working paper

9910

Timing and Real Indeterminacy in Monetary Models

by Charles T. Carlstrom and Timothy S. Fuerst

FEDERAL RESERVE BANK OF CLEVELAND

Working Paper 9910

Timing and Real Indeterminacy in Monetary Models

by Charles T. Carlstrom and Timothy S. Fuerst

Charles T. Carlstrom is an Economist at the Federal Reserve Bank of Cleveland. Timothy S. Fuerst is Associate Professor of Economics at Bowling Green State University, Bowling Green, Ohio, and a visiting consultant with the Federal Reserve Bank of Cleveland.

September, 1999

Timing and Real Indeterminacy in Monetary Models

By Charles T. Carlstrom and Timothy S. Fuerst

An increasingly common approach to the theoretical analysis of monetary policy is to ensure that a proposed policy does not introduce real indeterminacy and thus sunspot fluctuations into the model economy. Policy is typically conducted in terms of directives for the nominal interest rate. This paper uses a discrete-time money-in-the-utility function model to demonstrate how seemingly minor modifications in the trading environment result in dramatic differences in the policy restrictions needed to ensure real determinacy. These differences arise because of the differing pricing equations for the nominal interest rate.

Timing and Real Indeterminacy in Monetary Models

Charles T. Carlstrom[a]

Timothy S. Fuerst[b]

[a]Federal Reserve Bank of Cleveland, Cleveland, Ohio, 44101, USA.
[b]Department of Economics, Bowling Green State University,
Bowling Green, Ohio, 43403; and Consultant, Federal Reserve Bank of Cleveland.

December 1998

ABSTRACT: An increasingly common approach to the theoretical analysis of monetary policy is to ensure that a proposed policy does not introduce real indeterminacy and thus sunspot fluctuations into the model economy. Policy is typically conducted in terms of directives for the nominal interest rate. This paper uses a discrete-time money-in-the-utility function model to demonstrate how seemingly minor modifications in the trading environment result in dramatic differences in the policy restrictions needed to ensure real determinacy. These differences arise because of the differing pricing equations for the nominal interest rate.

1. Introduction.

An increasingly common approach to the theoretical analysis of monetary policy is to ensure that a proposed policy does not introduce real indeterminacy and thus sunspot fluctuations into the model economy. Policy is typically conducted in terms of directives for the nominal interest rate. For example, a simple Taylor (1993) rule posits that the central bank conducts policy according to the following rule: $R_t = B \cdot (\pi_t)^{\tau}$, where R_t and π_t denote the (gross) nominal interest and inflation rate (between t and t+1).[1] That is, the central bank varies the nominal rate in relation to movements in inflation with an elasticity of τ. In this context, an important policy question is what restrictions on τ are needed to ensure real determinacy.

Carlstrom and Fuerst (1998) took a first step to answering this question by analyzing a flexible price economy. They suggest that what is crucial in whether or not there is indeterminacy is how much the Fed increases the nominal rate with respect to increases in the economy's underlying real rate. Thus a more natural (although equivalent) way to rewrite the Taylor rule (in order to analyze indeterminacy) is $R_t = A \cdot (R_t / \pi_t)^{\gamma}$.[2]

The basic conclusion of the Carlstrom-Fuerst analysis is something of a ½ rule—to ensure real determinacy, we need $\gamma < ½$. That is, for a given 100 basis point movement in

[1] In a model with uncertainty this corresponds to targeting the expected inflation rate, a policy consistent with the practice of many central banks. Taylor's (1993) original rule has the central bank responding to past inflation rates. Since lagged inflation rates are good predictors of the future inflation rate, Taylor's empirical formulation of the rule can be viewed as a reduced form representation of a structural policy of targeting expected inflation. For the indeterminacy issues of this paper the structural version of the rule is more appropriate. The original Taylor Rule also had the central bank responding to output. This addition has no quantitative importance for the issues of this paper.

[2] There is a one-to-one mapping between a policy rule in terms of the real rate and a policy rule in terms of the inflation rate, $\tau = \gamma/(\gamma-1)$.

the real rate the central bank must limit the movement of the nominal rate to under 50 basis points.[3] The intuition for determinacy vs. indeterminacy goes something like this: Suppose that the real rate rises by 1%, and that the central bank allows the nominal rate to rise by γ%. This increase in the nominal rate depresses current real activity (i.e. consumption) thus leading to a higher real rate. This completed circle is suggestive of sunspots. Whether sunspots arise depends upon the elasticity, γ—with a small response, there is real determinacy, with a large response there are sunspots.

The contribution of this paper is to demonstrate the sensitivity of these stability conclusions to apparently small changes in the modeling structure. We utilize a money-in-the-utility function (MIUF) environment because of its generality. Feenstra (1986) demonstrates that any transactions cost (TC) economy can be written as a MIUF economy. Similarly, a shopping-time (ST) model can be rewritten as a MIUF economy. Finally, cash-in-advance (CIA) models are extreme versions of MIUF and TC economies. Thus, a MIUF environment is quite general. The TC, ST, and CIA assumptions simply imply particular functional forms for the MIUF economy.

We analyze a MIUF economy under differing assumptions about the money balances that enter into the utility function. In Model 1, we assume timing that is a direct extension of typical CIA timing. That is, the money available to satisfy consumption needs is the money the household has left after leaving the bond market and before entering the goods market. In contrast, in Model 2 we assume that goods market trading

[3] The exact quantitative details differ depending on the assumptions on the real environment. The ½ rule is approximately correct. In terms of the typical Taylor rule formulation, $\gamma < ½$ implies τ within the unit circle.

occurs first, and that bond trading occurs at the end of the period. Finally, in Model 3 we assume that end-of-period money balances enter the utility functional, net of current income and current consumption.

These differing assumptions lead to different pricing equations for the nominal interest rate. In a model in which the central bank operates monetary policy via the nominal interest rate these differences have important effects on the conditions for stability. Surprisingly in a model with production the ½ rule of Carlstrom and Fuerst holds for Model 1 timing irrespective of how money affects utility. With model 2's timing, however, the model is always indeterminate while model 3 is always determinate.

The paper proceeds as follows. Section 2 lays out the basic model. Section 3 presents the determinacy results for different modeling assumptions. Section 4 adds production to the model. Section 5 compares the results to the continuous time analysis of Benhabib, Grohe, and Uribe (1998). Section 6 concludes.

2. A MIUF Economy.

The economy consists of numerous infinitely-lived households with preferences given by

$$\sum_{t=0}^{\infty} \beta^t U(c_t, A_t/P_t),$$

where c_t and A_t/P_t denote consumption and real money balances, respectively. The key issue is what measure of money appears in the utility function. We will turn to this shortly. The intertemporal budget constraint is given by

$$M_{t+1} = M_t + X_t + B_{t-1}R_{t-1} - B_t - P_tc_t + P_ty_t,$$

where M_t denotes money balances at the beginning of time t, X_t denotes a monetary transfer from the government, B_{t-1} are bond-holdings acquired in period t-1, R_{t-1} denotes the nominal interest rate from t-1 to t, and the endowment is normalized to $y_t = 1$. Below we extend the analysis to an economy with endogenous production.

There are several possible ways to model A_t. *The central issue is what money balances aid in contemporaneous transactions.* The existing literature contains three possibilities:

Model 1:

$$A_t = M_t + X_t + B_{t-1} R_{t-1} – B_t$$

Model 2:

$$A_t = M_t + X_t$$

Model 3:

$$A_t = M_{t+1} = M_t + X_t + B_{t-1}R_{t-1} - B_t - P_tc_t + P_ty$$

Models 1 and 2 assume that what matters for time-t transactions is the money with which one enters the time-t goods market. The two models differ in the order in which bonds and goods trading occurs.

In Model 1, net bond trading is included in time-t money holdings since the bond market is assumed to open before (or concurrently with) the goods market, and therefore bonds can be transformed into money in advance of goods market trading. This assumption is typically used in CIA models (eg., Lucas (1982) and Lucas and Stokey (1987)).[4]

In contrast, Model 2 assumes that goods market trading occurs before the bond market opens, so that net bond trading is not included in current money balances. Lucas (1997) uses this convention in a MIUF setting, while Farmer (1993) uses it in a CIA model.

The traditional MIUF approach[5] is to assume Model 3 timing, i.e., that end-of-period balances, $A_t = M_{t+1}$, enter into the utility function. It is very difficult to justify this choice on theoretical grounds. Using end-of-period money implies that money at the *beginning of t+1* reduces transactions costs in *period t.* Equivalently, Model 3 implies that what matters for transactions purposes is the money you leave the goods market with, *net of current consumption and current income.* Including current income as part of current money balances violates Clower's dictum that "money buys goods, and goods buy money, but goods do not buy goods." One can imagine trading environments in which this violation is possible. But it is very difficult to defend the subtraction of current consumption from current money balances.

To see these differences another way, Model 1 and Model 2 both have the characteristic that under a reasonable MIUF specification, these models collapse to a CIA model as we drive the interest elasticity to zero. In sharp contrast, this is never possible

[4] Many CIA models never distinguish between models 1 and 2. This is because in equilibrium bonds are in zero net supply and are typically not included unless they are specifically priced.
[5] In fact, Model 3 is typically used in all monetary models (MIUF, TC, ST) except CIA models. The reason for this dichotomy is not clear.

in Model 3. Models 1 and 2 assume that what matters for transactions purposes is the cash one has in advance of goods trading, while Model 3 assumes that what matters is the cash one has after completing goods trading. Models 1 and 2 are thus models where "cash in advance" matters, while Model 3 assumes that "cash when I'm done" is what matters.

In any event, the Euler equations that define equilibrium in the three models are given by:

Model 1:

$$[U_m(t)+U_c(t)]/P_t = R_t\beta\, [U_m(t+1)+U_c(t+1)]/P_{t+1} \tag{1}$$

$$U_m(t)/U_c(t) = (R_t-1) \tag{2}$$

Model 2:

$$U_c(t)/P_t = R_t\, \beta\, U_c(t+1)/P_{t+1} \tag{3}$$

$$U_m(t+1)/U_c(t+1) = (R_t-1) \tag{4}$$

Model 3:

$$U_c(t)/P_t = R_t\beta U_c(t+1)/P_{t+1} \tag{5}$$

$$U_m(t)/U_c(t) = (R_t-1)/R_t \tag{6}$$

Note the differences between the Fisher equations in Models 1 and 2 (equations (1) and (3)). In Model 2, the household can substitute current goods for current bonds with no change in the money balances that enter the current utility function. Instead, purchasing a bond sacrifices *future* transactions facilitation. In contrast, under Model 1, an increase in bond purchases come at the expense of current money balances and the resulting ability to carry out *current* transactions. Thus, the time-t marginal utility of money enters into (1) but not (3). These differences in timing also manifest themselves in the timing differences in the money demand equations (2) and (4).

As for Model 3, the Fisher equation (5) is symmetric with Model 2 because (like Model 2) it is possible to substitute consumption for bonds with no change in the money balances that enter the current utility functional. The money demand equation is essentially the same as in Model 1 but with the added discounting that arises because end-of-period money balances enter the utility function. Model 3 is thus a peculiar combination of Models 1 and 2, a combination that (as argued above) is difficult to motivate intuitively.

These differences in timing across the models have no effect on equilibrium determinacy under some natural choices for monetary policy. For example, if the central bank engineers a constant money growth rate (or a money growth rule that depends on state variables) the conditions for determinacy in the three models are nearly equivalent. This arises because the Euler equation for money holdings are quite similar across the models. In Models 1 and 2 this Euler equation is given by

$$\frac{U_c(t)}{P_t} = \beta\left[\frac{U_c(t+1)+U_m(t+1)}{P_{t+1}}\right],$$

while in Model 3 we have

$$\frac{U_c(t)}{P_t} = \frac{U_m(t)}{P_t} + \beta\left[\frac{U_c(t+1)}{P_{t+1}}\right].$$

The timing difference between the first two models and Model 3 arises because in the latter model current consumption is subtracted from current money holdings.

However, these modeling differences have an important effect when one assumes that the central bank conducts policy according to a nominal interest rate rule. For example, suppose that the central bank conducts policy according to the following rule:

$$R_t = A \cdot (R_t / \pi_t)^{\gamma} \qquad (7)$$

That is, the central bank varies the nominal rate in relation to movements in the real rate with an elasticity of γ. The different Fisher equations across the models leads to the following money reaction functions (real money supply curves):

Model 1:

$$R_t^s = R_{ss}\left(\frac{U_c(t)+U_m(t)}{U_c(t+1)+U_m(t+1)}\right)^{\gamma} \qquad (8)$$

Model 2 and Model 3:

$$R_t^s = R_{ss}\left(\frac{U_c(t)}{U_c(t+1)}\right)^{\gamma} \qquad (9)$$

The bounds on γ for real determinacy are quite different across the models.

The economic intuition for real indeterminacy revolves around the slope of these real money supply curves. In the case of Model 1, equation (8) implies that (for a given m_{t+1}) an increase in m_t decreases the real rate if and only if $(U_{mm} + U_{cm}) < 0$. This is typically the case. For example, henceforth assume that preferences are given by

$$U(c,m) \equiv U[c\,h(\frac{m}{c})], \quad \textit{with } h' > 0 \textit{ and } h'' < 0\,.$$

The homotheticity assumption implies a unit consumption elasticity. The properties of h(.) determine the interest elasticity of money demand with concavity needed to ensure that money demand is decreasing in the nominal rate. These preferences are consistent with any sign for U_{cm}, but in either case we have $(U_{mm} + U_{cm}) < 0$. Thus, *regardless of the sign of* U_{cm}, the money supply curve (8) in Model 1 slopes up if $\gamma < 0$, and slopes down if $\gamma > 0$. Now suppose that real balances fall, implying an increase in the real rate of interest. If the supply curve slopes down so that the nominal rate rises by an appropriate amount, then the initial decline in real balances is rational. Hence, under Model 1 monetary policies with positive γ's will tend to generate sunspots.

In sharp contrast to Model 1, in Models 2 and 3 the slope of the supply curve *depends on the sign of* U_{cm}. If $U_{cm} > 0$, then equation (9) implies that (for a given m_{t+1}) a decrease in m_t decreases the real rate, so that the supply curve slope has the same sign as γ. Suppose that real balances fall. This implies that the real rate falls. If the nominal rate rises by an appropriate amount (a downwardly sloped supply curve), then the initial decline in real balances is rational. Hence, under Models 2 and 3 monetary policies with negative γ's will tend to generate sunspots—exactly the converse of Model 1. If instead $U_{cm} < 0$, then monetary policies with positive γ's will tend to generate sunspots—consistent with Model 1. In the next sections we develop the quantitative details of this basic intuition.

3. Real Indeterminacy in an Endowment Economy.

A. Model 1

Using the monetary policy rule (8), money supply equals money demand for model 1 can be collapsed into

$$R_t = R_{ss}\left(\frac{U_c(t)+U_m(t)}{U_c(t+1)+U_m(t+1)}\right)^{\gamma} = \frac{U_m(t)+U_c(t)}{U_c(t)}$$

Given this is an endowment economy we normalize $c_t = y = 1$. Let Δ denote the derivative dm_{t+1}/dm_t. Straightforward calculations yield

$$\Delta = -1 + \frac{\varepsilon}{\eta}$$

where $\eta = \dfrac{RU_c}{m(U_{mm} + U_{cm} - RU_{cm})} < 0$ is the (gross) interest elasticity of money demand

and $\varepsilon = \dfrac{RU_c}{m\gamma(U_{cm} + U_{mm})}$ is the (gross) interest elasticity of money supply (holding period t+1 money constant).[6] For real determinacy, we need Δ to be outside the unit circle. That is,

$$\varepsilon < 2\eta \qquad \text{or} \qquad \varepsilon > 0 .$$

Thus, a necessary and sufficient condition for indeterminacy is that

$$2\eta < \varepsilon < 0 .$$

A necessary condition for indeterminacy is that money supply, like money demand, slope down. The intuition for this can be broken into two parts. Suppose that $2\eta < \varepsilon < \eta$ ($\Delta>0$) so that money supply cuts demand from above. In this case an exogenous increase in future real balances rise shifts out the supply curve of money today leading to higher real balances today. Since $\Delta>0$ higher real balances today lead to higher real balances tomorrow thus completing the circle. The other possibility is for $\eta<\varepsilon<0$ ($\Delta<0$) so that supply cuts demand from below. In this case, things work the opposite of above: An increase in future real balances shifts today's supply curve out leading to *lower* current real balances. Now $\Delta<0$ so that *lower* real balances today lead to higher real balances tomorrow.

Solving for a condition in γ yields:

Proposition 1: Under Model 1 timing, a necessary and sufficient condition for real determinacy is

$$\gamma < \frac{1}{2} + \frac{-RU_{cm}}{2(U_{cm} + U_{mm})}.$$

Notice that for separable preferences the condition for determinacy is $\gamma < ½$. This is the same condition as in Carlstrom and Fuerst (1998). We will return to this similarity below.

B. Model 2.

Proceeding as before, under the posited policy rule Model 2's money supply equal money demand implies:

[6] The condition $\dfrac{d\ln(m_{t+1})}{d\ln(R_t)} = -\varepsilon$ is also used to calculate Δ.

$$R_t = R_{ss}\left(\frac{U_c(t)}{U_c(t+1)}\right)^{\gamma} = \frac{U_m(t+1)+U_c(t+1)}{U_c(t+1)}$$

Let Δ denote the derivative dm_{t+1}/dm_t:

$$\Delta = \frac{1}{\left(1+\frac{\varepsilon^s}{\eta^D}\right)}$$

where $\eta = \dfrac{RU_c}{m(U_{mm}+U_{cm}-RU_{cm})}$ is the *elasticity of m_{t+1} with respect to R_t* and

$\varepsilon = \dfrac{U_c}{m\gamma U_{cm}}$ is the (gross) interest elasticity of money supply (holding period t+1 money constant). Note that time-t money demand schedule is perfectly elastic with respect to R_t and shifts in m_{t+1} shift down time-t demand (η denotes this elasticity). The conditions for real indeterminacy are $\varepsilon < 0$ ($1>\Delta>0$) or $\varepsilon > -2\eta$ ($-1<\Delta<0$). The intuition is similar to before.

If $\varepsilon < 0$ then the downward sloping supply curve always cuts demand from above. Now increases in m_{t+1} shifts time-t supply to the right and time-t demand down. This increases m_t and since $\Delta>0$ completes the circle. If $\varepsilon>-2\eta$, however, then the upward sloping supply curve always cuts demand from below. Now increases in m_{t+1} shifts time-t supply to the right and time-t demand down. As long as supply compared to demand is sufficiently elastic this decreases m_t and since $\Delta < 0$ completes the circle.

We can now state:

Proposition 2: For model 2 timing if $U_{cm} > 0$ a necessary and sufficient condition for real determinacy is

$$\gamma > \frac{-U_c}{2\eta m U_{cm}}.$$

If $U_{cm} < 0$ a necessary and sufficient condition for real determinacy is

$$\gamma < \frac{-U_c}{2\eta m U_{cm}}.$$

If $U_{cm} = 0$, there is indeterminacy for all values of γ.

C. Model 3.

In the case of Model 3, the equilibrium condition is given by

$$q\left[R_{ss}\left(\frac{U_c(t)}{U_c(t+1)}\right)^{\gamma}\right] = \frac{U_m(t)+U_c(t)}{U_c(t)},\ where\ q(R) \equiv 1+\frac{(R-1)}{R}.$$

The function q( ) is needed because of the unusual form of the money demand equation (6) in Model 3. This modification is minor for small nominal rates since q is approximately equal to R. Proceeding as before, we have

$$\Delta = -1 + \frac{\varepsilon}{\eta}$$

where $\varepsilon = \dfrac{qU_c}{q'mR\gamma U_{cm}}$ is the (gross) elasticity of supply with respect to q and

$\eta = \dfrac{qU_c}{m(U_{mm}+U_{cm}-qU_{cm})}$ denotes the (gross) elasticity of demand with respect to q.

Since $\eta < 0$, a necessary and sufficient condition for determinacy is $\varepsilon > 0$, or $\varepsilon < 2\eta$. As before, a necessary condition for indeterminacy is that the supply curve slope down (the intuition is symmetric with Model 1). As with Model 2, the supply curve slope depends upon γ and the sign of U_{cm}. Thus we have two separate cases:

Proposition 3: If $U_{cm} > 0$ a necessary and sufficient condition for real determinacy is

$$\gamma > \frac{U_c qR}{2\eta m U_{cm}}.$$

If $U_{cm} < 0$ a necessary and sufficient condition for real determinacy is

$$\gamma < \frac{U_c qR}{2\eta m U_{cm}}.$$

If $U_{cm} = 0$, there is determinacy for all values of γ.

1. Real Indeterminacy in a Production Economy.

The previous sections developed the numerical details of the intuition discussed in section 2. In short, a necessary condition for indeterminacy is that the real money supply curve slope down, $\varepsilon < 0$.[7] Under the posited interest rate rule (7), the central bank moves the nominal rate in response to the real rate with elasticity γ. Thus the sign of ε depends upon the sign of γ and the effect of real balances on the real rate. One major reason why the conditions for determinacy differ across the models is because of differences in the effect that real balances have on the real rate. In Model 1, higher real balances unambiguously lower the real rate, so that ε has the opposite sign of γ. In Models 2 and 3, the effect of real balances on the real rate depends upon the sign of U_{cm}. If $U_{cm} > 0$, ε has the same sign as γ; if $U_{cm} < 0$, ε has the opposite sign of γ. The other major difference

[7] For model 2 there is indeterminacy for $\varepsilon > -2\eta$ as well as $\varepsilon < 0$.

is between models 1 and 2. The difference is whether today's interest rate determines money demand today (model 1) or money demand tomorrow (model 2). This difference is what flips the results of these two models on their heads.

Carlstrom and Fuerst (1998) examine a standard real business cycle model in which money is added to the environment with either a CIA constraint or a transactions cost (TC) function. In both cases, the timing used is Model 1's timing. Carlstrom and Fuerst demonstrate that indeterminacy arises if and only if $\gamma > ½$.[8] For the TC model the result that indeterminacy arises for positive γ's is to be expected from the results of Section 3.

However, in the case of the CIA model, this result is unexpected, or at least not predicted by the results of Section 3. In a CIA endowment economy with Model 1 timing one would expect that we would have determinacy for all values of γ. In such a model, bond pricing is given by the standard Fisherian decomposition (3) because the implicit Leontief transactions technology implies that $U_m = 0$ in equilibrium. That is, if $m < c$, it is impossible to carry out transactions; while if $m = c$, any additional cash has no transactions value. Thus, the real rate is constant and the real money supply curve is perfectly elastic ($\varepsilon = \infty$).

Carlstrom and Fuerst (1998) find a different result because they analyze a CIA economy *with production.* Although a CIA economy eliminates the velocity fluctuations that are the central story of this paper (since the Leontief technology implies $m = c$), once we add production to the model the basic indeterminacy logic reoccurs via the added

---

[8] This ½ result is exactly true for linear leisure. For more general preferences the numerical differences are trivial.

margin of production choice. This section explores how the indeterminacy results are affected by adding endogenous production to the model.

Assume that preferences are separable and linear in labor (L) and given by,

$$U(c, m, 1-L) \equiv V(c, m) - AL,$$

Production takes the standard Cobb-Douglas form:

$$y = K^{\alpha} L^{1-\alpha}.$$

The additional Euler equations for labor choice (10) and capital accumulation (11) are familiar:

$$\frac{U_L(t)}{U_c(t)} = f_L(t) \qquad (10)$$

$$U_c(t) = \beta U_c(t+1)[f_K(t+1) + (1-\delta)]. \qquad (11)$$

$$c_t = K_t^{\alpha} L_t^{1-\alpha} + (1-\delta)K_t - K_{t+1}. \qquad (12)$$

These Euler equations are common across all three models because consumption and output enter symmetrically in all three models. Real money balances indirectly enter both of these marginal conditions via the cross partials of the utility function. As a result the behavior of the nominal interest rate (and hence real balances) typically distorts the economy's behavior relative to an otherwise standard real business cycle (RBC) model.

As an example of this distortion, recall that Model 1 timing (equation (2)) implies that

$$U_c(t) = \left[\frac{U_c(t) + U_m(t)}{R_t}\right]. \qquad (13)$$

Note that $[U_c(t)+U_m(t)]$ is the marginal utility of an extra unit of real cash balances at the beginning of time-t. Substituting (13) into (10)-(11), the nominal interest rate in the denominator of (13) can be interpreted as a tax on real balances, so that we have an RBC economy with a distortionary tax. If in addition we assume a rigid CIA constraint so that U_m drops out of the system, this tax can be more directly interpreted as a consumption tax. As noted by Carlstrom and Fuerst (1998), a policy in which the central bank moves the nominal rate (consumption tax) too sharply with the real rate of interest (a large γ) is likely to produce real indeterminacy.

In sharp contrast to the previous example, if utility is separable between consumption and real balances then these monetary distortions have no effect on the RBC economy. Real indeterminacy will arise only in the behavior of real cash balances, an indeterminacy that does not spill over into the rest of the model because of the assumption of separability. This implies that the indeterminacy results from the previous section holds: with Model 1, we have determinacy for $\gamma < ½$; for Model 2, we never have determinacy; and for Model 3, we always have determinacy.

Surprisingly, if $U_{cm} \neq 0$, but leisure is both separable and linear, the determinacy results replicate those of an endowment economy with $U_{cm} = 0$.

Proposition 4: Assume that preferences are separable and linear in leisure, U(c,m,1-L) = V(c,m) – AL, and that the production technology is Cobb-Douglas. Then with Model 1 timing a necessary and sufficient condition for determinacy is $\gamma < ½$; with Model 2 timing, the equilibrium is indeterminate for all values of γ; and with Model 3 timing, the equilibrium is determinate for all values of γ.

Proof: see the appendix.

It is important to note that these conditions for determinacy are identical to an endowment economy with $U_{cm} = 0$. The additional restrictions on U_c implied by endogenous labor choice and capital accumulation cause the model to behave *as if* $U_{cm} = 0$. For example, consider a model without capital and with constant returns to labor (α = 0). In this case linear leisure and (10) implies that U_c is constant!

With capital and CRS Cobb-Douglas technology this basic logic carries through. The reason why the conditions for determinacy are identical to an endowment economy with $U_{cm} = 0$ can be seen if (using the above assumptions) we rewrite 10 and then substitute (10) into (11).

$$U_c(t) = \frac{x_t^{\alpha}}{(1-\alpha)}, \; where \; x_t = \frac{L_t}{K_t} \tag{10'}$$

$$x_t^{\alpha} = \alpha\beta x_{t+1} + \beta(1-\delta)x_{t+1}^{\alpha} \tag{11'}$$

The equilibrium marginal utility of consumption is not directly affected by real money because it is entirely determined by the capital-labor ratio. Although the proof of the proposition exploits the linearity in labor preferences, this assumption is theoretically convenient but computationally irrelevant. For example, if instead of linear leisure there was a constant labor supply elasticity of 0.1 then with plausible calibrations the bounds for determinacy are largely unchanged in all three models. Model 1 is determinate if and only if $\gamma < 0.5001$, Model 2 is determinate if $\gamma > 3740$, while a search for a γ that would produce indeterminacy in Model 3 proved futile. The assumption that leisure is separable in utility also proved to have no quantitative importance.

2. Comparison to Continuous Time Models.

Since at least the seminal work of Sidrauski (1967), many have used continuous time MIUF models. This is unfortunate as the continuous time assumption sweeps under the rug the important timing issues emphasized by this paper, i.e., the time interval between bond and goods market transactions collapses to zero in a continuous time setting.

For example, Benhabib, Grohe, and Uribe (1998) analyze a standard continuous-time money-in-the-utility function (MIUF) endowment economy.[9] The utility function depends on current money balances, while the budget constraint is a differential equation in money balances and bond holdings. The discrete time analog to this assumption is (*roughly*) Model 2 timing.

Benhabib et al. restrict the analysis in two ways. First, they only consider Taylor rules with non-negative coefficients on inflation, $\tau > 0$. Since $\tau = \gamma/(\gamma\text{-}1)$, this implies that they omit discussion of γ's between zero and one. Thus, $\tau > 1$ (an "active" policy) corresponds to $\gamma > 1$, while $0 < \tau < 1$ (a "passive" policy) corresponds to $\gamma < 0$. Second, the continuous time assumption implies that they restrict the equilibria to *continuous* time paths for real balances (along a perfect foresight path). This precludes oscillatory dynamics (complex roots are not possible in the flexible price setting as the system is

---

[9] Benhabib et al also analyze a Calvo-style (1978) money-in-the-production-function (MIPF) economy. As first noted by Feenstra (1986), such a model is isomorphic to a MIUF model with $U_{cm} < 0$. Hence, MIPF results are a direct extension of the MIUF results.

one-dimensional) and corresponds to restricting $\Delta = dm_{t+1}/dm_t$ to be nonnegative.

Benhabib et al. conclude that if $U_{cm} > 0$ determinacy occurs for $\tau > 1$ ($\gamma > 1$); and if $U_{cm} < 0$ determinacy occurs for $0 < \tau < 1$ ($\gamma < 0$). Recalling our earlier discussion this matches up with our results on Model 2 timing. The conditions for real indeterminacy in Model 2 were $\varepsilon < 0$ ($1>\Delta>0$) or $\varepsilon >-2\eta$ ($-1<\Delta<0$). The above discussion suggests that the region $\varepsilon >-2\eta$ ($-1<\Delta<0$) which produced indeterminacy in the discrete time problem will be determinate with continuous time. Therefore with continuous time determinacy will occur if the supply curve ($\varepsilon>0$) slopes up. This occurs for $\tau > 1$ ($\tau < 1$) if $U_{cm} > 0$ ($U_{cm} < 0$). Since Benhabib et al. do not consider negative τ's ($0 < \gamma < 1$) their continuous time model has essentially the same determinacy conditions as Model 2 with one important exception. In the case of $U_{cm} = 0$, they find that the system is always determinate (for <u>all</u> γ's), while in the deterministic framework Model 2 timing implies that there is always indeterminacy for $U_{cm} = 0$.

Continuous time "solves" the problem posed in this paper: what timing assumption should monetary modelers adopt? It solves the problem in a very artificial way by ignoring the important timing issue. Since these issues arise with any discrete but arbitrarily small time period this solution indeed appears artificial.

3. Conclusion.

Hippocrates advised the doctor to do no harm. This minimal advice is equally important to the central banker. In particular, a necessary condition for a good monetary policy is that the policy not introduce sunspot fluctuations into the real economy. This paper has demonstrated that the class of policies that are “good” in this regard depends on basic assumptions about the modeling environment. Hence, a central conclusion of this analysis is that we need to think much more carefully about basic modeling assumptions when writing down monetary models. A lot depends on apparently trivial assumptions.

One example will illustrate this point. King and Wolman (1996) analyze a sticky-price monetary model and conclude by advocating a price level target (this corresponds to a Taylor elasticity of $\tau = \infty$, or $\gamma = 1$). By pegging the price level the sticky price model is isomorphic to a flexible price model. This is the advantage of price-level targeting. King and Wolman do not encounter an indeterminacy problem under such a policy because they use Model 3 timing. If instead they had used either Model 1 or Model 2 timing, the price level peg would produce real indeterminacy. This illustrates the potential dangers of providing policy advice based on existing monetary models. Before such advice can be safely given deeper structural models of money must be investigated to see which, if any, of the three models explored above is most useful in giving monetary policy advice.

References

Benhabib, J., S. Schmitt-Grohe, M. Uribe, “Monetary Policy and Multiple Equilibria,” working paper, June 1998.

Calvo, Guillermo, “On Models of Money and Perfect Foresight,” *International Economic Review* 29 (1979): 83-103.

Carlstrom, C., and T. Fuerst, “Real Indeterminacy in Monetary Models with Nominal Interest Rate Distortions: The Problem with Inflation Targets" 1998, working paper.

Feenstra, Robert C., "Functional Equivalence between Liquidity Costs and the Utility of Money," *Journal of Monetary Economics* 17: 271-292 (1986).

Farmer, Roger E.A., *The Macroeconomics of Self-Fulfilling Prophecies* (MIT Press: 1993).

King, Robert G., and Alexander L. Wolman, “Inflation Targeting in a St. Louis Model of the 21st Century,” *Federal Reserve Bank of St. Louis Review* 78(3), 83-107, May/June 1996.

Lucas, Robert E., Jr., “On the Welfare Cost of Inflation,” 1998 University of Chicago Working Paper.

Lucas, Robert E., Jr., “Interest Rates and Currency Prices in a Two-Country World,” *Journal of Monetary Economics* (10), 1982: 335-359.

Lucas, Robert E., Jr., and Nancy L. Stokey, “Money and Interest in a Cash-in-Advance Economy,” *Econometrica* (55(3)) 1987: 491-513.

Taylor, John B. (1993), “Discretion versus Policy Rules in Practice,” *Carnegie-Rochester Series on Public Policy* 39, 195-214.

Appendix

Proposition 4: Assume that preferences are separable and linear in leisure, U(c,m,1-L) = V(c,m) – AL, and that the production technology is Cobb-Douglas. Then with Model 1 timing a necessary and sufficient condition for determinacy is $\gamma < ½$; with Model 2 timing, the equilibrium is indeterminate for all values of γ; and with Model 3 timing, the equilibrium is determinate for all values of γ.

Proof:

The proof for all three models proceeds by substituting equation (10) into (11)

$$x_t^{\alpha} = \alpha\beta x_{t+1} + \beta(1-\delta)x_{t+1}^{\alpha} \qquad \text{(A1)}$$

where x = L/K. Defining $z_t = U_c(t) + U_m(t)$, and using the fact that U_c depends only on x, the budget constraint can be written as

$$K_{t+1} = K_t x_t^{1-\alpha} + (1-\delta)K_t - c(x_t, z_t) \qquad \text{(A2)}$$

Model 1:

Substitute equation (13) into (10)

$$\frac{z_t}{R_t} = \frac{x_t^{\alpha}}{(1-\alpha)},\ where\ x_t = \frac{L_t}{K_t} \qquad \text{(A3)}$$

Substituting the monetary policy rule (8) into A3 yields

$$R_{ss} z_t^{(1-\gamma)} z_{t+1}^{\gamma} = \frac{x_t^{\alpha}}{1-\alpha} \qquad \text{(A3')}$$

The function for c used labor's f.o.c. (10). Equations (A1), (A2) and (A3') can be written as

$x_{t+1} = F(x_t)$

$K_{t+1} = G(x_t,z_t,K_t)$

$z_{t+1} = H(x_t,z_t)$

The characteristic matrix is

$$\begin{bmatrix} F_x - e_1 & 0 & 0 \\ G_x & G_k - e_2 & G_z \\ H_x & 0 & H_z - e_3 \end{bmatrix}$$

The three eigenvalues are

$$e_1 = F_x = \frac{\alpha}{1-\beta(1-\alpha)(1-\delta)} < 1,\ e_2 = G_K = \frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta} > 1,\ e_3 = H_z = \frac{\gamma - 1}{\gamma}.$$

Since there is only one predetermined variable, for the economy to be determinate two eigenvalues need to lie outside the unit circle. e_3 is within the unit circle for $\varepsilon \geq 1/2$.

Model 2

The proof mirrors that for model 1. Substituting (6) into the f.o.c. for labor yields

$$\frac{z_{t+1}}{R_t} = \frac{x_{t+1}^{\alpha}}{(1-\alpha)} \qquad \text{(A3)}$$

Substituting the monetary policy rule (9) and the definition of F into A1 gives

$$z_{t+1} = R_{ss}\left(\frac{x_t^{\alpha}}{(1-\alpha)}\right)^{\gamma}\left(\frac{F(x_t)^{\alpha}}{(1-\alpha)}\right)^{1-\gamma} \quad \text{(A3')}$$

Following above three eigenvalues are

$$e_1 = F_x = \frac{\alpha}{1-\beta(1-\alpha)(1-\delta)} < 1,\ e_2 = G_K = \frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta} > 1,\ e_3 = H_z = 0\ .$$

Since there is only one predetermined variable, for the economy to be determinate two eigenvalues need to lie outside the unit circle. Only one does so the system is always indeterminate.

Model 3:

From equation (6) defining $z_t = U_c(t) - U_m(t)$ yields

$$z_t = \frac{U_c}{R_t} \quad \text{(A3)}$$

From the monetary policy rule (9) we have

$$z_t = \frac{\left(\frac{x_t^{\alpha}}{(1-\alpha)}\right)^{1-\gamma}\left(\frac{F(x_t)^{\alpha}}{(1-\alpha)}\right)^{\gamma}}{R_{ss}} \quad \text{(A3')}$$

Substituting this into (A1) and (A2) gives

$x_{t+1} = F(x_t)$

$K_{t+1} = G(x_t, x_{t+1}, K_t)$

The eigenvalues are

$$e_1 = F_x = \frac{\alpha}{1-\beta(1-\alpha)(1-\delta)} < 1, \quad e_2 = G_K = \frac{1-\beta(1-\alpha)(1-\delta)}{\alpha\beta} > 1.$$

Since there is one predetermined variable the system is always determinate. QED