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FEDERAL RESERVE BANK OF CLEVELAND

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Abstract

The simple search-theoretic model of fiat money has three symmetric Nash equilibria: all agents accept money with probability 1; all agents accept money with probability 0; and all agents accept money with probability $y \in (0,1)$. Here we construct a nonsymmetric pure strategy equilibrium, payoff-equivalent to the symmetric mixed strategy equilibrium, where a fraction $N \in (0,1)$ of agents always accept money and $1-N$ never accept money . Counter to what has been conjectured previously, we find $N > y$. We also study evolutionary dynamics, and show that the economy converges to monetary exchange iff the initial proportion of agents accepting money exceeds N.

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1 Introduction

A simple search-theoretic model of monetary exchange based on the "double coincidence of wants" problem with direct barter is presented in Kiyotaki and Wright (1993). The model has exactly three symmetric, steady state, Nash equilibria: a pure strategy monetary equilibrium where every agent accepts fiat money with probability 1; a pure strategy nonmonetary equilibrium where every agent accepts fiat money with probability 0; and a mixed strategy equilibrium where every agent accepts fiat money with probability $y \in (0, 1)$, where y is a parameter representing the ease of direct barter. In the mixed strategy equilibrium agents are indifferent between trying to trade using money or direct barter. The following claim is made in Kiyotaki and Wright (1993), based on a similar claim in Kiyotaki and Wright (1991): the symmetric mixed strategy equilibrium is equivalent to a nonsymmetric pure strategy equilibrium in which a fraction y of the agents accept money with probability 1 and a fraction $1 - y$ of the agents accept money with probability 0 (i.e., it is claimed that there exists a nonsymmetric pure strategy equilibrium in which the fraction of agents who accept money is the same as the probability that each agent accepts it in the symmetric mixed strategy equilibrium).

It is shown here that this claim is wrong. It is indeed true that there exists a nonsymmetric pure strategy equilibrium that is payoff-equivalent to the symmetric mixed strategy equilibrium in which a fraction N of the agents accept fiat money with probability 1 and a fraction $1 - N$ accept fiat money with probability 0; but it turns out that $N > y$. That is, the fraction of the population who accept money in the pure strategy equilibrium exceeds the probability that any individual accepts money in the symmetric mixed strategy equilibrium. The intuition is simple. What is relevant to your decision to accept money is the probability that the next individual you meet accepts money given that he has goods for sale. In the nonsymmetric equilibrium, agents who accept money are less likely to have goods for sale than agents who do not accept money, and so the fraction of agents who accept money has to exceed y in order to make you indifferent between trading with money and direct barter. In a more general context, this example shows that purifying mixed strategy equilibria is not as straightforward in dynamic games as it is in static games.

Given that we explicitly derive the payoffs to following nonsymmetric strategies, we are then able to easily undertake an evolutionary dynamic analysis of monetary exchange. That is, we can arbitrarily assign the strategies of either using money or not using money to different subsets of the population, and then observe how the economy evolves under the assumption that types (i.e., strategies) with higher payoffs reproduce more quickly. The stability properties of the different equilibria are fairly intuitive; for example, it is more likely that monetary exchange will evolve if direct barter is difficult or the stock of money is not too big. What is interesting is that the economy needs to start off with a relatively high proportion of agents accepting fiat money – to be precise, the initial proportion has to be more than N, and not just more than y – in order for monetary exchange to survive in the long run. The reason is similar to the intuition given above for why the nonsymmetric pure strategy equilibrium had to have $N > y$: when some agents are always accepting money and others are never accepting money, the former group is less likely to have goods for sale, which tends to reduce the value of money.

2 The Model

The basic environment is a very simple version of the standard search-theoretic model of monetary exchange, and so the description will be relatively brief (for more details, see Kiyotaki and Wright [1991] and the references therein). There is a $[0, 1]$ continuum of agents. There are many varieties of indivisible consumption goods. There is an exogenous quantity $M \in (0, 1)$ of indivisible units of fiat money. Goods and money are costlessly storable, but each agent can hold no more than one indivisible unit of money or one consumption good (and not both) at a time. Initially, the money is randomly endowed across M agents, and the remaining $\gamma = 1 - M$ agents are endowed with goods. Agents cannot dispose of goods or money except by trade. We assume that agents do not consume the goods with which they are endowed, and therefore there is a desire to trade.

Trade occurs in a bilateral, anonymous, random matching process, where agents meet according to a Poisson process with parameter β. Suppose you have a good and meet someone who also has a good (as opposed to money). Then it is assumed that you want to consume what he has with probability $x \in (0, 1]$, and, conditional on this, he also wants to consume what you have with probability $y \in (0, 1)$. The probability of a double coincidence of wants is therefore xy.[1] An individual accepts a good when it is offered in trade if and only if it is a good that he wants to consume. When he gets such a good he immediately consumes it, which generates instantaneous utility u, and is then endowed with a new good. The interesting decision in the model is whether to accept fiat money when it is offered in trade. In this paper we only consider steady state equilibria, where this decision is constant over time (although other equilibria are possible).

We begin by looking for symmetric equilibria, where all agents follow the

[1]Many papers in this literature consider the special case where $y = x$, so that the probability of a double coincidence is x^2, or the case where $y = 0$, so that there is no direct barter.

same strategy. Let Π denote the probability that every other agent accepts money, let π denote the best response of a given individual, and let V_m and V_g denote the payoff or value functions for agents with money and goods, respectively. Then the usual Bellman's equations are

$$\begin{aligned} rV_m &= \beta(1-M)x\Pi(u+V_g-V_m) \\ rV_g &= \beta(1-M)xyu+\beta M \max_{\pi} \pi(V_m-V_g), \end{aligned}$$

where r is the rate of time preference. The first equation sets the flow return for an agent currently holding money, rV_m, equal to the rate at which he meets other agents, β, times the probability that they have a good, $1-M$, times the probability that it is a good the agent with money wants, x, times the probability that the agent with the good accepts money, Π, times the gain from trade, $u+V_g-V_m$. Similarly, the second equation sets the flow return for an agent with a good equal to the expected gain from a barter trade plus the expected gain from trading for money with probability π.

Let $\Delta = V_m - V_g$ be the gain from trading for money. Then a symmetric Nash equilibrium occurs at a value of $\pi = \Pi$ such that either: (a) $\Pi = 1$ and $\Delta \geq 0$; (b) $\Pi = 0$ and $\Delta \leq 0$; or (c) $0 < \Pi < 1$ and $\Delta = 0$. It is a simple matter to verify that, for all parameter values, there are exactly three such equilibria: a pure strategy equilibrium where $\Pi = 0$; a pure strategy equilibrium where $\Pi = 1$; and a mixed strategy equilibrium where $\Pi = y$. In the mixed strategy equilibrium, every agent accepts money and goods with the same probability, y, and so you are indifferent between trying to trade using barter or money, and therefore any $\pi \in [0,1]$ is a best response, including $\pi = y$.

We now want to construct an equilibrium in pure, but not necessarily symmetric, strategies. Let us divide the set of agents into a subset with measure N who accept money with probability 1, and a subset with measure $1-N$ who accept money with probability 0. Let γ_1 be the fraction of those in the former subset who currently have goods rather than money in their possession, and let γ_0 be the fraction of those in the latter subset who currently have goods in their possession (note that γ_0 may be less than 1 because even if an agent does not accept money he still may have it in his possession if he was endowed with it). Thus, the probability of a random agent being someone with goods who accepts money is $N\gamma_1$, while the probability of him being someone with goods who does not accept money is $(1 - N)\gamma_0$. Given that we assume agents cannot dispose of objects except by trade, in steady state we have:

$$\gamma_0 = \begin{cases} 1 & \text{for } M \leq N \\ \frac{1-M}{1-N} & \text{for } M > N \end{cases}$$

$$\gamma_1 = \begin{cases} \frac{N-M}{N} & \text{for } M \leq N \\ 0 & \text{for } M > N \end{cases}$$

Notice that when $M > N$ there are not enough agents who accept money in trade to accommodate the entire money supply, and some agents who do not accept money but were endowed with money end up stuck with it.[2]

Let V_{1m} and V_{1g} be the value functions for the type who accept money when they currently have money in their possession and when they currently have goods in their possession, respectively. Similarly, let V_{0m} and V_{0g} be the analogous value functions for the type who do not accept money. Then the Bellman equations are:

$$\begin{aligned} rV_{1m} &= \beta x N\gamma_1(u + V_{1g} - V_{1m}) \\ rV_{1g} &= \beta x[N\gamma_1 + (1-N)\gamma_0]yu + \beta x[1 - N\gamma_1 - (1-N)\gamma_0](V_{1m} - V_{1g}) \\ rV_{0m} &= \beta x N\gamma_1(u + V_{0g} - V_{0m}) \\ rV_{0g} &= \beta x[N\gamma_1 + (1-N)\gamma_0]yu. \end{aligned}$$

[2] If we alternatively assume that agents who do not accept money in trade dispose of it when they are endowed with it, then we must have $M \leq N$. It simplifies the presentation here to rule out disposing of objects (except by trade), but the basic points do not depend on this assumption.

For example, the first equation sets the flow return to an agent who accepts money and is currently holding money, rV_{1m}, equal to the rate at which he meets other agents who accept money and have a good that he wants, $\beta N\gamma_1 x$, times his gain from trade, $u + V_{1g} - V_{1m}$.

One can insert the steady state expressions for γ_0 and γ_1, and then solve the Bellman equations, to write payoffs as functions of parameters and N. In order to reduce notation, we normalize time (with no loss in generality) so that the arrival rate is $\beta = 1/x$. Then the solution is:

$$
\begin{aligned}
V_{1m} &= \begin{cases} \frac{[r+M+(1-M)y](N-M)u}{r(r+N)} & \text{for } M \leq N \\ 0 & \text{for } M > N \end{cases} \\
V_{1g} &= \begin{cases} \frac{[M(N-M)+(r+N-M)(1-M)y]u}{r(r+N)} & \text{for } M \leq N \\ \frac{(1-M)yu}{r+M} & \text{for } M > N \end{cases} \\
V_{0m} &= \begin{cases} \frac{(N-M)[r+(1-M)y]u}{r(r+N-M)} & \text{for } M \leq N \\ 0 & \text{for } M > N \end{cases} \\
V_{0g} &= \frac{(1-M)yu}{r} \quad \text{for all } M, N.
\end{aligned}
$$

For future reference, we also define the (average) welfare of the two types, $W_j = \gamma_j V_{jg} + (1-\gamma_j)V_{jm}$, $j = 0, 1$. Insertion of the value functions and simplification yields:

$$
\begin{aligned}
W_1 &= \begin{cases} \frac{(N-M)[M+(1-M)y]u}{rN} & \text{for } M \leq N \\ 0 & \text{for } M > N \end{cases} \\
W_0 &= \begin{cases} \frac{(1-M)yu}{r} & \text{for } M \leq N \\ \frac{(1-M)^2 yu}{(1-N)r} & \text{for } M > N \end{cases}
\end{aligned}
$$

Clearly there is an equilibrium where $N = 1$ and $\Delta \geq 0$ and also an equilibrium where $N = 0$ and $\Delta \leq 0$, since these correspond exactly to the two pure strategy equilibria found earlier. What we are interested in now is an equilibrium with $N \in (0, 1)$, which requires $\Delta = 0$ (so that agents are indifferent between accepting and rejecting money). In fact, we need

$N \in (M, 1)$, since otherwise $\Delta = 0$ is not possible. Solving $\Delta = 0$ for N yields $N^* = y + (1-y)M$. Notice that $N^* > M$, so there are necessarily more agents accepting money than there are units of money. Moreover, notice that $N^* > y$ for all $y < 1$. This last result contradicts the claim in Kiyotaki and Wright (1993) that there is a nonsymmetric pure strategy equilibrium where the fraction of the population that accepts money is $N = y$.

The genesis of the incorrect claim was the notion that what you care about is the probability that the next agent you meet in the random matching process will accept money, and not whether that probability comes from all agents randomizing or different agents using different strategies. But what this fails to take into account is that you only care about the probability that the next agent you meet accepts money if he is currently holding a good (and not if he is holding money). If some agents always and other agents never accept money, then the probability that a trader accepts money conditional on him having goods is different from the unconditional probability (since, e.g., an agent who accepts money is less likely to be holding goods than one who does not accept money). Hence, we need to set $N > y$ in order to yield the same probability of meeting an agent with goods who accepts money in the nonsymmetric pure strategy equilibrium as in the nonsymmetric mixed strategy equilibrium. This probability is in both cases $(1 - M)y$, and the fact that it is the same in the two equilibria makes them payoff-equivalent.[3]

[3] In the symmetric mixed strategy equilibrium, the a random agent has goods and accept money with probability $(1 - M)\Pi = (1 - M)y$, while in the nonsymmetric pure strategy equilibrium, it is $N\gamma_1 = (1 - M)y$. In both cases, $V_{1m} = V_{1g} = V_{0m} = V_{0g} = (1 - M)yu/r$.

3 Extensions

In this section, we do two things. First we add a cost of production $c > 0$, in terms of disutility suffered by a trader whenever he delivers a good to another agent, as well as a storage cost to holding money k, which can alternatively be interpreted as a dividend from holding money if $k < 0$. Then we analyze the evolution of monetary exchange.

Given the costs c and k, and given that all agents us the same strategy Π, Bellman's equations are

$$\begin{aligned} rV_m &= \beta(1-M)x\Pi(u+V_g-V_m)-k \\ rV_g &= \beta(1-M)xy(u-c)+\beta M\max_{\pi}\pi(V_m-V_g-c), \end{aligned}$$

using the normalization $\beta x = 1$. An equilibrium is defined as before, except that now the net gain from trading a good for money is $\Delta = V_m - V_g - c$. It is easy to verify that there is an equilibrium with $\Pi = 1$ iff $y \leq \hat{y}$; there is an equilibrium with $\Pi = 0$ iff $y \geq \hat{y} + 1$; and there is a mixed strategy equilibrium with $\Pi = \Pi^* \in (0, 1)$ iff $y \in (\hat{y} - 1, \hat{y})$, where $\hat{y} = 1 - \frac{rc+k}{(1-M)(u-c)}$ and

$$\Pi^* = y + \frac{rc+k}{(1-M)(u-c)}.$$

To recover the results in the previous section, note that $c = k = 0$ implies $\hat{y} = 1$ and $\Pi^* = y$.

Now alternatively suppose that N agents set $\Pi = 1$ and $1 - N$ set $\Pi = 0$. Bellman's equations for the two types are

$$\begin{aligned} rV_{1m} &= N\gamma_1(u+V_{1g}-V_{1m})-k \\ rV_{1g} &= [N\gamma_1+(1-N)\gamma_0]y(u-c)+[1-N\gamma_1-(1-N)\gamma_0](V_{1m}-V_{1g}-c) \\ rV_{0m} &= N\gamma_1(u+V_{0g}-V_{0m})-k \\ rV_{0g} &= [N\gamma_1+(1-N)\gamma_0]y(u-c). \end{aligned}$$

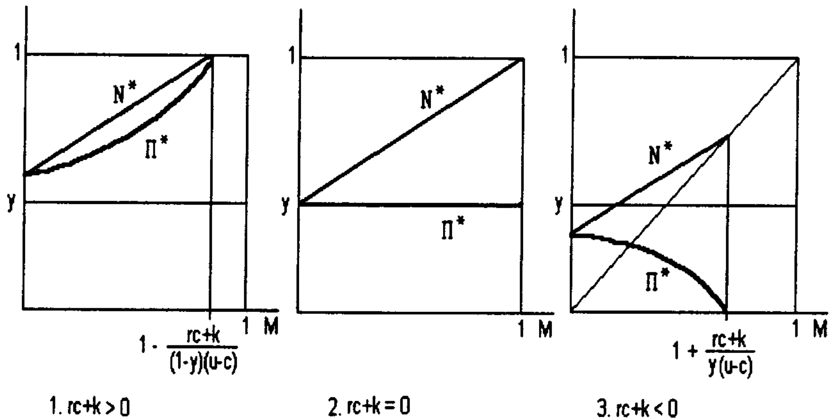

Figure 1: Equilibrium Π and N as Functions of M

Figure 1: Equilibrium Π and N as Functions of M

As above, we want to find an equilibrium where $N \in (M, 1)$ and $\Delta = 0$. Using the steady state conditions for γ_j (which are unaffected by adding production and storage costs), we find from setting $V_{1m} = V_{0g}$ and solving that

$$N^* = y + (1-y)M + \frac{rc+k}{(1-M)(u-c)}.$$

Note that $N^* \in (0,1)$ iff $y \in (\hat{y}-1, \hat{y})$, and that in the relevant range $N^* > \Pi^*$. One can show that the probability of meeting someone with goods who accepts money is the same in the two equilibria, and equal to $(1-M)y + (rc+k)/(1-M)$, and this makes them payoff equivalent. Figure 1 shows the equilibrium values of N^* and Π^* as functions of M in three cases: $rc+k$ positive, zero, and negative. Note that when $rc+k \neq 0$ monetary equilibria exist only for M below some threshold, which is required so that $y \in (\hat{y}-1, \hat{y})$.

We now consider the evolution of monetary exchange.[4] Purely for simplicity, let us return to the case $c = k = 0$ analyzed above, for which we have

[4] Evolution has been also analyzed in (much more complicated) search models of money

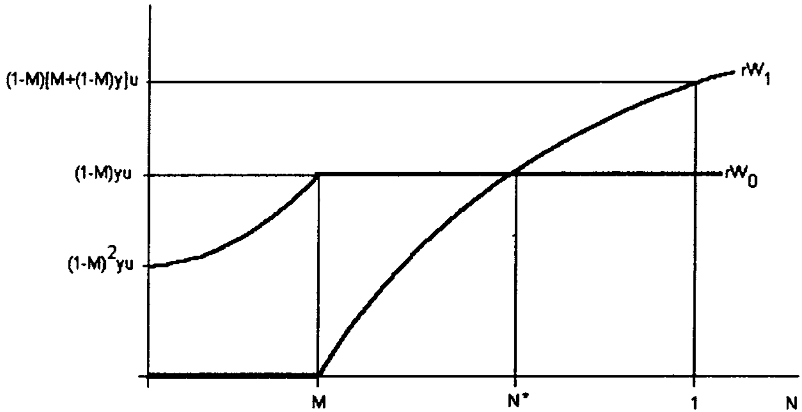

Figure 2: Payoffs as Functions of N

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already computed W_0 and W_1 as functions of N. Suppose that we forget about individual optimization and simply assign the strategies of accepting or rejecting money to different agents. One can then imagine the population evolving over time in some way that may be arbitrary, for our purposes, except that N should be increasing if and only if the steady state payoff to accepting money exceeds the steady state payoff to rejecting it, either because agents with higher payoffs reproduce more efficiently, or because agents imitate successful strategies, or for any other reason. Note that the use of the steady state payoffs (i.e., W_1 and W_0 computed from the steady state values of γ_0 and γ_1) amounts to assuming that evolution takes place on a much slower time scale than the exchange process.

As is standard (see, e.g., Mailath [1992]), an evolutionary stable state can be defined here as a value of N such that either: (a) $N = 1$ and $W_1 \geq W_0$; (b) $N = 0$ and $W_1 \leq W_0$; or (c) $0 < N < 1$ and $W_1 = W_0$. Figure 2 shows

in Sethi (1996) and Wright (1995). Related work includes Marimon, McGrattan and Sargent (1990), who apply genetic algorithms.

rW_0 and rW_1 versus N. From the picture it is clear that there are three evolutionary stable states, $N = 1$, $N = 0$, and $N = N^* = y + (1 - y)M$, which correspond to the three Nash equilibria described above.[5] It is also clear that, under any dynamic such that N increases if and only if $W_1 > W_0$, both $N = 1$ and $N = 0$ are stable while and $N = N^*$ is unstable. The economy converges to $N = 1$ if the initial measure of agents who accept money, N_0, exceeds N^*, and converges to $N = 0$ if N_0 is less than N^*.

In particular, it more likely for random initial conditions that the economy will evolve towards a universally accepted fiat currency if y and M are small (i.e., barter is not too easy and money is not too plentiful). The key point here is that it is not enough to start with $N_0 > y$ agents accepting money, we need to start with $N_0 > N^* > y$, if we want to avoid $N \to 0$. We interpret this as saying that a sizable fraction of agents must start off accepting fiat currency if it monetary exchange is to survive. As discussed above, it is the fact that agents who accept money end up holding goods with a lower probability than those who reject money that tends to reduce the payoff to using money. The more general point is that in games with state variables (such as the agent's inventory of either goods or money in this model), when we assign different strategies to different agents, the state variables can be affected in a way that has implications both for how we purify mixed strategy equilibria and for dynamics.

[5]Note that $N = N^*$ not only implies $W_0 = W_1$, it also implies $V_{1m} = V_{1g} = V_{0m} = V_{0g} = (1 - M)yu/r$. Also notice from the figure that the equilibria can be ranked in terms of W_j: the pure strategy monetary equilibrium ($N = 1$) dominates the mixed strategy monetary equilibrium ($N = N^*$) which dominates the nonmonetary equilibrium ($N = 0$). In terms of payoffs conditioned on current inventories, one can derive similar results: agents with money are strictly better off in the pure monetary equilibrium than in the mixed monetary equilibrium and strictly better off in the mixed monetary equilibrium than in the nonmonetary equilibrium; and agents with goods are strictly better off in the pure monetary equilibrium than in the mixed monetary equilibrium and just as well off in the mixed monetary and nonmonetary equilibria.

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