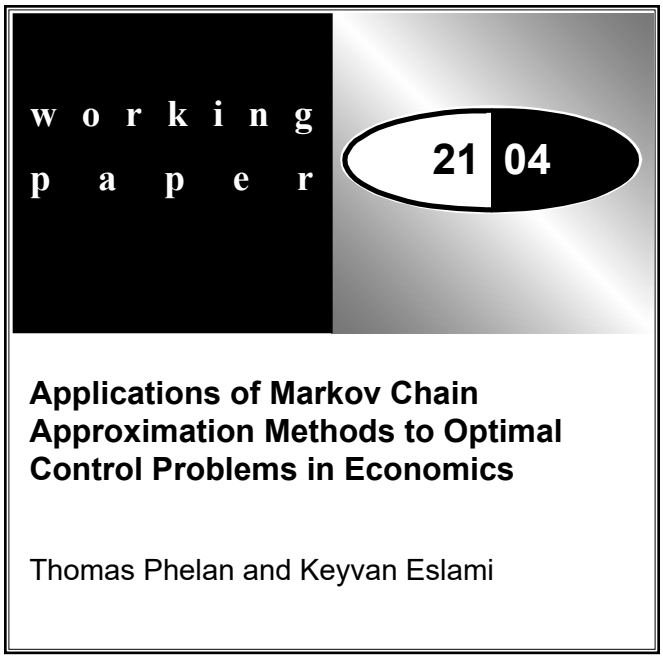

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In this paper we explore some of the benefits of using the finite-state Markov chain approximation (MCA) method of Kushner and Dupuis (2001) to solve continuous-time optimal control problems. We first show that the implicit finite-difference scheme of Achdou et al. (2017) amounts to a limiting form of the MCA method for a certain choice of approximating chains and policy function iteration for the resulting system of equations. We then illustrate the benefits of departing from policy function iteration by showing that using variations of modified policy function iteration to solve income fluctuation problems in two and three dimensions can lead to an increase in the speed of convergence of more than an order of magnitude. We then show that the MCA method is also well-suited to solving portfolio problems with highly correlated state variables, a setting that commonly occurs within general equilibrium models with financial frictions and for which it is difficult to construct monotone (and hence convergent) finite-difference schemes.

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1 Introduction

Dynamic optimization problems are ubiquitous in economics, and since closed form expressions for such problems are available only in isolated special cases, quantitative work requires the use of numerical methods for their solution. In this paper we solve a number of dynamic optimization problems that arise naturally in economic applications by employing the Markov chain approximation (MCA) method of Harold Kushner and Paul Dupuis. Although expositions and applications of this method already exist in the economics literature,[1] the method has several advantages over alternative approaches to continuous-time optimization problems that remain unexploited. To the best of our knowledge, this paper is the first to outline such advantages by means of examples taken from the economics literature.

The most common approach to solving continuous-time optimization problems is the method of finite-differences, which has recently been applied to a number of economic environments by Achdou et al. (2017). In this method, one first establishes that the value function is a (viscosity) solution of a partial differential equation known as the Hamilton-Jacobi-Bellman equation, before replacing derivatives with quotients and solving the ensuing finite system of equations. In contrast, the MCA method approximates the solution to the continuous-time control problem by replacing it with a simpler problem in which the state evolves according to a Markov chain assuming finitely many values. The method formalizes the intuitive idea that if the discrete-time process is "close" to the original process, then the value function of the discrete-state problem should also be close to the original value function.

The criteria necessary for the convergence of the value function of the latter problem to that of the original problem are referred to as *local consistency* conditions. These amount to the requirement that the increments of the chain possess the first- and second-order conditional moments of the original process, at least up to a term that is second-order in the time increment. One benefit to proceeding in this manner is that arguments from discrete-time dynamic programming already familiar to economists — such as the contraction mapping theorem and Blackwell's conditions — are applicable to this new problem and ensure the convergence of the numerical algorithms. Further, even in the presence of non convexities and in multiple dimensions, the Markov chain may

[1] See, e.g., Barczyk and Kredler (2014b), Barczyk and Kredler (2014a), and Golosov and Lucas Jr (2007).

often be chosen so as to eliminate the need for costly root-finding, without sacrificing the global convergence of the algorithm.

We first establish a connection between the above approaches by showing that a limiting case of one widely used finite-difference scheme is equivalent to a particular case of the MCA method. Formally, we show that a limiting version of the implicit finite-difference scheme of Achdou et al. (2017) is equivalent to using the MCA method for a certain chain with negligible timestep and solving the resulting Bellman equation using policy function iteration. Establishing this connection is useful since it means that the former algorithm amounts to making two choices, a choice of chain and a choice of solution method, neither of which may be optimal for a given problem. The two classes of examples given in this paper illustrate the benefits of a more general approach.

The first example shows the benefits of departing from policy function iteration. It is well known that policy function iteration converges at a quadratic rate near the solution, and so typically requires a small number of iterations for convergence. However, updating the value function using this algorithm requires solving a linear system of equations, which becomes very costly computationally as either the number of gridpoints increases or the sparsity structure of the matrix becomes more complex. One therefore expects that the implicit finite-difference method slows down rapidly as one increases the number of gridpoints or the dimension of the state variable. Section 3 confirms this point by considering variations of a problem common in economics, in which an infinitely lived risk-averse agent makes a consumption-savings choice in the presence of idiosyncratic risk and/or discrete choices over a durable good. We show that for standard parameters and moderate grid sizes, variations of the *modified* policy function iteration of Puterman and Shin (1978) can lead to an increase in the speed of convergence of more than an order of magnitude relative to policy function iteration, with no loss of accuracy.

The second example we consider applies the MCA method to (stationary) portfolio problems with highly correlated state variables, a situation that arises in many general equilibrium models with financial frictions. This property poses some difficulties, as it may be impossible to satisfy the local consistency requirements using only transitions to immediately adjacent points. We provide three algorithms of increasing complexity and accuracy for the parsimonious selection of non local points. The first two proceed in an elementary manner, while the third draws upon the abstract

construction in Bonnans et al. (2004), which has recently been applied to general equilibrium models by d'Avernas and Vandeweyer (2019). The purpose of this example is to again illustrate the flexibility of the MCA approach, and to emphasize that the choice of chain is separate from the choice of solution method for the resulting system of equations. Indeed, although we draw upon the aforementioned papers in the contruction of our locally consistent chains, for the stationary problems with which we are concerned, the solution to the individual portfolio problem may be found more rapidly using policy function iteration than with the "false transient" approach more commonly used for time-dependent problems.

As an illustration of this last point, we embed these portfolio problems into a general equilibrium macrofinance model with time-varying volatility in the spirit of Brunnermeier and Sannikov (2014). Given prices, the law of motion for the aggregate state, and the continuation values of the agent, we use the MCA method to discretize the individual problem and compute the policy functions. We then impose market-clearing requirements and consistency between individual and aggregate laws to obtain a single map on continuation values, which we iterate upon until convergence. As with the rest of the literature, we are unable to establish convergence of our algorithm to the competitive equilibrium. However, as we are primarily concerned with the solution to individual control problems we leave further investigation of the computation of equilibria to future work. For clarity and to aid the reader in the implementation of details, Appendix fA tests the speed and accuracy of all methods considered in the paper by applying them to linear-quadratic problems, for which policy and value functions are attainable in closed form. All code used in this paper is written in Python and available upon request.

2 Motivating example

In this section we outline the MCA method in the context of the stochastic one-sector neoclassical growth model. Although this example may be easily solved via a number of different numerical methods, it serves to give an intuitive account of how the method works and to contrast it with the finite-difference method. As we noted in the introduction, the basic idea here is to approximate the solution by solving a simpler problem in which the state evolves according to a chain that assumes only finitely many values. The value function associated with this simpler problem will be a good approximation to the original value function if for any given control vector, the increments of the

chain share the same first and second conditional moments as the original process.

2.1 Setup

Suppose that a social planner wishes to maximize the expected lifetime utility of an infinitely lived representative agent with preferences over consumption

$$U(c) = \mathbb{E}\left[\rho \int_0^\infty e^{-\rho t} u(c_t) dt\right].$$

We assume that capital and consumption goods may be costlessly transformed into one another, and so the sole state variable is the capital stock, which evolves according to the law of motion

$$dk_t = [f(k_t) - \delta k_t - c_t]dt + \sigma(k_t)dZ_t, \tag{1}$$

for some smooth function f satisfying the Inada conditions, constant $\delta > 0$, Brownian motion $(Z_t)_{t\geq 0}$ and smooth function σ. For simplicity suppose σ vanishes outside of some interval $[\underline{k}, \overline{k}]$ and that at these boundary points we impose $c(\underline{k}) \leq f(\underline{k}) - \delta\underline{k}$ and $c(\overline{k}) \geq f(\overline{k}) - \delta\overline{k}$, respectively, where $f(\underline{k}) > \delta\underline{k}$. Given any $k_0 \in [\underline{k}, \overline{k}]$ a natural way to solve this problem is to replace the law of motion (1) with the discrete counterpart

$$k_{t+\Delta_t} = k_t + (f(k_t) - \delta k_t - c_t)\Delta_t + \sqrt{\Delta_t}\sigma(k_t)X_t \tag{2}$$

for some $\Delta_t > 0$, where $(X_t)_{t=0}^\infty$ is an i.i.d. sequence of random variables with mean zero assuming the values ± 1. Using standard dynamic programming arguments,[2] one may show that the value function for the discretized problem is the unique fixed point of the functional equation $BV = V$, where

$$BV(k) = \max_{c\geq 0} \Delta_t u(c) + e^{-\rho\Delta_t}\mathbb{E}\Big[V\Big(k + (f(k) - \delta k - c)\Delta_t + \sqrt{\Delta_t}\sigma(k)X\Big)\Big] \tag{3}$$

subject to $c(\underline{k}) \leq f(\underline{k}) - \delta\underline{k}$ and $c(\overline{k}) \geq f(\overline{k}) - \delta\overline{k}$. One may show that T is a contraction on the space of continuous functions on $[\underline{k}, \overline{k}]$, so the fixed point may be found by applying it repeatedly to an arbitrary initial guess.

The finite-state Markov chain method of Kushner and Dupuis (2001) approximates the original problem in a fundamentally different way. Instead of (2), we consider an optimal control problem in which the capital stock assumes values in a finite grid $S := \{\underline{k}, \underline{k} + \Delta_k, \ldots, \overline{k} - \Delta_k, \overline{k}\}$, where $\Delta_k = (\overline{k} - \underline{k})/N$ for some $N > 1$. We construct the chain such that the increments possess the

[2] As outlined in, e.g., Stokey (1989).

same conditional mean and variance as (2). Specifically, if $c_t = c$, then the transition probabilities of the Markov chain given $k_t = k$ are supported on $\{k - \Delta_k, k, k + \Delta_k\}$ and given by

$$\begin{aligned} p(k, k + \Delta_k, c) &= \frac{\Delta_t}{\Delta_k^2}\left(\frac{\sigma^2(k)}{2} + \Delta_k[f(k) - c - \delta k]^+\right) \\ p(k, k - \Delta_k, c) &= \frac{\Delta_t}{\Delta_k^2}\left(\frac{\sigma^2(k)}{2} + \Delta_k[f(k) - c - \delta k]^-\right) \\ p(k, k, c) &= 1 - p(k, k - \Delta_k, c) - p(k, k + \Delta_k, c) \end{aligned} \tag{4}$$

where $x^{\pm} := \max\{\pm x, 0\}$ for $x \in \mathbb{R}$. Associated with (4) we have the Bellman operator

$$\tilde{B}V(k) = \max_{c \geq 0} \Delta_t u(c) + e^{-\rho \Delta_t} \mathbb{E}\big[V(k')\big] \tag{5}$$

where again we impose the requirements $c(\underline{k}) \leq f(\underline{k}) - \delta \underline{k}$ and $c(\overline{k}) \geq f(\overline{k}) - \delta \overline{k}$. Using the stochastic formulation of dynamic programming given in Stokey (1989), it is easy to show that the operator in (5) is a contraction and so the (unique) fixed point in the space of continuous functions on $[\underline{k}, \overline{k}]$ may be found simply by iterating successively on an arbitrary guess. Now note that the conditional mean of $\Delta k_t \equiv k_{t+\Delta_t} - k_t$ is

$$\mathbb{E}[k_{t+\Delta_t} - k_t | k_t = k] = -p(k - \Delta_k, c)\Delta_k + p(k + \Delta_k, c)\Delta_k = \Delta_t(f(k) - c - \delta k)$$

where for brevity we omit the dependence of probabilities on the original point. Further, the conditional variance $\mathbb{E}[(k_{t+\Delta_t} - k_t)^2 | k_t = k] - \mathbb{E}[(k_{t+\Delta_t} - k_t) | k_t = k]^2$ of Δk is

$$\Delta_t\Big(\sigma^2 k^2 + \Delta_k |f(k) - c - \delta k| - \Delta_t (f(k) - c - \delta k)^2\Big).$$

As $\Delta_t, \Delta_k \to 0$, the law of motion for this problem converges to the law of motion (2). One may then use weak convergence arguments to show formally that fixed points of the operators defined in (3) and (5) also converge to one another (and the value function of the original problem).

Why is this construction useful, given that both (3) and (5) lead to a discrete Bellman equation? The main point here is that in the second discretization, when the agent contemplates how varying the control (consumption) alters tomorrow's payoffs, she need only compare *local* continuation payoffs; so the shape or regularity of the value function is irrelevant. Using (4), the functional equation may be written

$$\frac{(1 - e^{-\rho \Delta_t})}{\Delta_t} V(k) = \max_{c \geq 0} \rho u(c) + e^{-\rho \Delta_t}\big([f(k) - c - \delta k]^+ V^F - [f(k) - c - \delta k]^- V^B\big) \tag{6}$$

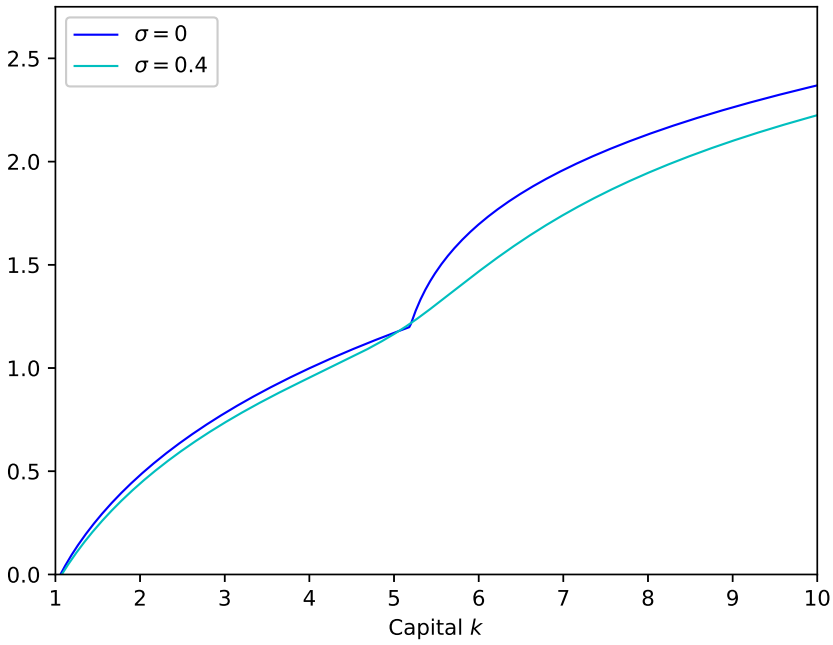

Figure 1: Value functions in a one-sector growth model with non concave technology

where V^F and V^B are approximate forward and backward derivatives. The crucial difference between (3) and (6) is that in the latter, optimal consumption is available in closed form, regardless of the shape of either the value function or production function and the non concavity in the problem. To illustrate, suppose that utility is logarithmic and the production function is $f(k) = \max\left\{\sqrt{k}, 5\sqrt{k-5}\right\}$. Figure 1 shows the computed value functions for the parameters ($\rho = 1, \delta = 0.05, \underline{k} = 1, \overline{k} = 20, N = 1000$), for both a deterministic ($\sigma = 0$) and a stochastic ($\sigma = 0.4$ in interior, vanishing at boundaries) case, and Figure 2 shows the corresponding policy functions. We used policy function iteration and imposed a tolerance of maximum error between successive iterations of 10^{-8}. In both cases, convergence occurs in less than 0.03 seconds using the standard languages in Python and an Intel Core i7-8650U processor, beginning with an initial guess of zero net saving.

We now compare the above approach with a class of finite-difference methods that have been applied to a number of economic problems of interest by Achdou et al. (2017), who in turn build upon the earlier application of Candler (2001). We first (heuristically) derive the appropriate partial

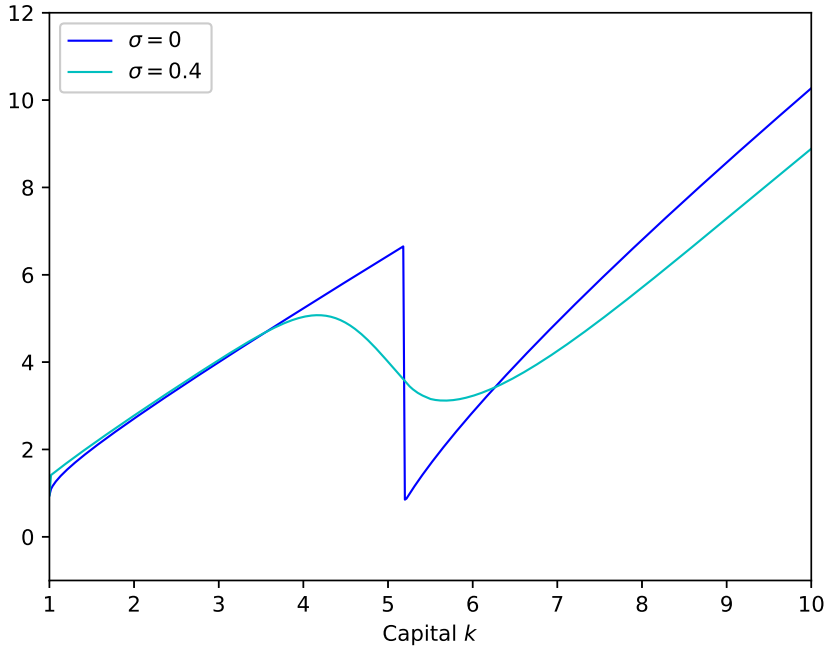

Figure 2: Policy functions in one-sector growth model with non concave technology

differential equation in order to motivate the algorithm.[3] Using the discrete-time version of the principle of optimality, for any $t, h > 0$ we have

$$V(k,t) = \max_c \left[\int_t^{t+h} u(c(s))ds + V(k(t+h), t+h) \right].$$

Subtracting $V(k,t)$ from both sides, dividing by h and using Ito's lemma gives

$$0 = \max_c u(c) + \frac{\partial V}{\partial t} + [f(k) - c - \delta k]\frac{\partial V}{\partial k} + \frac{\sigma(k)^2}{2}\frac{\partial^2 V}{\partial k^2}.$$

A common approach to solving the above partial differential equation is to approximate the partial derivatives with various choices of difference quotients and solve the resulting finite system of equations. Achdou et al. (2017) proceed in this manner and consider two formulations of the method, which they term *explicit* and *implicit*. In this section we focus only on the latter. To understand the algorithm, fix a rectangular grid S for $[\underline{k}, \overline{k}] \times [0, T]$ for some $T > 0$ with increments Δ_k and Δ_T in each dimension, and imagine we are given a terminal value $V(k, T)$. For each

[3]For details we refer to Achdou et al. (2017) and Tourin (2013) and the references therein. Our goal is not to recapitulate the theory of finite-difference methods but to instead relate it to Markov chain approximation methods.

$(k_j, t_n) \in S$ write $V^n(k_j) = V(k_j, t_n)$ and define V^{n+1} as the solution to

$$\begin{aligned}\left(\frac{1}{\Delta_t} + \rho\right)V^{n+1} = \frac{V^n}{\Delta_t} + u(c^n) + [f(k_j) - c_j^n - \delta k_j]^+(V^{n+1})^F \\ - [f(k) - c - \delta k]^-(V^{n+1})^B + \frac{\sigma(k)^2}{2}(V^{n+1})^C\end{aligned} \tag{7}$$

where c^n solves $\max_{c\geq 0} \Delta_t u(c^n) + [f(k) - c - \delta k]^+(V^n)^F - [f(k) - c - \delta k]^-(V^n)^B$ and F, B and C superscripts denote forward, backward and central differences, respectively,

$$\begin{aligned}V^F(k_j) &= \frac{V(k_{j+1}) - V(k_j)}{\Delta_k} \\ V^B(k_j) &= \frac{V(k_j) - V(k_{j-1})}{\Delta_k} \\ V^C(k_j) &= \frac{V(k_{j-1}) - 2V(k_j) + V(k_{j+1})}{\Delta_k^2}.\end{aligned}$$

In practice, for small grids convergence is rapid and insensitive to changes in the timestep when the latter is large. To understand why, note we may write the fixed point of (7) as $\max_{c\in\Gamma(k)} u(c) + T_{\text{IFD}}(c)V$, where

$$T_{\text{IFD}}(c) = [f(k) - c - \delta k]^+V^F(k) - [f(k) - c - \delta k]^-V^B(k) + \frac{\sigma(k)^2}{2}V^C(k) - \rho V(k). \tag{8}$$

If we set $\Delta_t = \infty$, then the implicit method may be written as follows: fix V_0; find c_0 solving $\max_{c\in\Gamma(k)} u(c) + T_{\text{IFD}}(c)V_0$; find V_1 solving $0 = u(c_0) + T(c_0)V_1$; replace V_0 with V_1 and repeat until convergence. The method fits within the framework of Puterman and Brumelle (1979), who establish the convergence of algorithms of this form when $T(c)^{-1} \leq 0$. Finally, if we denote the operator associated with the Markov chain defined by (4) by $T(c; \Delta_t) = [e^{-\rho\Delta_t}P(c) - I]/\Delta_t$, then the following allows us to understand both the convergence properties of the implicit method and its relationship to the Markov chain approximation method.

Lemma 2.1. *For any policy c we have* $\lim_{\Delta_t\to 0} T(c; \Delta_t) = T_{\text{IFD}}(c)$.

Proof. Simply compare the right-hand side of (8) with

$$T(c; \Delta_t)V = -\frac{(1 - e^{-\rho\Delta_t})}{\Delta_t}V + e^{-\rho\Delta_t}\left([f(k) - c - \delta k]^+V^F - [f(k) - c - \delta k]^-V^B + \frac{\sigma(k)^2}{2}V^C\right)$$

from which the result follows by taking limits. □

Lemma 2.1 shows that the implicit finite-difference method of Achdou et al. (2017) amounts to solving the original problem by considering a *particular Markov chain* and a *particular solution method* (policy function iteration) for the resulting system of equations. Although Achdou et al.

(2017) acknowledge a connection between Markov chain approximation methods and their finite-difference approach, we know of no analysis that provides the student of economics with guidance on which method to use in any particular situation. The remainder of this document is devoted to precisely this task by illustrating the benefits of departing from implicit finite-difference schemes for a number of economic problems. We first outline the method formally before turning to these examples.

2.2 Formal overview

This section provides a general overview of the theory developed in Kushner and Dupuis (2001). We focus only on fixing consistent notation and stating the relevant results and definitions necessary to understand subsequent examples and refer the reader to the text for details. We are interested in continuous-time control problems of the following form.

Definition 2.1. Let $B := (B_t)_{t\geq 0}$ be a standard n-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, and denote by $(\mathcal{F}_t)_{t\geq 0}$ the associated natural filtration. For a fixed compact set $U \in \mathbb{R}^m$ define $\mathcal{C}$ to be the set of *admissible controls*, the set of stochastic processes $(u_t)_{t\geq 0}$ adapted to $(\mathcal{F}_t)_{t\geq 0}$ such that $u_t \in U$ for all $t \geq 0$. For some functions $F : \mathbb{R}^n \times U \to \mathbb{R}$, $\mu : \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\mu : \mathbb{R}^n \times U \to \mathbb{R}^{n\times n}$ we consider the optimal control problem

$$\begin{aligned} V(x) = \max_{u\in\mathcal{C}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} F(x_t, u_t)dt\right] \\ dx_t = \mu(x_t, u_t)dt + \sigma(x_t, u_t)dB_t \\ x_0 = x. \end{aligned}$$

We refer to x and u as the *state* and *control* variables, F as the *payoff* function, and μ and σ as the *drift* and *diffusion* functions.

The law of motion for the state is a shorthand for

$$x_t = x_0 + \int_0^t \mu(x_s, u_s)ds + \int_0^t \sigma(x_s, u_s)dB_s \tag{9}$$

where the final term is to be interpreted as an Ito integral. For the examples of this paper we make the following assumption.

Assumption 2.1. The functions F, μ, and σ are continuous, bounded, and Lipschitz.

Assumption 2.1 can be weakened without affecting the validity of the approach. However, it covers many examples of interest to us and ensures that a weak solution to (9) exists and is unique for any admissible control, so that the value function in Definition 2.1 is well-defined. To reduce a problem of the form in Definition 2.1 to a finite-state problem, we must specify how to approximate the underlying *state* and *objective* function. The following introduces the notion of a locally consistent approximating Markov chain, which captures the requirement that the first and second conditional moments coincide with the drift and diffusion coefficients up to a term that vanishes with the timestep.

Definition 2.2. A finite-state Markov chain approximation to the processes $(x_t)_{t\geq 0}$ satisfying (9) for some admissible control $(u_t)_{t\geq 0}$ consists of a family of Markov chains $(\xi^h)_{h>0}$ with finite state spaces $(S_h)_{h>0}$, together with a family of time increment functions $(\Delta^h t(x,u))_{h>0}$ satisfying $\lim_{h\to 0}\sup_{x,u}\Delta^h t(x,u)=0$ and $\inf_{x,u}\Delta^h t(x,u)>0$ for all $h>0$. Defining $\Delta_n^h x=\xi_{n+1}^h-\xi_n^h$ the approximation is *locally consistent* if

$$\begin{aligned}\mathbb{E}^h_{x,n,u}[\Delta_n^h x] &= \Delta^h t(x,u)\mu(x,u)+o(\Delta^h t(x,u))\\ \mathbb{E}^h_{x,n,u}[(\Delta_n^h x-\mathbb{E}^h_{x,n,u}[\Delta_n^h x])^2] &= \Delta^h t(x,u)\sigma(x,u)\sigma(x,u)^t+o(\Delta^h t(x,u))\end{aligned} \tag{10}$$

where $\mathbb{E}^h_{x,u,n}$ denotes the conditional expectation of the chain ξ^h at time t_n given $(\xi_n^h,u_n^h)=(x,u)$, where $u_n^h:=u_{t_n^h}$ and $t_0=0$ and $\Delta^h t_n=t_{n+1}-t_n=\Delta^h t(\xi_n^h,u_n^h)$. In what follows we refer to (10) as the *mean* and *covariance* consistency requirements, respectively.

We will drop superscripts and subscripts for expectations, since the appropriate operator will be obvious from the context. For each ξ^h we approximate the objective in Definition 2.1 as

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t}F(x_t,u_t)dt\right]\approx\mathbb{E}\left[\sum_{n=0}^\infty e^{-\rho t_n^h}\Delta^h t_n F(\xi_n^h,u_n^h)\right]. \tag{11}$$

For each $h>0$, associated with the Markov chain ξ^h and control process $(u_t)_{t\geq 0}$, we define the continuous-time processes $\overline{\xi}^h$ and $\overline{u}^h$ as the right-continuous and piecewise constant processes that coincide with the above chains at the times $(t_n)_{n\geq 0}$. The sum on the right-hand side of (11) is approximately $\mathbb{E}\Big[\int_0^\infty e^{-\rho t}F(\overline{\xi}_t^h,\overline{u}_t^h)dt\Big]$ with the only difference being the continuous discounting on the intervals $[t_n,t_{n+1})$, which necessarily vanishes as $h\to 0$. The weak convergence arguments of Kushner and Dupuis (2001) are applied to these continuously interpolated processes, so that all approximations to the original process are defined on the same path space. However, for each $h>0$ the value function we solve numerically corresponds to the control problem with objective (11) and state evolving according to x^h, and so may be solved with discrete-time techniques.

Definition 2.3. Given a family of Markov chains $\{\xi^h\}_{h>0}$ locally consistent with (9) for each control, define the approximate value functions

$$V^h(x) = \max_{u \in \mathcal{C}} \mathbb{E}\left[\sum_{n=0}^{\infty} e^{-\rho t_n^h} \Delta^h t_n F(\xi_n^h, u_n^h)\right]$$
$$\xi_0^h = x$$

for any $h > 0$, where the maximum is once again over the set of all admissible controls.

The finite-state Markov chain approach applies discrete-time dynamic programming arguments problems of the form in Definition 2.3 rather than the original problem in Definition 2.1. This leads to the Bellman equation for the controlled Markov process being given by

$$V^h(x) = \max_{u \in U} \Delta_t F(x, u) + e^{-\rho \Delta_t} \mathbb{E}^u[V^h(x')] \tag{12}$$

where x evolves according to the given approximating Markov chain. We will consider problems with discounting and uniformly bounded payoff functions, so that there are no subtleties regarding the applicability of the principle of optimality, and the fixed point of (12) coincides with the sequence problem given in Definition 2.3. Familiar arguments, such as those outlined in Stokey (1989), show that the right-hand side defines a contraction on the space of continuous functions defined on some compact subset of the state space. Finally, Kushner and Dupuis (2001) show that under standard assumptions on the functions defined in the original problem Definition 2.1, local consistency ensures convergence of the approximate value functions to the true value function. The following is Theorem 5.2 on page 293 of Kushner and Dupuis (2001).

Theorem 2.2. *Under Assumption 2.1 we have $V^h(x) \to V(x)$ as $h \to 0$.*

The MCA method may be used to solve problems in which there are jumps in the state variable. Although we do not strive for the most general framework possible, we outline here the theory necessary to solve a problem of particular interest to economists, in which a risk-averse consumer faces a consumption-savings problem with fluctuating income and may consume discrete amounts of a durable consumption good. The state variable for the agent will consist of her wealth, her income, and the current value of the durable good. This necessitates a discussion of jump processes, since the purchase of the durable good will coincide with a fall in wealth that does not vanish with the length of the time interval. We therefore consider a jump-diffusion process of the form

$$dx_t = \mu(x_t, u_{Dt})dt + \sigma(x_t)dZ_t + dJ_t(u_{Jt}) \tag{13}$$

where $(J_t)_{t\geq 0}$ is a jump process defined by

$$J_t = \int_0^t \int_\Gamma q(x_{s-}, u_{Js}, \rho) N(dsd\rho). \tag{14}$$

In (14) one interprets the integrand $q(x_{s-}, u_{Js}, \rho)$ as the size of the jumps at time s, with ρ denoting the realization of uncertainty and supported in some compact set Γ. The quantity N is a Poisson random measure with intensity density $h(dtd\rho) = \lambda dt \times \Pi(d\rho)$, meaning $\mathbb{E}[N(A)] = \int_A h(dtd\rho)$ for all Borel A. The control vector is written $u_t := (u_{Dt}, u_{Jt})$ to illustrates that some components affect only the drift and others affect only the jumps. For the case of interest to us we may assume that ρ is supported at a single point and that the jumps in (13) correspond to purchases of the durable good. It may help to imagine that the "jumps" are arriving at a constant exogenous rate $\lambda > 0$, but that they coincide with a movement in the state variable only when $q(x_{t-}, u_{Jt}, \rho) \neq 0$.

To construct a locally consistent Markov chain for (13), one begins with a locally consistent Markov chain for the diffusion component and obtains the approximation by independently drawing from this and the jump component. The definition of a locally consistent finite-state Markov chain now includes an additional component, q_h, representing the jumps of the Markov chain. The manner in which the transition probability given in Definition 2.4 is constructed from the probabilities associated with the diffusion term has an intuitive interpretation. We may interpret it as arising by drawing from a jump process with probability $\lambda\Delta_t$ and drawing from the continuous part with probability $1 - \lambda\Delta_t$.[4]

Definition 2.4 (Local consistency with jumps)**.** A family of finite-state Markov chains $\{\xi^h\}_{h>0}$ with state spaces $\{S_h\}_{h>0}$ is locally consistent with the jump diffusion (13) if for each $h > 0$ there exist transition probabilities $\{p_D^h(x, x')\}_{x,x'\in S_h}$ and functions q_h such that:

1. The family of Markov chains defined by $\{p_D^h(x, x')\}_{x,x'\in S_h}$ is locally consistent with the diffusion process $(z_t)_{t\geq 0}$ defined by $dz_t = \mu(z_t, u_t)dt + \sigma(z_t)dZ_t$;

2. The functions $(q_h)_{h>0}$ satisfy $|q_h(x, u_J) - q(x, u_J)| \to 0$ as $h \to 0$ uniformly in (x, u_J); and

3. For some $\delta_h(x, u) = o(\Delta_t(x, u))$,

$$p^h(x, x', u) = (1 - \lambda\Delta_t - \delta_h(x, u))p_D^h(x, x', u) + (\lambda\Delta_t + \delta_h(x, u))1_{x+q_h(x,u_J)=x'}.$$

[4]The following definition is less general than that in Kushner and Dupuis (2001) but is sufficient to cover the example in Section 3.3.

The analogue of Theorem 2.2 for the case of controlled jump processes is outlined in Chapter 13 of Kushner and Dupuis (2001). To solve control problems of the form in Definition 2.1, we therefore need only solve the Bellman equation (12) for some choice of locally consistent approximating Markov chain. This is important because the literature on dynamic programming with finite state spaces contains a wealth of techniques for solving finite-state Markov decision problems. Section 3 illustrates the benefits of this viewpoint by solving an income fluctuation problem using modified policy function iteration. Section 4 deals with a more involved point that the MCA method is well-suited to solving problems with high correlation among multiple state variables, for which the construction of convergent finite-difference schemes is particularly difficult, and not considered in the analysis of Achdou et al. (2017).

3 Alternative solution methods for finite-state problems

In the one-sector growth model of Section 2, we constructed a locally consistent chain for the capital process and solved the resulting system of equations using policy function iteration. The algorithm converged in a small number of iterations, which is unsurprising given that policy function iteration is known to converge locally at a quadratic rate. However, the updating step in policy function iteration requires solving a linear system of equations. The computational cost of this operation grows rapidly when the number of gridpoints increases or the sparsity structure of the transition matrix becomes more complex, both of which occur naturally as the dimension grows.

In our first application we therefore illustrate the benefits of employing the *modified* policy function iteration algorithm of Puterman and Shin (1978). This algorithm generalizes value function iteration by updating the value function a fixed number of times between successive updates of the policy function. In this case convergence is known to occur only at a linear rate, and so will typically require more iterations than policy function iteration. However, crucially, at no point in the algorithm do we need to solve a linear system of equations. Further, we show that an analogue of the modified policy function remains applicable even when the timestep vanishes, and so is well suited to settings in which the state variable may change by a large amount instantaneously, as is the case in problems in which one must choose consumption in a fixed finite set. To the best of our knowledge, this *generalized* policy function iteration is novel, and in practice appears quite useful.

The construction of the approximating chain in the one-sector growth model of Section 2 required no delicate choices, since in one direction a chain can only move up, down, or stay at the same point. In higher dimensions, one may need to consider transitions to diagonal directions or even non local transitions. We return to this point in Section 4. For ease of exposition this section isolates the benefits of modified policy function iteration relative to policy function iteration, and so considers a problem for which a chain requires no subtle construction. Section 3.2 applies modified policy function iteration to an income fluctuation problem in which income is the product of two diffusion processes. Section 3.3 considers a variation of an income fluctuation problem with discrete choices over a durable good and uses the normalization used prior to Lemma 2.1 to establish that an analogue of modified policy function iteration is applicable even when the timestep is sent to zero. The appendix verifies the accuracy of these methods for linear-quadratic problems.

3.1 Modified and generalized policy function iteration

We briefly recapitulate here the arguments and algorithms of Puterman and Brumelle (1979) and Puterman and Shin (1978) in order to fix ideas and to explain our generalization. Suppose we have a finite-state Markov chain with state space S of cardinality $|S|$, and time increment $\Delta_t \in \mathbb{R}^{|S|}$ and that at each point $x \in S$ the control u may assume values in some subset U of Euclidean space, with the associated transition probabilities $P : S^2 \times U \to [0,1]$. The dynamic programming equation for the discrete-state problem with flow payoff function f and discount rate $e^{-\rho\Delta_t}$ is

$$V(x) = \max_{u \in U} \Delta_t(x) f(x,u) + e^{-\rho\Delta_t(x)} \sum_{x' \in S} P(x,x',u) V(x') \qquad x \in S. \tag{15}$$

Defining $\beta(x) := e^{-\rho\Delta_t(x)}$ and $F(x,u) = \Delta_t(x) f(x,u)$, we can (with some abuse of notation) write this in matrix form as $V = \max_{\hat{u} \in U^{|S|}} F(\hat{u}) + \beta P(\hat{u}) V$, where dependence of F, β, and V on the state is omitted and $P(\hat{u}) \in \mathbb{R}^{|S| \times |S|}$ is the matrix of transition probabilities. For any $\hat{u} \in U^{|S|}$ define $T(\hat{u}) := e^{-\rho\Delta_t} P(\hat{u}) - I$ and write the Bellman equation as

$$0 = \max_{\hat{u} \in U^{|S|}} F(\hat{u}) + T(\hat{u}) v =: B(v) \tag{16}$$

where the second equality defines $B : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$. Policy function iteration is then the following:

- Choose v_0 arbitrarily.
- Choose $\hat{u}(v_0)$ to solve $B(v_0) = F(\hat{u}(v_0)) + T(\hat{u}(v_0)) v_0 = \max_{\hat{u} \in U^{|S|}} F(\hat{u}) + T(\hat{u}) v_0$.
- Given $\hat{u}(v_0)$, define v_1 as the value of adhering to $\hat{u}(v_0)$, $v_1 = -T(\hat{u}(v_0))^{-1} F(\hat{u}(v_0))$.

- Replace v_0 with v_1 in the above and repeat until convergence.

We write $\hat{u}(v)$ for the control that attains the maximum in (16) and abbreviate $\hat{u}_n := \hat{u}(v_n)$. The updating law may be written

$$v_{n+1} = -T(\hat{u}(v_n))^{-1}F(\hat{u}(v_n)) = v_n - T(\hat{u}(v_n))^{-1}B(v_n). \tag{17}$$

A key insight of Puterman and Brumelle (1979) is that policy function iteration is essentially an abstract version of Newton's method and inherits some of the same properties, such as rapid (quadratic) convergence near the solution. However, as we noted earlier, the updating step in policy function iteration requires the solution of a linear system of equations of a size as large as the number of gridpoints. Computational time therefore grows very rapidly as one increases the grid size or dimension, motivating the search for alternatives to policy function iteration.

To this end, note that if $T = \beta P - I$ then $-T(\hat{u}_n)^{-1} = \sum_{j=0}^{\infty}(\beta P(\hat{u}_n))^j$ and (17) becomes

$$v_{n+1} = v_n + \sum_{j=0}^{\infty}(\beta P(\hat{u}_n))^j B(v_n). \tag{18}$$

Modified policy function iteration simply truncates this sum at a finite integer k,

$$v_{n+1} = v_n + \sum_{j=0}^{k}(\beta P(\hat{u}_n))^j B(v_n). \tag{19}$$

The non negativity of P ensures that $v_{n+1} \geq v_n$ if $B(v_n) \geq 0$. Further, the case $k = 0$ corresponds to value function iteration and policy function iteration arises as $k \to \infty$. Puterman and Shin (1978) establish rates of convergence in the case in which $T(\hat{u}) = \beta P(\hat{u}) - I$ for some $\beta \in (0, 1)$ and transition matrix P. However, for our purposes it is useful to note that there exist analogues of (19) for a general T that retain the monotonicity. For an arbitrary T and normalizing function $C : S \times U^{|S|} \to \mathbb{R}$ we define a *generalized* modified policy function iteration algorithm by

$$v_{n+1} = v_n + \sum_{j=0}^{k}(I + \tilde{T}(\hat{u}_n))^j \tilde{B}(v_n) \tag{20}$$

where $\tilde{T}(\hat{u}_n)(x, x') := T(\hat{u}_n)(x, x')/C(x, \hat{u}_n)$ and $\tilde{B}(v_n)(x) := B(v_n)/C(x, \hat{u}_n)$ for all $x, x' \in S$, and $\hat{u}_n$ is defined as in the above. The presence of the normalization $C(\cdot, \hat{u}_n)$ leaves the fixed point unchanged and may be chosen so that the resulting algorithm retains the monotonicity of policy function iteration.

Lemma 3.1. *If C is chosen so $I + \tilde{T}(\hat{u}_n) \geq 0$ for all $n \geq 0$, then $B(v_n) \geq 0$ implies $B(v_{n+1}) \geq 0$ for all $n \geq 0$. Consequently, the sequence defined by (20) is monotone increasing if $B(v_0) \geq 0$.*

Proof. For any $v, w \in \mathbb{R}^{|S|}$ we have $B(w) = \max_{\hat{u} \in U^{|S|}} F(\hat{u}) + T(\hat{u})w \geq F(\hat{u}(v)) + T(\hat{u}(v))w$ and so $B(w) \geq B(v) + T(\hat{u}(v))(w - v)$. For any $n \geq 1$ and $(v, w) = (v_n, v_{n+1})$ we then have

$$\begin{aligned} B(v_{n+1}) &\geq B(v_n) + T(\hat{u}_n)(v_{n+1} - v_n) \\ &= B(v_n) + T(\hat{u}_n) \sum_{j=0}^{k} (I + \tilde{T}(\hat{u}_n))^j \tilde{B}(v_n) \\ &= (I + \tilde{T}(\hat{u}_n))^{k+1} B(v_n) \geq 0 \end{aligned}$$

as claimed. □

The possibility of applying a version of the above algorithm to more general operators than those of the form $T = \beta P - I$ is mentioned in Puterman and Brumelle (1979).[5] Our interest in this generalization arises because we will apply the algorithm to a normalized operator similar to that appearing in Lemma 2.1. Specifically, if $P(\Delta_t) : U^{|S|} \to \mathbb{R}^{|S| \times |S|}$ denotes the transition probability functions arising from a discretization using the MCA method with constant timestep Δ_t, then we apply (20) to $T(\hat{u}) = \lim_{\Delta_t \to 0} (e^{-\rho \Delta_t} P(\hat{u}, \Delta_t) - I)/\Delta_t$ for $\hat{u} \in U^{|S|}$. In the examples that follow, we will always choose C to be the least value such that $I + \tilde{T}(\hat{u}_n) \geq 0$ always. This is convenient because it will avoid the need to choose the timestep so that the transition probabilities remain in the unit interval throughout the iterative process.

3.2 Income fluctuation problem

We first consider the problem of an infinitely lived agent who may borrow and lend at a risk-free rate and who faces idiosyncratic income risk. As emphasized by Achdou et al. (2017), this is a natural application for an economist, as such problems are an integral component of Bewley-Huggett-Aiyagari incomplete markets models, which form the backbone of much of modern macroeconomics. Suppose that preferences over consumption are given by

$$U(c) = \mathbb{E}\left[\rho \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt\right] \tag{21}$$

and that wealth evolves according to

$$da_t = [r a_t - c_t + y_t] dt \tag{22}$$

[5] See the bottom of their page 64. We unfortunately cannot access the references that elaborate on this point.

where $r > 0$ is fixed and $(y_t)_{t\geq 0}$ denotes the income of the agent. We will also assume that the agent faces a borrowing constraint of the form $a_t \geq \underline{a}$ for all $t \geq 0$ and some $\underline{a}$.[6] We assume $y_t = e^{z_t}$ where $z_t := z_{1t} + z_{2t}$ for z_1 and z_2 satisfying

$$dz_{it} = -\theta_i z_{it} dt + \sigma_i dZ_{it} \tag{23}$$

for $i = 1, 2$, where $Z := (Z_{1t}, Z_{2t})_{t\geq 0}$ is two-dimensional Brownian motion. When approximating (23) we assume the volatility vanishes near the boundary, but omit this from the notation for brevity. To construct our chain we must specify the state space, the transition probabilities, and the (possibly state-dependent) timestep. Write $\overline{a}$ for the maximum level of wealth in the discretization, and $\underline{z}_1, \overline{z}_1, \underline{z}_2$ and $\overline{z}_2$ for the lower and upper bounds for the income processes. For a vector of integers $N = (N_a, N_1, N_2)$, define the state increments

$$(\Delta_a, \Delta_1, \Delta_2) = ((\overline{a} - \underline{a})/N_a, (\overline{z}_1 - \underline{z}_1)/N_1, (\overline{z}_2 - \underline{z}_2)/N_2)$$

and the individual grids $S_a = \{\underline{a} + \Delta_a, \ldots, \overline{a} - \Delta_a\}$ and $S_i = \{\underline{z}_i + \Delta_i, \ldots, \overline{z}_i - \Delta_i\}$ for $i \in \{1, 2\}$ and define the state space $S_h := S_a \times S_1 \times S_2$. We choose our transition probabilities so that if the chain is at point $x := (a, z_1, z_2) \in S_h$ at time t then the possible values at time $t + \Delta_t$ lie in the set

$$\Delta(x) := \{(a, z_1, z_2), (a \pm \Delta_a, z_1, z_2), (a, z_1 \pm \Delta_1, z_2), (a, z_1, z_2 \pm \Delta_2)\}.$$

One may check that the following defines a locally consistent chain for any c,

$$\begin{aligned} p(a \pm \Delta_a, z_1, z_2) &= \frac{\Delta_t}{\Delta_a}[ra - c + e^{z_1+z_2}]^{\pm} \\ p(a, z_1 \pm \Delta_1, z_2) &= \frac{\Delta_t}{\Delta_1^2}\left(\frac{\sigma_1^2}{2}\chi_1(z_1) + \Delta_1[-\theta_1 z_1]^{\pm}\right) \\ p(a, z_1, z_2 \pm \Delta_2) &= \frac{\Delta_t}{\Delta_2^2}\left(\frac{\sigma_2^2}{2}\chi_2(z_2) + \Delta_2[-\theta_2 z_2]^{\pm}\right) \end{aligned} \tag{24}$$

where $\chi_i(z_i) := 1_{z_i \notin \{\underline{z}_i + \Delta_i, \overline{z}_i - \Delta_i\}}$ and $i = 1, 2$, provided the timestep is chosen so that the above probabilities lie in the unit interval. The borrowing constraint is imposed by requiring $c \leq ra + y$ when $a = \underline{a} + \Delta_a$. To ensure that the process remains on the grid, we impose $c \geq ra + y$ for $a = \overline{a} - \Delta_a$, although for a sufficiently large upper limit this will not bind. The Bellman equation is then

$$V(x) = \max_{c\geq 0} \Delta_t \frac{c^{1-\gamma}}{1-\gamma} + e^{-\rho\Delta_t}\mathbb{E}[V(x')] \tag{25}$$

[6]The state-dependence of the constraint set means that this does not fit into the framework of Section 2.2. This may be dealt with by introducing the notion of a reflecting boundary. For brevity of notation we omit these details.

for all $x \in S_h$, where the expectation operator is defined by (24). Optimal consumption solves

$$\max_{c \geq 0} \frac{c^{1-\gamma}}{1-\gamma} + e^{-\rho \Delta_t} \left([ra - c + e^{z_1+z_2}]^+ V^{Fa} - [ra - c + e^{z_1+z_2}]^- V^{Ba} \right)$$

which again requires no nonlinear root-finding.

Numerical illustration: We now compare computational times for the above algorithms for a fixed set of parameters. To the extent possible we adopt the parameters of Appendix F in Achdou et al. (2017), who consider a two-dimensional problem and compare the performance of their implicit finite-difference scheme with the endogenous grid method of Carroll (2006). Arguments analogous to those provided for the one-sector neoclassical growth model in Section 2 reveal that the implicit finite-difference method of Achdou et al. (2017) is asymptotically equivalent to using policy function iteration for the probabilities (24) as the timestep vanishes. To contrast the method with the literature, we therefore compare policy function iteration with the algorithms of Section 3.1.

Following Achdou et al. (2017) we fix $\gamma = 2$, $r = 0.03$, and $\rho \approx 0.0526$, corresponding to a discrete-time discount parameter of $\beta = 0.95$. With a single income state variable Achdou et al. (2017) target an annual autocorrelation of 0.95, which implies $\theta = -\ln(0.95) \approx 0.0513$. Since the stationary solution to (23) is Gaussian with mean zero and variance $\nu^2 := \sigma^2/(2\theta)$, their choice of $\nu = 0.2$ implies $\sigma \approx 0.064$. To illustrate the effect of changing the sparsity structure of the transition matrix on the performance of different methods, we consider two choices for the income process. In each case we choose the parameters of Achdou et al. (2017) for the first component of income. In the first case we set $z_2 = 0$ so that the problem becomes two-dimensional, while in the second we choose $(\theta_2, \sigma_2) = (\theta_1, \sigma_1)$. For each choice we solve the above problem using the modified and generalized modified policy function iteration algorithms of Section 3.1 for a number of grid sizes and relaxation steps. In the former case we must also specify a timestep with the property that the probabilities in (24) remain bounded within the unit interval. This can be found either by experimentation or by imposing *a priori* bounds on consumption and checking ex-post that they do not bind. We adopt the latter approach, and conjecture, (and verify ex-post) that consumption never exceeds two and a half times interest and labor income. Note that this experimentation is unnecessary in the case of generalized modified policy function iteration, in which the timestep has been sent to zero, and it is for this reason that we record the output of both exercises.

	PFI	VFI	k = 10	k = 50	k = 100
(200, 10)	0.090998	5.519102	0.720007	0.231040	0.131687
(300, 15)	0.102996	11.164892	1.498263	0.449940	0.386003
(400, 20)	0.228000	22.833399	3.744893	0.970725	0.785935
(500, 25)	0.462520	45.173126	6.354166	1.827667	1.178174

Table 1: Time until convergence: $2D$, MPFI

We now record the speed of convergence for policy function iteration (recall this is the implicit method of Achdou et al. (2017) for $\Delta_t = \infty$), value function iteration, and modified value function iteration with $k = \{10, 50, 100\}$, and four standard deviations in each dimension of the grid and a tolerance between successive iterations of 10^{-6}. All figures are in seconds and all calculations were performed using Python and an Intel Core i7-8650U processor. Table 1 gives the time until convergence (in seconds) for both modified and generalized policy function iteration in two dimensions. In this example there appears to be almost no gain from departing from policy function iteration. Table 3 repeats the analysis for a three-dimensional problem. For the third and fourth choices of grids, modified policy function iteration is easily an order of magnitude faster than policy function iteration. Table 4 gives the analogous results for the case of generalized modified policy function iteration. Note that the generalized modified policy function iteration has the advantage that one need not worry about the timestep being chosen such that the probabilities lie in the unit interval, which in more general settings than the above may be non-trivial to ensure. As a result, it does not appear that one approach is always better than the other and we believe both are of interest. We again emphasize that Appendix A tests the speed and accuracy of all methods considered in the paper by applying them to linear-quadratic problems, for which policy and value functions are attainable in closed form.

3.3 Durable consumption and discrete choice

One interesting property of the generalized modified policy function iteration given in Section 3.1 is that it remains applicable even as the timestep vanishes. This is useful for discrete-choice models in which there are large and instantaneous changes in wealth. To illustrate, we now consider

	PFI	k = 0	k = 10	k = 50	k = 100
(200, 10)	0.073000	1.999446	0.214006	0.088995	0.081000
(300, 15)	0.153181	5.468211	0.519973	0.194000	0.126002
(400, 20)	0.290641	13.241435	1.153003	0.329534	0.249995
(500, 25)	0.454972	23.043634	2.156318	0.652476	0.414068

Table 2: Time until convergence: $2D$, Generalized MPFI

	PFI	VFI	k = 10	k = 50	k = 100
(45, 15, 15)	2.091480	17.713820	2.058243	0.582666	0.379533
(60, 20, 20)	15.324209	59.655353	6.204724	1.677848	1.043586
(75, 25, 25)	57.241754	154.524114	18.712221	4.767470	3.716390
(90, 30, 30)	166.705059	396.583344	52.665829	14.400837	8.748964

Table 3: Time until convergence: $3D$, MPFI

	PFI	k = 0	k = 10	k = 50	k = 100
(45, 15, 15)	4.307563	15.962657	1.417314	0.493199	0.837331
(60, 20, 20)	16.961107	55.899013	5.327223	1.497001	1.622843
(75, 25, 25)	77.082336	207.281934	16.123330	4.166059	3.322334
(90, 30, 30)	220.458093	427.963283	42.920192	10.515165	7.004953

Table 4: Time until convergence: $3D$, Generalized MPFI

a variation of the income fluctuation problem of Section 3.2 in which the agent has preferences over non durable consumption that may assume a continuum of values, as well as a durable good that may assume only finitely many values. This is similar to a continuous-time version of the model of Fella (2014), who extends the endogenous grid method of Carroll (2006) to allow for both adjustment costs and discrete choices. The MCA method may be applied to this case without any delicate choices of grids or any need for interpolation of the function. In this case the amount by which that wealth changes upon purchase of the durable good does not vanish as the size of the grid tends to zero, and so we are not able to restrict attention to adjacent transitions. However, as we shall see, this does not cause any major problems.

Assume the agent has preferences over non durable and durable consumption given by

$$U(c, D) := \mathbb{E}\left[\rho \int_0^\infty e^{-\rho t} u(c_t, D_t) dt\right] \tag{26}$$

for some u and denote the values of durable consumption by $S_D := \{\underline{D}, \underline{D} + \Delta_D, \ldots, \underline{D} + N_D \Delta_D\}$ for some $\underline{D}, N_D$ and Δ_D. Note that in contrast to previous examples, this grid S_D is a primitive of the problem, and not a choice made in the discretization. We again suppose that income is of the form $y_t = e^{z_t}$ for some mean-reverting $(z_t)_{t \geq 0}$, and model the choice of the durable good as follows. At any instant the agent makes a binary choice indicating whether she wishes to change the durable good. However, the opportunities to change the durable good only arrive stochastically at some constant rate $\lambda > 0$. As $\lambda \to \infty$ this approximates a situation in which the durable good may change instantaneously. If $\overline{p}$ denotes the price of the durable good, then for some constants θ and σ with $\theta > 0$ the laws of motion for assets a_t, log income z_t and durable consumption D_t are

$$\begin{aligned} da_t &= [r a_t + e^{z_t} - c_t] dt - \overline{p} dD_t(q_t) \\ dz_t &= -\theta z_t dt + \sigma dZ_t \\ dD_t(q_t) &= dJ_t(q_t) \end{aligned} \tag{27}$$

where $(J_t)_{t \geq 0}$ is a jump process with arrival rate λ and $q_t \in \{(0,0,0), (-\overline{p}\Delta_a, 0, \Delta_D)\}$ indicates the desired change conditional on the jump's arrival. We first define the discretized problem for a positive timestep before considering operators that arise in the limit as we send this quantity to zero, as per the discussion following Lemma 3.1.

First, define the equispaced grids $S := S_a \times S_z \times S_D$, where $S_a := \{\underline{a} + \Delta_a, \ldots, \overline{a} - \Delta_a\}$ and $S_z := \{\underline{z} + \Delta_z, \ldots, \overline{z} - \Delta_z\}$ for some integers $N_a, N_z \geq 1$ and bounds $\underline{a}, \overline{a}, \underline{z}$ and $\overline{z}$, where

$(\Delta_a, \Delta_y) := ((\overline{a} - \underline{a})/N_a, (\overline{z} - \underline{z})/N_z)$. To ensure that income remains on the grid, we impose $\overline{p}\Delta_D = K\Delta_a$ for some $K \geq 1$. We first define the transition probabilities for wealth and income,

$$p(a \pm \Delta_a, z, D) = \frac{\Delta_t}{\Delta_a}[ra + e^z - c]^{\pm}$$
$$p(a, z \pm \Delta_z, D) = \frac{\Delta_t}{\Delta_z^2}\left(\frac{\sigma^2}{2}\chi(z) + \Delta_z[-\theta z]^{\pm}\right)$$

where $\chi(z) := 1_{z \notin \{\underline{z}+\Delta_z, \overline{z}-\Delta_z\}}$ and define the transitions for the durable good

$$p(a - \overline{p}\Delta_D, z, D + \Delta_a) = \lambda 1_{q_t = (-\overline{p}\Delta_a, 0, \Delta_D)}\Delta_t$$

where $q_t := (q_{at}, q_{zt}, q_{Dt}) \in \{0, (-\overline{p}\Delta_a, 0, \Delta_D)\}$. The Bellman equation is then

$$0 = \max_{c,q} u(c, D) + T(c, q; \Delta_t)V$$

where

$$T(c, q; \Delta_t) = u(c, D) + \frac{1}{\Delta_t}\big(e^{-r\Delta_t}\mathbb{E}[V(a', z', D')] - V(a, z, D)\big).$$

The optimal policy for the durable good is $q := (-\overline{p}\Delta_a, 0, \Delta_D)1_{V(a-K\Delta_a, z, D+\Delta_D) > V(a,z,D)}$. For our illustration we follow Fella (2014) and assume preferences of the form $u(c, D) = \ln c + \eta \ln(D + \iota)$ for some $\eta, \iota > 0$, so that the problem of finding optimal consumption is identical to the problem in Section 3.2 with $\gamma = 1$. We now define $\tilde{T}(c, q) = \lim_{\Delta_t \to 0} T(c, q; \Delta_t)$. Simplification gives

$$\begin{aligned}
\tilde{T}(c, q) = u(c, D) &+ \frac{1}{\Delta_a}[ra + e^z - c]^+[V(a + \Delta_a, z, D) - V(a, z, D)] \\
&+ \frac{1}{\Delta_a}[ra + e^z - c]^-[V(a - \Delta_a, z, D) - V(a, z, D)] \\
&+ \frac{1}{\Delta_z^2}\left(\frac{\sigma^2}{2}\chi(z) + \Delta_z[-\theta z]^+\right)[V(a, z + \Delta_z, D) - V(a, z, D)] \\
&+ \frac{1}{\Delta_z^2}\left(\frac{\sigma^2}{2}\chi(z) + \Delta_z[-\theta z]^-\right)[V(a, z - \Delta_z, D) - V(a, z, D)] \\
&+ \lambda(V(a - q_a, z + q_z, D + q_D) - V(a, z, D)) - rV(a, z, D).
\end{aligned} \tag{28}$$

Numerical illustration: For simplicity we retain the same parameters for the process for income as in Section 3.2. Our preferences are of the form used on page 339 of Fella (2014) and are ordinally equivalent to the value (in our notation) $\eta = 1/0.77 - 1$. We also follow Fella (2014) in our choice of interest rate $r = 0.06$, discount parameter $\rho = -\ln(0.93)$, and $\iota = 0.01$ and set the upper bound for durable consumption to be roughly 10 times the unconditional average of income. Table 5 gives the time until convergence for generalized modified policy function iteration, with a tolerance between successive iterations of 10^{-6}. Once again we find that modified policy function is significantly faster than policy function iteration for the larger grids.

	PFI	k = 0	k = 10	k = 50	k = 100
(50, 10, 10)	0.884064	2.792081	0.431957	0.194795	0.216002
(100, 20, 10)	8.636564	19.146504	2.899576	1.217550	0.790998
(150, 30, 10)	35.807196	77.942358	8.511809	3.947027	3.726132
(200, 40, 10)	103.424154	220.948103	31.568330	7.669892	5.873974

Table 5: Time until convergence for discrete choice problem

4 Problems with highly correlated state variables

This section applies the MCA method to a general equilibrium model with financial frictions in the spirit of Brunnermeier and Sannikov (2014). There is a unit mass continuum of infinitely lived agents with preferences over a single consumption good. Agents may be one of two types, termed *experts* and *households*. Experts have access to a linear production technology subject to aggregate depreciation shocks, and all agents have access to a risk-free technology with exogenous return (a storage technology). Capital may be transformed one-for-one into consumption and the sole tradable asset for all agents is a risk-free bond in zero net supply. To illustrate the flexibility of the MCA approach, we allow for time-varying volatility correlated with the depreciation shocks. A growing number of models in the macrofinance literature possess high correlation between multiple state variables and it poses difficulties for the construction of locally consistent chains.

Section 4.1 outlines an environment and Section 4.2 formulates the problem and equilibrium notions recursively. Section 4.3 outlines a general approach for solving a single decision problem with perfect correlation among multiple state variables. Section 4.4 uses the policy functions of Section 4.3 and imposes the requirements that the law of motion of the wealth share be consistent with individual decisions and bond market-clearing. Section 4.5 ties the above observations together to summarize the algorithm and computes an example.

4.1 Setup

Agents may be one of two types, indexed $i \in \{E, H\}$, and referred to as *experts* and *households*, respectively. There is a unit mass of each type indexed by $j \in [0, 1]$. Both types of agents are infinitely lived with the same flow utility function but they differ in their discount rates, with

preferences over sequences of a single consumption good represented by

$$U_i(c) = \mathbb{E}\left[\rho_i \int_0^\infty e^{-\rho_i t} \frac{c_t^{1-\gamma}}{1-\gamma} dt\right]$$

for some $\rho_H, \rho_E > 0$ and $\gamma \in (0,1)$. Aggregate capital in the economy at time $t \geq 0$ is denoted k_t, and the amount held by the jth agent of type i is denoted k_{it}^j. When the jth agent of type i invests a fraction ι_{it}^j of her capital in new capital, the flow output of consumption produced is $(\Pi_i - \iota_{it}^j)k_{it}^j dt$ and the law of motion of her capital stock is

$$dk_{it}^j = \iota_{it}^j k_{it}^j dt + \sigma_t k_{it}^j dZ_t \tag{29}$$

where $Z = (Z_t)_{t\geq 0}$ is a Brownian motion common to all agents. The increments of Brownian motion in (29) may then be thought of as representing stochastic depreciation shocks. The linearity in the investment production technology implies that the price of capital is constant (and here is unity). We also assume that agents have access to a risk-free storage technology with exogenous and constant real return $\underline{r}$. The volatility $(\sigma_t)_{t\geq 0}$ evolves over time according to

$$d\sigma_t = \theta(\overline{\sigma} - \sigma_t)dt + \sigma_\sigma dZ_t \tag{30}$$

for some positive $\theta, \overline{\sigma}$ and σ_σ, where the Brownian motions in (29) and (30) coincide. Agents may trade a risk-free bond in zero net supply with (endogenously determined) return denoted $(r_t)_{t\geq 0}$. An agent with wealth a_t must choose capital k_t, bond b_t and storage h_t holdings satisfying $k_t + b_t + h_t = a_t$. When an agent of type $i \in \{E, H\}$ adheres to choices $(c_{it}, h_{it}, k_{it})_{t\geq 0}$ for consumption, storage, and capital, her wealth evolves according to

$$da_t = [r_t a_t + (\underline{r} - r_t)h_{it} - c_{it} + (\Pi_i - r_t)k_{it}]dt + \sigma_t k_t dZ_t. \tag{31}$$

Note that market-clearing for investment requires the price of capital in terms of consumption to be unity, with investment ultimately determined by the consumption-savings decision. It is for this reason that investment is omitted from (31). We then have the following.

Definition 4.1. The problem of an agent of type $i \in \{E, H\}$ at time $t \geq 0$ with state (a, σ) is

$$\begin{aligned} V_{it}(a,\sigma) = \max_{c,k,h\geq 0} \mathbb{E}&\left[\rho_i \int_t^\infty e^{-\rho_i(\tau - t)} \frac{c_\tau^{1-\gamma}}{1-\gamma} d\tau\right] \\ da_\tau &= [r_\tau a_\tau + (\underline{r} - r_\tau)h_\tau - c_\tau + (\Pi_i - r_\tau)k_\tau]d\tau + \sigma_\tau k_\tau dZ_\tau \\ d\sigma_\tau &= \theta(\overline{\sigma} - \sigma_\tau)d\tau + \sigma_\sigma dZ_\tau \\ (a_t, \sigma_t) &= (a, \sigma). \end{aligned} \tag{32}$$

The notion of competitive equilibrium in this economy is standard: agents maximize utility and markets clear. We first formulate this in a sequential manner.

Definition 4.2 (Competitive equilibrium; sequential formulation)**.** Given an initial distribution of wealth (a^j_{i0}) for $i \in \{E, W\}$ and $j \in [0,1]$ and initial volatility σ_0, a competitive equilibrium consists of a stochastic process $r = (r_t)_{t\geq 0}$ for the risk-free rate, stochastic processes $(V^j_{it})_{t\geq 0}$ and $(c^j_{it}, h^j_{it}, k^j_{it})_{t\geq 0}$ for $i \in \{E, W\}$ and $j \in [0,1]$ such that V^j_i solves (32) with associated policy functions (c^j_i, h^j_i, k^j_i) given r and (a^j_i, σ_0), and the markets for consumption and bonds clear at all dates almost surely. The goods market-clearing condition is

$$\int_0^1 c^j_{Et} dj + \int_0^1 c^j_{Ht} dj = \int_0^1 [\Pi_E - \iota^j_{Et}] k^j_{Et} dj + \int_0^1 [\Pi_H - \iota^j_{Ht}] k^j_{Ht} dj$$

and the bond market-clearing condition is

$$0 = \int_0^1 (1 - h^j_{Et} - k^j_{Et}) dj + \int_0^1 (1 - h^j_{Ht} - k^j_{Ht}) dj.$$

The homotheticity of flow utility and the log-linearity of the law of motion for wealth then give the following, which motivates our subsequent search for a recursive formulation.

Lemma 4.1 (Homogeneity)**.** *For any process $(r_t)_{t\geq 0}$ and $i \in \{E, W\}$ there exist processes $(\overline{V}_{it})_{t\geq 0}$ and $(\overline{c}_{it}, \overline{h}_{it}, \overline{k}_{it})_{t\geq 0}$ such that $V_{it}(a, \sigma) = \overline{V}_{it}(\sigma) a^{1-\gamma}/(1-\gamma)$ and $c_{it}(a, \sigma) = \overline{c}_{it}(\sigma) a, h_{it}(a, \sigma) = \overline{h}_{it}(\sigma) a$ and $k_{it}(a, \sigma) = \overline{k}_{it}(\sigma) a$, respectively, for all $t, a, \sigma \geq 0$.*

In what follows we abuse notation slightly and write V_i for $\overline{V}_i$. Using the linearity of policy functions in Lemma 4.1, aggregate consumption, storage and capital demand may be written as functions of policy functions and the wealth share of experts,

$$x_t := \frac{\int_0^1 a^j_{Et} dj}{\int_0^1 a^j_{Et} dj + \int_0^1 a^j_{Ht} dj}. \tag{33}$$

Section B.2 shows that the wealth share evolves according to the law of motion

$$dx_t = \mu_x(x_t, \sigma_t) x_t dt + \sigma_x(x_t, \sigma_t) x_t dZ_t$$

for some μ_x and σ_x depending on the policy functions of each agent. We now focus on *Markov equilibria* in which all equilibrium quantities are time-independent functions of (x, σ).

4.2 Markov equilibria

We now suppose that $r_t = r(x_t, \sigma_t)$, $\mu_{xt} = \mu_x(x_t, \sigma_t)$ and $\sigma_{xt} = \sigma_x(x_t, \sigma_t)$ for all $t \geq 0$ almost surely for some functions r, μ_x and σ_x. The problem (32) may be written in the following form.

Definition 4.3. For any functions r, μ_x and σ_x, state (a, x, σ) and $i \in \{E, W\}$, the problem of the ith type of agent may be written

$$\begin{aligned} W_i(a,x,\sigma) = \max_{c_t,k_t \geq 0} \mathbb{E}\left[\int_0^\infty \rho_i e^{-\rho_i t} \frac{c_t^{1-\gamma}}{1-\gamma}\right] dt \\ da_t &= [r(x_t,\sigma_t) + (\underline{r} - r(x_t,\sigma_t))\overline{h}_t - \overline{c}_t]a_t dt + \overline{k}_t a_t dR_t \\ dx_t &= \mu_x(x_t,\sigma_t)dt + \sigma_x(x_t,\sigma_t)x_t dZ_t \\ d\sigma_t &= \theta(\overline{\sigma} - \sigma_t)dt + \sigma_\sigma Z_t \\ (a_0,x_0,\sigma_0) &= (a,x,\sigma) \end{aligned}$$

where $dR_t = (\Pi_i - r(x_t, \sigma_t))dt + \sigma_t dZ_t$.

Lemma 4.1 implies that the value functions in Definition 4.3 assume the form

$$W_i(a,x,\sigma) = V_i(x,\sigma)\frac{a^{1-\gamma}}{1-\gamma}$$

for some function V_i, with associated policy functions of the form

$$(c_i(a,x,\sigma), h_i(a,x,\sigma), k_i(a,x,\sigma)) = \big(\overline{c}_i(x,\sigma)a, \overline{h}_i(x,\sigma)a, \overline{k}_i(x,\sigma)a\big)$$

for all (a, x, σ).

Definition 4.4 (Markov equilibrium)**.** A Markov equilibrium consists of functions for the risk-free rate and drift and diffusion for the wealth share, together with value functions V_i and policy functions $(\overline{c}_i, \overline{h}_i, \overline{k}_i)$ for $i \in \{E, H\}$ solving (4.3), such that for all (x, σ) we have

$$0 = \big(1 - \overline{k}_E(x,\sigma) - \overline{h}_E(x,\sigma)\big)x + \big(1 - \overline{k}_H(x,\sigma) - \overline{h}_H(x,\sigma)\big)[1-x]$$

and the law of motion for the wealth share is consistent with individual policy functions.

For each agent the Hamilton-Jacobi-Bellman equation for the value function is of the form

$$\begin{aligned} \rho W = \max_{\overline{c},\overline{h},\overline{k} \geq 0} \frac{\rho(\overline{c}a)^{1-\gamma}}{1-\gamma} + \big[r + (\underline{r} - r)\overline{h} - \overline{c} + (\Pi - r)\overline{k}\big]aW_1 + \frac{\sigma^2\overline{k}^2}{2}a^2W_{11} + \mu_x x W_2 \\ + \frac{\sigma_x^2 x^2}{2}W_{22} + \sigma_x x \sigma \overline{k} a W_{12} + \sigma \overline{k} a \sigma_\sigma W_{13} + \sigma_x x \sigma_\sigma W_{23} + \theta(\overline{\sigma} - \sigma)W_3 + \frac{\sigma_\sigma^2}{2}W_{33}. \end{aligned} \tag{34}$$

We impose $r \geq \underline{r}$ and henceforth omit storage choice from the agent problems. We compute equilibria by solving a discretized version of the above problem for an arbitrary function for the risk-free rate on a grid and then iterate over this function until convergence. Prior to elaborating

upon the search for equilibria we explain how to solve the individual problem (34). This poses some difficulties, as standard constructions fail to work when there is a high correlation between the state variables. To see why, suppose that $(X_t)_{t\geq 0}$ is a two-dimensional process satisfying

$$dX_t = \mu(X_t)dt + \sigma(X_t)dZ_t$$

where $\mu : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma : \mathbb{R}^2 \to \mathbb{R}^{2\times m}$ and $(Z_t)_{t\geq 0}$ is m-dimensional Brownian motion. Now define coefficients $a_{ij}(X) = (\sigma\sigma^T)_{ij}$ for $i, j = 1, 2$. Suppose that μ and σ vanish outside of some domain $[-M, M]^2$, and set $S := S_1 \times S_2$ where S_1 and S_2 are arbitrary uniform grids with increments Δ_1 and Δ_2, respectively. Denote an arbitrary element of S by $x = (x_1, x_2)$ and consider a Markov chain such that if the chain is at point x at time t, then the possible values at time $t + \Delta_t$ are

$$\Delta(x) := \{(x_1, x_2), (x_1 \pm \Delta_1, x_2), (x_1, x_2 \pm \Delta_2), (x_1 \pm \Delta_1, x_2 \pm \Delta_2), (x_1 \pm \Delta_1, x_2 \mp \Delta_2)\}. \tag{35}$$

The set (35) is simply the set of *adjacent* points in $\mathbb{R}^2$. It is easy to check that if

$$a_{ii} - \sum_{j\neq i} |a_{ij}|\Delta_i/\Delta_j \geq 0, \tag{36}$$

for $i = 1, 2$, then for sufficiently small $\Delta_t > 0$ the following define a locally consistent chain

$$\begin{aligned}
p(x_1 \pm \Delta_1, x_2) &= \frac{\Delta_t}{\Delta_1^2}\left(\frac{1}{2}[a_{11} - |a_{12}|\Delta_1/\Delta_2] + \Delta_1\mu_1(x)^{\pm}\right) \\
p(x_1, x_2 \pm \Delta_2) &= \frac{\Delta_t}{\Delta_2^2}\left(\frac{1}{2}[a_{22} - |a_{12}|\Delta_2/\Delta_1] + \Delta_2\mu_2(x)^{\pm}\right) \\
p(x_1 \pm \Delta_1, x_2 \pm \Delta_2) &= \frac{\Delta_t}{\Delta_1\Delta_2}\frac{1}{2}a_{12}^{+} \\
p(x_1 \pm \Delta_1, x_2 \mp \Delta_2) &= \frac{\Delta_t}{\Delta_1\Delta_2}\frac{1}{2}a_{12}^{-}.
\end{aligned}$$

However, the above construction will fail to work whenever (36) fails for some point in the domain, since the expressions for probabilities may be negative, leading to the inapplicability of standard dynamic programming arguments and instability in the associated numerical algorithms. Indeed, when (36) fails it may be impossible to exactly match the moments of the increments of the underlying process using only local transitions. To illustrate, consider a drift-free diffusion process of the form $(dx_{1t}, dx_{2t}) = (\sigma_1(x)dZ_t, \sigma_2(x)dZ_t)$ for some functions σ_1 and σ_2, where $(Z_t)_{t\geq 0}$ is one-dimensional Brownian motion. In this case we have

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1(x)^2 & \sigma_1(x)\sigma_2(x) \\ \sigma_1(x)\sigma_2(x) & \sigma_2(x)^2 \end{bmatrix}.$$

In order for the above expressions for probabilities to be non negative, we need $\sigma_1(x)^2/\Delta_1 \geq |\sigma_1(x)\sigma_2(x)|/\Delta_2$ and $\sigma_2(x)^2/\Delta_2 \geq |\sigma_1(x)\sigma_2(x)|/\Delta_1$. This is only true if $|\sigma_1(x)|/\Delta_1 = |\sigma_2(x)|/\Delta_2$ and so cannot be assured to hold everywhere for arbitrary σ_1 and σ_2. A different construction is necessary for such a process, one that may call for non local transitions. Section 4.3 is devoted to this construction, explaining the general process before turning to the particular problem in (34). For expositional purposes we present three separate algorithms of increasing complexity, although in practice it appears that the third dominates the first two.

4.3 Chain construction with high correlation

We first illustrate the construction of a locally consistent chain for a drift-free process of the form $(dx_{1t}, dx_{2t}) = (\sigma_1(x)dZ_t, \sigma_2(x)dZ_t)$ on a domain of the form $[0, M_1] \times [0, M_2]$, for functions σ_1, σ_2 and constants $M_1, M_2 > 0$. We fix integers $N_1, N_2 \geq 1$ and define $\Delta_i = M_i/N_i$ for $i = 1, 2$ and

$$S_h = \{\Delta_1, \ldots, M_1 - \Delta_1\} \times \{\Delta_2, \ldots, M_2 - \Delta_2\}. \tag{37}$$

We also write $(\overline{\sigma}_1, \overline{\sigma}_2) := (\sigma_1(x_t)/\Delta_1, \sigma_2(x_t)/\Delta_2)$ and $w := \overline{\sigma}_1/\overline{\sigma}_2$. An arbitrary member of S_h is of the form $(i\Delta_1, j\Delta_2)$ for $i \in \{1, \ldots, N_1 - 1\}$ and $j \in \{1, \ldots, N_2 - 1\}$. We impose two restrictions on the possible transitions (m_1, m_2) from a point $(i\Delta_1, j\Delta_2)$: the state may only leave the grid from a point adjacent to the boundary; and the number of increments that the state may move in either direction cannot exceed a fixed integer $\overline{m} \geq 1$. This translates into the restriction

$$\begin{aligned} |m_1| &\leq \min\{\overline{m}, \min\{i - 1, N_1 - 1 - i\}\} \\ |m_2| &\leq \min\{\overline{m}, \min\{j - 1, N_2 - 1 - j\}\}. \end{aligned} \tag{38}$$

The set of non zero integer pairs satisfying (38) is denoted $\Gamma(i, j)$. In constructing our chain we consider two cases that differ in the number of points to which the state may travel. First, suppose that at any $(x_1, x_2) = (\Delta_1 i, \Delta_2 j) \in S_h$, the transitions assume only three values: for some (m_1, m_2),

$$(\Delta x_1, \Delta x_2) \in \{(0, 0), (\Delta_1 m_1, \Delta_2 m_2), (-\Delta_1 m_1, -\Delta_2 m_2)\}$$

If the non zero values occur with equal probability $\overline{p} \in (0, 1/2)$, then the mean consistency requirement is satisfied for all (m_1, m_2), and the covariance consistency requirements are

$$\begin{aligned} 2\overline{p}\Delta_1^2 m_1^2 &= \Delta_t \Delta_1^2 \overline{\sigma}_1^2 + o(\Delta_t) \\ 2\overline{p}\Delta_1\Delta_2 m_1 m_2 &= \Delta_t \Delta_1 \Delta_2 \overline{\sigma}_1 \overline{\sigma}_2 + o(\Delta_t) \\ 2\overline{p}\Delta_2^2 m_2^2 &= \Delta_t \Delta_2^2 \overline{\sigma}_2^2 + o(\Delta_t). \end{aligned} \tag{39}$$

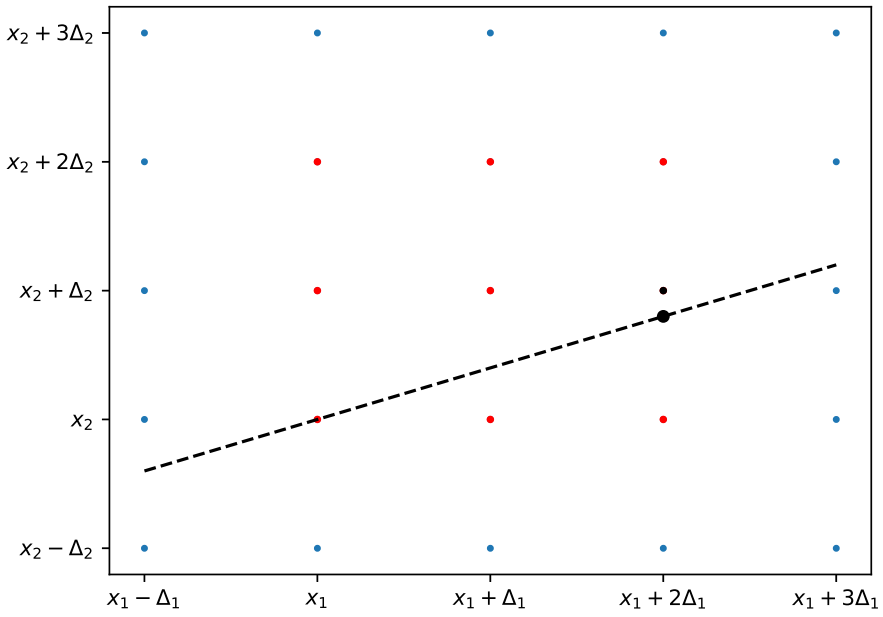

Figure 3: Optimal transition selection

First, note that if we can find n integers satisfying $m_2 = m_1\overline{\sigma}_2/\overline{\sigma}_1$, then (39) will be satisfied with zero o terms if $\Delta_t = 2\overline{p}m_1^2/\overline{\sigma}_1^2 = 2\overline{p}m_2^2/\overline{\sigma}_2^2$. In general we choose a non zero integer pair to minimize $|m_2 - m_1\overline{\sigma}_2/\overline{\sigma}_1|$ and adjust Δ_t such that either the first or third requirement in (39) holds. We proceed on a case-by-case basis to ensure that the timestep never vanishes: if $m_1 \geq m_2$, set $\Delta_t = 2\overline{p}m_1^2/\overline{\sigma}_1^2$ and if $m_2 > m_1$, set $\Delta_t = 2\overline{p}m_2^2/\overline{\sigma}_2^2$.

Figure 3 depicts this process. The slope of the line is $\overline{\sigma}_2/\overline{\sigma}_1$, the red dots represent the transitions satisfying (38), and the black dot represents the (m_1, m_2) selected. This picture suggests that the approximation may be made more accurate by placing some probability on a point adjacent to the black dot. To formalize this, define the candidate o terms in (39) as e_1, e_2 and e_3, respectively, and expand the possible transitions to five points:

$$(\Delta x_1, \Delta x_2) \in \{(0,0), \pm(\Delta_1 m_{11}, \Delta_2 m_{12}), \pm(\Delta_1 m_{21}, \Delta_2 m_{22})\}$$

for some quadruple $m = (m_{11}, m_{12}, m_{21}, m_{22})$. For some $\overline{p} \in (0, 1/2)$, we declare the probability of

$(\Delta x_1, \Delta x_2) \in \{(\Delta_1 m_{11}, \Delta_2 m_{12}), (\Delta_1 m_{21}, \Delta_2 m_{22})\}$ equal to $\overline{p}$, given $(i\Delta_1, j\Delta_2) \in S_h$ define

$$\begin{aligned} z &= \min(1, w)(m_2 - m_1/w) \\ (m_1, m_2) &= \operatorname{argmin} \left\{ |j' - i'/w| \mid (i', j') \in \Gamma(i, j) \right\} \end{aligned} \tag{40}$$

The first pair $m_1 = (m_1, m_2)$ will be the one selected in the three-point approximation. Geometrically, one may interpret z as the distance between the two black dots in Figure 3. The selection of our second point, m_2, will depend upon the signs of z and $w - 1$, as this determines whether the adjacent point lies to the left, right, above, or below the original point, and may be summarized as

$$\begin{aligned} m_{21} &= m_{11} + (w \leq 1)(2(z > 0) - 1) \\ m_{22} &= m_{12} + (w > 1)(2(z \leq 0) - 1). \end{aligned} \tag{41}$$

We place probability $\overline{p}(1 - |z|)$ on m_1 and $\overline{p}|z|$ on m_2, and the timestep is $\Delta_t = \overline{\Delta}_t \overline{p}$ where

$$\overline{\Delta}_t = (w > 1) \frac{(m_{11})^2}{\overline{\sigma}_1^2/2} + (w \leq 1) \frac{(m_{12})^2}{\overline{\sigma}_2^2/2}. \tag{42}$$

The description and accuracy of this five-point approximation is summarized in the following.

Lemma 4.2 (Five-point approximation)**.** *Given $\overline{p} \in (0, 1/2)$, define an approximation to the process $(dx_{1t}, dx_{2t}) = (\sigma_1(x_t)dZ_t, \sigma_2(x_t)dZ_t)$ by declaring, for $x \in S_h$, $p(x_1, x_2) = 1 - 2\overline{p}$ and*

$$\begin{aligned} p(x_1 \pm m_{11}\Delta_1, x_2 \pm m_{12}\Delta_2) &= \overline{p}|z| \\ p(x_1 \pm m_{12}\Delta_1, x_2 \pm m_{22}\Delta_2) &= \overline{p}(1 - |z|) \end{aligned}$$

where z and m are chosen according to (40) *and* (41)*, and Δ_t is given by* (42)*. Two of the three local consistency requirements may be satisfied exactly, and the remaining requirement has relative error proportional to $2\overline{p}\Delta_2^2 z(1 - z)/m_1^2$.*

The above two methods are intuitive to us as they require only minimization of errors in the local consistency requirements. To improve accuracy, however, we may draw upon the geometric analysis of Bonnans et al. (2004). Note that the local consistency requirements for our drift-free process will hold exactly if the transitions are symmetric about the origin and the non negative components are selected from a subset $\hat{\Gamma}(i, j) \subseteq \Gamma(i, j)$ with probabilities $\{\Delta_t \eta_\xi | \xi \in \hat{\Gamma}(i, j)\}$ satisfying

$$\sum_{\xi \in \hat{\Gamma}(i,j)} \eta_\xi \xi \xi^T = \begin{bmatrix} \overline{\sigma}_1^2/2 & \overline{\sigma}_1\overline{\sigma}_2/2 \\ \overline{\sigma}_1\overline{\sigma}_2/2 & \overline{\sigma}_2^2/2 \end{bmatrix}. \tag{43}$$

The set of sums of the form on the left-hand side of (43) is a convex cone. Bonnans et al. (2004) approximate a solution to (43) by projecting the right-hand side onto this cone. Denote the set of positive semi-definite $n \times n$ matrices by PSD_n and define $F : PSD_n \to \mathbb{R}^3$ and $R : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\begin{aligned} F(a) &= (a_{11}, \sqrt{2}a_{12}, a_{22})^T \\ R(z) &= ((z_1 - z_3)/\sqrt{2}, z_2, (z_1 + z_3)/\sqrt{2}). \end{aligned} \tag{44}$$

Note that the function R is a rotation and that the cone $C := \{R(F(a))|a \in PSD_2\}$ points "upwards" in $\mathbb{R}^3$. We define $H : C \to \mathbb{R}^2$ by $H(x) = (x_1/x_3, x_2/x_3)$ and identify the set of covariance matrices with the disc $D := H(R(F(PSD_2)))$. The boundary of the set D corresponds to those covariance matrices associated with perfectly correlated diffusion processes, in which case $a_{11}a_{22} = a_{12}^2$. To describe our selection criteria, we identify integer pairs (p, q) with the fraction q/p, and given q/p and q'/p' define the *child* $q''/p'' := (q + q')/(p + p')$. Given a point in the grid, we choose transitions $\hat{\Gamma}(i, j)$ by beginning near the origin before passing from adjacent points to their children. For any q/p write $\xi_{pq} = [p, q]^T$ and $X_{pq} = \xi_{pq}\xi_{pq}^T = [p, q]^T[p, q]$, and given q/p and q'/p' define $H(q/p, q'/p')$ to be the plane generated by X_{pq} and $X_{p'q'}$ and identify it with its range under F. The associated projection operator is $P_H(q/p, q'/p') = A(A^TA)^{-1}A^T$, where

$$A = \begin{bmatrix} p^2 & (p')^2 \\ \sqrt{2}pq & \sqrt{2}p'q' \\ q^2 & (q')^2 \end{bmatrix}.$$

We will be concerned only with the case in which $\sigma_1, \sigma_2 \geq 0$. For a fixed $\overline{m} \geq 1$ we approximate a solution to (43) as follows, where we write $a_{ih}^h = (\sigma\sigma^T)_{ij}/(\Delta_i\Delta_j)$.[7]

Algorithm 4.3. *Define* $(q/p, q'/p') = (1/1, 0/1)$ *if* $a_{11}^h \geq a_{22}^h$ *and* $(q/p, q'/p') = (1/0, 1/1)$ *otherwise. Notice that we then have* $q/p \geq a_{22}^h/a_{11}^h \geq q'/p'$. *We then update directions as follows:*

1. *If* $\max\{p + p', q + q'\} > \overline{m}$ *then stop. Otherwise, go to Step 2.*

2. *If* $a_{22}^h/a_{11}^h \geq (q+q')/(p+p')$, *return to Step 1 with* $q'/p' = (q+q')/(p+p')$. *Otherwise, return to Step 1 with* $q/p = (q + q')/(p + p')$.

Choose $S_h = \{(p, q), (p', q')\}$ *and* $\eta_{pq}, \eta_{p'q'} \geq 0$ *satisfying* $\eta_{pq}X_{pq} + \eta_{p'q'}X_{p'q'} = P_H(q/p, q'/p')a^h$.

[7]We are using some further properties that are specific to this problem to avoid redundancies. For example, our covariance matrix is never diagonally dominant, and for simplicity we omit reference to a tolerance level.

Finally, for the above weights we define probabilities $p_i = \eta_i \Delta_t$. This leaves us with one more degree of freedom for the timestep. For simplicity we proceed analogously to the earlier algorithms and choose the tiemstep so that $p_0 + p_1 = \overline{p}$ for some fixed $\overline{p} \in (0, 1/2)$. This amounts to choosing $\Delta_t = \overline{p}/(\eta_0 + \eta_1)$, with implied probabilities $p_i = \overline{p}\eta_i/(\eta_0 + \eta_1)$.

The accuracy of the above three methods across a number of different grids is illustrated in the context of linear-quadratic problems in Appendix A.2. We find that all three give approximations of increasing accuracy as the grid sizes increase. For simplicity, in what follows we apply the five-point approximation given in Lemma 4.2. Prior to applying this algorithm to the problem (34), we outline two simplifications. By Lemma 4.1 the value function of each agent satisfies $aW_{12}(a, x, \sigma) = (1 - \gamma)W_2(a, x, \sigma)$ and $aW_{13}(a, x, \sigma) = (1 - \gamma)W_3(a, x, \sigma)$. Substituting into (34) then gives

$$\begin{aligned}\rho W = \max_{\overline{c}, \overline{k} \geq 0} \frac{\rho(\overline{c}a)^{1-\gamma}}{1-\gamma} + \big[r - \overline{c} + (\Pi - r)\overline{k}\big] aW_1 + \frac{\sigma^2 \overline{k}^2}{2} a^2 W_{11} + [\mu_x x + \sigma_x x \sigma \overline{k}(1-\gamma)]W_2 \\ + \frac{\sigma_x^2 x^2}{2} W_{22} + [\theta(\overline{\sigma} - \sigma) + \sigma \overline{k} \sigma_\sigma (1-\gamma)]W_3 + \sigma_x x \sigma_\sigma W_{23} + \frac{\sigma_\sigma^2}{2} W_{33}.\end{aligned}$$

For convenience we define $y_t := \ln a_t$ and note that by Ito's lemma we obtain a control problem with state (y, x, σ), controls (c, k), flow payoffs $\rho \overline{c}^{1-\gamma} e^{(1-\gamma)y}/(1-\gamma)$ and law of motion

$$\begin{aligned} dy_t &= \Big(r - \overline{c}_t + (\Pi - r)\overline{k}_t - \sigma_t^2 \overline{k}_t^2/2\Big)dt + \sigma_t \overline{k}_t dZ_t^{(1)} \\ dx_t &= \big(\mu_x + \sigma_t \sigma_x (1-\gamma)\overline{k}_t\big) x_t dt + \sigma_x x_t dZ_t^{(2)} \\ d\sigma_t &= \big(\theta(\overline{\sigma} - \sigma_t) + \sigma_t \sigma_\sigma (1-\gamma)\overline{k}_t\big)dt + \sigma_\sigma dZ_t^{(2)} \end{aligned} \tag{45}$$

where $(Z^{(1)}, Z^{(2)})$ are now independent. Although this system is not the original one faced by the agent, the above homogeneity arguments show that it leads to the same value function. This does not allow us to completely eliminate the correlation between the state variables, and so the solution of the associated portfolio problem requires non local transitions. However, crucially, the diffusion terms exhibiting high correlation are not controlled by the agent. To outline our approximation to (45), first define the infinite grid $S_y = \{\dots, -\Delta_y, 0, \Delta_y, \dots\}$ for log wealth, and the finite grids

$$\begin{aligned} S_x &= \{0, \Delta_x, \dots, 1 - \Delta_x, 1\} \\ S_\sigma &= \{\underline{\Sigma}, \underline{\Sigma} + \Delta_\sigma, \dots, \overline{\Sigma} - \Delta_\sigma, \overline{\Sigma}\} \end{aligned} \tag{46}$$

for x and σ, where $\Delta_x = (1-0)/N_x$ and $\Delta_\sigma = (\overline{\Sigma} - \underline{\Sigma})/N_\sigma$, and define $S = S_y \times S_x \times S_\sigma$. To ensure that the process remains on (46), we alter (30) so that σ_σ vanishes at $\underline{\Sigma}$ and $\overline{\Sigma}$. For clarity,

we write the transition probabilities as the sum of the transitions for individual wealth, and the drift and diffusion for the aggregate state, respectively,

$$p(X') = p^{(1)}(X') + p^{(2)}(X') + p^{(3)}(X')$$

for any $X' = (y', x', \sigma') \neq X = (y, x, \sigma)$, with $p(X)$ chosen such that probabilities sum to unity. The transitions in log wealth are

$$\begin{aligned} p^{(1)}(y + \Delta_y, x, \sigma) &= \frac{\Delta_t}{\Delta_y^2}\Big(\sigma^2\overline{k}^2/2 + \Delta_y\big[r + (\Pi - r)\overline{k}\big]\Big) \\ p^{(1)}(y - \Delta_y, x, \sigma) &= \frac{\Delta_t}{\Delta_y^2}\Big(\sigma^2\overline{k}^2/2 + \Delta_y\Big[\overline{c} + \sigma^2\overline{k}^2/2\Big]\Big) \end{aligned} \tag{47}$$

for some Δ_t specified below. For the drift terms for the aggregate state we have

$$\begin{aligned} p^{(2)}(y, x + \Delta_x, \sigma) &= \frac{\Delta_t}{\Delta_x}\big([\mu_x x]^+ + \sigma_x x \sigma(1-\gamma)\overline{k}_t\big) \\ p^{(2)}(y, x - \Delta_x, \sigma) &= \frac{\Delta_t}{\Delta_x}[\mu_x x]^- \\ p^{(2)}(y, x, \sigma + \Delta_\sigma) &= \frac{\Delta_t}{\Delta_\sigma}\big([\theta(\overline{\sigma} - \sigma)]^+ + \sigma_\sigma \sigma(1-\gamma)\overline{k}_t\big) \\ p^{(2)}(y, x, \sigma - \Delta_\sigma) &= \frac{\Delta_t}{\Delta_\sigma}[\theta(\overline{\sigma} - \sigma)]^+. \end{aligned}$$

For the diffusion component we follow the non local selection criteria

$$\begin{aligned} p^{(3)}(y, x \pm m_{12}\Delta_x, \sigma \pm m_{13}\Delta_\sigma) &= \overline{p}|z| \\ p^{(3)}(y, x \pm m_{22}\Delta_x, \sigma \pm m_{23}\Delta_\sigma) &= \overline{p}(1 - |z|) \end{aligned}$$

where the transitions and weights are given by Lemma 4.2. Using homogeneity and eliminating probabilities independent of controls, the maximization in the discrete Bellman equation becomes

$$\begin{aligned} &\max_{\overline{c},\overline{k}\geq 0} \Delta_t \frac{\rho\overline{c}^{1-\gamma}}{1-\gamma} + e^{-\rho\Delta_t}\Big[p^{(2)}(y, x + \Delta_x, \sigma)\frac{\Delta_x V^{F,x}}{1-\gamma} + p^{(2)}(y, x, \sigma + \Delta_\sigma)\frac{\Delta_\sigma V^{F,\sigma}}{1-\gamma}\Big] \\ &+ e^{-\rho\Delta_t}\Big[p^{(1)}(y + \Delta_y, x, \sigma)[e^{(1-\gamma)\Delta_y} - 1] + p^{(1)}(y - \Delta_y, x, \sigma)[e^{-(1-\gamma)\Delta_y} - 1]\Big]\frac{V}{1-\gamma} \end{aligned}$$

Dividing by $\Delta_t e^{-\rho\Delta_t}$, eliminating terms independent of the controls and abbreviating gives

$$\begin{aligned} 0 = &\max_{\overline{c},\overline{k}\geq 0} e^{\rho\Delta_t}\frac{\rho\overline{c}^{1-\gamma}}{1-\gamma} + \frac{[e^{-(1-\gamma)\Delta_y} - 1]}{(1-\gamma)\Delta_y}V\overline{c} + (\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma})\sigma\overline{k} \\ &+ \frac{1}{\Delta_y^2}\Big([e^{(1-\gamma)\Delta_y} - 1]\Big(\sigma^2\overline{k}^2/2 + \Delta_y(\Pi - r)\overline{k}\Big) + (1 + \Delta_y)[e^{-(1-\gamma)\Delta_y} - 1]\sigma^2\overline{k}^2/2\Big)\frac{V}{1-\gamma}. \end{aligned} \tag{48}$$

We then have the following. The proof and expressions for the constants are in Appendix B.3.

Lemma 4.4. *For $i \in \{E, H\}$ the policy functions for consumption are $\overline{c}_i = \rho_i^{1/\gamma} V_i^{-1/\gamma} e^{\rho \Delta_t/\gamma} E_c(\Delta_y)$ and the capital policy function for the expert is*

$$\overline{k} = (E_1/E_2)\left(\frac{\Pi - r}{\gamma\sigma^2} + \frac{\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}}{E_1 \gamma \sigma V}\right)^+$$

where E_1, E_2 and E_c depend only upon Δ_y and tend to unity as $\Delta_y \to 0$.

4.4 Market-clearing and consistency

Lemma 4.4 calculated policy functions for a discretized version of the problem of an agent facing given continuation values, risk-free rate, and law of motion of the wealth share. We wish to construct a map that iterates upon the continuation values of each agent, and so we now impose two requirements: the market for bonds clears and the law of motion of the aggregate state is consistent with individual policy functions. Using Lemma 4.4, the bond market-clearing condition becomes

$$(E_1/E_2)\left(\frac{\Pi - r}{\gamma\sigma^2} + \frac{\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}}{E_1 \gamma \sigma V}\right)^+ x \leq 1 \tag{49}$$

with equality if $r > \underline{r}$. Note the inequality in (49) may be strict if the storage technology is utilized in equilibrium. The left-hand side of (49) is decreasing in r so there are two cases to consider: if $\overline{k}(\underline{r})x \leq 1$ then $r = \underline{r}$; otherwise r solves $\overline{k}(r)x = 1$. Rearranging gives

$$r = \max\left\{\underline{r}, \Pi + \frac{\sigma}{E_1 V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}) - \frac{\gamma\sigma^2}{x}(E_1/E_2)^{-1}\right\}. \tag{50}$$

Substituting (50) into the expression for capital in Lemma 4.4 gives

$$\overline{k} = \left(\min\left\{(E_1/E_2)\left(\frac{\Pi - \underline{r}}{\gamma\sigma^2} + \frac{E_1^{-1}}{\gamma\sigma V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma})\right), \frac{1}{x}\right\}\right)^+. \tag{51}$$

Expressions (50) and (51) give the risk-free rate and capital policy function consistent with bond market-clearing, given continuation values and law of motion for the wealth share. We now impose consistency between individual and aggregate laws of motion to obtain a single map defined on continuation values.

Lemma 4.5. *If $V_E^{F,x} \leq 0$ everywhere then*

$$\sigma_x x = \sigma x(1-x)\left(\min\left\{\frac{(E_1/E_2)[\Pi - \underline{r}]/\sigma^2 + E_2^{-1}\sigma_\sigma V_E^{F,\sigma}/[\sigma V_E]}{\gamma + (1-x)E_2^{-1}x[-V_E^{F,x}]/V_E}, \frac{1}{x}\right\}\right)^+, \tag{52}$$

and the drift in the wealth share is

$$\mu_x x = \left[\left(\rho_H^{1/\gamma} V_H^{-1/\gamma} - \rho_E^{1/\gamma} V_E^{-1/\gamma}\right) E_c(\Delta_y) + (\Pi - r)\overline{k} - \sigma^2 \overline{k}^2 x\right] x(1-x)$$

where $\overline{k}$ is given by (51).

Given continuation values and functions for the risk-free rate and drift and diffusion of the wealth share, Lemma 4.4 gives the policy functions for the discretized problem. Expression (50) gives the market-clearing interest rate as a function of the law of motion of wealth, and Lemma 4.5 imposes consistency between individual and aggregate laws of motion.

Prior to summarizing the equilibrium we show that one may proceed as in Section 2 and simplify our problem by considering the limit $\Delta_t, \Delta_y \to 0$. Subtracting the value function from both sides of the discrete Bellman equation and dividing by the timestep gives an equation of the form $0 = \max_{\overline{c},\overline{k}} \rho u(\overline{c}) + T(\overline{c}, \overline{k}; \overline{p}, \Delta_y)V$ for some T. As with the growth model in Section 2, when $\overline{p}, \Delta_y \to 0$ we have convergence to a non-trivial operator.

Lemma 4.6. *For any $\overline{c}, \overline{k}$ we have $\lim_{\overline{p},\Delta_y \to 0} T(\overline{c}, \overline{k}; \overline{p}, \Delta_y) = \overline{T}(\overline{c}, \overline{k})$ where*

$$\begin{aligned}
\overline{T}(\overline{c}, \overline{k})V = &-\Big(\rho - (1-\gamma)(r - \overline{c} + (\Pi - r)\overline{k} - \gamma\sigma^2\overline{k}^2/2)\Big)\frac{V}{1-\gamma} + \big([\mu_x x]^+ + \sigma_x x \sigma \overline{k}\big)\frac{V^{F,x}}{1-\gamma} \\
&+ \big([\theta(\overline{\sigma} - \sigma)]^+ + \sigma_\sigma \sigma \overline{k}\big)\frac{V^{F,\sigma}}{1-\gamma} + [\mu_x x]^- \frac{[-V^{B,x}]}{1-\gamma} + [\theta(\overline{\sigma} - \sigma)]^- \frac{[-V^{B,\sigma}]}{1-\gamma} \\
&+ \frac{(1-|z|)}{\overline{\Delta}_t(1-\gamma)}(V(x + m_{12}\Delta_x, \sigma + m_{13}\Delta_\sigma) + V(x - m_{12}\Delta_x, \sigma - m_{13}\Delta_\sigma) - 2V) \\
&+ \frac{|z|}{\overline{\Delta}_t(1-\gamma)}(V(x + m_{22}\Delta_x, \sigma + m_{23}\Delta_\sigma) + V(x - m_{22}\Delta_x, \sigma - m_{23}\Delta_\sigma) - 2V).
\end{aligned}$$

Further, if $\rho > (1-\gamma)\Big(r - \overline{c} + (\Pi - r)\overline{k} - \gamma\sigma^2\overline{k}^2/2\Big)$ then the above algorithm is globally convergent.

A proof of Lemma 4.6 is contained in Appendix B.3. We can now simplify the expressions in Lemma 4.4 and Lemma 4.5 by setting $E_1, E_2, E_c = 1$ and summarize them as follows.

Proposition 4.7. *The policy functions for consumption are $\overline{c}_i = \rho_i^{1/\gamma} V_i^{-1/\gamma}$ for $i \in \{E, H\}$ and the capital policy function for the expert is*

$$\overline{k} = \frac{1}{\gamma\sigma^2}\big(\Pi - r + [x\sigma_x\sigma V^{F,x} + \sigma\sigma_\sigma V^{F,\sigma}]/V\big)^+$$

If $V_E^{F,x} \leq 0$ everywhere then the volatility of the wealth share satisfies

$$\sigma_x x = \sigma x(1-x)\left(\min\left\{\frac{[\Pi - \underline{r}]/\sigma^2 + \sigma_\sigma V_E^{F,\sigma}/[\sigma V_E]}{\gamma + (1-x)x[-V_E^{F,x}]/V_E}, \frac{1}{x}\right\}\right)^+ \tag{53}$$

For this value, the interest rate is

$$r = \max\left\{\underline{r}, \Pi + \frac{\sigma}{V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}) - \frac{\gamma\sigma^2}{x}\right\}.$$

and the drift in the wealth share is

$$\mu_x x = \Big[\rho_H^{1/\gamma} V_H^{-1/\gamma} - \rho_E^{1/\gamma} V_E^{-1/\gamma} + (\Pi - r)\overline{k} - \sigma^2\overline{k}^2 x\Big]x(1-x).$$

4.5 Summary and example

We now summarize the algorithm for computing the competitive equilibrium. For the following write $S_{x\sigma} := S_x \times S_\sigma$ for S_x and S_σ given in (46).

Algorithm 4.8 (Individual problems)**.** *Given grid values for the risk-free rate and wealth share*

$$\{r(x,\sigma), \mu_x(x,\sigma), \sigma_x(x,\sigma)\}_{(x,\sigma)\in S_{x\sigma}},$$

for each $i \in \{E, H\}$ the value and policy functions are found as follows:

1. *Define consumption and expert investment using Proposition 4.7.*
2. *For these policy functions, solve $0 = \rho u(\overline{c}) + \overline{T}(\overline{c}, \overline{k})V$ for V, where $\overline{T}$ is given in Lemma 4.6.*
3. *Return to Step 1 and repeat until convergence.*

Algorithm 4.9 (Computation of competitive equilibrium)**.** *Given an initial guess*

$$V = (V_E(x,\sigma), V_H(x,\sigma))_{(x,\sigma)\in S_{x\sigma}}$$

for the value functions for each agent, an approximate equilibrium is found as follows:

1. *Given the value functions of the agents, calculate the risk-free rate and law of motion of the wealth share using Proposition 4.7.*
2. *Using the prices and law of motion found in Step 1, use Algorithm 4.8 to update V_E and V_H.*
3. *Return to Step 1 and repeat until convergence.*

Numerical illustration: We have calculated an example for the following parameters:

$$(\gamma = 0.5, \rho = (0.2, 0.1), \theta = 1, (\underline{\Sigma}, \overline{\Sigma}) = (0.1, 0.4), \overline{\sigma} = 0.25,$$
$$\sigma_\sigma = 0.2, \Pi_E = 0.065, \underline{r} = 0, N = (120, 60), \overline{m} = 4).$$

With a tolerance of 10^{-6} for both the consumer's problem and the updating step for the aggregate law, it converges in less than 3 seconds beginning from an initial guess in which r, μ_x and σ_x are identically zero, where we again use Python and an Intel Core i7-8650U processor. Unfortunately, it is difficult to assess the accuracy of the resulting solution, since we know of no closed form solutions to which we may compare the output. However, for the same degree of tolerance between successive iterations, the algorithm is an order of magnitude faster than the "false transient" approach employed in d'Avernas and Vandeweyer (2019), which is roughly analogous to employing value function iteration. Figure 4 gives the interest rate, Figure 5 gives the drift in the wealth share, and Figure 6 gives the volatility of the wealth share.

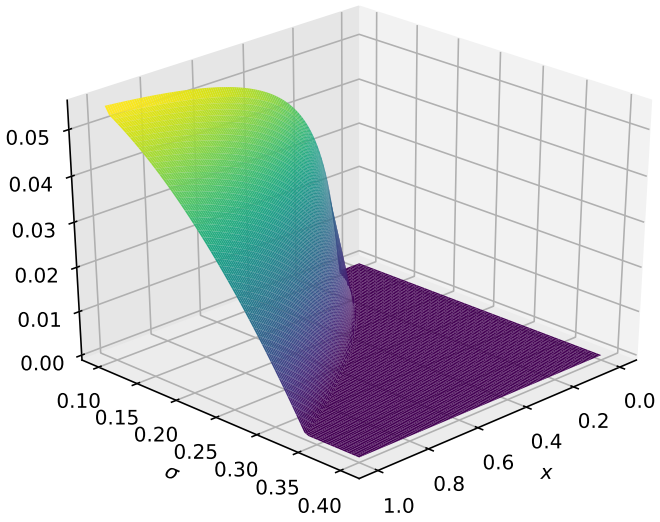

Figure 4: Interest rate

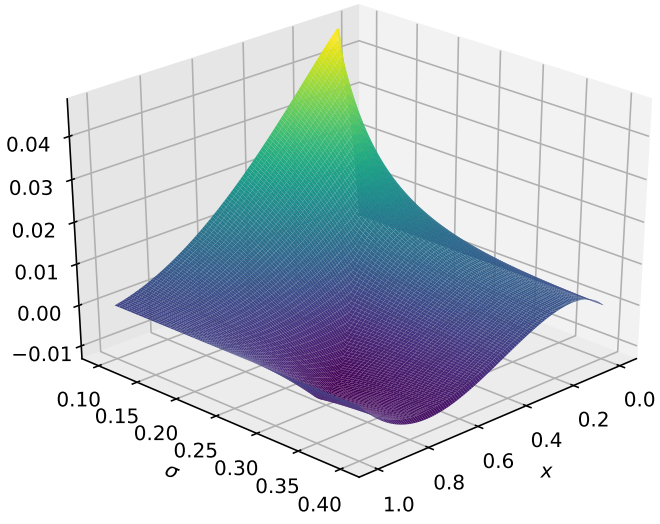

Figure 5: Drift in wealth share $\mu_x x$

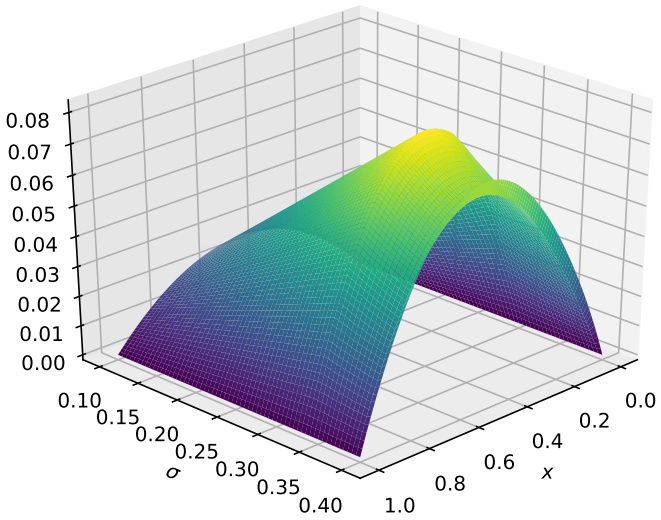

Figure 6: Drift in wealth share $\sigma_x x$

5 Conclusion

In this paper we explore several applications of the Markov chain approximation (MCA) method of Harold Kushner and Paul Dupuis to optimal control problems in economics, illustrating some unutilized benefits. We first show that for certain choices of the approximating chain, the MCA method with policy function iteration coincides with a limiting version of the implicit finite difference scheme of Achdou et al. (2017). We then demonstrate the benefits of a more general specification by means of two examples. In the first, we use variations of modified policy function iteration to solve income fluctuation problems, both with and without discrete choices. In the second, we show how the MCA method may be applied to problems with high correlation among state variables. In both cases, the MCA is robust and easy to apply and can result in an increase in speed of more than an order of magnitude over finite-difference methods.

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A Linear-quadratic regulator problem

To verify the accuracy of the algorithms, this section records closed form expressions for a class of linear-quadratic control problems. Lemma A.1 treats the standard case of an infinite-horizon linear-quadratic regulator problem, while Lemma A.3 treats the (slightly non standard) case in which volatility is linear in the state variable. The former will be useful for calculating the benefits associated with modified policy function iteration, while the latter will illustrate the applications of non local transitions. Suppose that the objective to be maximized is

$$\int_0^\infty e^{-\rho t}\mathbb{E}[F(x_t,u_t)]dt$$

where for some symmetric positive definite matrices Q and R the flow payoffs are given by

$$F(x,u) = -\frac{1}{2}x^TQx - \frac{1}{2}u^TRu \tag{54}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^q$ are the state and control vectors, respectively, for some $n, q \geq 1$. Now suppose that for some$A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times q}$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ the law of motion for the state is

$$dx_t = [Ax_t + Bu_t]dt + \sigma(x)dZ_t$$

where $Z := (Z_t)_{t\geq 0}$ is m-dimensional Brownian motion. Write $\mu(x,u) = Ax + Bu$ for the drift as a function of the state and controls, and note that the Hamilton-Jacobi-Bellman equation is

$$\rho V(x) = \max_{u\in\mathbb{R}^q} F(x,u) + \sum_{i=1}^n \mu_i(x,u)V_i(x) + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}V_{ij}(x) \tag{55}$$

where $a_{ij}(x) = (\sigma(x)\sigma(x)^T)_{ij}$. If σ is constant then we obtain the following.

Lemma A.1. *The solution to* (55) *is* $V(x) = -x^TPx/2 - d$*, where* P *is a symmetric matrix that solves*

$$\rho P = Q + PA + A^TP - P^TBR^{-1}B^TP \tag{56}$$

the constant term is $d = [2\rho]^{-1}\text{trace}(\sigma\sigma^TP)$*, and the policy function is*

$$u = -R^{-1}B^TPx. \tag{57}$$

Corollary A.2. *The drift in the state variables* $(A - BR^{-1}B^TP)x$.

Note that P solves (56) if and only if it solves the undiscounted problem

$$0 = Q + P(A - \rho I/2) + (A - \rho I/2)^T P - P^T B R^{-1} B^T P$$

and so the analysis for the discounted case follows from results from the undiscounted case, with A replaced by $A - \rho I/2$. The standard linear-quadratic regulator problem analyzed in Lemma A.1 assumes constant volatility. To illustrate the flexibility of the method in treating some cases that appear difficult with finite-differences, we consider here an extension of the standard linear-quadratic framework in which the volatility is linear in the state variables. The flow payoff remains (54) for some symmetric positive definite matrices Q and R, but the law of motion for the state variables is now $dx_t = [Ax_t + Bu_t]dt + \sigma x_t dZ_t$ where $\sigma \in \mathbb{R}^{n\times n}$ (and so $\sigma x_t \in \mathbb{R}^2$), and Z is scalar Brownian motion.

Lemma A.3. *If* σ *is a multiple of the identity, then the value function is* $V(x) = -x^T Px/2$ *where* P *solves*

$$0 = Q + P(A + [\sigma^2 - \rho]I/2) + (A + [\sigma^2 - \rho]I/2)^T P - P^T B R^{-1} B^T P. \tag{58}$$

Lemma A.3 will be useful to illustrate how the FSMC method may be used to deal with problems for which the covariance matrix is degenerate, a case that appears difficult to analyze via other means.

We now use the closed form expressions for value functions and policy functions in Lemma A.1 and Lemma A.3 to verify the accuracy of the Markov chain approximation method. There are many tests that we could conduct. We choose only a select few to illustrate the points highlighted in the main text and to provide confidence in the results recorded in the main text.

A.1 Independent noise

We first consider a three-dimensional problem with independent noise terms. This corresponds to the above with σ a diagonal matrix with main diagonal written $[\sigma_0\ \sigma_1\ \sigma_2]$. We consider domains in $\mathbb{R}^3$ of the form $[0, M_0] \times [0, M_1] \times [0, M_2]$ for constants M_0, M_1 and M_2. Now define $\Delta_i = M_i/N_i$ for $i = 0, 1, 2$ and

$$\begin{aligned} S_0 &= \{\Delta_0, \ldots, M_0 - \Delta_0\} \\ S_1 &= \{\Delta_1, \ldots, M_1 - \Delta_1\} \\ S_2 &= \{\Delta_2, \ldots, M_2 - \Delta_2\}. \end{aligned} \tag{59}$$

We then let our grids be $S = S_0 \times S_1 \times S_2$ and adopt the transition probabilities

$$\begin{aligned} p(x_0 \pm \Delta_0, x_1, x_2) &= \frac{\Delta_t}{\Delta_0^2}\left(\frac{\sigma_0^2}{2} + \Delta_0 (Ax + Bu)_0^{\pm}\right) \\ p(x_0, x_1 \pm \Delta_1, x_2) &= \frac{\Delta_t}{\Delta_1^2}\left(\frac{\sigma_1^2}{2} + \Delta_1 (Ax + Bu)_1^{\pm}\right) \\ p(x_0, x_1, x_2 \pm \Delta_2) &= \frac{\Delta_t}{\Delta_2^2}\left(\frac{\sigma_2^2}{2} + \Delta_2 (Ax + Bu)_2^{\pm}\right). \end{aligned} \tag{60}$$

	PFI	VFI	k = 10	k = 50	k = 100
(10, 10, 10)	0.236000	2.603990	0.295002	0.103997	0.113002
(20, 20, 20)	3.393991	17.368586	2.179996	0.569999	0.360012
(30, 30, 30)	34.720209	91.956503	10.794555	2.621202	1.228853
(40, 40, 40)	313.784663	388.433260	45.388144	10.365834	6.247165
(50, 50, 50)	1233.153091	1516.802365	155.727188	36.417583	20.135882

Table 6: Generalized MPFI: Time until convergence

We wish to avoid case-by-case technicalities and so consider problems with a single control. We normalize $R = 1$ and write $B = [b_0, b_1, b_2]^T$. The Riccati equation and policy function from Lemma A.1 become

$$\begin{aligned} \rho P &= Q + PA + A^T P - P^T B B^T P \\ u &= -B^T P x. \end{aligned} \tag{61}$$

We choose parameters for which the drift is always negative. Using (59) the maximization becomes

$$\max_{u \leq 0} \; -\frac{1}{2}u^2 + e^{-\rho \Delta_t}\big(b_0 V^{B0} + b_1 V^{B1} + b_2 V^{B2}\big)u$$

and so the optimal control is obvious $u = \min\{e^{-\rho\Delta_t}\big(b_0 V^{B0} + b_1 V^{B1} + b_2 V^{B2}\big), 0\}$. We will choose our timestep to be as large as possible while ensuring that the expressions for probabilities lie in the unit interval. If we restrict attention to controls such that $Ax + Bu \leq 0$ in each component the above probabilities will lie in the unit interval provided

$$1 \geq \Delta_t \bigg(\frac{\sigma_0^2}{\Delta_0^2} + \frac{\sigma_1^2}{\Delta_1^2} + \frac{\sigma_2^2}{\Delta_2^2} + \frac{(Ax + Bu)_0^-}{\Delta_0} + \frac{(Ax + Bu)_1^-}{\Delta_1} + \frac{(Ax + Bu)_2^-}{\Delta_2} \bigg).$$

To use the above to obtain an appropriate state-dependent timestep we require a bound on the control u. For this we will choose u to be $3 \times$ the true optimal policy function. Table 6 and Table 7 document the time until convergence for modified policy function iteration and the associated average percentage error with the closed form solution. Table 8 and Table 9 perform the analogous exercise for the generalized algorithm.

A.2 Perfectly correlated noise

We now verify the accuracy of the Markov chains constructed in Section 4.3. In all of the following cases we set Q equal to the identity, omit drift and controls, and set $(\sigma = 0.3, \rho = 0.15)$ and $M_1 = M_2 = 1$. Table 10 documents the average of the percent difference between the true and computed value with three points. Table 11 documents the same quantities for the case with five points, and Table 12 gives the same for the method of Bonnans et al. (2004). As can be seen, for all grids considered these methods are strictly increasing in their accuracy (3-point, then 5-point, then Bonnans et al. (2004)).

	PFI	VFI	k = 10	k = 50	k = 100
(10, 10, 10)	3.685199	3.685199	3.685199	3.685199	3.685199
(20, 20, 20)	1.368553	1.368552	1.368553	1.368553	1.368553
(30, 30, 30)	0.763641	0.763641	0.763641	0.763641	0.763641
(40, 40, 40)	0.507078	0.507076	0.507077	0.507078	0.507078
(50, 50, 50)	0.371107	0.371105	0.371107	0.371107	0.371107

Table 7: Generalized MPFI: Average absolute percentage error

	PFI	VFI	k = 10	k = 50	k = 100
(10, 10, 10)	0.131601	1.070645	0.115576	0.115925	0.099935
(20, 20, 20)	1.876517	9.621349	1.008876	0.269588	0.246597
(30, 30, 30)	27.004308	70.274081	7.364786	2.085560	1.201723
(40, 40, 40)	273.910850	398.572694	33.455023	8.473099	4.741551
(50, 50, 50)	1075.469271	1392.673882	147.142013	33.997009	20.956269

Table 8: Generalized MPFI: Time until convergence

	PFI	VFI	k = 10	k = 50	k = 100
(10, 10, 10)	1.921812	1.921812	1.921812	1.921812	1.921812
(20, 20, 20)	1.117398	1.117398	1.117398	1.117398	1.117398
(30, 30, 30)	0.781886	0.781885	0.781886	0.781886	0.781886
(40, 40, 40)	0.600415	0.600412	0.600415	0.600415	0.600415
(50, 50, 50)	0.487141	0.487137	0.487141	0.487141	0.487141

Table 9: Generalized MPFI: Average absolute percentage error

	m = 2	m = 4	m = 6	m = 8	m = 10
(50, 50)	5.445249	2.374563	2.161374	2.300301	2.575376
(100, 100)	4.681075	1.557819	1.164376	1.100234	1.143563
(150, 150)	4.427166	1.302681	0.880531	0.766900	0.740069
(200, 200)	4.314030	1.174924	0.741952	0.615596	0.569038

Table 10: Accuracy of correlated LQ problem: Three points

	m = 2	m = 4	m = 6	m = 8	m = 10
(50, 50)	2.497171	0.542278	0.330712	0.388588	0.500914
(100, 100)	2.438331	0.470257	0.195567	0.152025	0.170488
(150, 150)	2.420292	0.452827	0.168436	0.107166	0.098250
(200, 200)	2.411570	0.445115	0.157898	0.090526	0.072276

Table 11: Accuracy of correlated LQ problem: Five points

	m = 2	m = 4	m = 6	m = 8	m = 10
(50, 50)	0.533714	0.150560	0.103320	0.093319	0.091209
(100, 100)	0.487081	0.098067	0.044683	0.030831	0.026664
(150, 150)	0.476943	0.088402	0.034103	0.019453	0.014800
(200, 200)	0.472893	0.084929	0.030438	0.015540	0.010712

Table 12: Accuracy with Bonnans et al. (2004) method

B Macrofinance notes

B.1 Problems with highly correlated state variables

Proof of Lemma 4.2. We proceed on a case-by-case basis, depending upon the sign of $w-1$. The errors are

$$\begin{aligned} e_{11} &= \mathbb{E}[(\Delta x_1)^2] - \Delta_t \sigma_1^2 \\ e_{12} &= \mathbb{E}[(\Delta x_1)(\Delta x_2)] - \Delta_t \sigma_1 \sigma_2 \\ e_{22} &= \mathbb{E}[(\Delta x_2)^2] - \Delta_t \sigma_2^2. \end{aligned} \tag{62}$$

Condition $w > 1$ is equivalent to $\sigma_1/\Delta_1 > \sigma_2/\Delta_2$ or $\sigma_1/\sigma_2 > \Delta_1/\Delta_2$. In this case $m_{11} = m_{21} = m_1$ and $\Delta_t = 2\overline{p}\Delta_1^2 m_1^2/\sigma_1^2$, so the first term in (62) is $e_{11} = 2\overline{p}\Delta_1^2 m_1^2 - \Delta_t \sigma_1^2 = 0$. We also have $m_{12} = m_2$ and $m_{22} = m_{12}^{(1)} + 2(z \le 0) - 1$, so the second term in (62) becomes

$$\begin{aligned} e_{12} &= 2\overline{p}\Delta_1\Delta_2 m_1((1-|z|)m_2 + |z|[m_2 + 2(z \le 0) - 1]) - \Delta_t \sigma_1 \sigma_2 \\ &= 2\overline{p}\Delta_1\Delta_2 m_1(|z|[2(z \le 0) - 1] + z) = 0. \end{aligned}$$

Finally, the third error term in (62) simplifies to

$$\begin{aligned} e_{22} &= 2\overline{p}\Delta_2^2\big(\eta m_2^2 + (1-\eta)[m_2 + 2(z \le 0) - 1]^2\big) - \Delta_t \sigma_2^2 \\ &= 2\overline{p}\Delta_2^2\big[m_2^2 + (1-\eta)[2(2(z \le 0) - 1)m_2 + (2(z \le 0) - 1)^2] - \sigma_2^2(\Delta_1^2/\Delta_2^2)m_1^2/\sigma_1^2\big] \\ &= 2\overline{p}\Delta_2^2\big[|z|[2(2(z \le 0) - 1)m_2 + (2(z \le 0) - 1)^2] + m_2^2 - m_1^2/w^2\big]. \end{aligned}$$

Using $z = m_2 - m_1/w$ we write $(m_1/w)^2 = (m_2 - z)^2 = m_2^2 - 2m_2 z + z^2$ to note

$$e_{22} = 2\overline{p}\Delta_2^2\big[|z|[(4(z \le 0) - 2)m_2 + 1] + 2m_2 z - z^2\big] = 2\overline{p}\Delta_2^2(|z| - |z|^2)$$

as claimed. The case with $w \le 1$ is symmetric. □

B.2 Evolution of wealth shares

We now determine the law of motion for the wealth share x, by aggregating over the choices of experts and households. Lemma B.1 shows how the law of motion of the wealth share depends upon the law of motion of the wealth of the individual agents.

Lemma B.1. *Suppose that $da_{it}/a_{it} = \mu_{it}dt + \sigma_{it}dZ_t$ for $i \in \{E, H\}$ and that $x := N/(qK)$, where N is the aggregate wealth of experts. Then $dx_t = x_t\mu_x dt + x_t\sigma_x dZ_t$ where μ_x and σ_x are given by*

$$\begin{aligned} x\mu_x &= x(1-x)(\mu_E - \mu_H - (\sigma_E x + \sigma_H(1-x))(\sigma_E - \sigma_H)) \\ x\sigma_x &= x(1-x)(\sigma_E - \sigma_H). \end{aligned}$$

Proof. Aggregating over experts gives $dN_t/N_t = \mu_E dt + \sigma_E dZ_t$ and hence

$$\begin{aligned} d(q_tK_t) &= \mu_E N_t dt + \sigma_E N_t dZ_t + \mu_H(q_tK_t - N_t)dt + \sigma_H(q_tK_t - N_t)dZ_t \\ \frac{d(q_tK_t)}{q_tK_t} &= [\mu_E x_t + \mu_H(1-x_t)]dt + [\sigma_E x_t + \sigma_H(1-x_t)]dZ_t. \end{aligned} \tag{63}$$

Note that if $da_t/a_t = \mu_a dt + \sigma_a dZ_t$ and $db_t/b_t = \mu_b dt + \sigma_b dZ_t$, then using Ito's lemma $c_t := a_t/b_t$ satisfies $dc_t/c_t = (\mu_a - \mu_b - \sigma_b(\sigma_a - \sigma_b))dt + (\sigma_a - \sigma_b)dZ_t$. Applying to (63) gives

$$\begin{aligned}\frac{dx_t}{x_t} &= (\mu_E - [\mu_E x_t + \mu_H(1-x_t)] - (\sigma_E x_t + \sigma_H(1-x_t))(\sigma_E - [\sigma_E x_t + \sigma_H(1-x_t)])dt \\ &\quad + (\sigma_E - [\sigma_E x_t + \sigma_H(1-x_t)])dZ_t\end{aligned}$$

which simplifies as claimed. □

B.3 Individual problems

Proof of Lemma 4.4. Eliminating the terms independent of capital and dividing by V gives

$$\begin{aligned}&\frac{1}{V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma})\sigma\overline{k} + \frac{1}{\Delta_y(1-\gamma)}[e^{(1-\gamma)\Delta_y} - 1](\Pi - r)\overline{k} \\ &+ \frac{1}{\Delta_y^2(1-\gamma)}\Big(e^{(1-\gamma)\Delta_y} - 1 + (1+\Delta_y)[e^{-(1-\gamma)\Delta_y} - 1]\Big)\sigma^2\overline{k}^2/2.\end{aligned}$$

The first-order condition is then

$$\begin{aligned}0 = &\frac{1}{V}(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma})\sigma + \frac{1}{\Delta_y(1-\gamma)}[e^{(1-\gamma)\Delta_y} - 1](\Pi - r) \\ &- \frac{1}{\gamma(1-\gamma)}\left(\frac{1}{\Delta_y}[1 - e^{-(1-\gamma)\Delta_y}] - \frac{1}{\Delta_y^2}[e^{(1-\gamma)\Delta_y} - 2 + e^{-(1-\gamma)\Delta_y}]\right)\sigma^2\gamma\overline{k}\end{aligned}$$

which may be written $0 = (\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma})\sigma/V + E_1(\Delta_y)(\Pi - r) - E_2(\Delta_y)\sigma^2\gamma\overline{k}$, where

$$\begin{aligned}E_1(\Delta_y) &= \frac{1}{1-\gamma}[e^{(1-\gamma)\Delta_y} - 1]/\Delta_y \\ E_2(\Delta_y) &= \frac{1}{\gamma(1-\gamma)}\Big([2 - e^{-(1-\gamma)\Delta_y} - e^{(1-\gamma)\Delta_y}]/\Delta_y^2 + [1 - e^{-(1-\gamma)\Delta_y}]/\Delta_y\Big) \\ E_c(\Delta_y) &= \Big((1 - e^{-(1-\gamma)\Delta_y})/\Delta_y\Big)^{-1/\gamma}(1-\gamma)^{1/\gamma}\end{aligned}$$

from which rearrangement gives the result. □

Proof of Lemma 4.6. Using $\Delta_t(\overline{p}) = \overline{p}\Delta_t(1) =: \overline{p}\overline{\Delta}_t$, dividing all terms in the Bellman equation by Δ_t gives

$$\begin{aligned}0 = &\frac{1}{\Delta_t}(e^{-\rho\Delta_t}[1 - 2\overline{p}] - 1)\frac{V}{1-\gamma} + \frac{\rho\overline{c}^{1-\gamma}}{1-\gamma} + e^{-\rho\Delta_t}\frac{1}{\Delta_y}[e^{-(1-\gamma)\Delta_y} - 1]\frac{\overline{c}V}{1-\gamma} \\ &+ \frac{e^{-\rho\Delta_t}}{\Delta_y^2}\Big[\Big(\sigma^2\overline{k}^2/2 + \Delta_y(\Pi - r)\overline{k}\Big)[e^{(1-\gamma)\Delta_y} - 1] + (1+\Delta_y)[e^{-(1-\gamma)\Delta_y} - 1]\sigma^2\overline{k}^2/2\Big]\frac{V}{1-\gamma} \\ &+ e^{-\rho\Delta_t}\big(\sigma_x x V^{F,x} + \sigma_\sigma V^{F,\sigma}\big)\sigma\overline{k} + e^{-\rho\Delta_t}\frac{1}{\Delta_y}[e^{(1-\gamma)\Delta_y} - 1]\frac{rV}{1-\gamma} \\ &+ \frac{e^{-\rho\Delta_t}}{1-\gamma}\big([\mu_x x]^+ V^{F,x} + [\mu_x x]^-[-V^{B,x}] + [\theta(\overline{\sigma} - \sigma)]^+ V^{F,\sigma} + [\theta(\overline{\sigma} - \sigma)]^-[-V^{B,\sigma}]\big) \\ &+ e^{-\rho\Delta_t}\frac{(1 - |z|)}{\overline{\Delta}_t(1-\gamma)}(V(x + m_{12}\Delta_x, \sigma + m_{13}\Delta_\sigma) + V(x - m_{12}\Delta_x, \sigma - m_{13}\Delta_\sigma)) \\ &+ e^{-\rho\Delta_t}\frac{|z|}{\overline{\Delta}_t(1-\gamma)}(V(x + m_{22}\Delta_x, \sigma + m_{23}\Delta_\sigma) + V(x - m_{22}\Delta_x, \sigma - m_{23}\Delta_\sigma)).\end{aligned} \tag{64}$$

Note that as $\Delta_y \to 0$ we have

$$\frac{1}{\Delta_y^2}[e^{(1-\gamma)\Delta_y} - 2 + e^{-(1-\gamma)\Delta_y}] + \frac{1}{\Delta_y}[e^{-(1-\gamma)\Delta_y} - 1] \to (1-\gamma)^2 - (1-\gamma) = -\gamma(1-\gamma)$$

which gives the result upon substitution. The global convergence of the algorithm then follows from the fundamental results of policy function iteration recapitulated in Section 3.1. □