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I show, under intuitive conditions on the risk-averse utility function, the non-optimality of the Diamond and Dybvig (1983) contract in the Goldstein and Pauzner (2005) environment. If marginal utility at zero is low enough, then Goldstein and Pauzner (2005)'s claim about the optimality of the Diamond and Dybvig (1983) contract is true. When it is not, the optimal contract insures the patient depositor against a project default. The contract may exhibit risk-sharing with the impatient depositor. Unlike when Goldstein and Pauzner (2005)'s claim is correct, relative risk aversion greater than 1 does not necessarily make the optimal bank contract run-prone. I present a condition under which it is.

Keyword: Bank runs.

JEL Classification: D86, G21.

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1 Introduction

This paper computes the optimal bank contract in the Goldstein and Pauzner (2005) environment. Generally, the contract exhibits insurance to the patient depositor against a project default, and ***may*** exhibit risk-sharing. Goldstein and Pauzner (2005)'s claim about the optimality of the Diamond and Dybvig (1983) contract is only true when the marginal utility at zero is sufficiently low. Moreover, the optimal bank contract might not be run-prone. In the original Diamond and Dybvig (1983) environment, the optimal demand-deposit contract was always run-prone. Goldstein and Pauzner (2005)'s assumption that this contract is also optimal in their environment, suggests that this is also the case in their model. However, even under GP's assumptions on the environment, the more general optimal contract considered in this paper is not run-prone, unless marginal utility at 1 is sufficiently high. An example shows that the set of risk-averse utility functions considered in Goldstein and Pauzner (2005) includes some that are not run-prone.

Goldstein and Pauzner (2005) builds on Diamond and Dybvig (1983)'s seminal paper, which models banking as a mechanism for providing insurance against liquidity shocks. Ex-ante identical agents have access to a risk-free, long-term project that pays off only in the last period. It can be liquidated in the interim, foregoing the long-term return. Agents may experience an interim liquidity shock, making them value immediate consumption only. With no banks, only the patient agents benefit from the returns of the long-term project. Impatient agents liquidate all their holdings to consume interim, foregoing any benefit from the investment. Banks facilitate risk-sharing among both types. The Diamond and Dybvig (1983) contract (extended to Goldstein and Pauzner (2005)'s environment) exhibits the following characteristics: there is risk-sharing between the impatient and the patient depositors. In particular, an impatient depositor's utility is higher than in autarky. Moreover, the remaining proceeds of pooled resources after paying the impatient depositor are invested in the productive project. Henceforth, this contract will be called the "GP contract." Implementation of this contract in the Diamond and Dybvig (1983) environment has two equilibria.

In the good equilibrium, the patient depositors decide in the interim period to wait till the last period to withdraw from banks. In the bad one, they panic and run on the bank in the interim.

Goldstein and Pauzner (2005) add a very important twist to the Diamond and Dybvig (1983) story. The project is now risky, paying R with probability $p(\theta)$ and 0 otherwise. In the interim, each patient depositor receives a private noisy signal (θ_i) about the project's probability of default. The noise on these signals are then taken to zero. By developing global games techniques to handle the incentives of the patient depositors, Goldstein and Pauzner (2005) get a unique equilibrium, with a concurrent equilibrium run probability. This allows them to endogenize the effects of the run on ex-ante utility, and determine the optimal interim payment to the impatient depositor given that resulting probability. The interim payment is lower in the resulting optimal contract. They also analyze the viability of banks, given that depositors can foresee the run probability, and present a condition under which banks improve welfare.

Goldstein and Pauzner (2005)'s environment differs from Diamond and Dybvig (1983)'s in two crucial ways. First, they exclude utility functions satisfying Inada conditions. Second and more importantly, they assume that there is a range of fundamentals (θ) under which the project defaults with a positive probability. Goldstein and Pauzner (2005) claim that the optimal contract in their new environment is the Diamond and Dybvig (1983) contract discussed before. But unlike the Diamond and Dybvig (1983) environment, where the project always pays R, their environment includes a risky project that pays zero in default. An optimal bank contract would insure the risk-averse patient agent against that contingency. Even under autarky, the patient agent would prefer to insure himself against a project default by liquidating some of his investment in the interim. This insurance necessarily decreases the total investments in the risky project. In what remains of this paper, Goldstein and Pauzner (2005) and Diamond and Dybvig (1983) would be referred to as GP and DD respectively.

A synopsis of this paper's results follow. The optimal first-best contract is not generally a GP contract. When the project is risky, the remaining proceeds of pooled resources after paying the impatient depositor are not all invested in the productive project. More liquidation of the long-term project occurs to insure the risk-averse patient depositor against a project default. The general problem set-up allows the patient depositor to liquidate part of his investments in the interim to insure himself. Solving the first-best problem is similar to solving a problem where an optimal transfer of resources from the impatient to the patient depositor is determined. Then, given that transfer, the patient depositor acts as if he is in autarky, but with resources boosted by the transfer. Thinking of the first-best problem in this intuitive way, makes the problem much easier to interpret. Risk-sharing, a transfer of resources to the impatient depositor, only occurs when marginal utility at 1 exceeds a multiple of marginal utility when the project pays in autarky with no transfer. It is important to note that there is a clear and direct link between risk-sharing and the contract being run-prone. When we have risk-sharing, the contract is run-prone. When we do not, the contract is not run-prone. An example, satisfying GP's assumptions on the utility function, shows an optimal contract that is not run-prone. Next, it is shown that when marginal utility at zero is low enough, then the remaining resources after paying the impatient agents are invested in the project. In this case, the planner does not find it optimal to insure the patient depositor. It is here that GP's contract is actually optimal. But in general, utility functions with low marginal utility at zero are unintuitive, and contrary to the spirit of the Inada conditions. The Inada conditions are assumed in DD's paper. It is enough that marginal utility at zero is high enough to violate the exact upper bound of the condition in Section 4, for GP's contract to violate optimality. This is not saying that it **has** to go to infinity at zero.

Banks are then introduced, where they compete among each other, netting zero. They maximize the depositor's ex-ante utility subject to the resource constraint considered in the first-best contract, and two incentive constraints specifying that no type would be better off mimicking the other type. When the first-best contract is not run-prone, it satisfies the incentive constraints, and is the optimal contract offered by banks. When the first-best contract is run-prone, it sometimes satisfies the incentive constraints and is the optimal bank contract. However, it sometimes does not. When the first-best contract is run-prone and does not satisfy the incentive constraints, I show that more resources are transferred to the patient depositor. This raises his consumption in both states of the world: when the project defaults and when it does not. Even though the impatient depositor's consumption goes down in the optimal bank contract, but it is still necessarily more than 1 (there still is risk-sharing in bank's contract). Hence, the optimal bank contract is still run-prone. To sum up, when the first-best contract is run-prone, so is the optimal bank contract. When it is not run-prone, then the optimal bank contract is not either.

In the special case when the utility is exponential, I analyze how the optimal bank contract depends on the perceived probability of default. In recessions and weak recoveries, this perceived probability of default goes up. Because of the increased risk he faces, the patient depositor's insurance goes up. Both the impatient depositor's consumption, and the investments in the project, go down. The model predicts that banks hold more liquid assets in recessions and in weak recoveries than in booms, foregoing investments in risky projects (decreasing bank lending). This is supported by data that shows that U.S. banks have been carrying an exploding amounts of cash and safe assets relative to pre-recession levels.

The remainder of the paper is organized as follows. Section 2 computes the first-best contract. Section 3 presents the condition when risk-sharing is optimal. Section 4 analyzes when the GP contract is optimal. Section 5 presents the optimal bank contract. Section 6 discusses the relationship between the optimal contract and the perceived probability of default. Section 7 concludes with future research to pursue.

2 The Economy with No Private Signals and No Public Signals

The set-up follows GP's. Time is discrete, with three periods: 0, 1, 2. There is one good, and a continuum of agents $[0, 1]$, each endowed with 1 unit of the good in period 0. Agents are either patient or impatient and learn their type at the beginning of period 1. With probability λ, an agent is impatient; with $1 - \lambda$, he is patient. An impatient agent only values consumption in period 1 (with a utility function $u(c_1)$); a patient agent views consumption in periods 1 and 2 as perfect substitutes (with a utility function $u(c_1 + c_2)$). The utility function $u(.)$ is increasing, concave, and twice continuously differentiable, with coefficient of relative risk aversion greater than 1 for $c \geq 1$.

2.1 The Technology

Agents have access to a productive but risky technology. For each unit of input in period 0, the technology generates one unit of output if liquidated in period 1 and R units of output in period 2 with probability $p(\theta)$ and 0 with $1 - p(\theta)$. θ is the state of the economy drawn from a uniform distribution $[0, 1]$ and is unknown to agents before period 2. p(.) is strictly increasing in θ indicating that a higher value of θ is viewed as a higher chance of the project paying back. Assumptions on the technology are made in the relevant sections of this paper.

2.2 First-Best Contract

2.2.1 Autarky

In autarky, an impatient agent consumes one unit of the good. A patient agent decides to liquidate part of his investments in the project to insure himself against a project default. Denote an impatient agent's consumption by c_1; and a patient agent's guaranteed consumption by c_2 and his investment in the project by i_2. The patient agent solves the following problem:

$Max_{c_2,i_2} E_\theta[p(\theta)]u(c_2 + i_2 R) + (1 - E_\theta[p(\theta)])u(c_2)$ s.t. $i_2 + c_2 = 1$ and $0 \leq c_2 \leq 1$.

The FOC follow: $\frac{u'(R-c_2(R-1))}{u'(c_2)} = \frac{1-E_\theta[p(\theta)]}{E_\theta[p(\theta)](R-1)}$. Denote c_2^{aut} a solution to the preceding equation. Existence and uniqueness of this solution is guaranteed by continuity and concavity of the utility function which translate into the expected utility function.

If $E_\theta[p(\theta)]R \leq 1$, then $c_2^{aut} = 1$. If the project does not pay better in expected terms than consuming the endowment, then the patient agent does not invest in the project, and consumes his endowment getting the same utility as the impatient agent. To make the problem interesting, I assume that $E_\theta[p(\theta)]R > 1$ implying that $c_2^{aut} < 1$, the patient agent invests some of his endowment in the risky project. I also assume that $u'(0) > \frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]}u'(R)$, this is intuitive in the spirit of Inada conditions. It just says that marginal utility at 0 is high enough to guarantee that $c_2^{aut} > 0$. Violation of a similar condition is linked to the optimality of the GP contract in Section 4. In autarky, an agent's ex-ante expected utility will be: $\lambda u(1) + (1-\lambda)[(1 - E_\theta[p(\theta)])u(c_2^{aut}) + E_\theta[p(\theta)]u(R - c_2^{aut}(R-1))]$.

2.2.2 Uninformed Planner's Problem

In this section, I follow GP's assumption that the planner can not observe the true θ and only uses the prior on default. At this stage, we can just assume that there are no signals about the state of the economy. Following is the planner's problem:

$Max_{c_1,c_2} \lambda u(c_1) + (1-\lambda)\{E_\theta[p(\theta)]u(c_2 + i_2 R) + (1 - E_\theta[p(\theta)])u(c_2)\}$

s.t.

$(1-\lambda)i_2 = 1 - \lambda c_1 - (1-\lambda)c_2$

and $c_1 \geq 0$, $c_2 \geq 0$, $i_2 \geq 0$.

I first note that there are no participation constraints in the above problem. The solution to the planners problem improves on the ex-ante autarkic allocation because the optimal autarkic allocation is feasible. To clarify, as is standard, I assume that the depositors write enforceable contracts with the planner ex-ante. It is possible that ex-post a depositor might find himself doing worse than autarky and might entertain the idea of reneging on the contract, but contract enforceability prevents that. The following conditions govern the planner's allocation:

$$\frac{u'(c_1)}{u'(\frac{R}{1-\lambda} - c_2(R-1) - \frac{R\lambda}{1-\lambda}c_1)} = RE_\theta[p(\theta)] \tag{1}$$

and

$$\frac{u'(c_2)}{u'(\frac{R}{1-\lambda} - c_2(R-1) - \frac{R\lambda}{1-\lambda}c_1)} = \frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]} \tag{2}$$

From Equation 1 and Equation 2 we get that:

$$\frac{u'(c_1)}{u'(c_2)} = \frac{R(1-E_\theta[p(\theta)])}{R-1} \tag{3}$$

Denote the solution to the problem above by c_1^{FB} and c_2^{FB}. Let $i_2^{FB} = \frac{1-\lambda c_1^{FB}-(1-\lambda)c_2^{FB}}{1-\lambda}$. Note that $E_\theta[p(\theta)]R > 1$ implies that $\frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]} > RE_\theta[p(\theta)] > 1$ and $\frac{R(1-E_\theta[p(\theta)])}{R-1} < 1$.

I start with a simple lemma that determines the relationship between c_1^{FB} and c_2^{FB}.

Lemma 1. *The consumptions in the interim period can not be zero for both types. That is, $c_1^{FB} = 0$ and $c_2^{FB} = 0$ is impossible. This implies that $c_1^{FB} > c_2^{FB} \geq 0$.*

Proof. Assume $c_2^{FB} = 0$ and $c_1^{FB} = 0$, then $u'(0) \leq RE_\theta[p(\theta)]u'(\frac{R}{1-\lambda})$ and $u'(0) \leq \frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]}u'(\frac{R}{1-\lambda})$. But $u'(0) > u'(1) \geq Ru'(R) > RE_\theta[p(\theta)]u'(R) > RE_\theta[p(\theta)]u'(\frac{R}{1-\lambda})$. The middle weak inequality follows from the fact that the coefficient of relative risk aversion is more than 1 for $c \geq 1$ which implies that $cu'(c)$ is decreasing. Therefore, $u'(0) \leq RE_\theta[p(\theta)]u'(\frac{R}{1-\lambda})$ can not hold and $c_1^{FB} > 0$. If $c_2^{FB} = 0$, then $c_1^{FB} > c_2^{FB}$. When $c_2^{FB} > 0$, Equation 3 implies that $u'(c_1^{FB}) < u'(c_2^{FB})$ and $c_1^{FB} > c_2^{FB}$. □

Lemma 1 restricts how we can get bank-runs in the first-best. This is expanded on in the next section. Moreover, it shows that the impatient depositor would not envy the patient depositors consumption in the first-best. This is helpful for analyzing the incentive constraints in the optimal bank contract.

3 When is Risk-Sharing Optimal?

In the more general problem considered in this paper, risk-sharing could either mean a transfer from the patient type to the impatient one or viceversa. I will use a somewhat restricted meaning here. There is risk-sharing when the planner finds it optimal to transfer extra resources to the impatient depositor, that is when $c_1^{FB} > 1$. In GP's world, where the patient depositor is not insured against a project default, the assumption on relative risk aversion guaranteed that there is always risk-sharing. That is not the case here. Risk-sharing is important because of its intimate link to bank-runs. For bank-runs to occur either c_1^{FB} or c_2^{FB} should be higher than 1, the liquidation value of a unit of the project. Lemma 1 already showed that $c_1^{FB} > c_2^{FB}$. Because of the resource constraint, c_2^{FB} can not be greater than 1. This leaves us with risk-sharing as the only possibility for bank-runs. Unlike in GP's contract with no patient depositor insurance, the optimal contract is not always run-prone.

In this section, I will interpret the planner's problem as determining an optimal transfer from the impatient depositor to the patient depositor. Given that transfer, the patient depositor determines his consumption optimally as if he is in autarky. It is important to note the simple fact that this transfer is the only link between the two types of the depositor in this setting. To get the risk-sharing condition, I will first solve an autarky problem with transfer, and then find the condition on the transfer that makes it optimally less than 0. Lemma 2 shows that risk-sharing can occur outside GP's computed contract. Lemma 3 shows what happens to consumptions of the patient depositor in autarky as the transfer to him increases his available resources. Proposition 1 and Corollary 1 determine the condition on utility function for risk-sharing to occur.

Lets begin with the obvious.

Lemma 2. $c_2^{FB} = 0$ *implies risk-sharing. However, risk-sharing does not imply that* $c_2^{FB} = 0$.

Proof. By relative risk aversion coefficient greater than 1 assumption, we have that $u'(1) > Ru'(R) > Rpu'(R)$. Hence Equation 1 means that the marginal benefit of increasing consumption (c_1) at 1 is greater than the cost when $c_2^{FB} = 0$. Proof of second part of lemma is by example: Let $u(c) = 1 - \frac{1}{e^c}$ and $R = 2, E_\theta[p(\theta)] = 0.7, \lambda = 0.1$ then $0 < c_2^{aut} = 0.576 < 1$ and $c_1^{FB} = 1.078$, $c_2^{FB} = 0.568$. $\square$

To prepare the way for the risk-sharing condition, consider the following autarky problem with a transfer ϵ of resources to the patient depositor. The patient depositor solves the following:

$Max_{c_2} E_\theta[p(\theta)]u(c_2 + R(1+\epsilon - c_2)) + (1 - E_\theta[p(\theta)])u(c_2)$ s.t. $0 \le c_2 \le 1 + \epsilon$

FOC yields:

$$\frac{u'(R(1+\epsilon) - c_2^\epsilon(R-1))}{u'(c_2^\epsilon)} = \frac{1 - E_\theta[p(\theta)]}{E_\theta[p(\theta)](R-1)}. \tag{4}$$

From now on, we focus on transfers that give interior solutions. For any two transfers ϵ and ϵ', we have that: $\frac{u'(R(1+\epsilon')-c_2^{\epsilon'}(R-1))}{u'(c_2^{\epsilon'})} = \frac{u'(R(1+\epsilon)-c_2^{\epsilon}(R-1))}{u'(c_2^{\epsilon})} = \frac{1-E_\theta[p(\theta)]}{E_\theta[p(\theta)](R-1)}$

This means

$$\frac{u'(R(1+\epsilon) - c_2^\epsilon(R-1))}{u'(R(1+\epsilon') - c_2^{\epsilon'}(R-1))} = \frac{u'(c_2^\epsilon)}{u'(c_2^{\epsilon'})} \tag{5}$$

Lemma 3. *Raising the transfer, raises the patient depositor's consumption in both states of the world: in a project default, and when it pays. If $\epsilon > \epsilon'$ then $c_2^\epsilon > c_2^{\epsilon'}$ and $R(1+\epsilon) - c_2^\epsilon(R-1) > R(1+\epsilon') - c_2^{\epsilon'}(R-1)$.*

Proof. By contradiction. Assume $\epsilon > \epsilon'$ but $c_2^\epsilon \le c_2^{\epsilon'}$. Then $\frac{u'(c_2^\epsilon)}{u'(c_2^{\epsilon'})} \ge 1$. By Equation 5, we have that: $\frac{u'(R(1+\epsilon)-c_2^\epsilon(R-1))}{u'(R(1+\epsilon')-c_2^{\epsilon'}(R-1))} = \frac{u'(c_2^\epsilon)}{u'(c_2^{\epsilon'})} \ge 1$. Therefore, $R(1+\epsilon') - c_2^{\epsilon'}(R-1) \ge R(1+\epsilon) - c_2^\epsilon(R-1) \implies R(\epsilon' - \epsilon) \ge (R-1)(c_2^{\epsilon'} - c_2^\epsilon) \implies c_2^{\epsilon'} < c_2^\epsilon$. Hence, $\epsilon > \epsilon' \implies c_2^\epsilon > c_2^{\epsilon'}$.

For second part, again proof by contradiction. Assume $\epsilon > \epsilon'$ but $R(1+\epsilon) - c_2^\epsilon(R-1) \le R(1+\epsilon') - c_2^{\epsilon'}(R-1)$. This means: $u'(c_2^{\epsilon'}) > u'(c_2^\epsilon)$ and $u'(R(1+\epsilon) - c_2^\epsilon(R-1)) \ge u'(R(1+\epsilon') - c_2^{\epsilon'}(R-1))$. Therefore, $\frac{u'(c_2^{\epsilon'})}{u'(R(1+\epsilon')-c_2^{\epsilon'}(R-1))} \ge \frac{u'(c_2^{\epsilon'})}{u'(R(1+\epsilon)-c_2^{\epsilon}(R-1))} > \frac{u'(c_2^{\epsilon})}{u'(R(1+\epsilon)-c_2^{\epsilon}(R-1))}$. $\square$

By Lemma 3, $\epsilon > 0 \implies c_2^\epsilon > c_2^{aut}$ and $R(1-\epsilon) - c_2^\epsilon(R-1) > R - c_2^{aut}(R-1)$. When $\epsilon < 0$, we have both that $c_2^\epsilon < c_2^{aut}$ and $R(1-\epsilon) - c_2^\epsilon(R-1) < R - c_2^{aut}(R-1)$.

The meaning of this is clear, as resources get transferred away from a patient depositor, he decreases his consumptions in both states of the world: when the project defaults and when it does not. As more resources are transferred away, his consumption in both states drops even more. And viceversa.

Comparing Equation 4 with Equation 2 shows that for any fixed c_1, we can interpret Equation 2 as a solution to a problem with a transfer $\epsilon = \frac{\lambda(1-c_1)}{1-\lambda}$. This transfer is decreasing in c_1. It is positive for $c_1 < 1$, negative for $c_1 > 1$, and zero at $c_1 = 1$. We know from the autarky problem with transfers that increasing the transfer, increases $R(1+\epsilon) - c_2^\epsilon(R-1)$, decreasing $RE_\theta[p(\theta)]u'(R(1+\epsilon) - c_2^\epsilon(R-1))$.

Here is a procedure to find the optimal c_1^{FB}. Start at any point c_1, compute the transfer $\epsilon = \frac{\lambda(1-c_1)}{1-\lambda}$, and then compute $RE_\theta[p(\theta)]u'(R(1+\epsilon) - c_2^\epsilon(R-1))$ from the autarky problem with transfer. If $u'(c_1) > RE_\theta[p(\theta)]u'(R(1+\epsilon) - c_2^\epsilon(R-1))$, then increase c_1, decreasing the LHS, decreasing the transfer, and increasing the RHS, until equality is got. Similarly if $u'(c_1) < RE_\theta[p(\theta)]u'(R(1+\epsilon) - c_2^\epsilon(R-1))$, then decrease c_1, increasing the LHS, increasing the transfer and decreasing the RHS, until equality is got. This gives us a good insight into the condition for risk-sharing.

Proposition 1. *If* $u'(1) > RE_\theta[p(\theta)]u'(c_2^{aut} + R(1-c_2^{aut}))$ *then risk-sharing between the patient depositor and the impatient depositor occurs (i.e.* $c_1^{FB} > 1$*).*

Proof. Let $u'(1) > RE_\theta[p(\theta)]u'(c_2^{aut} + R(1-c_2^{aut}))$, but $c_1^{FB} \leq 1$. We will apply the procedure above. At $c_1 = 1$, $\epsilon = 0$, and optimal $c_2^\epsilon = c_2^{aut}$. Let $\epsilon' = \frac{\lambda(1-c_1^{FB})}{1-\lambda}$, when $c_1^{FB} \leq 1$ we have $\epsilon' \geq 0 = \epsilon$. By Lemma 3, we have that $c_2^{\epsilon'} \geq c_2^\epsilon = c_2^{aut}$ and $R(1+\epsilon') - c_2^{\epsilon'}(R-1) \geq R(1+\epsilon) - c_2^\epsilon(R-1) = R - c_2^{aut}(R-1)$. Now, $u'(c_1^{FB}) \geq u'(1) > RE_\theta[p(\theta)]u'(c_2^{aut} + R(1-c_2^{aut})) \geq RE_\theta[p(\theta)]u'(R(1+\epsilon') - c_2^{\epsilon'}(R-1))) = RE_\theta[p(\theta)]u'(R\frac{1-\lambda c_1^{FB}}{1-\lambda} - c_2^{FB}(R-1))$. But then optimality condition can not hold. □

Corollary 1. *If* $u'(1) < RE_\theta[p(\theta)]u'(c_2^{aut} + R(1-c_2^{aut}))$ *then* $c_1^{FB} < 1$*. When* $u'(1) = RE_\theta[p(\theta)]u'(c_2^{aut} + R(1-c_2^{aut}))$ *then* $c_1^{FB} = 1$.

It is worth noting that under Log utility, the planner can not improve on autarky.

3.1 Are there Utility Functions Satisfying GP's Conditions but not Run-Prone?

This section shows a utility function that satisfy increasing relative risk aversion greater than 1 for $c \geq 1$ and $u'(1) < Rpu'(c_2^{aut} + R(1 - c_2^{aut}))$. Therefore, the optimal contract might be bank run-proof in equilibrium. Unlike in Diamond and Dybvig (1983) and in the contract computed in Goldstein and Pauzner (2005), relative risk aversion coefficient greater than 1 for $c \geq 1$ does not guarantee the contract is run-prone.

Consider the family of constant relative risk aversion coefficient (CRAA) utility functions: $u(c) = \frac{c^{1-\beta}-1}{1-\beta}$. $u'(c) = c^{-\beta} > 0$ and $u''(c) = -\beta c^{-\beta-1} < 0$. $RRA = \beta$. Consider the following composite utility function: $u(c) = -\frac{c^{-2}-1}{2}$ i.e. with $\beta = 3$ $\forall c < 1$, and $u(c) = -\frac{c^{-0.5}-1}{0.5}$ i.e. with $\beta = 1.5$ $\forall c \geq 1$. At 1, utility function is continuous and has a continuous derivative, with $u(1) = 0$ and $u'(1) = 1$. c_2^{aut} solves the FOC: $p(R-1)(R-c(R-1))^{-1.5} - (1-p)c^{-3} = 0$ For $R = 2$ and $p = .51$, we get that: $c_2^{aut} = 0.9911233$. Hence, $Rpu'(c_2^{aut} + R(1 - c_2^{aut})) = \frac{Rp}{(R-c_2^{aut}(R-1))^{1.5}} = 1.00657 > u'(1) = 1$.

This showed that there are risk-averse utility functions that satisfy GP's assumptions, but with first-best contracts that are not run-prone.

4 When is GP's Contract Optimal?

GP's contract is optimal when the marginal utility at zero is bounded above. The following lemma presents a sufficient condition for GP contract to be optimal when there is risk-sharing. It shows, in the case of risk-sharing considered by GP, one upper-bound on marginal utility at zero under which GP's contract is optimal.

Lemma 4. *Assume there is risk-sharing with the impatient depositor ($c_1^{FB} \geq 1$). If $u'(0) \leq \frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]}u'(R)$ then $c_2^{FB} = 0$.*

Proof. When there is risk-sharing, $R\frac{1-\lambda c_1^{FB}}{1-\lambda} \leq R$. Hence, $u'(R\frac{1-\lambda c_1^{FB}}{1-\lambda}) \geq u'(R)$. We have,

$u'(0) \leq \frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]}u'(R) \leq \frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]}u'(R\frac{1-\lambda c_1^{FB}}{1-\lambda})$. The foc of the planner's problem shows that the marginal benefit of increasing patient depositor's consumption c_1^{FB}, at $c_2 = 0$ is less than the marginal cost. Therefore, $c_2^{FB} = 0$. □

The following upper bound works in both cases when there is risk-sharing and when there is none.

Proposition 2. *Let c^* be the c that solves: $u'(c) = E_\theta[p(\theta)]Ru'(R\frac{1-\lambda c}{1-\lambda})$. Note that c^* is the c_1^{FB} computed by GP.*

$u'(0) \leq \frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]}u'(R\frac{1-\lambda c^*}{1-\lambda})$ if and only if $c_2^{FB} = 0$.

Proof. Straightforward from planner problem's first order conditions. □

5 Bank's Contract

Banks face competition and offer the contract that maximizes the depositors expected utility, subject to incentive constraints that guarantee that no depositor has an incentive to misreport his type. If the first-best contract satisfies the incentive constraints, then the bank offers the first-best contract. When it does not, the optimal bank contract provides less consumption for the impatient depositor than the first-best. But the bank contract still offers risk-sharing, since resources are still transferred to the impatient depositor, raising his consumption above 1. The patient depositor's consumption on the other hand, is more than the first-best in both states of the world. He is also indifferent between saying he is the patient or the impatient type. Moreover, when the first-best contract is not run-prone, then it satisfies the incentive constraints and is the optimal bank's contract. When it is run-prone, I show that the optimal bank's contract is as well.

In the current context, unlike most of the literature, a bank-run does not just mean that the patient depositors visit the bank in period 1 to withdraw funds. This is because the consumption of the patient depositor c_2^* might be given out in period 1. A bank-run means that the patient masquerade as the impatient. The patient depositor visits the bank in period 1, but instead of asking for c_2^*, he asks for c_1^* and foregoes the last period project return. Therefore, observing how many depositors line up in the queue in the first period does not tell us that there is a run. A run depends on the actual withdrawals that the depositors are making. This draws a small distinction between this paper and other papers that argue that observing a long line infront of a bank, provides information to patient depositors that a run is in order.

The optimal bank contract solves the following problem.

$Max_{c_1,c_2} \lambda u(c_1) + (1-\lambda)\{E_\theta[p(\theta)]u(c_2 + i_2 R) + (1 - E_\theta[p(\theta)])u(c_2)\}$ s.t.

$(1-\lambda)i_2 = 1 - \lambda c_1 - (1-\lambda)c_2$

$u(c_1) \geq u(c_2)$

$E_\theta[p(\theta)])u(c_2 + i_2 R) + (1 - E_\theta[p(\theta)])u(c_2) \geq u(c_1)$

and $c_1 \geq 0$, $c_2 \geq 0$, $i_2 \geq 0$.

Denote the optimal bank contract by the pair (c_1^*, c_2^*). Let $i_2^* = \frac{1-\lambda c_1^* - (1-\lambda)c_2^*}{1-\lambda}$.

Proposition 3. *If $u'(1) \leq RE_\theta[p(\theta)]u'(c_2^{aut} + R(1 - c_2^{aut}))$ then the optimal bank contract is the first-best contract, $(c_1^*, c_2^*) = (c_1^{FB}, c_2^{FB})$. Moreover, implementation exhibits the unique efficient equilibrium and there are no runs.*

Proof. By Corollary 1, $u'(1) \leq RE_\theta[p(\theta)]u'(c_2^{aut} + R(1 - c_2^{aut}))$ means that $c_1^{FB} \leq 1$. This means that resources are transferred to the patient depositor and hence he would not want to act as if he is impatient. Lemma 1 allows us to say $c_2^{FB} < c_1^{FB} \leq 1$. This implies that the impatient depositor would not want to act as if he is patient. To see that there are no runs is easy. $c_2^{FB} < c_1^{FB} \leq 1$ makes clear that even when all depositors demand payment in first period, the bank would not fail. □

When the first-best contract is run-proof, it satisfies the incentive constraints, and is the bank's contract. Therefore, it is enough to look at the first-best contract to realize if the bank's optimal contract is run-proof. How about when the first-best contract is not run-proof?

Proposition 4. *Assume $u'(1) > RE_\theta[p(\theta)]u'(c_2^{aut} + R(1 - c_2^{aut}))$. If the first-best allocation satisfies the incentive constraints, then the first-best contract is the optimal bank contract. When it does not, then the optimal bank contract is a (c_1^*, c_2^*) such that: $1 < c_1^* < c_1^{FB}$, $c_2^* > c_2^{FB}$ and $c_2^* + Ri_2^* > c_2^{FB} + Ri_2^{FB}$ and*
$E_\theta[p(\theta)])u(c_2^* + R\frac{1-\lambda c_1^* - (1-\lambda)c_2^*}{1-\lambda}) + (1 - E_\theta[p(\theta)])u(c_2^*) = u(c_1^*)$. Implementation of this contract has two equilibria: the efficient equilibrium and another run equilibrium.

Proof. When the first-best contract does not satisfy the incentive constraints, by Lemma 1 it is definitely the patient depositor preferring to be impatient. That can only happen when resources are taken away from him (risk-sharing). The only way to raise his expected utility is by taking less resources away from him. Lemma 3 showed that this raises both his consumptions.

All that remains to be shown is that $c_1^* > 1$, and noting that this means the contract is run-prone. Here is a useful way to understand what the optimal bank contract is. Start at first-best contract. In moving to the constrained optimum, the optimal contract decreases the consumption of the impatient depositor (c_1) and increases the patient depositor's consumption in both states of the world. This process continues, until the violated constraint is satisfied with equality. It is definitely the case that $c_1^* \geq 1$, since at autarky the incentive constraints are definitely satisfied. Assume it is exactly 1. The autarky problem of the patient depositor should definitely give him a strictly higher utility than $u(1)$ and hence the incentive constraint is slack, which can not be. $c_1^* > 1$ has two important implications, first that the impatient depositor would not want to act as a patient depositor after we decrease the impatient depositors consumption to get to optimality. Second, even though the impatient depositors consumption has been decreased, but the optimal bank contract is still run-prone. □

Proposition 4 shows that to determine if the optimal bank contract is run-prone, it is enough to look at the first-best contract. When the first-best contract is run-prone, then so is the bank's contract.

6 Discussion of Bank's Contract

The optimal bank contract shows that even though banks themselves are risk neutral, but depositor risk-aversion with coefficient of relative risk aversion greater than 1, coupled with high marginal utility at zero forces banks to scale back bank lending (this is interpreted as risky project in the model). The contract restricts what the bank can do. Do we see this in reality? $c_2^* > 0$ can be interpreted in a couple of interesting ways. What bank assets could serve as c_2^*, protecting depositors against a spike in loan defaults? Two interpretations are presented here. First, c_2^* could be the cash and safe assets that the banks generally hold. These are held for two reasons, to accommodate "normal" withdrawals, but can also act as a promise of some return if risky bank loan defaults spike. Second, c_2^* could be interpreted as the reserve requirements that a bank has to hold by law. These are a fraction of deposits that has to be invested in very safe, secure assets. It is interesting to note that these reserve requirements have failed to mitigate bank-runs in the pre-federal reserve world. This is also supported by the model presented here. Under some conditions, we do see a run-prone contract, even with the existence of enough liquidity to cover the liquidity needs of impatient depositors (c_1^*), and the insurance needs of patient ones (c_2^*). It is interesting to see how the optimal contract responds to changes in the perceived default probability of the projects $1 - E_\theta[p(\theta)]$. The next subsection does that.

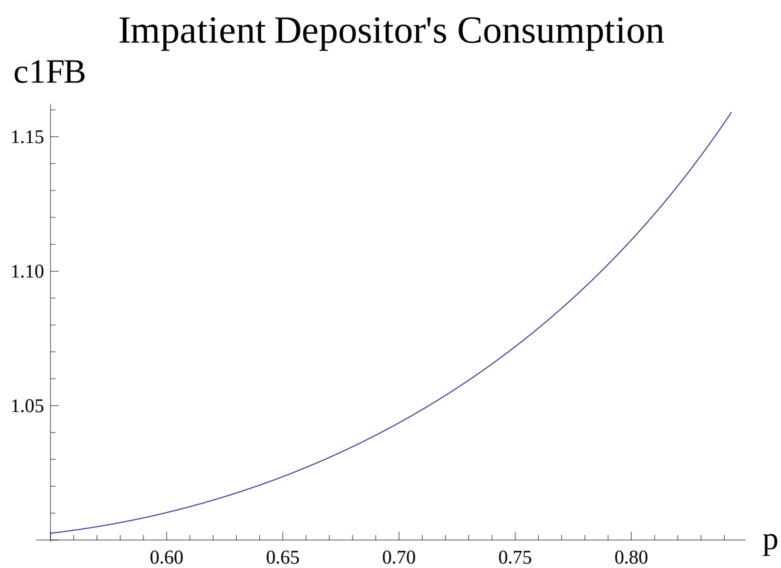
6.1 Contract as a Function of Expected Probability of Paying

To get concrete results, I work out how the optimal contract changes with $E_\theta[p(\theta)]$ in the case when $u(c) = 1 - e^{-c}$. Maximizing the planner's problem, assuming an interior solution and solving the foc yields: $c_1^{FB} = 1 - (1-\lambda)Log(E_\theta[p(\theta)]R) + \frac{(R-1)(1-\lambda)}{R}Log(\frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]})$

and $c_2^{FB} = 1 + \lambda Log(E_\theta[p(\theta)]R) - \frac{\lambda(R-1)+1}{R} Log(\frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]})$. This gives us that $i_2^{FB} = \frac{1}{R}(Log[\frac{E_\theta[p(\theta)]}{1-E_\theta[p(\theta)]}] + Log[R-1])$.

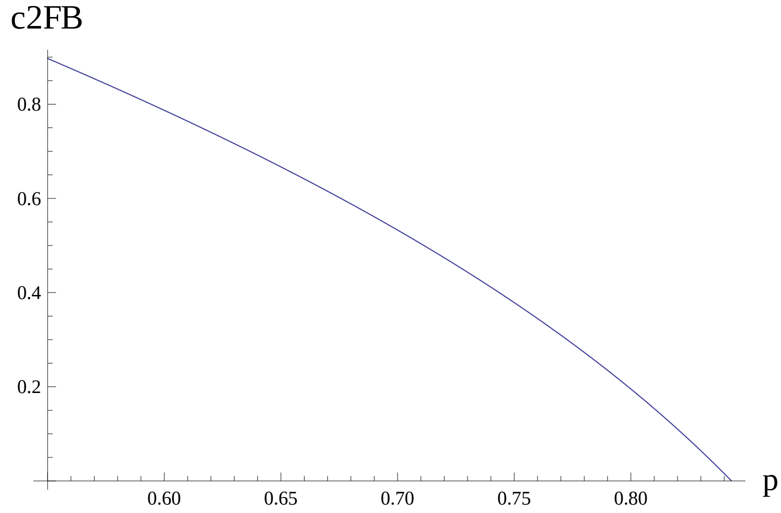
The following table shows the derivative of optimal quantities of interest w.r.t. $E_\theta[p(\theta)]$.

Derivative	Value	Sign	Interpretation
$\frac{d}{dE_\theta[p(\theta)]} c_1^{FB}$	$\frac{(1-\lambda)}{R} \frac{RE_\theta[p(\theta)]-1}{E_\theta[p(\theta)](1-E_\theta[p(\theta)])}$	Positive	Increases with $E_\theta[p(\theta)]$
$\frac{d}{dE_\theta[p(\theta)]} c_2^{FB}$	$-\frac{\lambda(RE_\theta[p(\theta)]-1)+1}{RE_\theta[p(\theta)](1-E_\theta[p(\theta)])}$	Negative	Decreases with $E_\theta[p(\theta)]$
$\frac{d}{dE_\theta[p(\theta)]} i_2^{FB}$	$\frac{1}{RE_\theta[p(\theta)](1-E_\theta[p(\theta)])}$	Positive	Increases with $E_\theta[p(\theta)]$
$\frac{d}{dE_\theta[p(\theta)]} x_2^{FB}$	$\frac{R-1-\lambda(RE_\theta[p(\theta)]-1)}{(1-E_\theta[p(\theta)])E_\theta[p(\theta)]R}$	Positive	Increases with $E_\theta[p(\theta)]$

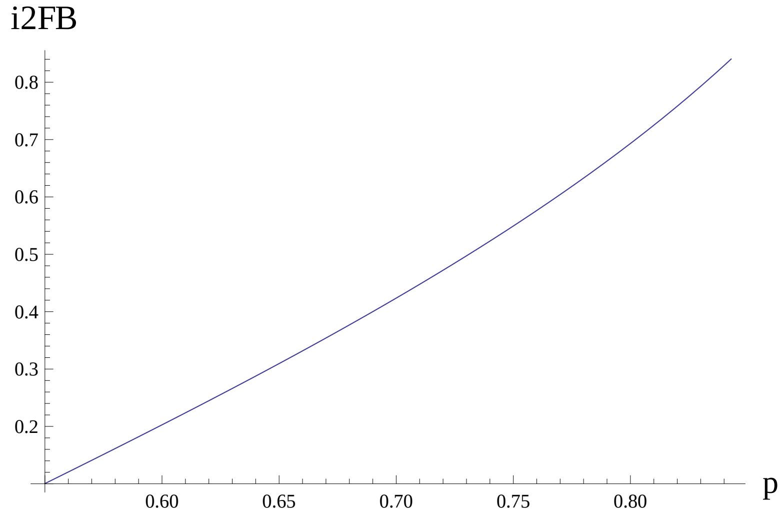


As $E_\theta[p(\theta)]$ goes up, the project becomes less risky. c_2^{FB}, the patient depositor's insurance, goes down since he faces less risk now. Both c_1^{FB}, the impatient depositor's consumption, and i_2^{FB}, the investments in the project, go up. Since c_2^{FB} goes down, while both c_1^{FB} and i_2^{FB} go up, it is interesting to see which force dominates in determining the direction of consumption of the patient depositors when the project pays: $x_2^{FB} = R\frac{1-\lambda c_1^{FB}}{1-\lambda} - c_2^{FB}(R-1)$. This consumption increases with $E_\theta[p(\theta)]$. Hence, bank lending (i_2^{FB}) and risk-sharing (c_1^{FB}) increase in a cyclical manner, when projects have a higher chance of paying and $E_\theta[p(\theta)]$ is

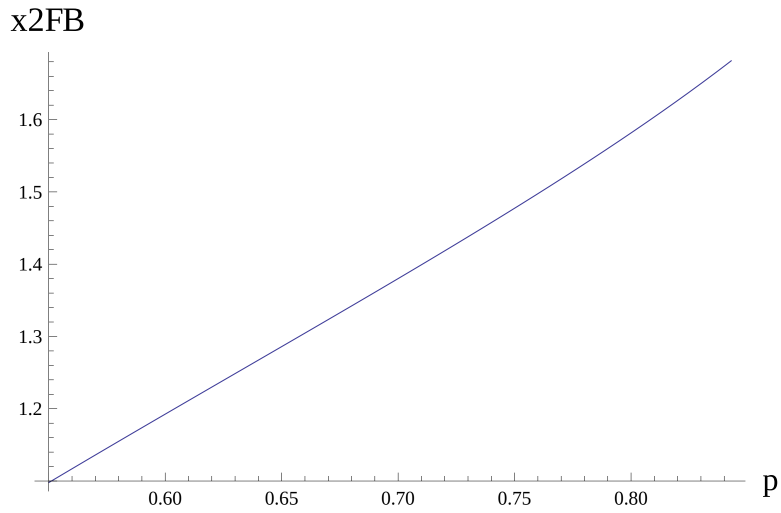
Patient Depositors Consumption



Impatient Depositor's Investment

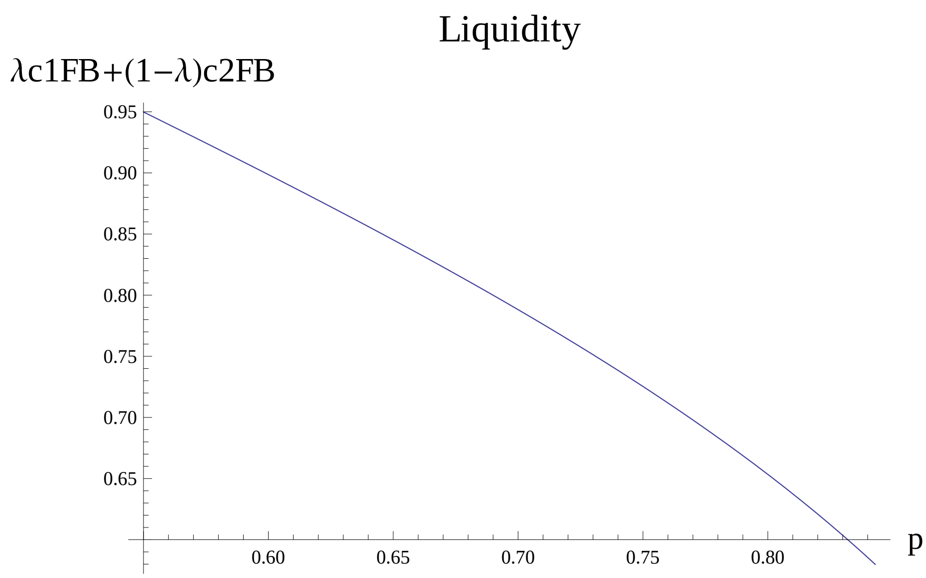


Consumption When the Project Pays



high. While (c_2^{FB}) is counter-cyclical. Plots of these consumptions as a function of $E_\theta[p(\theta)]$ are provided.

To understand the effect on total liquidity held by the bank, we first note that as $E_\theta[p(\theta)]$ increases, c_2^{FB} goes down, but c_1^{FB} goes up. So are banks required to hold more liquidity or less liquidity as perceived probability of default $(1 - E_\theta[p(\theta)])$ increases? Liquidity here is defined to be $L = \lambda c_1^{FB} + (1-\lambda)c_2^{FB}$. In the current environment, $L = 1 - \frac{1-\lambda}{R} Log[\frac{E_\theta[p(\theta)](R-1)}{1-E_\theta[p(\theta)]}]$. This is obviously declining in $E_\theta[p(\theta)]$ The following graph shows that required bank liquidity goes up in recessions, and declines in booms.



7 Further Research

The author is currently working on using the results of this paper and the Goldstein and Pauzner (2005) results on the use of global games techniques in this environment, to find the effect on the probability of bank-runs and the effects of bank contracts on welfare.

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