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Some Benefits of Cyclical Monetary Policy

by Ricardo de O. Cavalcanti and Ed Nosal

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In this paper, we present a simple random-matching model in which different seasons translate into different propensities to consume and produce. We find that the cyclical creation and destruction of money is beneficial for welfare under a wide variety of circumstances. Our model of seasons can be interpreted as providing support for the creation of the Federal Reserve System, with its mandate of supplying an elastic currency for the nation.

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1 Introduction

Should monetary policy be cyclical? Although this is an old question in monetary economics, there is no general consensus as to the correct answer. Recent research on the "pure theory of money" has contributed very little, if anything, to the debate that surrounds this question. By pure theory of money, we refer to that line of research where money arises endogenously as a solution to a trading problem, instead of being treated as a primitive of the economic environment, like preferences and technology. Perhaps it is not so surprising that modern theories of money have remained silent on the desirability of cyclical monetary policy: Although the environments that are suitable for modeling a role for fiat money—environments with infinite horizons and diverse trading opportunities—are quite tractable when they are stationary, they become quite intractable when the stationarity assumption is relaxed. In this paper, we explore a simple departure from the standard model of money as a medium of exchange; in particular, we construct a model with seasonal fluctuations in output, where money is essential and the cyclical creation and destruction of money can be welfare-enhancing.

We claim that cyclical policies *ought to* have a role when the standard model is generalized to account for seasonal movements in output and consumption. Our argument is twofold: First, because money is essential, there necessarily exist some sorts of frictions in the economy. But the very existence of these frictions means that the standard welfare theorems will not apply. Hence, monetary economies will be relegated to the world of the second-best. In a second-best world, activist government policies may be beneficial. However, in the context of a monetary environment, activist government policies should not be able to implement the first-best allocation, for this would imply that government policies can somehow "neutralize" the fundamental frictions that exist in the economy. For example, monetary policy should not be able to overcome the fact that money holdings are a less-than-perfect substitute for credit. Because of this, a distribution of money holdings should persist after central bank interventions, so long as full insurance against idiosyncratic risk is unattainable. Second, and consequently, any monetary policy aimed at improving the efficiency of monetary trades should attempt to make adjustments in the distribution of money over economic cycles. In this paper, we study a simple framework in which a beneficial role for cyclical monetary policy is derived; we believe that this result will remain valid for any generalization to the model environment that preserves the second-best aspect of fiat money.

The task that we set for ourselves is to construct a model where the social role of money varies over a cycle and where the monetary authority can "react" to the cyclical nature of money by using a limited set of policy instruments, namely lump-sum creation and destruction. In addition, we want to maintain a reasonably sized state space for the economy. We find that a simple alternating movement in preferences, studied from the perspective of mechanism design, within a set of cyclical but otherwise stationary allocations, can be addressed without difficulties when money holdings are limited to either zero or one unit. We wish to emphasize that the limited holdings of money are used for analytical tractability and do not drive the key results.

The model, the creation of the Fed, and some literature In our model economy, individual agents experience seasonal preference shocks, and trade between pairs of agents is characterized by a lack of double coincidence of wants. Agents in the economy belong to one of two equally sized groups. When one group has a production opportunity, the other group has a consumption opportunity; on a period-by-period basis, each group alternates between having production and consumption opportunities. In pairwise meetings, the consumer faces an idiosyncratic preference shock that affects his desire to consume. The notion of seasons is introduced by having the (economywide) distribution of consumer-preference shocks differ over even and odd dates. For example, the even period will be a high-demand season and the odd period a low-demand season, if the total number of consumer agents who actually want to consume in even periods exceeds the total number of consumer agents who want to consume in odd periods.

Monetary policy is restricted to take the form of a reccurring pattern of taxing money holdings in one period and injecting the proceeds in a lump-sum fashion in the subsequent period. If taxes and subsidies are non-zero, then monetary policy will be cyclical; if taxes and subsidies are equal to zero, then the money supply will be constant. We first show that under a constant monetary policy rule, the seasonal frequency of trade is constant. When we compare a cyclical monetary policy with a constant-money-supply policy, we find two basic effects. First, cyclical policies may reduce the return on money and, hence, reduce the producers' desire to supply output. This *intensive margin* effect may reduce the social surplus associated with each trade. Second, if there is sufficient asymmetry in the distribution of aggregate preference shocks, so that one season has a higher desire or demand for consumption than the other season, then a cyclical monetary policy will increase the average frequency of trades, or the *extensive margin*, compared to a constant-money-supply policy. We find that under a wide variety of circumstances, the optimal monetary policy will be cyclical. So, although a cyclical monetary policy may result in a lower and inefficient level of production at the match level, an increase in the economywide frequency of trades implies that a cyclical monetary policy can deliver a higher level of social welfare than a constant-money-supply policy.

The results from our model can be loosely interpreted as providing some support for the creation of the Federal Reserve System in 1913. The preamble to the Federal Reserve Act states that the Reserve banks were established to "furnish an elastic currency," among other things. According to Meltzer (2003), *elasticity* has two meanings. One refers to a central bank's ability to pool reserves and lend them out in the event of a banking or financial crisis. The second refers to seasonal fluctuations, the topic of this paper. In practice, the two meanings of elasticity are related because the data indicate that seasonal fluctuations in money demand can exacerbate a (potential) banking or financial panic. For example, before the Fed was founded, farmers needed cash in the autumn to harvest their crops but, given the structure of the banking system, there was essentially only a fixed amount of reserves to go around. As a result, the increase in demand for cash in the autumn could potentially turn a quite independent and manageable liquidity problem in financial markets into a financial panic or banking crisis.

Miron (1986) concludes that the founding of the Fed had positive welfare consequences for the economy because its policy of furnishing an elastic currency greatly reduced the possibility of financial panics, which formerly were commonplace and sometimes quite severe. Note that two meanings of elasticity are at play here: By consolidating reserves at a central place, the Fed could provide reserves to banks that needed them in a time of financial stringency. Furthermore, by discounting real bills, the Fed could provide (additional) liquidity to farmers, implying that their increase in demand for money need not exacerbate a potential liquidity problem in financial markets. Miron (1986) points out, however, that if an economy has deposit insurance, then an elastic currency policy would not improve welfare improving because the existence of deposit insurance would greatly reduce, if not eliminate, the possibility of financial panics, which is the source of the welfare gain in his analysis. In this paper, we completely abstract any notion of financial panics and find that there are other sources of welfare gains associated with an elastic currency (cyclical monetary) policy and that providing an elastic currency can increase the average frequency of trade in the economy.

Since the "fine tuning" of monetary policy is a broad topic with a voluminous literature, it is important to relate our model to some well-known papers at the outset, in order to highlight the particular debt our work owes to them. Lucas (1972) was the first to present a pure theory of the short-run effects of monetary policy, but an important ingredient in his analysis is an exogenous and random supply of money. In a competitive environment, the optimal monetary policy invariably leads to the Friedman (1969) rule in the form of a deflation that eliminates the opportunity cost of holding money. Bewley (1980), Levine (1991), and Sheinkman and Weiss (1986), among others, departed from a representative-consumer structure and found that there are welfare gains associated with an ongoing inflation. In these models, traders face uninsurable shocks and can benefit from some redistribution of wealth generated by inflation. The literature that has followed the seminal work of Kiyotaki and Wright (1989) on the media of exchange has more or less been limited to reproducing these inflation gains.[1]

The rest of the paper is divided as follows: In section 2, we describe the environ-

[1] See Molico (1999) and Deviatov and Wallace (2001). There has also been work on the effects of inflation on search intensity, such as Li (1995) and Shi (1999), among others.

ment with two seasons. In section 3, we define symmetric and stationary allocations as well as the welfare criteria that guide the discussion of optimal monetary policies. In section 4, we define an *implementable allocation*. Section 5 analyzes extensive margin effects associated with a cycle monetary policy, and section 6 analyzes intensive margin effects. Section 7 characterizes the optimal monetary policy, and section 8 concludes.

2 The environment

Time is discrete and the horizon is infinite. There are two types of people, each defined on a $[0, 1]$ continuum. Each type is specialized in consumption and production: A type e person consumes even-date goods and produces odd-date goods, whereas a type d person consumes odd-date goods and produces even-date goods. We find it convenient to refer to a type e individual in an even (odd) date, or a type d individual in an odd (even) date, as a *consumer* (*producer*). Each type maximizes expected discounted utility, with a common discount factor $\beta \in (0, 1)$. Let $s \in \{e, d\}$ indicate the season and/or the type of person. We find it useful to have a notation for the two-period discount factor, $\delta \equiv \beta^2$.

The utility function for a consumer in season $s \in \{e, d\}$ is $\varepsilon_s u_s(y_s)$, where ε_s is the idiosyncratic shock affecting this consumer and $y_s \in R_+$ is the amount consumed. The shock ε_s is Bernoulli and the probability that $\varepsilon_s = 1$, $\pi_s \in (0, 1)$ is indexed by the season s. A producer in season s can produce any choice of $y_s \geq 0$ at a utility cost normalized to be y_s itself. Utility in a period is thus $\varepsilon_s u_s(y_s)$ when consuming and $-y_s$ when producing. The function u_s is assumed to be increasing, twice differentiable, and to satisfy $u_s(0) = 0$, $u_s'' < 0$, $u_s'(0) = \infty$ and $u_s'(\infty) < 1$. We assume that $\pi_e \geq \pi_d$ and $u_e' \geq u_d'$, so that even dates feature a higher desire for consumption—both at the individual and aggregate levels—than odd dates. It should be emphasized that a strict inequality for either of these gives rise to a cyclical demand for liquidity.

In every period, a type e person is matched randomly with a type d person. During meetings, the realization of preference shocks occurs and production may take place. All individuals are anonymous, in the sense that they all have private histories. We also assume that people cannot commit to future actions, so that those who produce must get a tangible (future) reward for doing so. In this paper, the reward takes the form of fiat money. To keep the model simple, we assume that each person can carry from one meeting to the next either 0 or 1 units of fiat money. A consequence of this assumption, which makes the distribution of people tractable, is that trade will take place only when the consumer realizes $\varepsilon_s = 1$ and has money, and the producer has no money.

Monetary policy takes the simple form of a choice of the pair (σ, τ), where σ is the probability that a person without money gets one unit of money before meetings, and τ is the probability that a person with money loses the money before meetings. Let M_e denote the measure of individuals holding money in even periods and M_d the measure of individuals holding money in odd ones. We restrict attention to cases in which either $\tau = \sigma = 0$ in all dates, or $\sigma > 0$ in even dates and $\tau > 0$ in odd dates. This simple formulation is designed to limit our analysis to the specific question of whether periods of high desire for consumption should have an increase in the supply of money, which is offset by a reduction of economywide money balances in the subsequent period.

3 Stationarity and welfare criteria

We let the measure of consumers with money during meetings in season s be denoted by q_s and consumers without money denoted by $1 - q_s$. We let the measure of producers without money during meetings in season s be denoted by p_s and producers with money denoted by $1-p_s$. In order to save on notation, let $y = (y_e, y_d)$ denote the list of output levels, let $x \equiv (p_e, q_e, p_d, q_d)$ denote an arbitrary *distribution*, and use the superscript $+$, as in $x^+ \equiv (p_e^+, q_e^+, p_d^+, q_d^+)$, when the qualification that $\sigma > 0$ for that distribution becomes essential. A distribution $x \in [0,1]^4$ is considered *invariant* if and only if there exists $(\sigma, \tau) \in [0,1]^2$ such that

$$p_e = (1-\sigma)(1 - q_d + \pi_d p_d q_d), \tag{1}$$

$$p_d = (1-\tau)(1 - q_e + \pi_e p_e q_e) + \tau, \tag{2}$$

$$q_e = (1-\sigma)(1 - p_d + \pi_d p_d q_d) + \sigma, \tag{3}$$

and

$$q_d = (1-\tau)(1 - p_e + \pi_e p_e q_e), \tag{4}$$

where the distribution x is described *after* money is created or destroyed.

The stationarity requirement (1) can be explained as follows: During odd-date meetings, trade takes place after money is destroyed. The measure of consumers with money is q_d, and the measure of producers without money is p_d. Consumers without money, whose number is $1 - q_d$, cannot buy goods; each of them faces a probability σ of finding money at the beginning of the *next* date. Hence, $1-\sigma$ times $1 - q_d$ is the total flow of consumers who become producers without money in the next (even) date. Similarly, the measure of consumers with money in the odd date is q_d. Only a fraction π_d of these consumers will want to consume in the odd date, and only a fraction q_d of them will meet a producer without money. Therefore, $\pi_d p_d q_d$ represents the measure of consumers with money that will trade in date d, and $(1-\sigma)\pi_d p_d q_d$ represents the number of these consumers that become producers without money in the next (even) period, after money creation takes place.

Likewise, regarding requirement (2), we first notice that a measure $1 - q_e + \pi_e p_e q_e$ producers arrive at the beginning of date d without money. Adding to that the mass of money destroyed from date e consumers with money who did not trade, $\tau q_e(1 - \pi_e p_e)$, yields the right-hand side of (2). The same principle explains requirement (3). The measure of consumers with money at date e consists of the measure of producers who leave date d with money, $1 - p_d + \pi_d p_d q_d$, plus the measure of producers who leave date d without money but obtain some when additional money is created at the beginning of date e, $\sigma p_d(1 - \pi_d q_d)$. Finally, requirement (4) follows from imposing stationarity on the measure of consumers with money arriving at date d, $1 - p_e + \pi_e p_e q_e$, after the destruction of money takes place with probability τ.

Our notion of stationarity amounts to restricting that output, y_s, as well as the measures p_s and q_s, to be constant functions of the season, s, only. These functions are used symmetrically in a measure of welfare as follows: We adopt an *ex ante* welfare criterion, with an expected discounted utility computed according to an invariant distribution and output function. Whenever trade takes place in a season, it is because money is changing hands from a fraction p_s of the mass of consumers $\pi_s q_s$ in position to trade. Since there is a producer for each consumer, the flow of total utility in season s is $\pi_s p_s q_s[u_s(y_s) - y_s]$. We call the term $\pi_s p_s q_s$ the *extensive margin* at s, and $u_s(y_s) - y_s$ the *intensive margin* at s. The extensive margin is a property of the distribution x, and the intensive margin is a property of outputs y. An *allocation* is a pair (x, y), where x and y are invariant and y has non-negative coordinates. The *welfare* U attained by an allocation is defined as the present discounted value

$$U(x, y) = \frac{1}{(1-\beta)} \sum_s \pi_s p_s q_s[u_s(y_s) - y_s].$$

The intensive margin at s is maximized at y_s^*, where $u_s'(y_s^*) = 1$, which is uniquely defined by assumption. We refer to $y^* = (y_e^*, y_d^*)$ as the *first-best* output list.

4 Implementable allocations

The definition of the values of y consistent with incentive compatibility follows the notion of sequential individual rationality employed by Cavalcanti and Wallace (1999) and Cavalcanti (2004). Underlying their definition of participation constraints is the idea that a social planner proposes an allocation but anonymous individuals may defect from that proposal by not trading in a given meeting. If individual(s) defect, then they do not lose any money holdings that were brought into the meeting. We adopt the same concept here, with the exception of the taxation of money holdings, which we assume cannot be avoided by individuals with money. The participation constraints are then defined by a set of allocations, according to the expected discounted utilities implied by the allocations. To be able to represent these constraints, we first need to describe the Bellman equations of the economy.

The value functions will be computed *before* the realization of the effects of creation and destruction of money for each individual in a given date. (Recall that money is created at the beginning of even dates and is destroyed at the beginning of odd dates.) The value function for consumers *with* money at s is v_s, and that for producers *without* money is w_s. We define $\bar{v}_s$ as the value for consumers without money at s and $\bar{w}_s$ as that of producers with money. The Bellman equations for $(v, w) = (v_e, v_d, w_e, w_d)$ are defined by

$$\begin{aligned} v_e &= \pi_e p_e(u_e + \beta w_d) + (1 - \pi_e p_e)\beta \bar{w}_d \\ w_e &= \sigma \beta v_d + (1 - \sigma)[\pi_e q_e(-y_e + \beta v_d) + (1 - \pi_e q_e)\beta \bar{v}_d] \\ v_d &= \tau \beta w_e + (1 - \tau)[\pi_d p_d(u_d + \beta w_e) + (1 - \pi_d p_d)\beta \bar{w}_e] \\ w_d &= \pi_d q_d(-y_d + \beta v_e) + (1 - \pi_d q_d)\beta \bar{v}_e, \end{aligned} \tag{5}$$

where u_e and u_d, by an abuse of notation, stand for $u_e(y_e)$ and $u_d(y_d)$, respectively. The definition is completed by substituting for the values of $(\bar{v}, \bar{w})$ given by

$$\begin{aligned} \bar{v}_e &= \sigma v_e + (1 - \sigma)\beta w_d \\ \bar{w}_e &= \beta v_d \\ \bar{v}_d &= \beta w_e \\ \bar{w}_d &= \tau w_d + (1 - \tau)\beta v_e \end{aligned} \tag{6}$$

into the previous system.

The participation constraint for producers at even dates is simply

$$-y_e + \beta v_d \geq \beta \bar{v}_d = \delta w_e, \tag{7}$$

since an even-date producer is bringing no money into a meeting and only has the option of leaving the meeting and becoming a producer two periods later. Producers at odd dates must take into account that if they disagree with producing the planned output y_d and walk away from a trade, then they have a chance of receiving money in the next period from the money-creation policy. Thus, the participation constraint for producers at odd dates can be stated as

$$-y_d + \beta v_e \geq \beta \bar{v}_e = \beta \sigma v_e + \delta(1 - \sigma) w_d. \tag{8}$$

For completeness, we state the participation constraint for consumers, which can be shown to be implied by the participation constraints of producers. They are

$$u_e + \beta w_d \geq \beta \bar{w}_d \tag{9}$$

and

$$u_d + \beta w_e \geq \beta \bar{w}_e. \tag{10}$$

An allocation (x, y) is said to be *implementable* if $x \equiv (p_e, q_e, p_d, q_d)$ is invariant for some policy (σ, τ) such that there exist (v, w) and $(\bar{v}, \bar{w})$, for which (5)-(10) hold. An allocation is said to be *optimal* if it maximizes $U(x, y)$ among the set of implementable allocations.

5 Extensive-margin effects

Monetary policy can be viewed as a choice of an invariant distribution x. Changes in x resulting from changes in (σ, τ) have direct effects on extensive margins, $\pi_s p_s q_s$, and indirect effects on intensive margins, $u_s(y_s) - y_s$, through the participation constraints; i.e., y depends on x. Note that the latter effects can be ignored if, for both $s = e$ and $s = d$, the maximizer of $u_s(y_s) - y_s$, y_s^* satisfies participation constraints. In this section, we investigate whether the maximizer of the sum $\sum_s \pi_s p_s q_s$, among all invariant distributions x, is a *cyclical* policy x^+, i.e., one with a positive σ. We shall see that a cyclical monetary policy tends to increase the extensive margin at e and to decrease that at d. Since $u_e' \geq u_d'$, it will follow that if y^* satisfies participation constraints and the maximizer of the sum $\sum_s \pi_s p_s q_s$ is cyclical, then the optimal allocation is indeed a cyclical monetary policy.

Acyclical distributions We start by pointing out an important property of the invariant distributions when the money supply is constant, i.e., when $\sigma = 0$. If x is invariant when $\sigma = 0$, we will say that x is *acyclical*, a label motivated by the following lemma:

Lemma 1 *Assume that x is acyclical. Then the extensive margin, $\pi_s p_s q_s$, is constant in s.*

Proof. Set $\sigma = \tau = 0$ in equations (1) and (4). It follows that $\pi_e p_e q_e = \pi_d p_d q_d$. ■

Interestingly, the property of constant extensive margins holds regardless of the relative values of π_s. We can offer an intuitive explanation for this property as follows: Let us consider the inflow and outflow of money for a set of individuals of the same type, say type e. Then, on one hand, the stationary measure of consumers of this type spending money is $(\pi_e p_e)q_e$, an event taking place at even dates. On the other hand, the stationary measure of producers of this type acquiring money is $(\pi_d q_d)p_d$, an event taking place at odd dates. Since the quantity of money in the hands of this group must be stationary, and all seasons have the same frequency, these two margins must be equalized, as stated in the lemma.

Some useful observations about acyclical distributions can be made with regard to the relative values of $p_s$and q_s.

Lemma 2 *Assume that x is acyclical. Then (i) $p_e - q_e = p_d - q_d$, and (ii) $p_e \leq p_d$ if and only if $\pi_d \leq \pi_e$.*

Proof. (i) Set $\sigma = \tau = 0$ in equations (1) and (2). Since, by lemma 1, $\pi_e p_e q_e = \pi_d p_d q_d$, equations (1) and (2) imply that $p_e - q_e = p_d - q_d$. (ii) By lemma 1, $\pi_e p_e q_e = \pi_d p_d q_d$, so $\pi_e \geq \pi_d$ if and only if $p_e q_e \leq p_d q_d$. Part (i) of this lemma implies that if $p_e q_e \leq p_d q_d$, then $p_e \leq p_d$ and $q_e \leq p_d$. ■

There is an alternative way to think about part (i) of lemma 1. The measure of individuals that hold money in period s, M_s, is the sum of consumers with money, q_s, and producers with money, $1 - p_s$. When $\sigma = \tau = 0$, the measures of individuals that hold money in odd and even periods are the same, i.e., $M_e = M_d$, which implies that $1 - p_e + q_e = 1 - p_d + q_d$, or that $p_e - q_e = p_d - q_d$.

An application of lemma 2 allows us to describe in rather simple terms the set of acyclical distributions when $\pi_e = \pi_d$.

Lemma 3 *Assume $\pi_e = \pi_d = \pi$. Then the set of acyclical distributions is fully described by $p_e = p_d = p$, $q_e = q_d = q$, and $p = 1 - q + \pi pq$ for $q \in [0, 1]$.*

Proof. Since $\pi_e = \pi_d$ then, by lemma 2, $p_e = p_d$, and consequently, by lemma 1, $q_e = q_d$. Equation (1) with $\sigma = 0$ thus proves the lemma. ■

The one-dimensional set described by lemma 3 is the symmetric set of distributions that appears in Cavalcanti (2004). The equation $p = 1 - q + \pi pq$ defines a strictly concave function for $q \in [0, 1]$, and the extensive margin πpq is maximized when $p = q = [1 - (1 - \pi)^{\frac{1}{2}}]/\pi$.

Properties similar to those described by lemma 3 also obtain when $\pi_e > \pi_d$; for example, every acyclical x can be indexed by a one-dimensional choice of q_d.

Lemma 4 *When $\pi_e > \pi_d$ there exists, for each q_s, a unique acyclical x. Moreover, x can be solved for analytically. The statement holds for any s in $\{e, d\}$.*

Proof. See appendix 1. ■

The extensive margin is maximized when the measure of consumers with money equals the measure of producers without money.

Proposition 1 *When $\pi_e > \pi_d$, the maximizer of $\pi_s p_s q_s$, among the set of acyclical distributions, is the unique x such that $p_s = q_s$ for $s \in \{e, d\}$.*

Proof. See appendix 2. ■

Hence, when the money supply is constant, the distribution that maximizes the extensive margin is characterized by $p_d = q_d$ and $p_e = q_e$. This result echoes a standard result in many search models of money, namely, that it is optimal for half of the population to hold money. Such a distribution of money holdings maximizes the number of productive matches. To see that our model also has this feature, note that when $\sigma = \tau = 0$ and when $\pi_s p_s q_s$ is maximized, i.e., $p_s = q_s$ for $s \in \{e, d\}$, then the measure of individuals holding money at date s is $1 - p_s + q_s = 1$. Since the total measure of individuals in the economy is 2, having half the population holding money maximizes the extensive margin when $\sigma = \tau = 0$. Note that the value of x is easily computed when the extensive margin is maximized,.

Lemma 5 *If x is acyclical and $p_s = q_s$, then*

$$p_d = \frac{1 + \sqrt{\theta} - \sqrt{(1 + \sqrt{\theta})^2 - 4\pi_d}}{2\pi_d},$$

and

$$p_e = 1 - p_d + \pi_d p_d^2,$$

where $\theta = \pi_d / \pi_e$.

Proof. Since by lemma 1, $\pi_e p_e q_e = \pi_d p_d q_d$, equation (2) with $\tau = 0$ yields $p_d = 1 - q_e + \pi_e p_e q_e$. Because $q_s = p_s$, then $p_e = \sqrt{\theta} p_d$ and $p_d = 1 - p_e + \pi_d p_d^2$. The last two expressions yield a quadratic equation in p_d whose only relevant solution is as stated. The value for p_e can be computed from the last expression once p_d is determined. ■

This completes our discussion of acyclical distributions, i.e., a constant money supply. We can now move on to cyclical money policy and cyclical distributions.

Cyclical distributions We now consider small perturbations in the quantity of money. We consider cyclical distributions x^+ in a neighborhood of a given acyclical x. Our ultimate goal is to describe and sign the derivative of the sum $\sum_s \pi_s p_s q_s$ with respect to σ, evaluated at $\sigma = 0$ and $p_s = q_s$. It follows, by force of proposition 1, that if this derivative is positive, then the maximizer of the sum must be cyclical. Clearly, the system (1)-(4) that defines x^+ depends on σ and τ. The existence of x^+ follows from a simple fixed-point argument.

Lemma 6 *Let $(\tau, \sigma) \in (0,1)^2$ be fixed. Then there exists an invariant distribution x^+.*

Proof. The right-hand side of (1)-(4) defines a continuous function of x^+, with domain on the compact and convex set $[0,1]^4$. The result then follows from Brower's fixed-point theorem. ■

If x^+ is invariant, then the quantity of money destroyed in season d must equal the quantity created in e, i.e.,

$$\tau(1 - p_e^+ + q_e^+) = \sigma(1 - q_d^+ + p_d^+). \tag{11}$$

It can be shown that the equality (11) is implied by the system (1)-(4). The quantity of money during season e meetings, just before trade, is given by the mass $1 - p_e^+$ with producers, plus the mass q_e^+ with consumers. Since trade itself does change this quantity of money, and each money holder at the beginning of next season faces a probability τ of losing his money, then the total amount of money destroyed is given by the left-hand side of (11). Likewise, the measure of individuals without money at the end of season d is $1 - q_d^+ + p_d^+$, and since each of those finds money at the beginning of season e with probability σ, then the quantity of money created is expressed in the right-hand side of (11).

While there is a continuum of acyclical distributions, i.e., when $\sigma = \tau = 0$, for each q_d there is a unique x (lemma 4), the same does not hold for cyclical distributions.

When σ and τ are strictly positive, there is an inflow of money that must be matched by an outflow of the same quantity. Our numerical experiments indicate that only one level of q_d^+ produces quantities of money that are capable of equalizing inflows and outflows for a given pair (σ, τ). We can, however, pin down the neighborhood in which q_d^+ lies as follows: Because we want to associate x^+ with a given x, we find it useful to define the constant ϕ with the property that, for $\tau = \phi\sigma$, x^+ converges to x as σ approaches zero. Since the pair (σ, τ) must be consistent with the stationary quantities of money in the economy, expressed above by equation (11), the desired ratio of τ to σ, for a given $x = (p_e, q_e, p_d, q_d)$, is

$$\phi = \frac{1 - q_d + p_d}{1 - p_e + q_e}. \tag{12}$$

By lemma 1 and proposition 1, the maximizer of the sum $\sum_s \pi_s p_s q_s$ among the set of *acyclical* distributions is the unique x for which $\phi = 1$. We assess the effects of perturbations by differentiating the system (1)-(4) with respect to σ for ϕ fixed.

The lemma can be viewed as generalizing lemma 2; in other words, the difference between the measures of consumers with money and producers without money will be equalized between seasons only if the distribution is acyclical.

Lemma 7 *If x is invariant and $\tau = \phi\sigma$, then $p_s - q_s = f_s(\sigma)$, where $f_e(\alpha) = \frac{\phi - \phi\alpha - 1}{1 + \phi - \phi\alpha}$ and $f_d(\alpha) = \frac{\phi + \phi\alpha - 1}{1 + \phi - \phi\alpha}$.*

Proof. See appendix 3. ■

Note that f_s does not depend on the fraction of consumers who desire to consume in season s, π_s.

The next proposition, which is the main result of this section, characterizes the sign of the derivative of the sum $\sum_s \pi_s p_s q_s$, evaluated at $p_s = q_s$, and $\sigma = 0$ (the latter two equalities characterize the optimal constant-money-supply policy).

Proposition 2 *The maximizer of sum $\sum_s \pi_s p_s q_s$ is cyclical if and only if $\pi_d \in [0, \bar{\pi}]$, where $\bar{\pi} \in (0, \pi_e)$ can be solved for analytically as a function of π_e.*

Proof. See appendix 4. ■

The maximizer is therefore acyclical if $\pi_d = \pi_e$. The intuition behind which policy—acyclical or cyclical—maximizes the average extensive margin is straightforward. Suppose that $\pi_d = \pi_e = \pi$. Then, the policy that maximizes the average extensive margin is given by $p_e = p_d = q_e = q_d \equiv t$. Now if σ is slightly increased from zero, there will be a stationary cyclical distribution x^+ in the neighborhood of x. When $\sigma > 0$, then $M_d = 1 - p_d^+ + q_d^+ < 1 < M_e = 1 - p_e^+ + q_e^+$. Hence, it must be the case that q_e increases by more than p_e decreases and q_d decreases by more than p_d increases when σ (and τ) is increased from zero. Therefore, $p_e^+ q_e^+ > t^2$ and $p_d^+ q_d^+ < t^2$: For a cyclical monetary policy, the extensive margin will *increase* in season e and *decrease* in season d, compared to the acyclical policy. Since a constant stock of money is optimal in a world with "no seasons," i.e., when $\pi_d = \pi_e = \pi$, it must be the case that the negative extensive margin effect associated with season d outweighs the positive extensive margin effect associated with season e. Another way of thinking about this result is that when $p_s q_s$ is "equally weighted," i.e., $\pi_e = \pi_d$, the (negative) odd-season effect dominates the (positive) even-season effect. Suppose now that $\pi_d < \pi_e$. It will still be the case that $p_e^+ q_e^+ > p_e q_e$ and $p_d^+ q_d^+ < p_d q_d$, where (p_e, q_e, p_d, q_d) is the distribution associated with the optimal acyclical monetary policy. However, since the differences between "p" and "q" do not depend upon π_e and π_d (see lemma 7) it may now be the case that the (positive) even-season effect dominates the (negative) odd-season effect. This is because the even-season matching probability of a consumer with money meeting a producer without money, $p_e q_e$, is weighted more heavily than the odd-season matching probability, $p_d q_d$, i.e., $\pi_e > \pi_d$. Hence, if the fraction of potential consumers in odd periods is sufficiently smaller than the fraction of potential consumers in even periods—or if demand in the "high" season is sufficiently greater than demand in the "low" season, then a cyclical monetary policy will deliver a higher-average extensive margin than the optimal acyclical policy.

6 Intensive-margin effects

The only participation constraints that are relevant, given our notion of stationarity, are those of producers. In this section, we derive representations of producer constraints as functions of preference parameters, policy parameters, and allocations, without reference to value functions. Although the first-order effect of cyclical interventions is a tightening of participation constraints, these negative effects can be negligible or even absent if the discount factor is sufficiently high.

Proposition 3 *The participation constraints are satisfied if and only if*

$$u_d(y_d) \geq \frac{y_e}{\beta}\left[\frac{1}{(1-\tau)\pi_d p_d} - (1-\sigma)\delta\frac{1-\pi_d p_d}{\pi_d p_d}\right] \tag{13}$$

and

$$u_e(y_e) \geq \frac{y_d}{\beta}\left[\frac{1}{(1-\sigma)\pi_e p_e} - (1-\tau)\delta\frac{1-\pi_e p_e}{\pi_e p_e}\right]. \tag{14}$$

Proof. See appendix 5. ■

Inequalities (13) and (14) indicate that cyclical policies have a potentially negative effect on intensive margins, since the right-hand side of both inequalities is increasing in σ and τ. The intuition behind these potential negative effects is straightforward: In either case—whether money is injected or withdrawn from the economy—the value of money in a trade will fall compared to the situation where $\sigma = \tau = 0$. In the case where the money supply is contracted after production and trade, the value of currency falls because there is a chance that the producer will be unable to use his unit of currency in a future trade because it will be taken away; in the case where the money supply is expanded after production and trade, the fact that a producer may receive a unit of currency if he does not produce reduces the value of a unit of currency for a producer who does. A fall in the value of money implies that the amount of output received per unit of currency is reduced. If, however, β is sufficiently high, then inequalities (13) and (14) will not bind at $y = y^*$, the output levels that maximize the intensive margins; hence, the potential effects on the intensive margins do not materialize for small monetary interventions.

Suppose that neither participation constraint binds when $\sigma = 0$. Then, it turns out that if β is reduced, the first participation constraint to be violated is the participation constraint for date-e producers, (13). Hence,

Lemma 10 *If the participation constraint for date-e producers is satisfied for x acyclical and $y = y^*$, then (x, y) is implementable.*

Proof. Since $u_e' \geq u_d'$ and $u_e(0) = u_d(0)$, then $u_e'(y_d^*) \geq 1$, so that $y_e^* \geq y_d^*$ and $u_e^*(y_e^*) \geq u_d^*(y_d^*)$. Now, it has been established in the previous section that, if x is acyclical, then $\pi_d \leq \pi_e$ implies $q_d \leq q_e$. As a result, since the equality $\pi_e p_e q_e = \pi_d p_d q_d$ holds for all acyclical x, $\pi_d \leq \pi_e$ implies $\pi_d p_d \leq \pi_e p_e$. Since the right-hand side of (13) or (14) is increasing in $\pi_s p_s$, and since $u_e^*(y_e^*)/y_d^* \geq u_d^*(y_d^*)/y_e^*$, then the result follows. ■

Lemma 10 indicates that it suffices to look at the participation constraint for date-e producers in order to find a value of β such that small interventions have no negative effects on intensive margins; the following proposition characterizes the critical β for the optimal *acyclical* distribution such that the participation constraint for the date-e producer "just" binds.

Proposition 3 *Let x take the value of the acyclical distribution with $p_s = q_s$, and let $\beta > \bar{\beta}$, where*

$$\bar{\beta} = \frac{-\frac{u_d(y_d^*)}{y_e^*} + \sqrt{\left(\frac{u_d(y_d^*)}{y_e^*}\right)^2 + 4\frac{1-\pi_d p_d}{(\pi_d p_d)^2}}}{2\frac{1-\pi_d p_d}{\pi_d p_d}}.$$

Then, if σ is sufficiently small, the cyclical allocation (x^+, y^*) for x^+, in a neighborhood of x, is implementable.

Proof. The cutoff value $\bar{\beta}$ was constructed so that (x, y^*) is implementable for $\beta = \bar{\beta}$. Since the participation-constraint sets vary continuously with (σ, τ), the result follows. ■

7 Optimal policies

On one hand, our results regarding extensive-margin effects show that there exists a cutoff value for π_d, called $\bar{\pi}$, such that the maximizer of the average extensive margin is cyclical if and only if $\pi_d < \bar{\pi}$. On the other hand, our results on intensive margins show that there exists a cutoff value of β, called $\bar{\beta}$, such that for $\beta > \bar{\beta}$, small interventions around the allocation (x, y^*), where $p_s = q_s$, are implementable. It follows, therefore, that the optimum is cyclical for a large set of parameters, including π_s and β such that $\pi_d < \bar{\pi}$ and $\beta > \bar{\beta}$.

Proposition 4 *If $\pi_d < \bar{\pi}$ and $\beta \geq \bar{\beta}$, then the optimum monetary policy is cyclical.*

Proof. Welfare is proportional to $\sum_s E_s I_s$, where E_s is the extensive margin at s, $\pi_s p_s q_s$, and I_s is the intensive margin at s, $u_s(y_s) - y_s$. By lemma 1, $E_e = E_d$ for all acyclical policies, so that for fixed (I_e, I_d), the acyclical x that maximizes welfare features $p_s = q_s$. Since $\beta \geq \bar{\beta}$, y^* satisfies participation constraints evaluated at this maximizer, so that the allocation that attains the highest welfare among acyclical policies is (x, y^*). Since a small intervention increases E_e and $E_e + E_d$ when $\pi_d < \bar{\pi}$, and $I_e \geq I_d$ for $y = y^*$, and such intervention is implementable according to our last proposition, then the optimal cannot be acyclical. ■

The proof of this proposition holds even when $u_e = u_d$. When $u_e = u_d$, $\pi_d = \bar{\pi}$, and $\beta = \bar{\beta}$, proposition 4 implies that

$$\sum_s \pi_s p_s^+ q_s^+ \left[u_s\left(y_s^+\right) - y_s^+\right] = \sum_s \pi_s p_s q_s \left[u_s\left(y_s^*\right) - y_s^*\right],$$

where x^+ is a cyclical distribution, x is the (optimal) acyclical distribution and, by construction, $y_e^+ = y_d^+ = y_d^* = y_e^*$. However, if $u_e' > u_d'$, $\pi_d = \bar{\pi}$, and $\beta = \bar{\beta}$, then

$$\sum_s \pi_s p_s^+ q_s^+ \left[u_s\left(y_s^+\right) - y_s^+\right] > \sum_s \pi_s p_s q_s \left[u_s\left(y_s^*\right) - y_s^*\right], \tag{15}$$

since $u_e\left(y_e^+\right) - y_e^+ = u_e\left(y_e^*\right) - y_e^* > u_d\left(y_d^+\right) - y_d^+ = u_d\left(y_d^*\right) - y_d^*$, $\pi_e p_e^+ q_e^+ > \pi_e p_e^* q_e^*$, and $\sum_s \pi_s p_s^+ q_s^+ = \sum_s \pi_s p_s q_s$. Therefore, when $u_e' > u_d'$ there exist (non-unique) numbers $\hat{\beta} < \bar{\beta}$ and $\hat{\pi} < \bar{\pi}$ such that for any $\beta \in \left(\hat{\beta}, \bar{\beta}\right)$ and $\pi \in (\hat{\pi}, \bar{\pi})$, inequality (15) holds. Therefore, proposition 4 describes the conditions that are sufficient, but not necessary, for the optimal money policy to be cyclical. (We have documented these properties with numerical simulations, which are available upon request.) As a result, cyclical monetary policy may be optimal for some economies where the conditions of proposition 4 do not hold.

8 Conclusion

We have constructed a random-matching model of seasons, where different seasons are characterized by the buyer's differing desires and intensities to consume. Even when the buyer's intensity to consume is constant over seasons—and only the desire to consume varies over seasons—we show that a monetary policy that injects money into the economy when the desire to consume is high and withdraws it when the desire is low may be beneficial. Compared to a constant monetary policy, a cyclical policy increases the chances of single-coincidence meetings in the high season and decreases their chances in the low season. A cyclical policy will be beneficial if the proportion of consumers who want to consume is smaller in the low season than in the high season. In this situation, the *average* number of successful matches over both seasons will increase—which in turn increases welfare—because the measure of single-coincidence matches is weighted by a larger factor in the high season than in the low season. When the seasons are characterized by the buyer's differing desires *and* intensities to consume, a cyclical monetary policy can be optimal even when the difference between the proportion of buyers that want to consume in the high versus the low season is not very large. Our theory provides some additional support for the founding of the Fed. Previous explanations relied on the reduction in financial panics that came about after the Fed was founded; ours relies on the improved production and consumption allocations that result when the Fed follows a cyclical policy that alters the amount of money in the economy on a seasonal basis.

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Appendix 1

Lemma 4 *There exists, for each q_s, a unique acyclical x. Moreover, x can be solved for analytically. The statement holds for any s in $\{e, d\}$.*

Proof. We shall make repeated use of the system (1-4) with $\sigma = \tau = 0$. According to lemma 2, $p_s = q_s + a$ for some a that does not depend on s. We shall first solve for a analytically. For this purpose, let $A \equiv 1 + \pi_s p_s q_s$, which, by force of lemma 1, does not depend on s as well. Equation (1) now reads (i) $p_e = A - q_d$. Using (ii) $p_d = q_d + a$, we can write (2) as (iii) $q_e = A - (a + q_d)$. The equality $p_e q_e = \theta p_d q_d$ for $\theta = \pi_d/\pi_e$, can be written, using (i), (ii), and (iii), as $(A - q_d)^2 - a(A - q_d) - \theta q_d(a + q_d) = 0$. The only relevant solution of this quadratic equation is given by (iv) $2(A - q_d) = a + \sqrt{a^2 + 4\theta b}$, where $b = q_d(a + q_d)$. Since $A = 1 + \pi_d q_d p_d = 1 + \pi_d b$, we can rewrite (iv) as (v) $a^2 + 4\theta b = [2\pi_d b + 2(1 - q_d) - a]^2$. Expanding (v) as a quadratic equation in b, we find that the only relevant solution is given by (vi) $2\pi_d^2 b = \theta + a\pi_d - 2\pi_d(1 - q_d) + \sqrt{\theta^2 - 4\pi_d(1 - q_d)\theta + \pi_d^2 a^2 + 2\theta\pi_d a}$. Substituting in (vi) the expression $b = q_d(a + q_d)$, produces a quadratic equation in a as a function of q_d. The only relevant solution of the latter is (vii) $a = [-k_2 - \sqrt{k_2^2 - 4k_1k_3}]/(2k_1)$, where $k_1 = \pi_d^2[(2\pi_d q_d - 1)^2 - 1]$, $k_2 = 2\pi_d\{(2\pi_d q_d - 1)[2(\pi_d q_d)^2 + 2\pi_d(1 - q_d) - \theta] - \theta\}$, and $k_3 = [2(\pi_d q_d)^2 + 2\pi_d(1 - q_d) - \theta]^2 - \theta^2 + 4\pi_d(1 - q_d)\theta$. If q_d is fixed, then $p_d = q_d + a$ determines p_d. Using (1) and $p_e = q_e + a$, the values of p_e and q_e are also determined. Since the system (1-4) is symmetric in e and d, when $\sigma = \tau = 0$, similar conclusions follow when q_e is given, instead of q_d. $\blacksquare$

Appendix 2

Proposition 1 *The maximizer of $\pi_s p_s q_s$, among the set of acyclical distributions, is the unique x such that $p_{s'} = q_{s'}$. The statement holds for any s and s' in $\{e, d\}$.*

Proof. The set of acyclical distributions is closed, and $\pi_s p_s q_s$ is continuous in x for each s, so that a maximizer exists. Let us fix $x = x^1$, with $p_s^1 \neq q_s^1$ for some s, and show that x^1 cannot be the maximizer. Note that, by lemma 2, $p_s^1 \neq q_s^1$ if and only if $p_{s'}^1 \neq q_{s'}^1$. We start by constructing x^2, the "mirror image" of x^1, with the equalities $p_s^2 = q_s^1$ and $q_s^2 = p_s^1$ for $s \in \{e, d\}$. Also, for $\alpha \in (0, 1)$, let $x^\alpha \equiv \alpha x^1 + (1 - \alpha)x^2$.

It is clear that, for all s, $\alpha p_s^1 q_s^1 + (1-\alpha)p_s^2 q_s^2 < p_s^\alpha q_s^a$. Thus the distribution of x^α attains a higher extensive margin than that of x^1, although x^α is not invariant if it does not satisfy (1-4) with equality. However, using now lemma 1, one can rewrite each equation in the system (1-4), when $\sigma = \tau = 0$, as $p_s + q_{s'} = 1 + \pi_s p_s q_s$ or $p_{s'} + q_s = 1 + \pi_s p_s q_s$, where $s' \neq s$, so that each right-hand side is increasing in the extensive margin. Since $p_s^\alpha + q_{s'}^\alpha < 1 + \pi_s p_s^\alpha q_s^\alpha$ and $p_{s'}^\alpha + q_s^\alpha < 1 + \pi_s p_s^\alpha q_s^\alpha$, then there exists an acyclical $\bar{x}$, with $\bar{x} \geq x^\alpha$, that attains a higher extensive margin than that of x. The proof is now complete. ■

Appendix 3

Lemma 7 *If x is invariant and $\tau = \phi\sigma$, then $p_s - q_s = f_s(\sigma)$, where $f_e(\alpha) = \frac{\phi - \phi\alpha - 1}{1 + \phi - \phi\alpha}$ and $f_d(\alpha) = \frac{\phi + \phi\alpha - 1}{1 + \phi - \phi\alpha}$.*

Proof. The system (1-4) can be rewritten as

$$\hat{p}_e = 1 - (1-\tau)\hat{q}_d + \pi_d p_d q_d, \tag{16}$$

$$\hat{p}_d = 1 - (1-\sigma)\hat{q}_e + \pi_d p_e q_e + \frac{\tau}{1-\tau}, \tag{17}$$

$$\hat{q}_e = 1 - (1-\tau)\hat{p}_d + \pi_d p_d q_d + \frac{\sigma}{1-\sigma}, \tag{18}$$

and

$$\hat{q}_d = 1 - (1-\sigma)\hat{p}_e + \pi_e p_e q_e, \tag{19}$$

where $\hat{p}_e = p_e/(1-\sigma)$, $\hat{p}_d = p_d/(1-\tau)$, $\hat{q}_e = q_e/(1-\sigma)$, and $\hat{q}_d = q_d/(1-\tau)$. Eliminating $\pi_d p_d q_d$ between equations (16) and (18), and $\pi_e p_e q_e$ between (17) and (19), yields

$$\hat{p}_e - \hat{q}_e = (1-\tau)(\hat{p}_d - \hat{q}_d) - \frac{\sigma}{1-\sigma}$$

and

$$\hat{p}_d - \hat{q}_d = (1-\sigma)(\hat{p}_e - \hat{q}_e) + \frac{\tau}{1-\tau},$$

which can now be solved as

$$\hat{p}_e - \hat{q}_e = \frac{(1-\sigma)\tau - \sigma}{(1-\sigma)[1-(1-\tau)(1-\sigma)]} \tag{20}$$

and

$$\hat{p}_d - \hat{q}_d = \frac{\tau - (1-\tau)\sigma}{(1-\tau)[1-(1-\tau)(1-\sigma)]}. \tag{21}$$

One can now multiply both sides of (20) by $1-\sigma$ to obtain the expression $p_e - q_e = f_e(\sigma)$, and multiply both sides of (21) by $1-\tau$ to obtain the expression $p_d - q_d = f_d(\sigma)$. ■

Appendix 4

Before we can provide a proof for proposition 2, the following two lemmas are needed. From lemma 7 we can use the expression $q_s = p_s - f_s$ to reduce (1)-(4) to a system in (p_e, p_d), which allows us to write the derivatives of p_s with respect to σ as follows.

Lemma 8 *If x is invariant and $\tau = \phi\sigma$, then the derivatives of p_s with respect to σ, evaluated at $\sigma = 0$, satisfy*

$$\begin{bmatrix} 1 & 1-\pi_d(2p_d - f_d) \\ 1-\pi_e(2p_e - f_e) & 1 \end{bmatrix} \begin{bmatrix} p_e' \\ p_d' \end{bmatrix} = \begin{bmatrix} (1-\pi_d p_d)f_d' - p_e \\ (1-\pi_e p_e)f_e' - \phi p_d + \phi \end{bmatrix}.$$

Proof. Equations (1) and (2) can be written as

$$\frac{p_e^+}{1-\sigma} = 1 - p_d^+ + f_d + E_d \tag{22}$$

and

$$\frac{p_d^+}{1-\phi\sigma} - \frac{\phi\alpha}{1-\phi\sigma} = 1 - p_e^+ + f_e + E_e, \tag{23}$$

where $E_d = \pi_d p_d^+(p_d^+ - f_d)$ and $E_e = \pi_e p_e^+(p_e^+ - f_e)$. Taking derivatives on both sides of (22) and (23), with respect to σ, yields, for $\sigma = 0$,

$$p_e + p_e' = -p_d' + f_d' + E_d' \tag{24}$$

and

$$\phi p_d + p_d' - \phi = -p_e' + f_e' + E_e', \tag{25}$$

where $E_d' = \pi_d p_d'(2p_d - f_d) - \pi_d p_d f_d'$ and $E_e' = \pi_e p_e'(2p_e - f_e) - \pi_e p_e f_e'$. Substituting the expressions for E_d' and E_e' into equations (24) and (25) yields the result. ■

The total effect of changes in σ on extensive margins can also be expressed in a compact form.

Lemma 9 *If x is invariant and $\tau = \phi\sigma$, then the derivative of the sum $\sum_s \pi_s p_s q_s$, with respect to σ, evaluated at $\sigma = 0$, is equal to $p_e + \phi p_d - \phi - f_e' - f_d' + 2(p_e' + p_d')$.*

Proof. Using equations (24) and (25), derived in the proof of the previous lemma, yields the results because the derivative of the sum $\sum_s \pi_s p_s q_s$ is precisely $E'_d + E'_e$. ■

Using lemmas 7, 8 ,and 9, we can characterize the sign of the derivative of the sum $\sum_s \pi_s p_s q_s$, for $p_s = q_s$, as follows:

Proposition 2 *The maximizer of sum $\sum_s \pi_s p_s q_s$ is cyclical if and only if $\pi_d \in [0, \bar{\pi}]$, where $\bar{\pi} \in (0, \pi_e)$ can be solved for analytically as a function of π_e.*

Proof. Lemmas 7, 8, and 9 allow the substitution of expressions for $p'_e + p'_d$ and $f'_e + f'_d$ into the expression of the derivative of $\sum_s \pi_s p_s q_s$, evaluated at $\sigma = 0$, $p_s = q_s$, and $\phi = 1$. Substituting also the analytical solution for p_e and p_d, when $p_s = q_s$ and $\sigma = 0$ from lemma 5, yields an expression for the derivative involving only parameters. After some tedious but straightforward algebra, the condition according to which this derivative is positive can be written as

$$2\pi_d \leq (1-\theta)\sqrt{2} - (1-\sqrt{\theta})^2,$$

where $\theta = \pi_d/\pi_e$. The inequality is not satisfied for $\theta = 1$ and $\pi_d > 0$. Hence, the cutoff value of π_d for which the derivative is positive must be below π_e. Imposing equality in this expression and substituting for the value of θ yields, after solving for the unique relevant solution of the implied quadratic equation in π_d^2,

$$\bar{\pi} = \frac{1}{4}\left[\frac{2/\sqrt{\pi_e} + \sqrt{4/\pi_e - 4(2+(1+\sqrt{2})/\pi_e)(1-\sqrt{2})}}{2+(1+\sqrt{2})/\pi_e}\right]^2,$$

which has the properties stated in the proposition. ■

Appendix 5

Before providing a proof of proposition 3, we will first rewrite (v, w) in a convenient form and will then introduce two lemmas that will be needed in the proof. Substituting the values of $(\bar{v}, \bar{w})$ from equation (6) into equation (5) allows us to work with two independent systems of Bellman equations in (v, w), represented in matrix format as

$$\begin{bmatrix} v_s \\ w_{s'} \end{bmatrix} = \frac{1}{\det(M_{ss'})} M_{ss'} \begin{bmatrix} \mu_{us}\pi_s p_s u_s \\ -\mu_{ys'}\pi_{s'} q_{s'} y_{s'} \end{bmatrix}, \tag{26}$$

where $s, s' \in \{e, d\}$, $s' \neq s$, $\mu_{ue} = \mu_{yd} = 1$, $\mu_{ud} = 1 - \tau$, $\mu_{ye} = 1 - \sigma$, and

$$M_{ss'} = \left[\begin{array}{cc} 1 - (1 - \pi_{s'} q_{s'})\delta(1 - \sigma) & \tau\beta + (1 - \tau)\pi_s p_s \beta \\ \sigma\beta + (1 - \sigma)\pi_{s'} q_{s'} \beta & 1 - (1 - \pi_s p_s)\delta(1 - \tau) \end{array} \right].$$

We start with the following lemma, which allows us to ignore $\det(M_{ss'})$ in the algebra that follows.

Lemma 8 *The determinant of $M_{ss'}$ is positive.*

Proof. For $a_d \equiv 1 - \pi_d q_d$ and $a_e \equiv 1 - \pi_e p_e$, the determinant of M_{ed} equals

$$(1 - \delta a_d + \sigma \delta a_d)(1 - \delta a_e + \tau \delta a_e) - \delta(\pi_d q_d + \sigma a_d)(\pi_e p_e + \tau a_e),$$

which can be written as the sum of two terms, k_0 and k_1, where k_0 contains all the terms without σ or τ, and k_1 contains the other terms. The expression for k_0 is

$$k_0 = [1 - \delta(1 - \pi_d q_d)][1 - \delta(1 - \pi_e q_e)] - \delta \pi_d q_d \pi_e q_e.$$

After some simple algebra, that expression becomes

$$k_0 = (1 - \delta)(1 - \delta + \delta \pi_d q_d + \delta \pi_e q_e - \delta \pi_d q_d \pi_e q_e),$$

which is positive if x is invariant. Likewise, since for $a_d \equiv 1 - \pi_d q_d$ and $a_e \equiv 1 - \pi_e p_e$, one can write k_1 as

$$\begin{aligned} &\tau \delta a_e (1 - \delta a_d - \pi_d q_d) + \sigma \delta a_d (1 - \delta a_e - \pi_e p_e) + \delta \sigma a_d \tau a_e (\delta - 1), \text{ or} \\ &\tau \delta a_e (1 - \delta)(1 - \pi_d q_d) + \sigma \delta a_d (1 - \delta)(1 - \pi_e p_e) - \sigma \delta a_d (1 - \delta) \tau a_e, \text{ or} \\ &\tau \delta a_e (1 - \delta)(1 - \pi_d q_d) + \sigma \delta a_d (1 - \delta)(1 - \pi_e p_e)(1 - \tau), \end{aligned}$$

which is nonnegative. A similar argument shows that $\det(M_{de})$ is also positive. ■

Next, we use the Bellman equation for $w_e$to write (7) in an equivalent format that does not depend on y_e explicitly.

Lemma 9 *The participation constraint for date-s producers is equivalent to $[1 - (1 - \sigma)\delta] w_s \geq \sigma \beta v_{s'}$.*

Proof. Let $s = e$. The Bellman equation for w_e can be written as

$$[1 - (1 - \sigma)\delta] w_e - \sigma \beta v_d = (1 - \sigma)\pi_e q_e(-y_e + \beta v_d - \beta \bar{v}_d),$$

then the result follows directly from (7). The argument for $s = d$ follows from the same steps. ■

We now use the previous two lemmas to write the slack in the producer constraint in matrix algebra as

$$\begin{bmatrix} -\sigma\beta & 1-(1-\sigma)\delta \end{bmatrix} \begin{bmatrix} v_s \\ w_{s'} \end{bmatrix} = \frac{1}{\det(M_{ss'})} \begin{bmatrix} m_{us} & m_{ys'} \end{bmatrix} \begin{bmatrix} \mu_{us}\pi_s p_s u_s \\ -\mu_{ys'}\pi_{s'} q_{s'} y_{s'} \end{bmatrix} \quad (27)$$

where the scalars m_{us} and $m_{ys'}$ are to be computed so that the sign of the participation constraint does not depend on the magnitude of $\det(M_{ss'})$. After some straightforward algebra is used to produce a simple expression for m_{us} and $m_{ys'}$, the desired inequalities are derived as follows.

Proposition 3 *The participation constraints are satisfied if and only if*

$$u_d(y_d) \geq \frac{y_e}{\beta}\left[\frac{1}{(1-\tau)\pi_d p_d} - (1-\sigma)\delta\frac{1-\pi_d p_d}{\pi_d p_d}\right] \quad (28)$$

and

$$u_e(y_e) \geq \frac{y_d}{\beta}\left[\frac{1}{(1-\sigma)\pi_e p_e} - (1-\tau)\delta\frac{1-\pi_e p_e}{\pi_e p_e}\right]. \quad (29)$$

Proof. The steps for deriving inequality (28) are simple; we omit the proof for inequality (29) because it is identical to the proof of inequality (28). Regarding participation constraint for date-e producers, we find it useful to set $\rho = \pi_d p_d$ and $\xi = \pi_e q_e$, so that the expression for m_{ud} can be written as

$$\begin{aligned} -m_{ud} &= \sigma\beta - \sigma\beta(1-\sigma)(1-\xi)\delta - \sigma\beta - (1-\sigma)\xi\beta + \sigma\beta(1-\sigma)\delta + \\ &\quad (1-\sigma)\xi\beta(1-\sigma)\delta \\ &= -(1-\sigma)\xi\beta + (1-\sigma)\xi\beta\delta \\ &= -(1-\delta)(1-\sigma)\xi\beta. \end{aligned}$$

The expression for m_{ye} is

$$\begin{aligned}
-m_{ye} &= \sigma\beta\tau\beta + \sigma\beta(1-\tau)\rho\beta - 1 + (1-\tau)(1-\rho)\delta + (1-\sigma)\delta + \\
&\quad -(1-\sigma)\delta(1-\tau)(1-\rho)\delta \\
&= \sigma\beta\tau\beta + \sigma(1-\tau)\rho\delta - 1 + (1-\sigma)\delta + \\
&\quad -(1-\tau)(1-\rho)\delta[(1-\sigma)\delta - 1] \\
&= -1 + \delta - \sigma\delta[1-\tau-(1-\tau)\rho] + \sigma\delta(1-\tau)(1-\rho)\delta + \\
&\quad (1-\delta)(1-\tau)(1-\rho)\delta \\
&= -1 + \delta - \sigma\delta(1-\tau)(1-\rho) + \sigma\delta(1-\tau)(1-\rho)\delta + \\
&\quad (1-\delta)(1-\tau)(1-\rho)\delta \\
&= -1 + \delta - (1-\delta)\sigma\delta(1-\tau)(1-\rho) + (1-\delta)(1-\tau)(1-\rho)\delta \\
&= -(1-\delta)[1-(1-\sigma)\delta(1-\tau)(1-\rho)].
\end{aligned}$$

Thus, the right-hand side of (27) equals

$$\frac{(1-\delta)(1-\sigma)\pi_e q_e}{\det(M_{de})} \begin{bmatrix} \beta & 1-\delta(1-\sigma)(1-\tau)(1-\pi_d p_d) \end{bmatrix} \begin{bmatrix} (1-\tau)\pi_d p_d u_d \\ -y_e \end{bmatrix},$$

so that (28) follows. ■

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