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This paper demonstrates that in a plausibly calibrated monetary model with explicit production, exogenous money growth rules ensure real determinacy and thus avoid sunspot fluctuations. Although it is theoretically possible to construct examples in which real indeterminacy does arise, these examples rely on implausible money demand elasticities or ignore the effect of production on the model's dynamics.

Money Growth Rules and Price Level Determinacy

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ABSTRACT: This paper demonstrates that in a plausibly calibrated monetary model with explicit production, exogenous money growth rules ensure real determinacy and thus avoid sunspot fluctuations. Although it is theoretically possible to construct examples in which real indeterminacy does arise, these examples rely on implausible money demand elasticities or ignore the effect of production on the model's dynamics.

I. Introduction.

It has become conventional wisdom that central banks should target interest rates and not monetary aggregates. There are at least two reasons for this preference. First, following the classic arguments of Poole (1970), the apparent evidence of exogenous shocks to velocity leads to a preference for interest rate targeting. Second, a more recent line of research suggests that even in the absence of velocity shocks, money growth targeting may be problematic because it is more prone to real indeterminacy. For example, Matsuyama (1990) and Woodford (1994) show that money growth targeting can allow extrinsic uncertainty ("sunspots") to be introduced into an otherwise determinate real economy.

The purpose of this paper is to challenge the assertion that real indeterminacy is likely with money growth targeting. Although it is theoretically possible for an exogenous money growth policy to introduce sunspot equilibria, this paper demonstrates that in a reasonably calibrated monetary model with explicit production, money growth rules produce real determinacy. That is, money growth rules avoid the possibility of sunspot equilibria. We see the avoidance of sunspots as a necessary condition for any good monetary policy rule. Hence, exogenous money growth rules satisfy this minimalist criterion.

In contrast, interest rate rules do not generally satisfy this minimalist criterion. Interest rate rules are prone to sunspot equilibria because money growth is *endogenous* under such a policy. For example, consider the extreme case of an interest rate peg in which the money supply is passively varied to hit an interest rate directive. The well-known nominal indeterminacy under such a rule means that sunspot fluctuations in the price level naturally arise. In environments with nominal rigidities, these nominal fluctuations induce real fluctuations and are welfare-reducing.[1]

In contrast, we show that money growth rules ensure determinacy in a general monetary environment for all plausible calibrations. We utilize a generic money-in-the-utility function (MIUF) model because of its generality as it encompasses rigid cash-in-advance (CIA) models, transactions cost models (see Feenstra (1986)), shopping time models, and the cash-credit model pioneered by Lucas and Stokey (1983,1987).[2] These models differ in the micro details of the trading arrangements, but since we calibrate the models to aggregate monetary data (eg., the interest elasticity of money demand), these micro differences are irrelevant.

We restrict the analysis to an infinitely-lived representative agent economy because this has become the workhorse in theoretical monetary policy analysis.[3] Since we are concerned with issues of determinacy without loss of generality we limit the discussion to a deterministic model. As is well known, if the deterministic dynamics are not unique, then it is possible to construct sunspot equilibria in the model economy. Below we will use the terms "real indeterminacy" and "sunspot equilibria" interchangeably.

Under the assumption of an exogenous money growth rate, we consider two types of real indeterminacy. First we analyze the possibility of self-fulfilling hyperinflations in

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[1] It is possible to design more complex interest rate operating procedures that avoid these problems. For example, Carlstrom and Fuerst (2000) argue that if the central bank uses an interest rate operating procedure, then the only way of ensuring real determinacy in a sticky price model is for the central bank to respond aggressively to lagged inflation. For a related analysis see Benhabib, Schmitt-Grohe and Uribe (2000).

[2] For the case of the cash-credit model see footnote 7 of Lucas and Stokey (1983).

[3] There is a vast literature on real indeterminacy in overlapping generations models of money. See, for example, Azariadis (1981).

which the economy becomes de-monetized in the limit. We show that these can only arise if the limiting elasticity of money demand is quite high. This is in contrast to the classic contribution of Obstfeld and Rogoff (1983) in which hyperinflations arise for all money demand elasticities. The difference arises because, following Carlstrom and Fuerst (2000), we use cash-in-advance (CIA) timing in which the money that facilitates transactions is the money the economic agent has in advance of entering the goods market. In contrast, Obstfeld and Rogoff assume that money balances held at the end of the period facilitates trading earlier in the period or what we call cash-when-I'm-done (CWID) timing.

The remainder of the analysis focuses on the second form of real indeterminacy; the possibility of stationary multiple equilibria. A key innovation is that we add a standard CRS production technology to the environment. In this case, multiple stationary equilibria arise only with implausibly low money demand interest elasticities. This contrasts with the high elasticities needed for rational hyperinflations.

To understand our results in the case of stationary equilibria, it is helpful to compare them with the work of Matsuyama (1990) and Woodford (1994). Matsuyama analyzes an endowment MIUF model with an exogenous money growth policy.[4] He demonstrates that a necessary condition for stationary sunspot equilibria is that the cross-partial of the utility function U_{cm} be sufficiently negative.

Woodford (1994) analyzes a Lucas-Stokey (1983,1987) cash-credit economy. Surprisingly, Woodford's analysis is consistent with the existence of sunspot equilibria in

[4] Another difference between the current paper and Matsuyama (1990) is that we utilize CIA timing, while he uses CWID timing. This difference has a small effect on the existence of multiple stationary equilibria, and is footnoted when appropriate. See Carlstrom and Fuerst (2000) for a complementary analysis for interest rate rules.

a model in which the isomorphic MIUF has a *positive* cross partial, $U_{cm} > 0$. Below we will show that this discrepancy between Matsuyama (1990) and Woodford (1994) arises because the former uses CWID timing, while the latter uses CIA timing. More importantly, the paper demonstrates that these sunspot equilibria with $U_{cm} \geq 0$ arise only with implausibly low interest elasticities. As for the sunspots when $U_{cm} < 0$, a second contribution of the paper is to demonstrate that when a standard CRS production technology is added to the model this possibility disappears.

The paper proceeds as follows. The next section considers an endowment economy, and develops conditions for indeterminacy. Section three extends the analysis to an environment with production and demonstrates that once we restrict the analysis to the plausible parameter space that the sunspots of Matsuyama (1990) and Woodford (1994) disappear, and money growth rules ensure real determinacy. Section four concludes.

II. A MIUF Endowment Economy.

The economy consists of numerous infinitely-lived households with preferences given by

$$\sum_{t=0}^{\infty} \beta^t U(c_t, A_t/P_t),$$

where c_t and A_t/P_t denote consumption and real money balances, respectively.

The household begins the period with M_t cash balances and B_{t-1} holdings of nominal bonds. Before proceeding to the goods market, the household visits the financial market where it carries out bond trading and receives a cash transfer of $M_t^s(G_t - 1)$ from the monetary authority where M_t^s denotes the per capita money supply and G_t is the gross money growth rate. Hence, before entering goods trading, the household has cash balances given by

$$A_t \equiv M_t + M_t^s(G_t - 1) + B_{t-1}R_{t-1} - B_t,$$

where R_{t-1} denotes the gross nominal interest rate from t-1 to t. Notice that following Carlstrom and Fuerst (2000) we utilize CIA timing. That is, the money balances that aid in transactions are the money balances that the household has upon entering goods market trading. In contrast, Matsuyama (1990) utilizes end-of-period money balances, what Carlstrom and Fuerst (2000) call CWID timing. We will comment on these differences below.

After engaging in goods trading, the household ends the period with cash balances given by the intertemporal budget constraint.

$$M_{t+1} = M_t + M_t^s(G_t - 1) + B_{t-1}R_{t-1} - B_t - P_t c_t + P_t y_t,$$

where $y_t = y$ denotes real household endowment income. We will endogenize production in the next section.

The first order conditions to the household's problem include the following:

$$[U_m(t)+U_c(t)]/P_t = R_t\beta\, [U_m(t+1)+U_c(t+1)]/P_{t+1} \qquad (1)$$

$$U_m(t)/U_c(t) = (R_t-1). \qquad (2)$$

Equation (1) is the Fisherian interest rate determination in which the nominal rate varies with expected inflation and the real rate of interest on bonds. Equation (2) is the model's money demand function.[5] Money demand elasticity $\eta \equiv -d\ln m/d\ln i$ is given by

$$\eta = \frac{-U_m}{m[U_{mm} - iU_{cm}]} > 0,$$

where i = R –1 is the net nominal rate.

Suppose that the central bank expands the money supply at a constant (gross) growth rate of $M_{t+1}/M_t = G > \beta$. Since the nominal interest rate is endogenous, one can combine (1)-(2) to yield the following difference equation in real balances $m_t \equiv M_t/P_t$.

$$\frac{G}{\beta} m_t U_c(m_t) = m_{t+1}[U_c(m_{t+1}) + U_m(m_{t+1})]. \qquad (3)$$

An equilibrium consists of a non-negative m_t sequence that satisfies (3) and the standard transversality condition.

Expressing m_{t+1} as an implicit function of m_t, $m_{t+1} = g(m_t)$, we note first that g is non-negative. Under the assumption that money demand slopes down ($\eta > 0$), there is a unique positive steady-state solution to (3) given by the fixed point $g(m_{ss}) = m_{ss}$. One equilibrium is of course $m_t = m_{ss}$ for all t. The key issue is whether there are other equilibria. There are three possibilities.

First, hyperdeflations in which m_t explodes and goes to infinity in the limit. These paths are typically not equilibria as they violate the household's transversality condition (see Obstfeld and Rogoff (1986)).[6]

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[5] In the case of "cash-when-I'm done" timing, the corresponding equations are $U_c(t)/P_t = R_t\beta U_c(t+1)/P_{t+1}$, and $U_m(t)/U_c(t) = (R_t-1)/R_t$.

[6] A necessary condition for ruling out these equilibria is that G > 1, ie., that money growth is positive.

Second, hyperinflations where m_t goes to zero (see Obstfeld and Rogoff (1983)). We will discuss these in the next subsection.

Third, and finally, stationary multiple equilibria in which for all starting values of m, the path converges to m_{ss} in the limit. The bulk of our analysis will revolve around these equilibria.

Self-fulfilling hyperinflations:

Self-fulfilling hyperinflations (paths with m_t converging to zero) are possible if and only if $\lim_{m \to 0} g(m) = 0$, that is if there exists a non-monetary steady-state. We adopt the mild assumption that $\sup_{m \geq 0} U_c(y,m)$ is positive and finite. Given this assumption both sides of equation (3) go to zero (as m goes to zero) if and only if

$$\lim_{m \to 0} mU_m(y,m) = 0. \qquad (4)$$

From the money demand relationship (2) and the assumption that $\sup_{m \geq 0} U_c(y,m)$ is finite and positive, (4) is equivalent to

$$\lim_{m \to 0} mi(m) = 0,$$

where i(m) denotes the inverted money demand curve. This condition has an elasticity interpretation: in the limit, money demand interest elasticity must exceed unity so that the decline in real balances can occur without too large a movement in the nominal rate.

Obstfeld and Rogoff's (1983) analysis of hyperinflations is quite different. In particular, money demand interest elasticity has no role in their analysis. The essential difference is that Obstfeld and Rogoff use CWID timing (utility depends on end-of-period money, $A_t \equiv M_{t+1}$) so that the counterpart to (3) is

$$\frac{G}{\beta} m_t [U_c(m_t) - U_m(m_t)] = m_{t+1} U_c(m_{t+1}) . \qquad (5)$$

Let $m_{t+1} = h(m_t)$ denote this mapping. Obstfeld and Rogoff assume that $U_{cm} = 0$, and that there exists an $\hat{m} > 0$ such that $h(\hat{m}) = 0$. This latter assumption arises from the reasonable assertion that as m decreases U_m eventually exceeds the constant U_c.[7] In this case h becomes negative for small $0 < m < \hat{m}$, and there are hyperinflationary equilibria if and only if h(0) = 0, or

$$\lim_{m \to 0} m U_m(y, m) = 0. \qquad (6)$$

If condition (6) holds, then there are a countable infinity of equilibria that have as a penultimate point $\hat{m}$. Afterwards the economy jumps discontinuously to a completely demonetized economy where m = 0. All of these equilibria are found by backing up the transition path from $\hat{m}$ a countable number of periods.

Although condition (6) is mathematically the same as condition (4), the economics are quite different. Condition (6) does not have an elasticity interpretation. This is because the economy has become demonetized in the previous period when $m_T = \tilde{m}$ and nominal rates are infinite. After this money demand ceases to hold as both interest rates and prices are infinite. Why then is money held in this penultimate period (T) if nominal rates are infinite and the price level tomorrow (T+1) is infinite? Equivalently,

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[7] If U_m never exceeds the constant U_c, then h(m) is always nonnegative, and there are a continuum of equilibria in which real balances go to zero only in the infinite limit. Since $i(m) = [U_m / (U_c - U_m)]$, these equilibria have the property that nominal rates are typically finite even with zero real balances. In the hyperinflationary equilibria considered by Obstfeld and Rogoff (1983), real balances "jump" to zero in finite time, and in the penultimate period nominal rates are infinite.

why are money balances held at the end of this penultimate period even though they can never be used for transactions? Because under the peculiar assumption of CWID timing households receive utility from end-of-period money, that is, transactions in time-t are facilitated with the nominal money balances the household has at the beginning of time t+1. Condition (6) is thus a restriction devoid of economic content. The usual argument that hyperinflationary paths are always possible in infinite horizon monetary models results from a very peculiar timing assumption. In contrast, in the case of CIA timing a continuum of hyperinflationary paths are possible if and only if the limiting interest elasticity is quite large, in excess of unity.[8]

Multiple Stationary Equilibria:

Our primary focus for the remainder of the paper is on the third equilibrium possibility: multiple stationary equilibria. We find these more compelling than the hyperinflationary equilibria because, as noted by Obstfeld and Rogoff (1983), even if the interest rate elasticity of money demand exceeds unity, these hyperinflations can still be ruled out under the mild assumption that the government guarantees a minimal real redemption value for money. In contrast, the existence of stationary sunspot equilibria is much more troubling. Since all of these paths converge to the monetary steady-state, simple limiting arguments cannot rule them out.

Returning to condition (3) and the implicit g-mapping, it is straightforward to calculate the slope of g at m_{ss}:

[8] Typical estimates of money demand elasticity are far below unity. However, in the case of hyperinflations the evidence is less clear. Cagan (1956) estimates a semi-elasticity of about 4.5, implying an elasticity of 4.5 times the nominal rate. During the high inflation periods this elasticity will exceed unity.

$$g'(m_{ss}) = \left[1 - \frac{i/(1+i)}{\eta(1+mU_{cm}/U_c)}\right]^{-1}, \tag{7}$$

where η = -dlnm/dlni > 0 is the interest elasticity of money demand. Recall that

$$\eta = \frac{-U_m}{m[U_{mm} - iU_{cm}]}. \tag{8}$$

For what follows, it is helpful to note that the values of η and U_{cm} are logically distinct. Although economic theory and empirical evidence implies that η is positive, Feenstra (1986) demonstrates that there is no theoretical restriction on the sign of U_{cm}.[9] For any given value of U_{cm}, there exists a value of $U_{mm} < 0$ that maps into any estimated η. Holding U_{cm} fixed, η varies inversely with the absolute value of U_{mm}.

A necessary and sufficient condition for local real indeterminacy is that $g'(m_{ss})$ is within the unit circle. The expression for $g'(m_{ss})$ thus implies that there exist stationary sunspot equilibria in this endowment model only if $(1+mU_{cm}/U_c) < 0$ or if η is sufficiently small.

What is the intuition for these multiple stationary equilibria? Suppose that real balances begin below steady state, $m_t < m_{ss}$. Since the path is stationary, it must be the case that $m_t < m_{t+1}$, ie., the path is moving back to the steady-state. For $m_t < m_{ss}$, it must be the case that the nominal rate at time-t is *above* steady-state. But given a constant money growth rule it also must be the case that inflation is *below* steady-state. Therefore the real rate of interest must be sufficiently above steady-state. Hence, stationary sunspot

[9] For example, using Feenstra's (1986) transactions cost model, c denotes total consumption expenditures, including transactions costs. These expenditures are turned into actual consumption (ac) with the assistance of real cash balances, ac = ϕ(c,m). Utility is thus given by U(ac) = U(ϕ(c,m)). Since U is concave, assuming that the cross partial of ϕ is positive does not guarantee that the cross partial of U is positive.

equilibria are possible only if lower real balances ($m_t < m_{ss}$) lead to increases in the real rate of interest. From (1), the effect of real balances on the real rate of interest depends upon the sign of ($U_{mm} + U_{cm}$). If this term is sufficiently negative, then there are sunspot equilibria.

There are thus two cases, one corresponding to U_{cm} being sufficiently negative and the other to U_{mm} being sufficiently negative. First, if U_{cm} is sufficiently negative so that $(1+mU_{cm}/U_c) < 0$, then we have non-oscillatory sunspot equilibria ($0 < g'(m_{ss}) < 1$) for all values of η. These are akin to those discussed in Matsuyama (1990).[10]

Second, even if $(1+mU_{cm}/U_c) > 0$, there are sunspot equilibria if U_{mm} is sufficiently negative. Recalling that η varies inversely with U_{mm}, this corresponds to an η that is sufficiently small. These equilibria are oscillatory because with an extremely small money demand elasticity a given movement in real balances requires an extremely large movement in the nominal rate and hence the real rate. Such a large real rate movement requires real balances between t and t+1 to be sufficiently different ($m_t < m_{ss} < m_{t+1}$), that is for real balances to be oscillatory. In particular, if η satisfies

$$\eta < \left(\frac{i}{2(1+i)}\right)\left(\frac{1}{1+mU_{cm}/U_c}\right), \qquad (9)$$

then $-1 < g'(m_{ss}) < 0$ and we have oscillatory sunspot equilibria. As η increases and $g'(m_{ss})$ falls below -1, the Hopf-Bifurcation theorem implies that the two-period cycles associated with an Eigenvalue of –1 eventually increase until the cycles become infinite,

[10] Matsuyama (1990) assumes CWID timing, which implies $g'(m_{ss}) = \left[1 + \frac{i/(1+i)}{\eta(1+mU_{cm}/U_c)}\right]$. Matsuyama thus concludes that $(1+mU_{cm}/U_c) < 0$ is necessary for real indeterminacy but not sufficient. In the case of CIA timing, this negativity condition is sufficient, but

that is chaotic dynamics emerge.[11] In these cases sunspot equilibria are also possible although the economy is not locally stable. If η becomes large enough this possibility disappears. For example, given $(1+mU_{cm}/U_c) > 0$, a sufficient condition for determinacy is

$$\eta > \left(\frac{i}{1+i}\right)\left(\frac{1}{1+mU_{cm}/U_c}\right), \qquad (10)$$

so that $g'(m_{ss}) > 1$. For the remainder of the paper our focus will be on local analysis, but the reader may note that there is a small range between conditions (9) and (10) in which the local determinacy conditions are not sufficient for global determinacy.

In summary, there exist stationary sunspot equilibria if and only if (i) U_{cm} is sufficiently negative, or (ii) U_{mm} is sufficiently negative. In the next section we will demonstrate that the former equilibria disappear in a model with explicit production as the optimization conditions for production constrain the behavior of U_c. As for the latter, these equilibria arise only under implausibly low interest elasticities. The following example will provide a precise bound.

An Example: Suppose preferences are given by

$$U = \frac{1}{1-\sigma}\left[c^{1-\rho} + Am^{1-\rho}\right]^{\frac{1-\sigma}{1-\rho}}.$$

In this case there is a unit consumption elasticity, and $\eta = 1/\rho$ is the interest elasticity. The sign of U_{cm} is given by the sign of $\rho-\sigma$. As ρ goes to infinity the utility function

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not necessary.

[11] See Fukuda (1993), Matsuyama (1991), and Michener and Ravikumar (1998).

becomes Leontief and the model collapses to a rigid cash-in-advance constraint.

Straightforward calculations imply:

$$g'(m_{ss}) = \left[1 - \frac{i\rho(v+i)}{R(v+i(1+\rho-\sigma))}\right]^{-1}$$

where v = c/m denotes steady-state velocity. Henceforth we will typically assume v > 1. (However, as ρ goes to infinity, v will converge to one.) There are two cases:

(1) If $v + i(1+\rho-\sigma) < 0$, then $0 < g'(m_{ss}) < 1$, and we have non-oscillatory sunspot equilibria. Since velocity is large relative to the nominal rate, we need σ quite large for these sunspots to arise, ie., U_{cm} must be sufficiently negative.

(2) If $v + i(1+\rho-\sigma) > 0$, then we can have oscillatory sunspot equilibria ($-1 < g'(m_{ss}) < 0$) if and only if v-1-R > 0 *and* if ρ is sufficiently large (η is sufficiently small):

$$\rho > \frac{2R[v+i(1-\sigma)]}{i[v-1-R]}.$$

This region is quite small. For σ =10, v = 3, R = 1.02, this region is $\rho > 293$, or η < .003. This is an implausibly low interest elasticity. A sufficient condition for global determinacy (and hence to rule out chaotic equilibria) is $\rho < 147$, or η > .006 (see (10) vs. (9)). Notice that for a rigid cash-in-advance constraint, ρ goes to infinity, but v goes to one, so that the requirement v > 1+R is not satisfied, and there is determinacy. Similarly, as we shrink the time period between visits to the bank, then v declines until the condition v > 1+R is not satisfied so that the oscillatory sunspots disappear.

III. A Production Economy.

In this section we will add a standard production technology to the analysis. Assume that preferences are separable and linear in labor (L) and given by

$$U(c,m,1-L) \equiv V(c,m) - B\frac{L^{1+\gamma}}{1+\gamma},$$

and that production takes the standard Cobb-Douglas form:

$y = K^{\alpha}L^{1-\alpha}$ with a constant depreciation rate of δ.

We will consider more general preferences over labor below.

The additional Euler equations for labor choice (12) and capital accumulation (13) are familiar:

$$\frac{U_L(t)}{U_c(t)} = f_L(t) \tag{12}$$

$$U_c(t) = \beta U_c(t+1)[f_K(t+1) + (1-\delta)]. \tag{13}$$

$$c_t = K_t^{\alpha}L_t^{1-\alpha} + (1-\delta)K_t - K_{t+1}. \tag{14}$$

Real money balances indirectly enter both of these marginal conditions via the cross partials (U_{cm}) of the utility function. As a result the behavior of real balances typically alters the economy's behavior relative to an otherwise standard real business cycle (RBC) model.

For present purposes, a critical issue is that (12)-(13) place restrictions on the behavior of U_c. This is particularly clear in the case of linear leisure (γ=0). Let x_t = (L_t/K_t) denote the labor-capital ratio. Exploiting the linearity in leisure preferences, we can use (12) to rewrite (13) as:

$$x_t^{\alpha} = \alpha\beta x_{t+1} + \beta(1-\delta)x_{t+1}^{\alpha}.$$

Since (12) implies that U_c depends only on x, then real balances, m, depend only on c and x so that we can rewrite (14) as

$$K_{t+1} = K_t x_t^{1-\alpha} + (1-\delta)K_t - c(x_t, m_t)$$

Collecting these results, we can express (3) and (13)-(14) as the following linearized functions:

$$x_{t+1} = q^1(x_t)$$

$$m_{t+1} = q^2(x_t, m_t)$$

$$K_{t+1} = q^3(x_t, m_t, K_t)\,.$$

It is immediately obvious that we have a block-recursive system, with eigenvalues given by the diagonal elements. For determinacy we need two explosive roots. The first and third eigenvalues are given by

$$e_1 = \frac{\alpha}{1-\beta(1-\alpha)(1-\delta)} < 1,$$

$$e_3 = \frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta} > 1,$$

so that we have determinacy if and only if q_m^2 is outside the unit circle. Equivalently, we need to evaluate the slope of the g-function given by (3), holding x_t, that is $U_c(t)$, constant. This restriction imposes a relationship between c and m, c = c(m), with dc/dm = -Ucm/Ucc. Imposing this restriction on the analysis of the previous section we have:

$$g'(m_{ss}) = \frac{1}{1 + \dfrac{m}{(1+i)U_{cc}}\left[U_{mm}U_{cc} - U_{mc}^2\right]}.$$

Concavity implies $\left[U_{mm}U_{cc} - U_{mc}^{2}\right] > 0$ and $U_{cc} < 0$. Hence, the only possible equilibria are oscillatory. The non-oscillatory equilibria have disappeared because the implicit conditions on U_c make it impossible for $(1+mU_{cm}/U_c) < 0$.[12] This suggests that we have sunspots only if η is sufficiently small. This is the case, as an example will demonstrate:

An Example: Suppose preferences are given by

$$U = \frac{1}{1-\sigma}\left[c^{1-\rho} + Am^{1-\rho}\right]^{\frac{1-\sigma}{1-\rho}} - BL \, .$$

Recall that the consumption elasticity is unity and that $\eta = 1/\rho$ is the interest elasticity. We then have:

$$g'(m_{ss}) = \left[1 - \frac{i\rho\sigma(i+v)}{R(\sigma v + i\rho)}\right]^{-1}.$$

Notice that the only type of real indeterminacy possible is of the oscillatory type.

As a special case let ρ go to infinity, that is, η goes to zero so that the utility function becomes Leontief. With v=1 the model collapses to a rigid CIA constraint. In this case $g'(m_{ss}) = \frac{1}{1-\sigma}$ so that the model is indeterminate if and only if $\sigma > 2$. This is exactly the result derived by Farmer (1999) in his text and extended by Carlstrom and Fuerst (1999) to a model with capital. The result is also implicit in Woodford (1994) in the Lucas-Stokey model. These results suggested that money growth rules were likely

[12] In the case of CWID timing we have

$$g'(m_{ss}) = 1 - \frac{(1+i)m}{U_{cc}}\left[U_{mm}U_{cc} - U_{mc}^{2}\right]$$

which always exceeds one so that we never have real indeterminacy.

prone to sunspots. But this result does not hold up under more reasonable calibrations of money demand elasticity.

In general we have local indeterminacy if and only if $\sigma(i+v) > 2R$ *and* if ρ is sufficiently large:

$$\rho > \frac{2R\sigma v}{i[\sigma(i+v)-2R]}.$$

Notice that as in the case of a strict CIA constraint indeterminacy becomes more likely the larger is σ. Accordingly we choose σ =10 the upper end of plausible estimates to demonstrate the implausibility of sunspots. Calibrating to quarterly data we choose R=1.02 (8% annualized). Given these choices the larger is v the easier it is to get sunspots. Therefore we interpret money to mean the monetary base so that quarterly velocity is 3. Given these choices the indeterminacy region is $\rho > 108$, so that there are sunspots only if $\eta < .009$! This is an implausibly low money demand elasticity.[13] As before a sufficient condition for global determinacy is $\rho < 54$ or $\eta > .018$.

Notice that we calibrated according to quarterly data. Most would contend that if money is being held to facilitate transactions that the model should be calibrated to an even higher frequency. Calibrating to a higher frequency, however, makes indeterminacy even less likely since v and i decline so that the condition $\sigma(i+v) > 2R$ no longer is satisfied.

Moving away from linear leisure ($\gamma > 0$) has no quantitative effect on our results. Remarkably, even with an extremely small labor supply elasticity (eg., $\gamma = 100$) the cutoff for local indeterminacy is unchanged to the first decimal point. We conclude that for all reasonable calibrations there is real determinacy under an exogenous money growth process.

IV. Conclusion.

One of the first papers to integrate money into a real business cycle model is Cooley and Hansen (1989). That paper assumed log preferences over consumption, linear preferences over leisure, imposed a strict cash-in-advance constraint, and assumed an exogenous money growth rate. The model as written is determinate. However, if the risk aversion coefficient is greater than 2 (an entirely reasonable assumption), then the real economy is indeterminate. This has led Woodford (1994) and others to argue for the inherent instability of money growth rules.

A surprising contribution of this paper is that even though estimated money demand elasticities are fairly small, the absolute zero elasticity inherent in the cash-in-advance constraint is critical for the existence of stationary sunspots in the Cooley-Hansen model. For all plausible money demand elasticities and risk aversion coefficients within the reasonable range, the Cooley-Hansen model is determinate.

For self-fulfilling hyperinflationary equilibria to be rational, however, we show that the interest elasticity of money demand must be quite large –exceeding unity. This contrasts with the results of Obstfeld and Rogoff who use CWID timing.

The paper's analysis was conducted within the context of a generic aggregative MIUF model, an environment that is incredibly general, encompassing transactions cost models, shopping time models, rigid cash-in-advance models, and cash-credit models.

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[13] This is low even for short-run elasticities. The relevant elasticity for stability analysis, however, is the

Hence, it is hard to imagine a plausibly calibrated monetary environment in which money growth rules are prone to stationary sunspots. In contrast, stationary sunspots are endemic under most interest rate operating procedures. This result provides some theoretical support for those who favor money growth targeting.

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long-run elasticity which is substantially greater.

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