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A Note on Sunspots with Heterogeneous Agents

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This paper studies sunspot fluctuations in a model with heterogeneous households. We find that wealth inequality reduces the degree of increasing returns needed to produce indeterminacy, while wage inequality increases it. When the model is calibrated to match the joint distribution of hours, income, and wealth, the required degree of increasing returns to scale is still much too high to be supported empirically (although smaller than similar homogeneous agent economies). We also find that the model robustly predicts only one sunspot, despite having 1,262 predetermined state variables.

Keywords: Heterogeneity, Sunspots.

JEL codes: E21, E25, E62

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1. Introduction

Benhabib and Farmer (1994) and Farmer and Guo (1994) show that a homogeneous-agent business cycle model with socially-increasing returns to scale can have steady states that are sinks; since transversality can no longer be invoked to uniquely determine current consumption, the model has a continuum of equilibrium paths. Furthermore, random iid variables can generate business cycles – called sunspots, animal spirits, or self-fulfilling expectations by various authors – as agents coordinate their consumption decisions on these fundamentally-irrelevant variables. Econometric evidence can be found that supports a wide range of increasing returns, from very small (Burnside, Eichenbaum, and Rebelo 1995, Bartelsman 1995, Basu and Fernald 1995) to very large (Hall 1990, 1991, Bartelsman, Caballero, and Lyons 1994); none support the degree needed for the basic model to display indeterminacy, however. Our objective in this paper is to consider the presence of sunspots in models of heterogeneous agents.

We find that heterogeneity in wealth reduces the required degree of increasing returns to scale needed for sunspots to appear, while heterogeneity in wages increases it. In a simple two agent economy we are able to reduce the required increasing returns to scale to 0.275 by setting wealth inequality near its upper bound while setting wage inequality to its lower bound. While this parametrization is not reasonable empirically (permanent wage inequality is significant in US data, as shown in Flodén and Lindé 2001), it does work in the "right" direction since the distribution of wealth in the US is much more unequal than the distribution of wages; of course, the required returns to scale is still too large. When we carefully calibrate the model to the US distribution of wealth, income, and hours – following the approach in Carroll (2009) – we find that it requires a higher degree of increasing returns to scale, but not as large as the models of Benhabib and Farmer (1994) or Farmer and Guo (1994) do. The introduction of other features that have been shown to reduce the required increasing returns to scale – such as home production, multiple sectors, or elastic capacity utilization – may very well push the model into plausible ranges.

We also explore the number of additional stable eigenvalues that appear. In particular, we want to know whether the eigenvalues "move en masse" across the unit circle or somewhat independently – that is, can the model produce only one sunspot, many sunspots, or does it have to have the same number of sunspots as predetermined wealth levels? We find that the model robustly produces only one sunspot variable, despite having 1242 different capital stocks. That is, while each agent can have a shock to their Euler equation, $n - 1$ of the shocks must be linear functions of the shock

that hits the n th individual (Benhabib and Nishimura 1998 show a similar result in a model with multiple sectors and irreversible investment). In other words, the model can be parameterized to match any volatility for individual consumption, since the scale factors are not pinned down (this result is similar to the homogeneous agent case where the volatility of the sunspot is arbitrary and frequently calibrated to match the volatility of output). Furthermore, the sunspot disappears when the degree of increasing returns to scale gets too large, a phenomenon not observed in homogeneous agent economies.

In summary, we see our contribution here as pointing out the ability of heterogeneity to influence the dimension of the stable dynamics of the growth model with increasing returns to scale. There are two related contributions that we want to reference here. Herrendorf, Valentinyi, and Waldmann (2000) use the model from Matsuyama (1991) to study the effects of heterogeneity for multiplicity, concluding that heterogeneous agent models make the empirical importance of multiplicity more untenable. However, Ghiglini and Olszak-Duquenne (2005) note that the direction of the effect is actually ambiguous in the model of Matsuyama (1991) – heterogeneity could reduce or increase the required returns to scale. The Matsuyama (1991) model is substantially different than the standard growth model – it contains overlapping generations of households with geometric life who must make an irreversible occupational choice at birth, for example – so our work constitutes an independent contribution.

2. Model

The model economy is adapted from Carroll and Young (2009). Preferences are

$$E_0 \sum_{t=0}^{\infty} \beta_i^t \left(\log(c_{it}) + \frac{B_i (1 - h_{i,t})^{1-\sigma}}{1 - \sigma} \right) \quad (2.1)$$

where $\beta_i \in (0, 1)$ and $B_i > 0$ are heterogeneous preference parameters. $\sigma^{-1} \geq 0$ is the common intertemporal elasticity of substitution for leisure. The budget constraint is

$$(1 + \tau_c) c_{i,t} + a_{i,t+1} \leq (1 + r_t - \delta) a_{i,t} + w_t \varepsilon_i h_{i,t} - \tau ((r_t - \delta) a_{i,t} + w_t \varepsilon_i h_{i,t}) - \chi_t ((r_t - \delta) a_{i,t} + w_t \varepsilon_i h_{i,t}) + T_t; \quad (2.2)$$

$c_{i,t}$ is consumption, $a_{i,t}$ are shares of the production technology, ε_i is the permanent productivity level for hours $h_{i,t}$. $r_t - \delta$ is the return on shares, and w_t is the aggregate wage index. $\tau(y)$ is the

income tax function from Gouveia and Strauss (1994):

$$\tau(y) = \nu_0 \left(y - (y^{-\nu_1} + \nu_2)^{-\frac{1}{\nu_1}} \right). \quad (2.3)$$

This functional form is flexible enough to capture a wide range of tax functions; we assume that the parameters (ν_0, ν_1, ν_2) are such that the tax schedule is marginal-rate progressive (the marginal tax rate is an increasing function of income) because this setting is the empirically-relevant one.¹ T_t is a lump-sum transfer, τ_c is a constant consumption tax, and χ_t is an additional flat tax on income.

The production firm rents capital and labor and generates net output. The production technology displays aggregate increasing returns to scale. Gross output is given by

$$Y_t = S_t K_t^\alpha N_t^{1-\alpha} \quad (2.4)$$

where

$$S_t = \left(\bar{K}_t^\alpha \bar{N}_t^{1-\alpha} \right)^v \quad (2.5)$$

is the productive externality and \bar{K} and \bar{N} are the economy-wide values for capital and labor input. The factor prices are therefore

$$r_t = \alpha K_t^{\alpha-1+v\alpha} N_t^{(1-\alpha)(1+v)} \quad (2.6)$$

$$w_t = (1-\alpha) K_t^{\alpha(1+v)} N_t^{-\alpha+v(1-\alpha)}. \quad (2.7)$$

In equilibrium averages equal aggregates and the factor markets must clear at these prices:

$$K_t = \sum_i a_{i,t} \psi_i \quad (2.8)$$

$$N_t = \sum_i \varepsilon_i h_{i,t} \psi_i \quad (2.9)$$

where ψ_i is the measure of type i (consisting of a triple $(\varepsilon_i, \beta_i, B_i)$ for each i). The parameter $v \geq 0$ measures the strength of the externality; since $(1+v)\alpha + (1-\alpha)(1+v) \geq 1$, the model features increasing returns to scale at the aggregate level whenever $v > 0$. We restrict attention to the case $\alpha(1+v) < 1$, so that the economy does not display unbounded growth.

¹Consistency with balanced-growth requires ν_2 to grow at rate $g_y^{-\nu_1}$, where g_y is the growth rate of income. As it plays no important role in the presence of indeterminate equilibria, we ignore growth here.

The government collects taxes to finance government consumption according to the budget constraint

$$G_t + T_t = \tau_c \sum_i c_{i,t} \psi_i + \sum_i \tau (y_{i,t}) \psi_i + \chi_t \sum_i y_{i,t} \psi_i; \quad (2.10)$$

we ignore government debt. The final market is for goods:

$$\sum_i c_{i,t} \psi_i + \sum_i a_{i,t+1} \psi_i + G_t = K_t^{\alpha(1+\nu)} N_t^{(1-\alpha)(1+\nu)} + (1 - \delta) \sum_i a_{i,t} \psi_i. \quad (2.11)$$

We assume that transfers and government spending are constant over time, so that χ adjusts to clear the government budget constraint. As noted in Schmitt-Grohé and Uribe (1997), this assumption typically leads to indeterminacy at smaller returns to scale, so we are biasing our investigation in favor of finding sunspots.

3. Two Agent Economy

The full model is difficult to analyze due to its immense size, so we develop intuition using a simpler model with two agents and no fiscal policy; as a result, the wealth distribution is indeterminate and we can discuss how inequality affects indeterminacy by exogenously varying the fraction of assets held by each type of household. This economy has only one predetermined variable: the aggregate capital stock. Indeterminacy then arises whenever the number of eigenvalues smaller than 1 in absolute value exceeds 1. As a preview of the results, wealth inequality makes it more likely that sunspots occur while wage inequality makes it less likely (in the sense that the critical value of ν is lower or higher).

Consider the economy above but with only two agents, no taxes, and homogeneous preferences.

The equilibrium is characterized by the equations

$$\begin{aligned}
C_{1t}^{-1} &= \beta E_t [C_{1t+1}^{-1} (r_{t+1} + 1 - \delta)] \\
C_{2t}^{-1} &= \beta E_t [C_{2t+1}^{-1} (r_{t+1} + 1 - \delta)] \\
B(1 - H_{1t})^{-\sigma} &= C_{1t}^{-1} w_t \varepsilon_1 \\
B(1 - H_{2t})^{-\sigma} &= C_{2t}^{-1} w_t \varepsilon_2 \\
r_t &= \alpha K_t^{\alpha-1+v\alpha} N_t^{(1-\alpha)(1+v)} \\
w_t &= (1 - \alpha) K_t^{\alpha(1+v)} N_t^{-\alpha+(1-\alpha)v} \\
C_{1t} &= (r_t + 1 - \delta) A_{1t} + w_t \varepsilon_1 H_{1t} - A_{1t+1} \\
C_{2t} &= (r_t + 1 - \delta) A_{2t} + w_t \varepsilon_2 H_{2t} - A_{2t+1} \\
K_t &= \psi A_{1t} + (1 - \psi) A_{2t} \\
N_t &= \psi \varepsilon_1 H_{1t} + (1 - \psi) \varepsilon_2 H_{2t}.
\end{aligned}$$

The distribution of assets is not determined by the economy, so we specify it exogenously. We set $\varepsilon_1 = \mu$ and $\varepsilon_2 = 1$ and then set $A_{1t} = \eta K_t$ and $A_{2t} = (1 - \eta) K_t$ for each t .² We use this simple system to illustrate how the presence of sunspots depends on wage heterogeneity μ and wealth heterogeneity η . We set $\psi = 0.5$, although the results we obtain are not sensitive to this number.

When linearized around the unique interior steady state, the fundamental difference equation takes the form

$$\mathcal{A}E_t[\mathbf{X}_{t+1}|\mathcal{F}_t] = \mathcal{B}\mathbf{X}_t \quad (3.1)$$

where \mathcal{A} and \mathcal{B} are 10×10 matrices. We calibrate the parameters $\{\delta, \beta, B\}$ to match a capital-output ratio of 11.5, an investment-output ratio of 25 percent, and total hours equal to 33 percent of the time endowment. We set $\sigma = 0.5$, since it is known that sunspot equilibria require highly elastic labor supply, and set $\alpha = 0.36$ in line with estimates of capital's share of income.³ Indeterminacy arises when the number of generalized eigenvalues of the matrix pencil $|A - Bz|$ that lie within the unit circle is larger than 1.⁴

²Setting $\varepsilon_2 = 1$ is an innocuous normalization that serves only to choose units of steady state output.

³The system (3.1) contains a unitary eigenvalue due to the indeterminacy of the steady-state wealth distribution. The full model will not have this feature because progressive taxation is sufficient to deliver determinate wealth distributions – see Carroll and Young (2009).

⁴It is possible that the number of eigenvalues inside the unit circle is actually 0, implying that the steady state is locally a source. However, it also may be surrounded by stable cycles, as in Coury and Wen (2008). We discuss such issues in the conclusion.

For each value of (η, μ) we calculate the critical value of v that induces indeterminacy. As shown in Figure (1), increasing μ increases the required increasing returns to scale while increasing η tends to decrease this value. What is interesting to note is that the required increasing returns to scale can be small if wealth inequality is extreme and wage inequality is moderate. We also point out that the model parametrization $\mu = 1$ and $\eta = 0.5$ is equivalent to a homogeneous agent economy. In this case, our model requires that v exceed 0.5 in order for sunspot equilibria to appear, consistent with results in Benhabib and Farmer (1994) and Farmer and Guo (1994). These results suggest that one route to indeterminacy may be through heterogeneity; given that the US distribution has extreme wealth inequality (Budría *et al.* 2002 compute a Gini coefficient of 0.8) and more moderate wage inequality (Gini coefficient of earnings is 0.6 and hours are highly uniform across individuals who work), it seems that sunspot equilibria may not be implausible. To examine this possibility more carefully the next section presents an economy that matches these distributions by construction.

Before moving on to the full model, however, we want to provide some intuition about how wealth and wage inequality work to produce indeterminacy. As noted in Benhabib and Farmer (1994), the key to generating indeterminacy is highly elastic aggregate labor input. Consider a static labor-leisure choice

$$\max_{c,h} \left\{ \log(c) + B \frac{(1-h)^{1-\sigma}}{1-\sigma} \right\}$$

subject to the budget constraint

$$c = a + wh,$$

where a is total nonhuman wealth and w is the after-tax wage. The optimal tradeoff between consumption and leisure implies the condition

$$(a + wh)^{-1} w = B(1-h)^{-\sigma}.$$

The derivative of hours with respect to the wage is

$$\frac{dh}{dw} = \frac{1 - wh(a + wh)^{-1}}{\sigma w(1-h)^{-1} + w^2(a + wh)^{-1}};$$

note that the response of hours is decreasing in σ , so small values of σ imply large labor supply

elasticities. Larger a implies that hours become more responsive to changes in wages:

$$\frac{d^2h}{dwda} = (1 + h(\sigma - 1)) \frac{1 - h}{(\sigma(a + wh) + w(1 - h))^2} > 0$$

since $1 > h(1 - \sigma)$ holds for any $\sigma \geq 0$ and any $h < 1$. Furthermore, large w means that hours are less responsive to changes in wages:

$$\frac{d^2h}{dw^2} = -\frac{a(1 - h)}{w^2} \frac{\sigma(a + 2wh) + 2w(1 - h)}{(\sigma(a + wh) + w(1 - h))^2} < 0.$$

Thus, an economy where wealth is concentrated in the hands of the highly productive will feature larger movements in aggregate labor input; the economy computed here satisfies this requirement and so displays indeterminacy at relatively-low increasing returns to scale.⁵ In US data, there is a positive correlation between wages and financial wealth, but whether this correlation is strong enough to significantly reduce the required returns to scale is a quantitative issue. We now turn to examining this question.

4. The Full Model

We now present and calibrate our full model. As noted in Carroll and Young (2009), without preference heterogeneity the distribution of income and wealth in a complete market model is fundamentally inconsistent with the data in the presence of progressive taxation. We therefore calibrate our model by using data to infer characteristics that would equate the model's steady state to empirical observations. Specifically, the deterministic steady state is used to obtain estimates for the individual parameters $(\beta_i, \varepsilon_i, B_i)$ using the formulae

$$\begin{aligned} \beta_i &= \frac{1}{1 + (1 - \tau'(y_i) - \chi)(r - \delta)} \\ \varepsilon_i &= \frac{y_i - (r - \delta)a_i}{wh_i} \\ B_i &= \frac{(1 - \tau'(y_i) - \chi)w\varepsilon_i}{(1 + \tau_c)c_i} (1 - h_i)^\sigma \end{aligned}$$

where

$$c_i = \frac{(1 - \chi)y_i + T - \tau(y_i)}{1 + \tau_c}$$

⁵Jaimovich (2008) also notes the importance of the wealth effect on labor supply, although within the context of a representative agent model.

is steady-state consumption.⁶ We ignore agents with zero hours, since their presence would not affect the near-steady-state dynamics. We also discard from the sample any households whose reported income y_i is smaller than their reported wealth a_i times the model interest rate $r - \delta$, since those households would have negative productivity values. To simplify computation of the near-steady-state dynamics, we combine households into a smaller number of types and normalize units so that steady state income is 1, steady state average hours are 0.33 of the total time endowment, and steady state wealth is 3; types that have mass less than 10^{-5} are discarded.⁷ Figures (2)- (4) display the cumulative distribution functions from the SCF data and the approximated distribution functions. Only the distribution of wealth deviates significantly from that found in the data, and even this fit is not unreasonable given the simplicity of the model.⁸

We calibrate the other parameters to match some targets in US data. We choose δ to match an investment-output ratio of 15 percent and G to match a government spending-output ratio of 20 percent. The tax parameters are $\nu_0 = 0.268$ and $\nu_1 = 0.768$, with ν_2 set to clear the government budget constraint. We set $\chi = 0.1$ and T to generate transfers equal to 10 percent of output and to imply that progressive taxes account for 68 percent of government revenue. We set $\alpha = 0.36$ to match capital's share of income and set $\tau_c = 0.05$. We set $\sigma = 0.5$, so that labor supply is highly elastic. A steady state can be computed then as the solution to three equations in three unknowns: aggregate capital K , aggregate labor input N , and the tax parameter ν_2 .

Denote by n the number of household types that survive elimination and the eigenvalues of the linear system by λ ; $\#\{|\lambda| < 1\}$ represents the number of eigenvalues with modulus less than 1. The model is saddle-stable if $\#\{|\lambda| < 1\} = n$ and indeterminate if $\#\{|\lambda| < 1\} > n$.⁹ If $\#\{|\lambda| < 1\} = 2n$ the economy has a sink-stable steady state, as all combinations of capital stocks and consumption choices converge to the steady state; otherwise we will refer to the indeterminate case as a nonunique saddle since the combinations of capital and consumption that converge to the steady state has lower dimension than the system as a whole. Although in principle n could vary with v since r is endogenous, we find that the number of household types is constant at 1242.

Table 1 summarizes the results from varying v (with recalibration for each value). We find

⁶These equations are simply the steady state conditions for optimality at the individual level, inverted to solve for parameters instead of quantities.

⁷See Appendix A.

⁸In particular, we note that rich households in the data earn above-average returns on their portfolios due to the presence of stocks.

⁹ n is equal to the number of predetermined variables, namely a_{it} . Unlike the model in the previous section, the determinacy of the steady state wealth distribution eliminates the unitary eigenvalue.

that the critical level of v is smaller in our heterogeneous-agent economy than in similar economies studied by Benhabib and Farmer (1994) and Farmer and Guo (1994), despite the fact that we use a smaller labor supply elasticity (those papers use $\sigma = 0$) – we note again that the minimum required returns needed for our model to display indeterminacy when agents are homogeneous is above 0.5, as in those papers.¹⁰ However, because the wage distribution is also relatively concentrated, the full model still requires large increasing returns to generate indeterminacy. In the homogeneous agent economy of Benhabib and Farmer (1994), a necessary condition for indeterminacy is that the labor demand curve slopes upward and crosses the labor supply curve from below; this condition is satisfied if $\sigma < (1 - \alpha)(1 + \eta)$, meaning that the intertemporal elasticity of labor supply must be sufficiently high. In our model, what matters is roughly the elasticity of output with respect to aggregate labor input, not labor supply – that is, one must weight each agent’s labor supply curve by their efficiency unit before summing. And the full model implies that aggregate labor input behaves quite similarly to aggregate labor hours because the correlation between wealth and wages is not sufficiently strong. We conclude that a calibrated model of income and wealth heterogeneity is unlikely to make sunspot equilibria more plausible.¹¹

The other thing to note about Table 1 is that the number of sunspots is at most 1. When the economy passes into the indeterminacy region, it robustly predicts only one expectational variable can affect the equilibrium. In terms of dynamics, this result implies that the stable dynamics unfold on an $n + 1$ -dimensional manifold where the exact path is chosen by the single sunspot realization. Of course, with heterogeneous agents the assumption that all agents coordinate on a single random variable to pin down their expectations strains credulity; however, if they do this random variable must be unidimensional.¹² As in Benhabib and Nishimura (1998), one should interpret the model as having each agent individual’s Euler equation perturbed by a sunspot shock that is perfectly correlated with every other sunspot; thus, the model does not pin down the volatility of any individual’s consumption. Furthermore, unlike Benhabib and Farmer (1994) and Farmer and Guo (1994), the sunspot disappears when increasing returns get too large. It seems

¹⁰We do not use $\sigma = 0$ because it implies a labor supply indivisibility (see Rogerson 1988). With heterogeneous agents, indivisible labor supply generates corner solutions for almost every household (see Maliar and Maliar 2004); for our economy with a discrete number of types, the aggregate labor supply elasticity would then be zero.

¹¹The business cycle dynamics of our calibrated heterogeneous agent model are very similar to the representative agent version. We defer a complete study of those dynamics as they would substantially lengthen the paper without adding any significant contribution.

¹²Heterogeneous expectations in the complete market model typically lead to degeneracy of the wealth distribution, as formally they are equivalent to differences in discount factors (see Carroll and Young 2009 for an explicit definition of degeneracy). Since heterogeneous sunspots would be difficult to reconcile with complete markets, we do not pursue this direction.

likely that the steady state is surrounded by stable cycles – global sunspots – even when the local results indicate saddle-stability, however, and we make some comments in the conclusion about this issue.

5. Conclusion

We have shown that heterogeneity is a potential route to indeterminacy – a complete market model with wage and wealth inequality displays indeterminate equilibria at lower levels of social increasing returns to scale than do homogeneous agent models. A large-scale model designed to fit the US distribution of income, wealth, and hours worked shows a reduction in the required level of returns to scale relative to the homogeneous agent model of Farmer and Guo (1994), but it is still quite high relative to numbers the data would support. Other researchers have added elements to the basic model that reduce these required increasing returns, such as capacity utilization (Wen 1998), multiple production sectors (Benhabib and Farmer 1996), and home production (Perli 1998). One extension of this note would be to introduce these ingredients into the heterogeneous agent model; since our purpose is not to find a model that generates sunspots with quantitatively-reasonable parameter values we do not pursue these extensions.

A more challenging extension would be to investigate the global dynamics of the heterogeneous-agent model. Guo and Lansing (2002) and Coury and Wen (2009) show that the growth model with increasing returns to scale can display complicated global dynamics even when the equilibrium is locally determinate. Those models are limited in their potential to deliver exotic dynamics by their low dimension; in contrast, the full heterogeneous-agent model that we use could deliver a large range of wildly-complicated dynamics that are simply not possible in low-dimensional systems. In particular, Guo and Lansing (2002) caution against using policy prescriptions derived from linear approximations, as they can generate global indeterminacy in the form of limit cycles and chaos. We hope to explore these issues in future work; our work would be complementary to Dromel and Pintus (2008), who study a stylized model of workers and capitalists and show that progressive taxation can rule out local but not global indeterminacy. Solving for the global dynamics of our model will be computationally-intensive, however, and so we do not pursue it further here.

Finally, we note the extensive literature that rejects complete markets using individual consumption data.¹³ Extending our investigation to models of incomplete insurance markets would

¹³This literature is so extensive and well-known we refrain from choosing one contribution to cite.

be technically difficult (since incomplete markets introduce unit roots into the evolution of asset holdings) but would provide a more natural setting for coordination failures; when agents do not coordinate on a single aggregate variable, heterogeneous beliefs arise and markets become effectively incomplete (see Graham and Wright 2007). It is known that incomplete markets can generate cycles and bubbles when they would be impossible under complete markets (see Kocherlakota 1992,1996), so moving to such settings may indeed make expectational equilibria more empirically tenable. We leave this investigation for future work.

A. Appendix

The calibration exercise backs out preferences (β_i, B_i) and labor productivity ε_i from household level data on income, wealth, and hours by using the steady state Euler equations and the definition of income in the model. In this way, one may calibrate (β_i, B_i) and ε_i so that the long-run distribution from the model closely matches the data. There are 15,961 households in the pooled SCF data (we deflate each year by the GDP deflator for 1992 to express all variables in real terms). We then coarsen the distribution to reduce the number of types by pooling types that are similar enough. The following steps are used to calibrate the steady state (for additional detail see Carroll 2009).

1. Fix a range of income and wealth values over which to place grid points and partition the income, wealth, and hours intervals into (n_y, n_k, n_h) segments, respectively. While it is permissible to make these grid points evenly spaced, because of the skewness of the data we find that a better approximation can be achieved by bunching more grid points at the lower ends of the interval for income and wealth. We use grid points generated according to the rule

$$z_{i+1} = z_i + \exp\left(c + d\frac{\sqrt{i}}{n}\right) \quad (\text{A.1})$$

where c and d are constants and n is the number of grid points. For income we set $c = -2$ and $d = 0.5$, and for wealth we set $c = -0.8$ and $d = 5$. We use 20 grid points for income and wealth, and 10 evenly-spaced points for hours. Take every cube defined by

$$\left\{ \begin{array}{l} (x_y, x_k, x_h) : [a_j \leq x_y < a_{j+1}] \cup [b_m \leq x_k < b_{m+1}] \cup [c_n \leq x_h < c_{n+1}], \\ j \in \{1, \dots, n_y\}, m \in \{1, \dots, n_k\}, n \in \{1, \dots, n_h\} \end{array} \right\}$$

where (a, b, c) are the grid points for income, wealth, and hours, respectively. Sum the

population weights for observations that lie within the cube and assign this measure to the center point (y_j, k_m, h_n) . The collection of all such center points is the support for the joint distribution.

2. Normalize the type weights so that $\sum_{i=1}^{n_y n_k n_h} \psi_i = 1$. Because some income/wealth/hours combinations do not appear in the data, if $\psi_i < 1.0 * 10^{-5}$ set $\psi_i = 0$. Let the number of types with non-zero weight be $n_t \leq n_y n_k n_h$. Normalize (A, B, C) such that $\sum_{i=1}^{n_y} \psi_i y_i = 1$, $\sum_{i=1}^{n_k} \psi_i k_i = 11.5$, and $\sum_{i=1}^{n_h} \psi_i h_i = 0.33$.
3. Because wealth in the data may be composed of many types of assets each yielding a different return, while the model has only one asset, it is possible for some wealth levels in the data to imply negative income at r . These observations are removed. The total number of remaining types after consolidation of small measure types and dropping of negative income types is 1262.
4. Having obtained $\{k_i\}_{i=1}^{n_k}$, $\{y_i\}_{i=1}^{n_y}$, and $\{h_i\}_{i=1}^{n_h}$, solve for $(\beta_i, \varepsilon_i, c_i, B_i)$:

$$\begin{aligned} \beta_i &= [(1 - \tau_y(y_i) - \chi)(r - \delta) + 1]^{-1} \\ \varepsilon_i &= \frac{y_i - (r - \delta)k_i}{wh_i} \\ c_i &= \frac{y_i - \tau(y_i) + T}{1 + \tau_c} \\ B_i &= \frac{(1 - \tau_y(y_i) - \chi)w\varepsilon_i}{(1 + \tau_c)c_i} (1 - h_i)^\sigma. \end{aligned}$$

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Table 1

Returns to Scale and Local Dynamics

v	n	$\#\lambda < 1$	Local Dynamics
0.0 – 0.3420	1242	1242	Unique Saddle
0.3420 – 0.5430	1242	1243	Nonunique Saddle
0.5430+	1242	1242	Unique Saddle

Figure 1

Critical Level of ν , $\psi = 0.5$

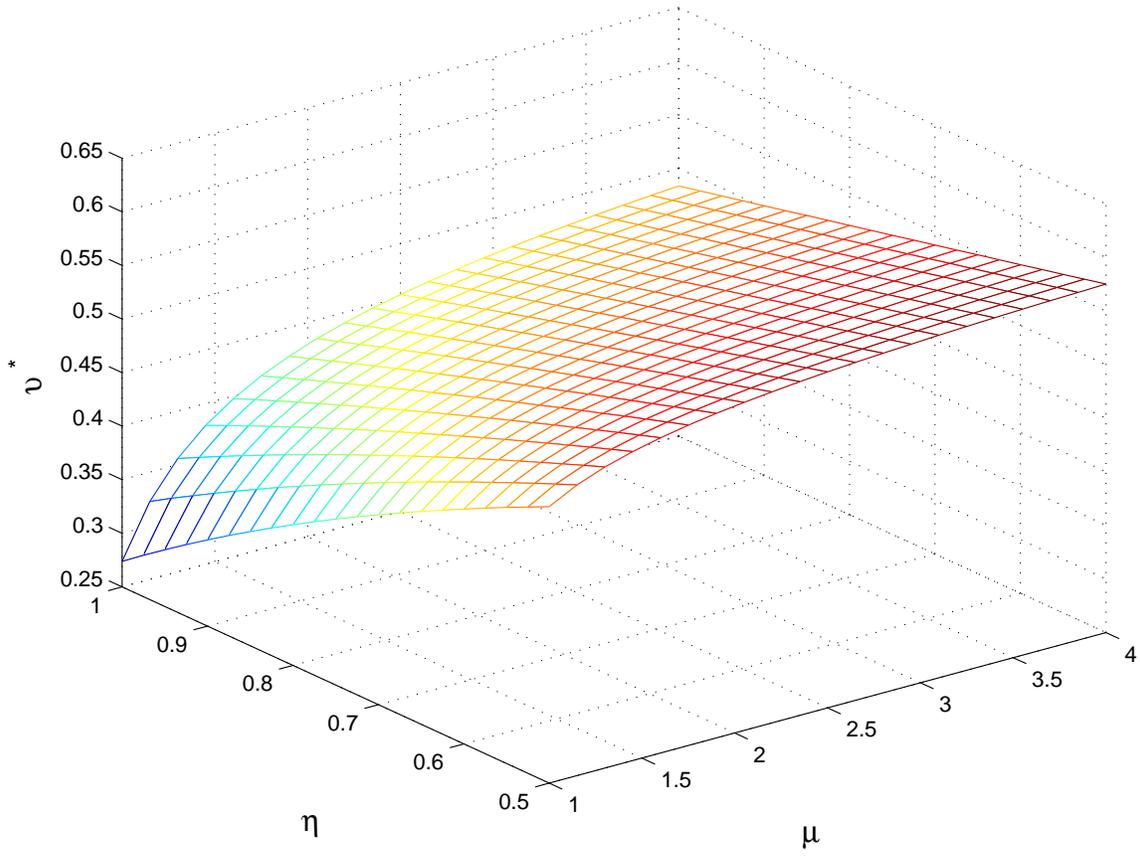


Figure 2
Wealth Distribution

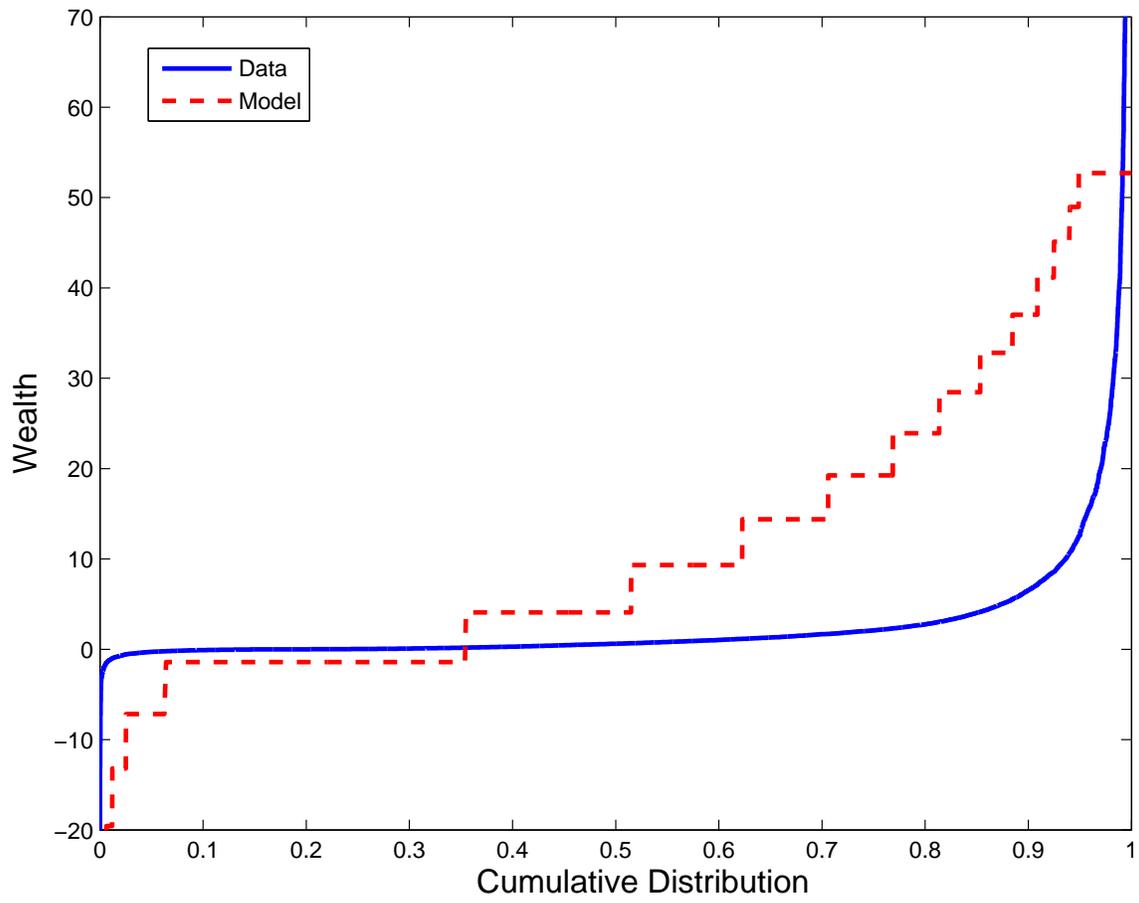


Figure 3
Income Distribution

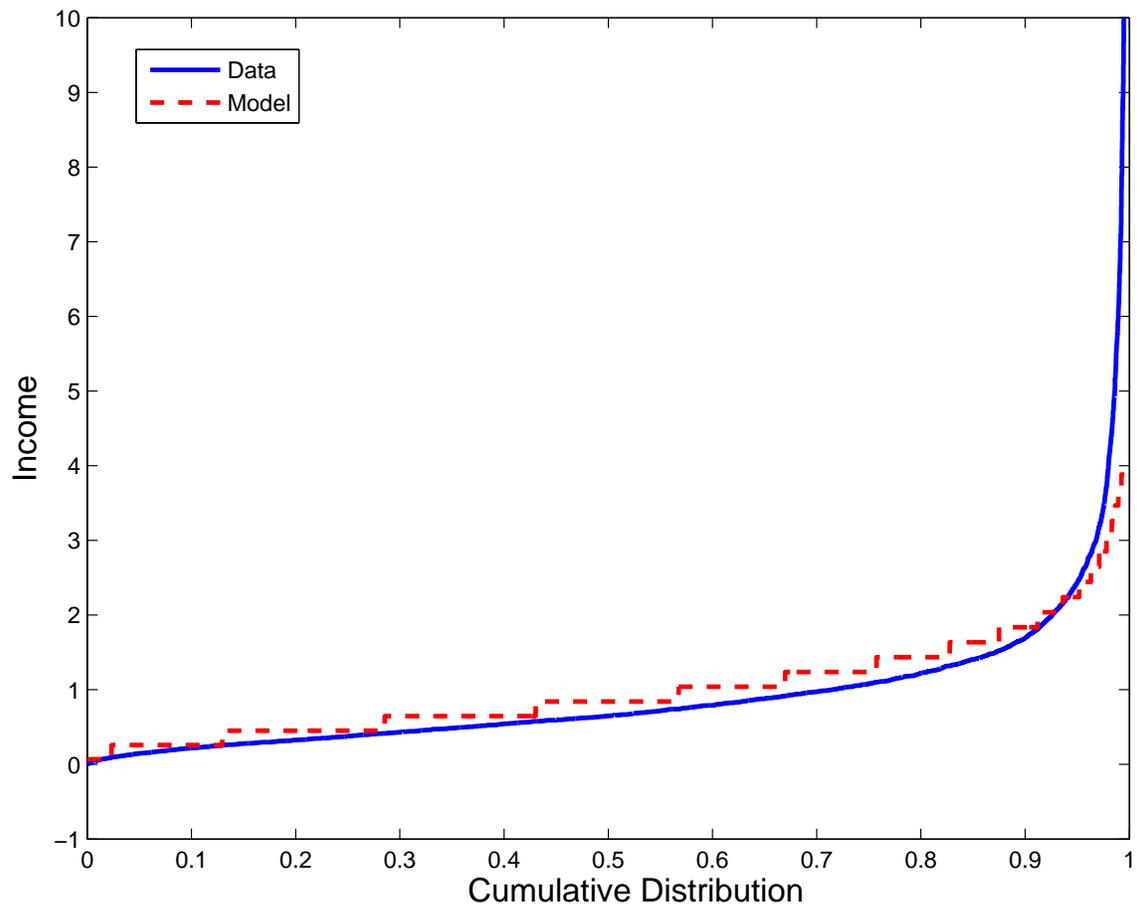


Figure 4
Hours Distribution

