frac{\gamma^2\theta^2\delta^2}{2}} - e^{(\gamma-1)\theta + \frac{(\gamma-1)^2\theta^2\delta^2}{2}}\right) p_t$$

4.2.3. Perturbations around the DSS

To draw a comparison with conventional perturbations, we focus on the case $\rho \to 1$, which has an exact solution. Here while our approximation recovers the exact solution as $\rho \to 1$, conventional perturbations do so only as the order of approximation goes to infinity. Therefore, even if in this example the FRSS turns out to be nested in conventional perturbations, it is less costly to use our approximation.

Indeed, even in the simple case with time-separable preferences ($\gamma = \rho$), as $\rho \to 1$ a conventional ℓth-order perturbation of the risk-free rate yields:

$$r_t = -\ln(\beta e^{-\mu}) - \theta p_t - \sum_{j=1}^{\ell} \frac{\kappa_{j,t}}{j!} \overset{\ell\to\infty}{\to} -\ln(\beta e^{-\mu}) - \frac{\sigma^2}{2} - (e^{\theta + \theta^2\frac{\delta^2}{2}} - 1)p_t$$

where $\kappa_{j,t}$ is the jth conditional cumulant of $\theta\varepsilon_{t+1}^{\xi} - \sigma\varepsilon_{t+1}^{c}$, hence $\sum_{j=1}^{\infty}\frac{\kappa_{j,t}}{j!} = \ln E_t e^{\theta\varepsilon_{t+1}^{\xi} - \sigma\varepsilon_{t+1}^{c}}$ by definition of the ccgf. Conventional perturbations yield the exact solution only as the

[9] Coefficient ψ_v solves a quadratic equation and we retain the negative root as it is the only one that implies $\psi_v = 0$ when disasters have no impact—$\theta = 0$.

order of approximation goes to infinity.

4.2.4. Numerical example

We calibrate the model using the parameter values in Wachter (2013) for the mean and standard deviation of the distribution of consumption drops during a disaster fitted to OECD country data. Table 1 reports the parameter values. Figure 3 compares the exact solution to our approximation by plotting the map from the value of the state variable (disaster intensity) into the utility-consumption and the wealth-consumption ratios for different values of the elasticity of intertemporal substitution. We consider logutility ($\rho = 1$), a value of 3 ($\rho = 1/3$) (higher than typically considered by the literature), and expected utility ($\rho = \gamma$). The approximation around the FRSS recovers the exact solution at $\rho = 1$ and remains extremely close to the global solution around the expansion point for specifications departing from logutility. The accuracy degrades only in tail regions of the state space, where nonlinearities become particularly relevant (see also Pohl et al., 2018).

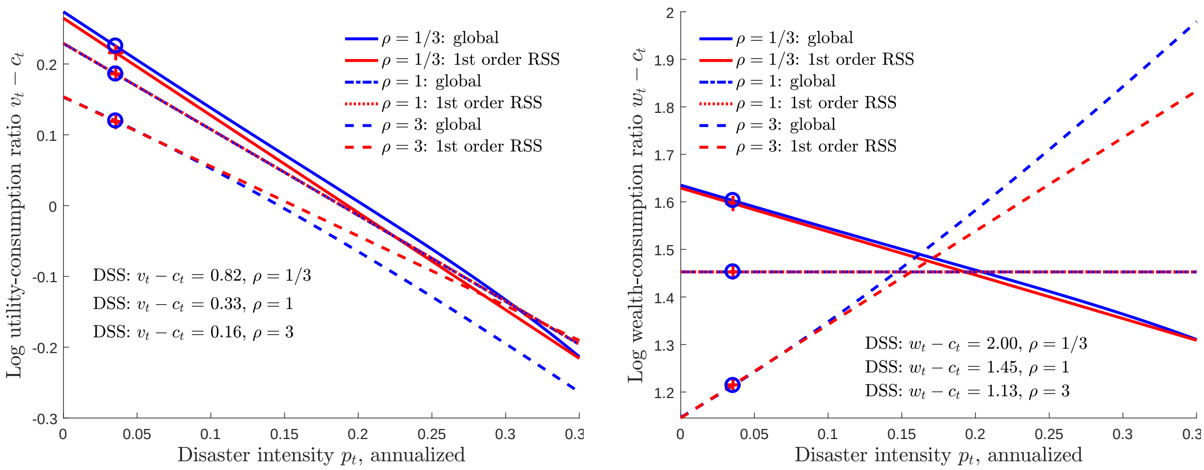
Note that a first-order approximation around the DSS recovers constant functions for the utility-consumption and the wealth-consumption ratios that lie dramatically far from the global functions, as indicated in Figure 3 (tagged by 'DSS').

Figure 7 complements Figure 3 by comparing the global solution with our approximate solution. The figure reports the term structures of equilibrium risk premia and realized return volatilities of zero-coupon equities and bonds. Dividends are defined as in Wachter (2013) as levered consumption $D_t = C_t^{2.6}$. Relative to the projected solution, the first-order perturbation around the FRSS manages to capture the level, amplitude, and shape of the term structures.

4.3. Production economy with habits

Consider next a version of the production economy with the habit formation and capital adjustment costs in Jermann (1998) extended to incorporate a Campbell-Cochrane specification for habits to give a larger role to risk, as studied in Chen (2017). In particular, these habits can fix the counterfactually large interest rate variation in Jermann's specification through a large precautionary savings motive.

A representative consumer with Campbell-Cochrane habits in consumption lives in a production economy and chooses output $Y_t = A_t^{1-\alpha} K_t^{\alpha}$ and the trajectory of capital, whose



(a) *Wachter (2013)*. (b) *Wachter (2013)*.

Figure 3: Comparison of solutions for the map of surplus consumption into the wealth-consumption ratio in Wachter's (2013) model with variable rare disasters. Global solution and first-order perturbations around the FRSS for different values of the elasticity of intertemporal substitution. Global solutions use Smolyak collocation of Chebyshev polynomials of up to degree 8 and 10-point Gauss-Hermite quadrature.

accumulation is subject to adjustment costs:

$$K_{t+1} = \left[1 - \delta + \Phi\left(\frac{I_t}{K_t}\right)\right] K_t = e^{\mu} K_t + \frac{\bar{i}}{1 - \frac{1}{\xi}} \left[\left(\frac{I_t}{\bar{i} K_t}\right)^{1-\frac{1}{\xi}} - 1\right] K_t$$

where $\bar{i} \equiv \frac{\delta}{1+1/\xi}$ is the DSS investment-capital ratio. Output is devoted to consumption or to investment, $Y_t = C_t + I_t$. Technology and habits are driven by:

$$a_{t+1} = \mu + a_t + \sigma \varepsilon_{t+1}$$
$$s_{t+1} = \phi s_t + \Lambda(s_t)(c_{t+1} - E_t c_{t+1})$$

where $\varepsilon_t \sim Niid(0, 1)$.

Joint optimality of consumption, investment, and capital accumulation implies:

$$\frac{i_t - k_t - \log(\bar{i})}{\xi} + \Delta k_{t+1} = \ln\left(e^{rk_t^{(N)}} + \sum_{n=1}^{N} e^{pk_t^{(n)}}\right)$$
$$e^{rk_t^{(n)}} = E_t e^{m_{t+1} + \Delta k_{t+1} + rk_{t+1}^{(n-1)}}, \qquad e^{pk_t^{(n)}} = E_t e^{m_{t+1} + \Delta k_{t+1} + pk_{t+1}^{(n-1)}}$$

with initial conditions $pk_t^{(0)} = d_t - k_t$ and $rk_t^{(0)} = (i_t - k_t - \log(\bar{i}))/\xi + \Delta k_{t+1}$, and marginal

Parameter	Habit formation
Frequency	quarterly
Subjective discount factor, β	.987
Utility curvature parameter γ	2
Habit persistence, ϕ	.98
Steady-state surplus consumption ratio, S	.073
Mean growth rate (in %), μ	1.80/4
Standard deviation of tfp innovations (in %), σ_c	$1.20/\sqrt{4}$
Capital share, α	0.35
Investment-capital ratio, $\bar{i} = \frac{\delta}{1+1/\xi}$	0.0205
Capital adjustment cost curvature, $\frac{1}{\xi}$	0.4

Table 2: Deep parameters and their calibration (quarterly frequency) in the RBC model with Campbell-Cochrane habits.

product of capital net of new investment

$$\frac{D_t}{K_t} = \alpha \frac{Y_t}{K_t} - \frac{I_t}{K_t}$$

This specification is a version of the model in Jermann (1998) with Campbell-Cochrane habits explored by Chen (2017).

4.3.1. Numerical example

Under the specification $\Lambda(s_t) = S^{-1}\sqrt{1-2s_t} - 1$, we calibrate the model using the values listed in Table 2. We let the sensitivity function of surplus consumption vary to avoid a risk-free rate puzzle, and set β and S to achieve a stable risk-free rate around the mean reported by Chen. The rest of the parameterization is the same as in Chen (2017).

Figure 4 plots the policy function of the equilibrium investment-capital ratio and the consumption-productivity ratio as a function of the states. Deterministic and risky steady state values of states differ, especially for detrended capital K/A. Each plot sets the other state to its steady-state value. In particular, the approximation around the FRSS is close to the global solution at the expansion point, while the DSS approximation is inaccurate. For completeness we also plot the standard affine approximation exemplified by Malkhozov (2014), which disregards the volatility in the sensitivity function of surplus consumption due to the presence of endogenous (consumption) risk and linearizes around the DSS. The first-order approximation around the FRSS offers a better approximation. We argue that the rooting of our approximation in perturbation theory makes it superior to ad hoc risk-adjustment strategies, as illustrated by this example.

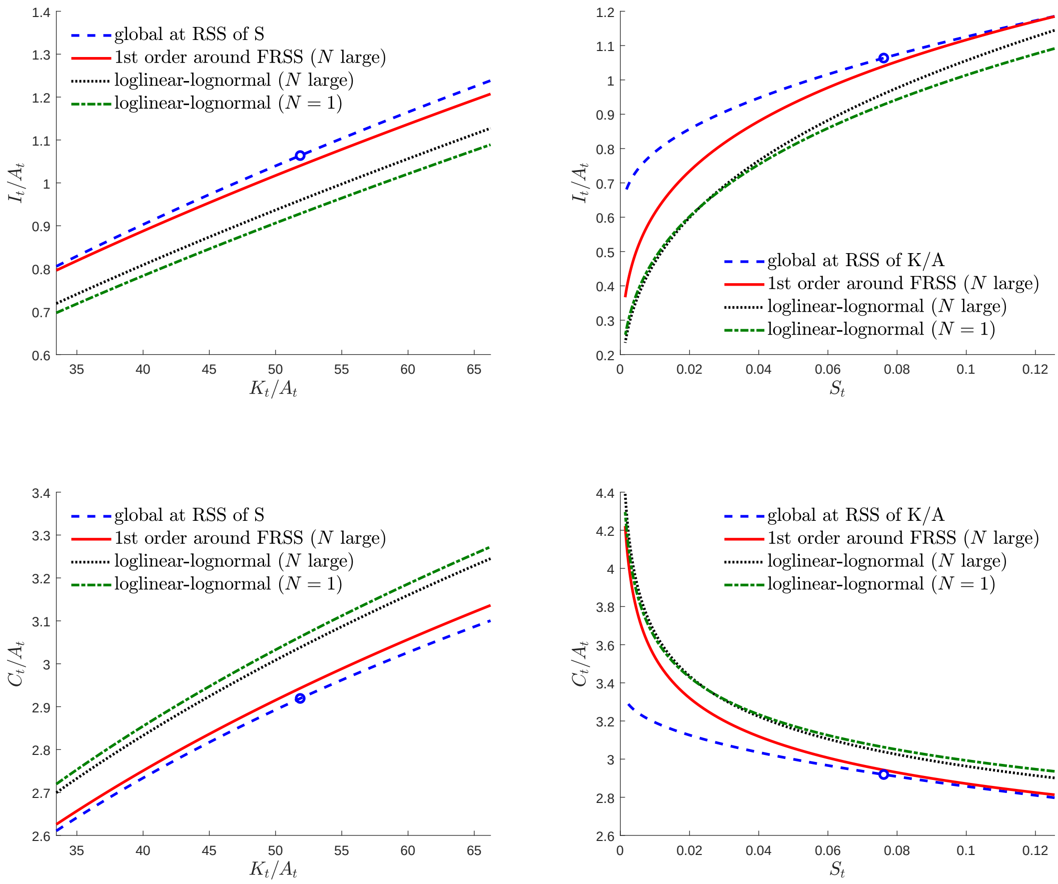


Figure 4: Comparison of the standard loglinear-lognormal affine approximations in Malkhozov (2014) and first-order perturbations around the FRSS for the map of states into detrended investment and consumption in the model with capital accumulation and Campbell-Cochrane habits. The projected solution uses Smolyak collocation of Chebyshev polynomials of up to degree 8 and 10-point Gauss-Hermite quadrature. The state space consists of surplus consumption (S) and detrended capital (K/A). Blue circles denote the RSS values. For a meaningful comparison, we plot solutions as we vary one state and set the other at its RSS value.

Figure 8 plots term structure implications; in this context, the linearization around the FRSS performs similar to the global solution. Risk pricing is accurate at all horizons.

Figure 9 shows multiperiod Euler equation errors. The accuracy of our global solution in terms of conventional one-step ahead Euler equation errors is consistently lower than -2, and remains with maximums of around -2 over arbitrarily long horizons. These values are considerably lower than under the global solution but remain relatively small; values of around -3 are typically retained as acceptable in the extant literature (e.g., Fernández-Villaverde et al., 2015).

5. Conclusion

We developed a theory of first-order perturbations around a specific stochastic steady state—what we call the first-order risky steady state. Importantly, this first-order approximation captures time variation in risk premia; in this precise sense, this approximation method is a natural choice for models that imply risk premia of first-order importance. The resulting approximation technique offers explicit formulas and numerical routines to approximate equilibrium quantities and asset prices in a large class of dynamic macro-finance models as well as conditions for the existence and uniqueness of the approximate local dynamics. We have also provided a flexible and user-friendly Matlab code available online that can be integrated in Dynare.

Appendix

A. Proof of Proposition 1

We follow Klein (2000) and consider the generalized Schur factorization of Γ and Υ, with unitary $Q, Z \in \mathbb{C}^{n_y+n_z \times n_y+n_z}$ and upper triangular matrices $S, T \in \mathbb{C}^{n_y+n_z \times n_y+n_z}$ such that:

$$Q\Gamma Z = S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \quad Q\Upsilon Z = T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \quad Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}, \; Z^* = \begin{bmatrix} Z_{11}^* & Z_{21}^* \\ Z_{12}^* & Z_{22}^* \end{bmatrix}$$

with Z^* the conjugate transpose of Z, where $S_{11}, T_{11} \in \mathbb{C}^{n_z \times n_z}$, $S_{22}, T_{22} \in \mathbb{C}^{n_y \times n_y}$, $Z_{11} \in \mathbb{C}^{n_z \times n_z}$, $Z_{12} \in \mathbb{C}^{n_z \times n_y}$, and matrices S, T are sorted with generalized eigenvalues $\alpha(\Gamma, \Upsilon) = \{t_{ii}/s_{ii}, i = 1, ..., n_y + n_z\}$ in increasing order as $|t_{ii}/s_{ii}| < 1, i = 1, ..., n_z$ and $|t_{ii}/s_{ii}| > 1, i = n_z + 1, ..., n_z + n_y$. The dependence of Q, S, T, Z on q is not denoted explicitly for simplicity.

We rewrite the matrix equation that describes the solution (6) as:

$$\Gamma \begin{bmatrix} I_{n_z} \\ \Psi \end{bmatrix} [g_1(y,z)\Psi + g_2(y,z)](z_t - z) = \Upsilon \begin{bmatrix} I_{n_z} \\ \Psi \end{bmatrix} (z_t - z)$$

$$\text{or:} \quad Q\Gamma ZZ^* \begin{bmatrix} I_{n_z} \\ \Psi \end{bmatrix} E_t(z_{t+1} - z) = Q\Upsilon ZZ^* \begin{bmatrix} I_{n_z} \\ \Psi \end{bmatrix} (z_t - z) \quad \Leftrightarrow \quad SE_t \begin{bmatrix} x_{z,t+1} \\ x_{y,t+1} \end{bmatrix} = T \begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix} \tag{A.1}$$

with

$$\begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix} \equiv Z^* \begin{bmatrix} I_{n_z} \\ \Psi \end{bmatrix} (z_t - z), \qquad x_{z,t} \in \mathbb{R}_t^{n_z}, \; x_{y,t} \in \mathbb{R}_t^{n_y} \tag{A.2}$$

Note that the upper triangular matrices S_{11} and T_{22} are invertible, as their respective eigenvalues $\{s_{ii}, i = 1, ..., n_z\}$ and $\{t_{ii}, i = n_z + 1, ..., n_z + n_y\}$ are nonzero by the assumption about eigenvalues.

By the stability requirement $\lim |E_t z_{t+N}| < \infty$, equation (A.1) implies:

$$x_{y,t} = T_{22}^{-1} S_{22} E_t x_{y,t+1} = (T_{22}^{-1} S_{22})^N E_t x_{y,t+N} \overset{N \to \infty}{\to} 0$$

as the eigenvalues of the upper triangular matrix $T_{22}^{-1} S_{22}$ coincide with $\{s_{ii}/t_{ii}, i = n_z + 1, ..., n_z + n_y\}$, and hence lie within the unit circle. Therefore, $x_{y,t}$ is determined uniquely and is a bounded process if and only if $\{s_{ii}/t_{ii}, i = n_z + 1, ..., n_z + n_y\}$ lies within the unit circle.

Next, using definition (A.2), it follows that $\Psi = -(Z_{22}^*)^{-1} Z_{12}^* = Z_{21} Z_{11}^{-1}$, where the last equality and invertibility are due to the orthonormality of matrix Z. The orthonormality of Z also implies $Z_{11}^* - Z_{21}^* (Z_{22}^*)^{-1} Z_{12}^* = Z_{11}^{-1}$. Therefore, equation (A.1) implies:

$$E_t x_{z,t+1} = S_{11}^{-1} T_{11} x_{z,t}, \quad x_{z,t} = (Z_{11}^* + Z_{21}^* \Psi)(z_t - z) = Z_{11}^{-1}(z_t - z)$$

hence $E_t(z_{t+1} - z) = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} (z_t - z)$, so the spectrum of matrix $g_1(y,0)\Psi + g_2(y,0)$ is:

$$\left\{\lambda \in \mathbb{C} : \det[Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} - \lambda I_{n_z}] = 0\right\} = \left\{\frac{t_{ii}}{s_{ii}}, i = 1, ..., n_z\right\}$$

Therefore, the state vector has stable dynamics if and only if $\{t_{ii}/s_{ii}, i = 1, ..., n_z\}$ lies within the unit circle.

B. Proof of Proposition 2

FRSS is a saddle point ⇒ Locally unique and differentiable implicit functions. The goal is to show that maps $[y, z]$ of $[\varepsilon, q, \tau]$ are defined uniquely and are differentiable on a sufficiently small neighborhood of $[0, q, 0]$. The proof follows from the implicit function theorem, provided we can invoke it. To be able to invoke the implicit function theorem in Banach spaces (e.g., Lang, 1993, p.364), we have to prove that the derivative operator around the expansion point is invertible as a continuous (and hence bounded) linear operator.[10]

In turn, we have invertibility—i.e., an a.s.-bounded process $\{u_t\}_{t=0}^{\infty}$ maps into unique a.s.-bounded processes $\{\hat{y}_t; \hat{z}_t\}_{t=0}^{\infty}$—if and only if the expansion point is a saddle point. To prove this claim, we write the derivative as:

$$Q\ D_{F,t}[\hat{y}; \hat{z}] = S \begin{bmatrix} E_t x_{z,t+1} \\ E_t x_{y,t+1} \end{bmatrix} - T \begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix}, \text{ with } \begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix} \equiv Z^* \begin{bmatrix} \hat{z}_t \\ \hat{y}_t \end{bmatrix}$$

where Q, S, T, Z constitute the Schur factorization of Γ and Υ. The dependence of Q, S, T, Z on q is not denoted explicitly for simplicity. We then note that the derivative operator in equation:

$$D_F[\hat{y}; \hat{z}] = u \quad \Leftrightarrow \quad S \begin{bmatrix} E_t x_{z,t+1} \\ E_t x_{y,t+1} \end{bmatrix} = T \begin{bmatrix} x_{z,t} \\ x_{y,t} \end{bmatrix} + v_t, \quad \begin{bmatrix} v_{z,t} \\ v_{y,t} \end{bmatrix} \equiv Q u_t$$

can be inverted as:

$$x_{y,t} = T_{22}^{-1} S_{22} E_t x_{y,t+1} - T_{22}^{-1} v_{y,t} \overset{N\to\infty}{\to} -\sum_{j=0}^{\infty} (T_{22}^{-1} S_{22})^j T_{22}^{-1} E_t v_{y,t+j}$$

$$E_t x_{z,t+1} = S_{11}^{-1} T_{11} x_{z,t} + S_{11}^{-1}(T_{12} x_{y,t} - S_{12} E_t x_{y,t+1}) + S_{11}^{-1} v_{z,t}$$

if and only if T_{22} and S_{11} are invertible and $T_{22}^{-1} S_{22}$ and $S_{11}^{-1} T_{11}$ have eigenvalues inside the unit circle; this property defines the FRSS as a saddle point. Orthonormal matrices Q and Z map v and $[x_z; x_y]$ back into the original processes u and $[y; z]$.

The invertibility of the derivative operator evaluated at the expansion point implies

[10] We also require $[y, z]$ to be in an open set of the topology of a.s.-bounded functions. As in the case of linear perturbations around the DSS, we can guarantee this property in the topology of essentially bounded functions if exogenous shocks have a.s.-bounded support (Jin and Judd, 2002). Note that the reasoning is local; in particular, for a z_t in a neighborhood of $\tilde{z}$ we have that z_{t+1} is in the same neighborhood only under a sufficiently small $q > 0$. Whether $q = 1$ is sufficiently small will in turn depend on whether $\sigma(\tilde{z})$ is and will be a practical question about the quality of the approximation.

that we can rely on the implicit function theorem to characterize the functions of the history of shocks with the target form $y_t = y(z_t, q, \tau)$ and $z_{t+1} = z(z_t, q, \varepsilon_{t+1}, \tau)$ that solve $F([y, z], \varepsilon, q, \tau) = 0$. Namely, these functions are unique and differentiable in a neighborhood of the expansion point $(z_t, \tau) = (\tilde{z}, 0)$.

Locally unique and differentiable implicit functions ⇒ Coefficients from first-order Taylor approximation equal coefficients from heuristic approximation. The uniqueness and differentiability of the implicit functions imply that we can now approximate the local solution around the FRSS $(z_t, \tau) = (\tilde{z}, 0)$ via the Taylor theorem. (Note that no expansion in q will take place.) We are looking to identify the approximate functions:

$$\begin{aligned}
y_t &= y(\tilde{z}, q, 0) + y_1(\tilde{z}, q, 0)(z_t - \tilde{z}) + y_3(\tilde{z}, q, 0)\tau \\
z_{t+1} &= z(\tilde{z}, q, \varepsilon_{t+1}, 0) + z_1(\tilde{z}, q, \varepsilon_{t+1}, 0)(z_t - \tilde{z}) + z_4(\tilde{z}, q, \varepsilon_{t+1}, 0)\tau \\
x_{t+1} &= x(\tilde{z}, q, \varepsilon_{t+1}, 0) + x_1(\tilde{z}, q, \varepsilon_{t+1}, 0)(z_t - \tilde{z}) + x_4(\tilde{z}, q, \varepsilon_{t+1}, 0)\tau
\end{aligned}$$

It is useful to define the derivative of a differentiable matrix $\lambda : \mathbb{R}^{n_z} \to \mathbb{R}^{n_y \times n_x}$ as:

$$\lambda_1(0) = \begin{bmatrix} \frac{\partial \lambda_{(1,:)}(0)}{\partial \hat{z}_t} \\ \vdots \\ \frac{\partial \lambda_{(n_y,:)}(0)}{\partial \hat{z}_t} \end{bmatrix} \in \mathbb{R}^{n_z n_y \times n_x}, \quad \frac{\partial \lambda_{(i,:)}(0)}{\partial \hat{z}_t} \equiv \begin{bmatrix} \frac{\partial \lambda_{(i,1)}(0)}{\partial \hat{z}_{1,t}} & \cdots & \frac{\partial \lambda_{(i,n_x)}(0)}{\partial \hat{z}_{1,t}} \\ \vdots & & \vdots \\ \frac{\partial \lambda_{(i,1)}(0)}{\partial \hat{z}_{n_z,t}} & \cdots & \frac{\partial \lambda_{(i,n_x)}(0)}{\partial \hat{z}_{n_z,t}} \end{bmatrix} \in \mathbb{R}^{n_z \times n_x}$$

for each row $i = 1, ..., n_z$.

A Taylor approximation of the equilibrium conditions around point $[z_t, \tau] = [\tilde{z}, 0]$ yields:

$$\begin{aligned}
z(\tilde{z}, q, \varepsilon_{t+1}, 0) &= g[y(\tilde{z}, q, 0), \tilde{z}] + \lambda(\tilde{z})(E_{t+1} - E_t)y[z(\tilde{z}, q, \varepsilon_{t+1}, 0), q, 0] + q\sigma(\tilde{z})\varepsilon_{t+1} \\
&= g[y(\tilde{z}, q, 0), \tilde{z}] + \sigma_z(\tilde{z}, q)\varepsilon_{t+1}, \quad \sigma_z(z_t, q) \equiv q[I_{n_z} - \lambda(z_t)y_1(\tilde{z}, q, 0)]^{-1}\sigma(z_t) \\
z_1(\tilde{z}, q, \varepsilon_{t+1}, 0)(z_t - \tilde{z}) &= [g_1 y_1(\tilde{z}, q, 0) + g_2](z_t - \tilde{z}) + [I_{n_z} \otimes (z_t - \tilde{z})']\lambda_1(\tilde{z})(E_{t+1} - E_t)y[z(\tilde{z}, q, \varepsilon_{t+1}, 0), q, 0] \\
&\quad + \lambda(\tilde{z})(E_{t+1} - E_t)y_1[z(\tilde{z}, q, \varepsilon_{t+1}, 0), q, 0]z_1(\tilde{z}, q, \varepsilon_{t+1}, 0)(z_t - \tilde{z}) \\
&\quad + [I_{n_z} \otimes (z_t - \tilde{z})']\sigma_1(\tilde{z})q\varepsilon_{t+1} \\
&= [g_1 y_1(\tilde{z}, q, 0) + g_2](z_t - \tilde{z}) + [I_{n_z} \otimes (z_t - \tilde{z})']\sigma_{1,z}(\tilde{z}, q)\varepsilon_{t+1} \\
z_4(\tilde{z}, q, \varepsilon_{t+1}, 0) &= g_1 y_3(\tilde{z}, q, 0) + [I_{n_z} - \lambda(z_t)y_1(\tilde{z}, q, 0)]^{-1}\lambda(\tilde{z})(E_{t+1} - E_t)y_3(\tilde{z}, q, 0) \\
&= g_1 y_3(\tilde{z}, q, 0)
\end{aligned}$$

where $g_1 \equiv g_1[y(\tilde{z},q,0),\tilde{z}]$ and $g_2 \equiv g_2[y(\tilde{z},q,0),\tilde{z}]$, with the auxiliary variable:

$$
\begin{aligned}
x(\tilde{z},q,\varepsilon_{t+1},0) &= h[y(\tilde{z},q,0),\tilde{z}] + f_3 y[z(\tilde{z},q,\varepsilon_{t+1},0),q,0] + f_4 z(\tilde{z},q,\varepsilon_{t+1},0) \\
&= h[y(\tilde{z},q,0),\tilde{z}] + f_3 y(\tilde{z},q,0) + f_4\tilde{z} + [f_3 y_1(\tilde{z},q,0) + f_4]\sigma_z(\tilde{z},q)\varepsilon_{t+1} \\
x_1(\tilde{z},q,\varepsilon_{t+1},0)(z_t-\tilde{z}) &= [f_1 y_1(\tilde{z},q,0) + f_2 + f_3 y_1[z(\tilde{z},q,\varepsilon_{t+1},0),q,0] z_1(\tilde{z},q,\varepsilon_{t+1},0) + f_4 z_1(\tilde{z},q,\varepsilon_{t+1},0)]\,(z_t-\tilde{z}) \\
&= [f_1 y_1(\tilde{z},q,0) + f_2 + [f_3 y_1(\tilde{z},q,0) + f_4]\,(g_1 y_1(\tilde{z},q,0) + g_2)]\,(z_t-\tilde{z}) \\
&\quad + [f_3 y_1(\tilde{z},q,0) + f_4][I_{n_z}\otimes(z_t-\tilde{z})']\sigma_{1,z}(\tilde{z},q)\varepsilon_{t+1} \\
x_4(\tilde{z},q,\varepsilon_{t+1},0) &= f_1 y_3(\tilde{z},q,0) + f_3 y_3[z(\tilde{z},q,\varepsilon_{t+1},0),q,0] + [f_3 y_1[z(\tilde{z},q,\varepsilon_{t+1},0),q,0] + f_4]\,z_4(\tilde{z},q,\varepsilon_{t+1},0) \\
&= [f_1 + f_3 + [f_3 y_1(\tilde{z},q,0) + f_4]g_1]\,y_3(\tilde{z},q,0)
\end{aligned}
$$

where $f_1 \equiv f_1[y(\tilde{z},q,0),\tilde{z}]$ and $f_2 \equiv f_2[y(\tilde{z},q,0),\tilde{z}]$. In the derivation we used the property of the approximate solution:

$$
\begin{aligned}
y[z(\tilde{z},q,\varepsilon_{t+1},0),q,0] &= y(\tilde{z},q,0) + y_1(\tilde{z},q,0)\sigma_z(\tilde{z},q)\varepsilon_{t+1} \\
y_1[z(\tilde{z},q,\varepsilon_{t+1},0),q,0] &= y_1(\tilde{z},q,0)
\end{aligned}
$$

that follows from $y[z(z_t,q,\varepsilon_{t+1},0),q,0] = y(\tilde{z},q,0) + y_1(\tilde{z},q,0)[z(z_t,q,\varepsilon_{t+1},0)-\tilde{z}]$.

Next, we evaluate entropy using the local solution:

$$
w(\tilde{z},q,0) = L\left[e^{x(\tilde{z},q,\varepsilon_{t+1})+x_1(\tilde{z},q,\varepsilon_{t+1})(z_t-\tilde{z})}|\tilde{z}\right] = \kappa\left[(f_3 y_1(\tilde{z},q,0)+f_4)\sigma_z(\tilde{z},q);\tilde{z}\right] \tag{B.3}
$$

and hence we identify $[y(\tilde{z},q,0), y_1(\tilde{z},q,0)]$ using equation $E_t x_{t+1} + \tau w(z_t,q,\tau) + (1-\tau)\tilde{L}(z_t,q) = 0$ and matching coefficients as:

$$
\begin{aligned}
0 &= h[y(\tilde{z},q,0),\tilde{z}] + f_3 y(\tilde{z},q,0) + f_4\tilde{z} + \tilde{L}(\tilde{z},q) &\text{(B.4)}\\
0 &= f_1 y_1(\tilde{z},q,0) + f_2 + [f_3 y_1(\tilde{z},q,0) + f_4][g_1 y_1(\tilde{z},q,0) + g_2] + \tilde{L}_1(\tilde{z},q) &\text{(B.5)}\\
0 &= [f_1 + f_3 + [f_3 y_1(\tilde{z},q,0) + f_4]g_1]\,y_3(\tilde{z},q,0) + w(\tilde{z},q,0) - \tilde{L}(\tilde{z},q) &\text{(B.6)}
\end{aligned}
$$

Matrix equations (B.4) and (B.5) coincide with matrix equation (6). It follows that $z = \tilde{z}$, $y = y(\tilde{z},1,0)$ and $\Psi = y_1(\tilde{z},1,0)$. Therefore, matrix equations (B.4) and (B.5) coincide with matrix equation (6); the heuristically derived coefficients can be interpreted as the coefficients from a first-order perturbation around the FRSS $(z_t,\tau) = (\tilde{z},0)$ evaluated at $q \in [0,1]$ and $\varepsilon_{t+1} = 0$.

Finally, $z = \tilde{z}$ and $\Psi = y_1(\tilde{z},1,0)$ imply $w(\tilde{z},q,0) = \tilde{L}(\tilde{z},q)$ by equation (B.3). It follows that $y_3(\tilde{z},q,0) = 0$ by equation (B.6). The local *slope* of the solution with respect to τ is zero.

C. Numerical considerations

The nonlinear system of equations (6) in the unknowns $[y, z, \Psi]$ is amenable to standard numerical solution methods. Still, an educated initial guess is often needed to select the saddle path.

When the DSS can be solved for, it offers a natural initial guess. Namely, we can often start by finding the point $[\bar{y}, \bar{z}]$:

$$0 = f(\bar{y}, \bar{z}) + f_3\bar{y} + f_4\bar{z}, \quad \bar{z} = g(\bar{y}, \bar{z})$$

and, by a QZ decomposition (see Appendix A), the DSS slope $\bar{\Psi}$ that solves:

$$0 = f_1(\bar{y}, \bar{z})\bar{\Psi} + f_2(\bar{y}, \bar{z}) + (f_3\bar{\Psi} + f_4)[g_1(\bar{y}, \bar{z})\bar{\Psi} + g_2(\bar{y}, \bar{z})]$$

We propose two algorithms that start from the output of a first-order approximation around the DSS.

C.1. Continuation algorithm

A simple continuation algorithm to solve system (6) numerically can be based on the observation that the solution $[y, z, \Psi]$ to system:

$$\begin{aligned} 0 &= h(y, z) + f_3y + f_4z + q\tilde{L}(z, q), \quad z = g(y, z) \\ 0 &= f_1\Psi + f_2 + (f_3\Psi + f_4)(g_1\Psi + g_2) + q\tilde{L}_1(z, q) \end{aligned} \tag{C.7}$$

coincides with a linear approximation around the DSS at $q = 0$ and around the FRSS at $q = 1$.

Algorithm 2. *Set a sequence of N scalars $q_0 < ... < q_N$ with $q_0 = 0$ and $q_N = 1$ and solve system (C.7) at $q_0 = 0$, which yields the DSS $(\bar{y}, \bar{z})$ and the associated first-order coefficients $\bar{\Psi}$. Then, for each $n = 1, ..., N$, solve system (C.7) at $q = q_n$ by a numerical solver using the solution of system (C.7) at $q = q_{n-1}$ as the initial guess.*

When the DSS is not a valid expansion point but the FRSS is, the system should be solved directly at $q > 0$, so the algorithm can be started at some value of $q \in (0, 1]$ for which a solution can be obtained.

C.2. Iterative algorithm

A second, faster iterative algorithm to solve system (6) numerically also starts from the DSS.

Algorithm 3. *Initialize the algorithm at $y_0 = \bar{y}$, $z_0 = \bar{z}$, $\Psi_0 = \bar{\Psi}$, and iterate to convergence the following steps:*

1. *Find $[y_n, z_n]$ in:*

$$
\begin{aligned}
0 &= f(y_n, z_n) + f_3 y_n + f_4 z_n + \tilde{L}(z_{n-1}; \Psi_{n-1}) \\
z_n &= g(y_n, z_n)
\end{aligned}
$$

   using $\tilde{L}(z_t; \Psi) = \kappa[(f_3\Psi + f_4)[I_{n_z} - \lambda(z_t)\Psi]^{-1}\sigma(z_t); z_t]$.

2. *By a QZ decomposition find Ψ_n in:*

$$
0 = f_1(y_n, z_n)\Psi_n + f_2(y_n, z_n) + (f_3\Psi_n + f_4)[g_1(y_n, z_n)\Psi_n + g_2(y_n, z_n)] + \tilde{L}_1(z_{n-1}; \Psi_{n-1})
$$

While we advocate for running the algorithm to convergence to solve for the FRSS, an alternative, faster procedure is to run it in simply two stages, thereby stopping at the point $[y_1, z_1]$ and with coefficients Ψ_1. Such a procedure would extend to our setup a strategy similar in spirit to de Groot (2013), who first uses a conventional second-order approximation to approximate the FRSS as the SDSS and then linearizes the equations around the SDSS to derive the linear coefficients of the solution.

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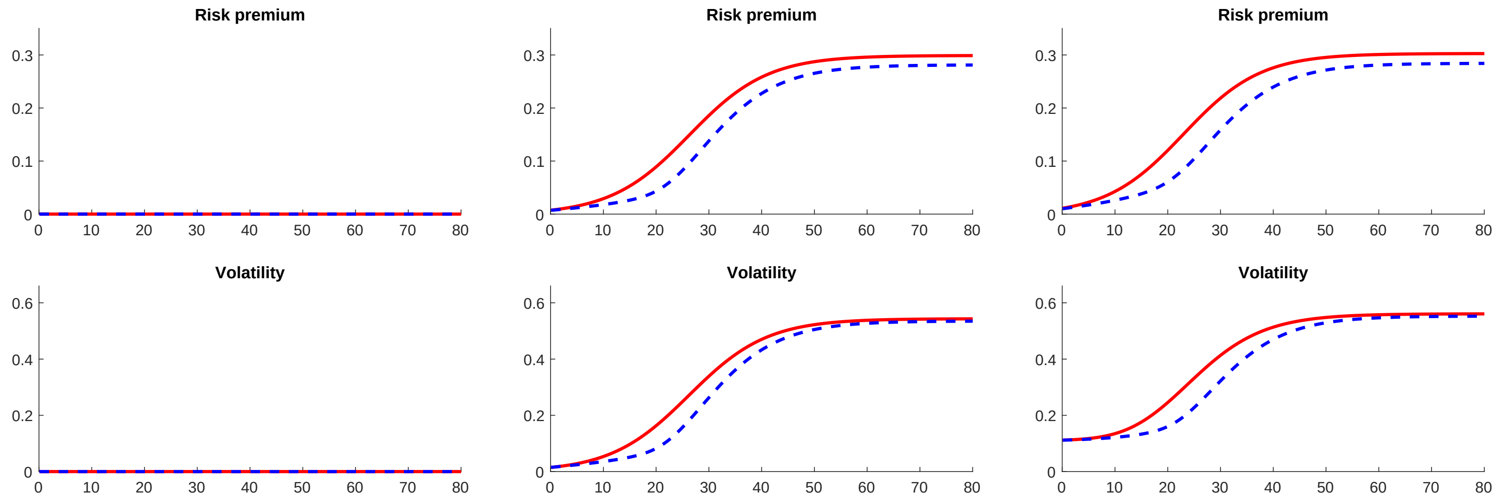
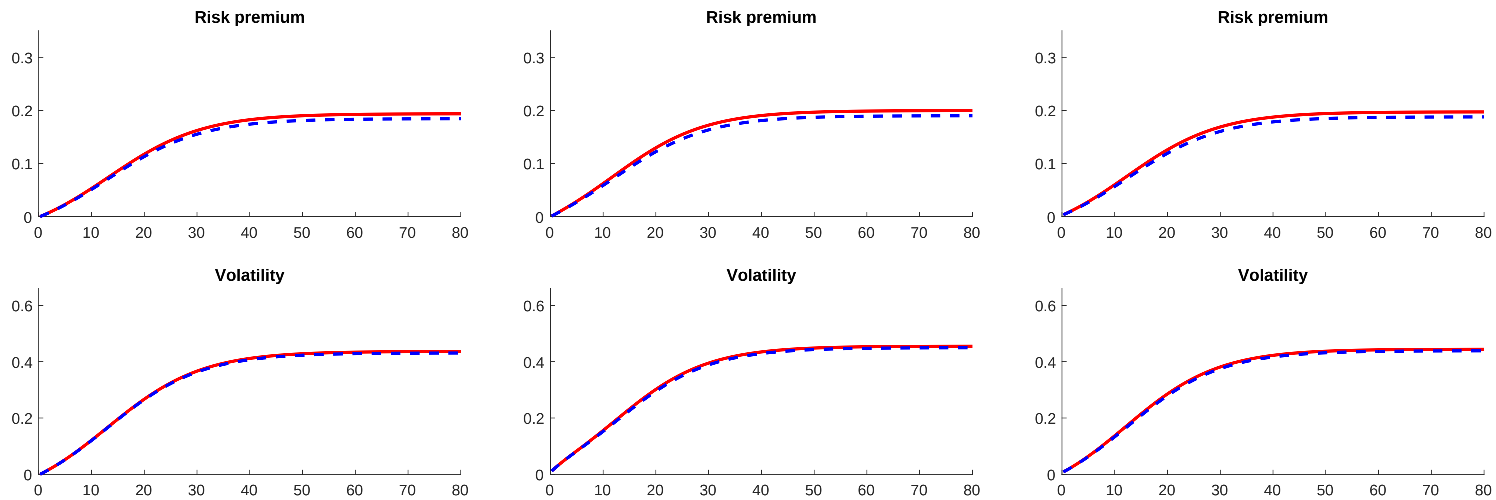


(a) *Real bonds.* (b) *Consumption strips.* (c) *Dividend strips.*

Figure 5: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R_{t+1}^{e,(n)}\}$ and volatilities $\{std_t(r_{t+1}^{(n)})\}$ in Campbell and Cochrane (1999). Linearization around the FRSS (solid red) and projected solution using cubic splines collocated over 200 Chebyshev nodes and 20-point Gauss-Hermite quadrature (dashed blue).



(a) *Real bonds.* (b) *Nominal bonds.* (c) *Consumption strips.*

Figure 6: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R^{e,(n)}_{t+1}\}$ and volatilities $\{std_t(r^{(n)}_{t+1})\}$ in Wachter (2006). Linearization around the FRSS (solid red) and projected solution using cubic splines collocated over 200 Chebyshev nodes and 20-point Gauss-Hermite quadrature (dashed blue).

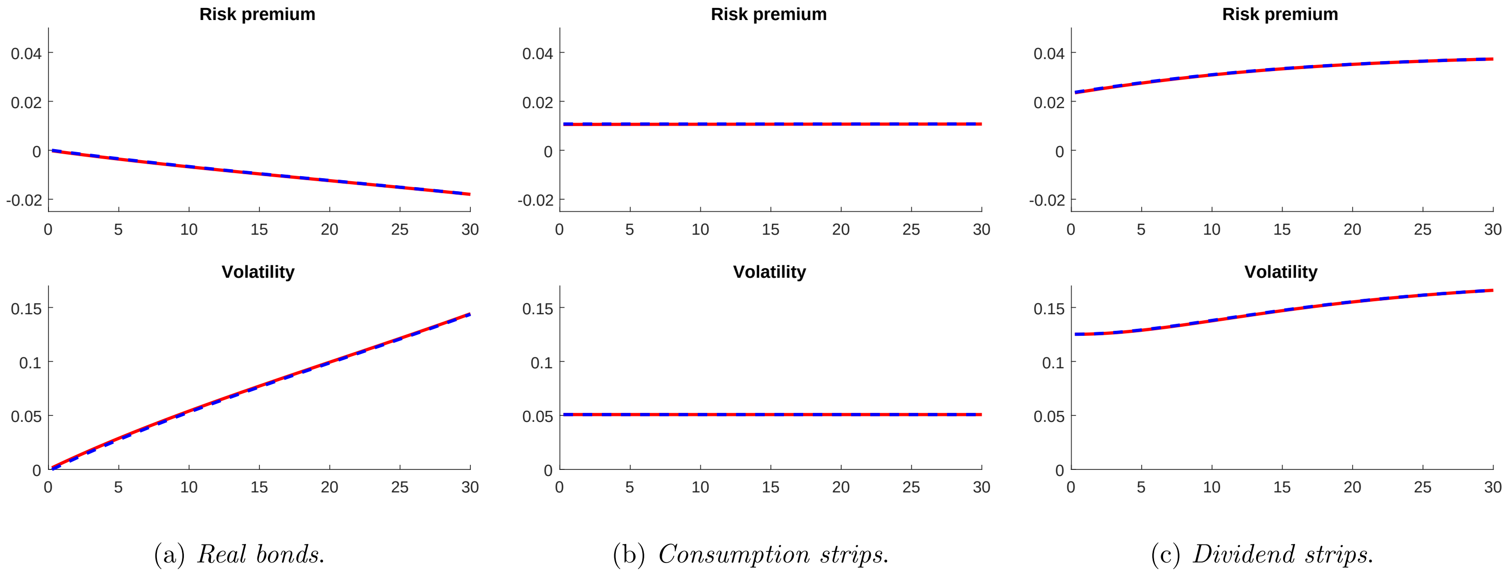
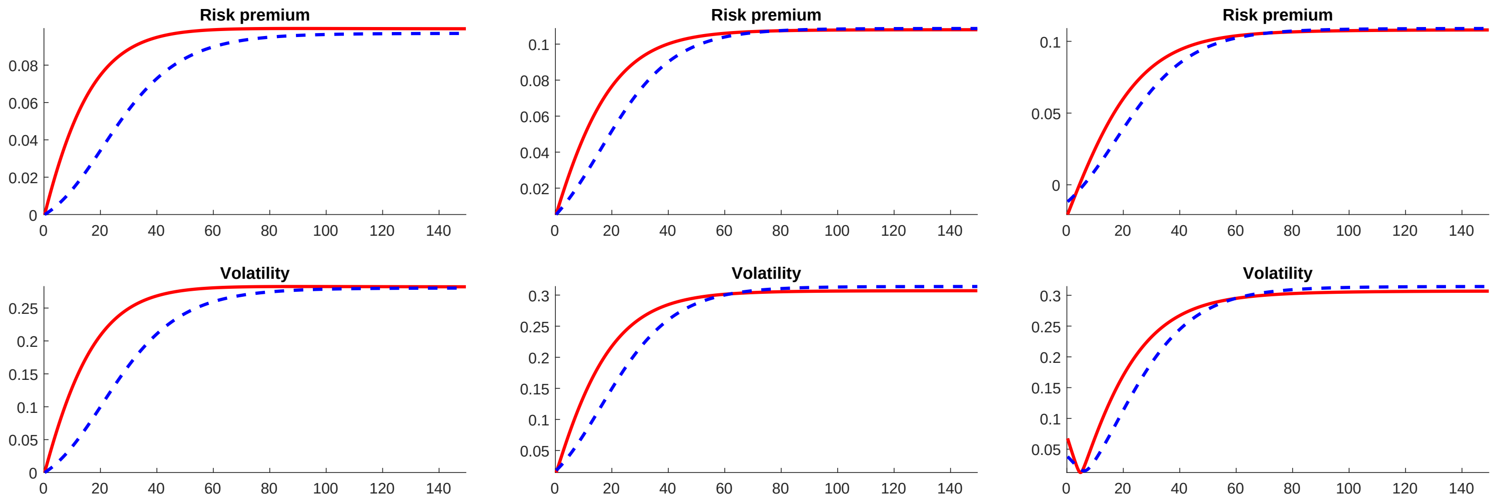
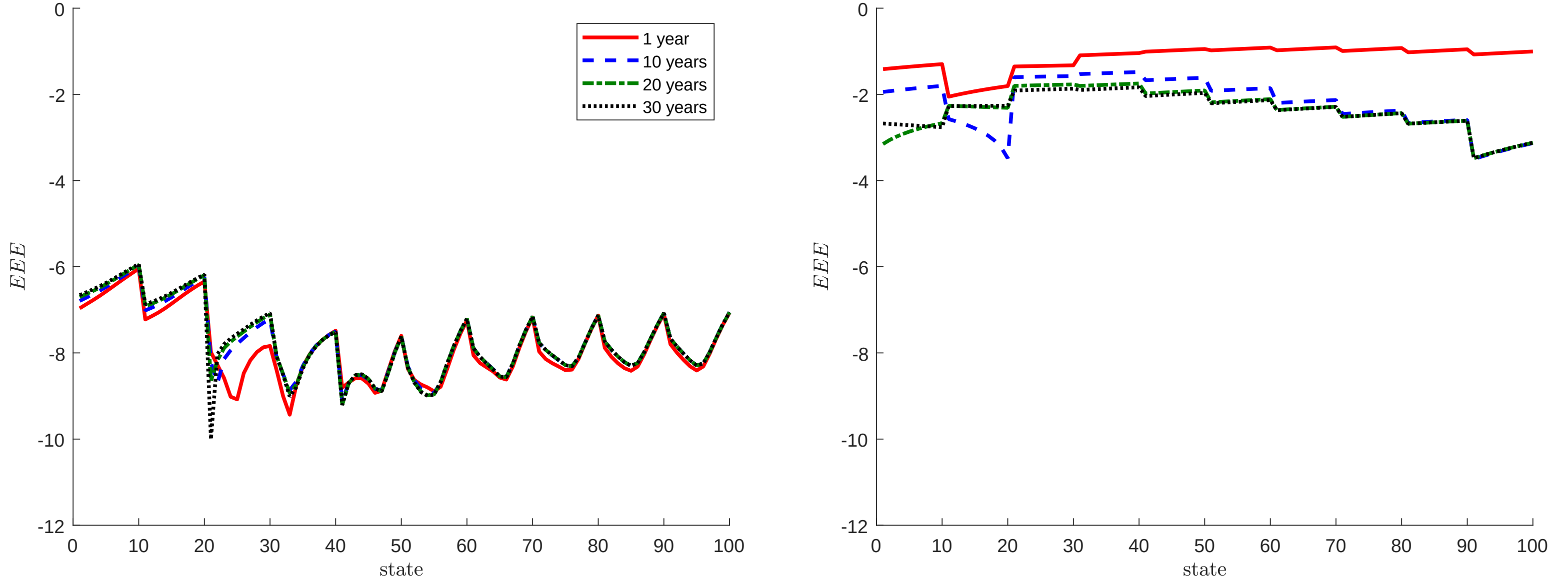


Figure 7: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R_{t+1}^{e,(n)}\}$ and volatilities $\{std_t(r_{t+1}^{(n)})\}$ in Wachter (2013). Linearization around the FRSS (solid red) and projected solution using Chebyshev polynomials of up to degree eight collocated over a Smolyak grid and 10-point Gauss-Hermite quadrature (dashed blue).



(a) *Real bonds.* (b) *Consumption strips.* (c) *Dividend strips.*

Figure 8: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R^{e,(n)}_{t+1}\}$ and volatilities $\{std_t(r^{(n)}_{t+1})\}$ in Jermann (1998)/Chen (2017). Linearization around the FRSS (solid red) and projected solution using Chebyshev polynomials of up to degree eight collocated over a Smolyak grid and 10-point Gauss-Hermite quadrature (dashed blue).



(a) *Projected solution for quantities.*

(b) *Linearization around the FRSS for quantities.*

Figure 9: Multiperiod Euler equation errors in Jermann (1998)/Chen (2017). Errors are expressed in $\log_{10}$. Values in the state dimension index different pairs $[ka_t, \sqrt{1-2\hat{s}_t}]$ built as the Cartesian product of 10 equidistant points along each dimension. The projected solution uses Chebyshev polynomials of up to degree eight collocated over a Smolyak grid. Expectations are evaluated using 10-point Gauss-Hermite quadrature.