Appendix to
"Recessions and the Trend in the US Unemployment Rate" 2021. Kurt G. Lunsford. Federal Reserve Bank of Cleveland, *Economic Commentary*, 2021-01

In this appendix, I describe how I compute both unconditional and conditional forecasts. Let u_t denote the unemployment rate and x_t denote the recession indicator. I use a VAR model with these two variables and six lags to compute the forecasts. The VAR model is

$$\begin{bmatrix} x_t \\ u_t \end{bmatrix} = \Gamma_0 + \Gamma_1 \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} + \cdots + \Gamma_6 \begin{bmatrix} x_{t-6} \\ u_{t-6} \end{bmatrix} + \begin{bmatrix} e_{x,t} \\ e_{u,t} \end{bmatrix}, \qquad \text{(A.1)}$$

where $e_{x,t}$ and $e_{u,t}$ are the VAR innovations. In the VAR model, Γ_0 is a 2×1 matrix and Γ_1, …, Γ_6 are each 2×2 matrices. The matrices Γ_0, Γ_1,…, Γ_6 contain the parameters of the VAR model. I assume that the VAR innovations have the following properties

$$E\left(\begin{bmatrix} e_{x,t} \\ e_{u,t} \end{bmatrix}\right) = 0_{2\times1}, \qquad \text{(A.2)}$$

$$E\left(\begin{bmatrix} e_{x,t} \\ e_{u,t} \end{bmatrix} \begin{bmatrix} e_{x,t} & e_{u,t} \end{bmatrix}\right) = \begin{bmatrix} \sigma_{e,x}^2 & \sigma_{e,ux} \\ \sigma_{e,ux} & \sigma_{e,u}^2 \end{bmatrix}, \qquad \text{(A.3)}$$

and

$$E\left(\begin{bmatrix} e_{x,t} \\ e_{u,t} \end{bmatrix} \begin{bmatrix} e_{x,t-i} & e_{u,t-i} \end{bmatrix}\right) = 0_{2\times2} \quad for\ i \neq 0, \qquad \text{(A.4)}$$

where $0_{m\times n}$ denotes an $m \times n$ matrix of zeros.

I use $\{u_{-5}, u_{-4}, u_{-3}, u_{-2}, u_{-1}, u_0, u_1 \ldots, u_T\}$ to denote the sample of unemployment rate values so that there are a total of $T + 6$ observations. Similarly, the sample for the recession indicator is $\{x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, x_1 \ldots, x_T\}$. I use these data samples to estimate Γ_0, Γ_1, …, Γ_6 with least squares, and I denote the estimates with $\hat{\Gamma}_0$, $\hat{\Gamma}_1$, …, $\hat{\Gamma}_6$. Then, I estimate the VAR residuals with

$$\begin{bmatrix} \hat{e}_{x,t} \\ \hat{e}_{u,t} \end{bmatrix} = \begin{bmatrix} x_t \\ u_t \end{bmatrix} - \hat{\Gamma}_0 - \hat{\Gamma}_1 \begin{bmatrix} x_{t-1} \\ u_{t-1} \end{bmatrix} - \cdots - \hat{\Gamma}_6 \begin{bmatrix} x_{t-6} \\ u_{t-6} \end{bmatrix}. \qquad \text{(A.5)}$$

I estimate the parameters in (A.3) with

$$\hat{\sigma}_{e,x}^2 = T^{-1} \sum_{t=1}^{T} \hat{e}_{x,t}^2, \quad \hat{\sigma}_{e,u}^2 = T^{-1} \sum_{t=1}^{T} \hat{e}_{u,t}^2, \quad \hat{\sigma}_{e,ux} = T^{-1} \sum_{t=1}^{T} \hat{e}_{u,t} \hat{e}_{x,t}. \qquad \text{(A.6)}$$

For the purposes of forecasting, I define the following matrices

$$A_0 = \begin{bmatrix} \Gamma_0 \\ 0_{2\times1} \\ 0_{2\times1} \\ 0_{2\times1} \\ 0_{2\times1} \\ 0_{2\times1} \end{bmatrix}, \quad A_1 = \begin{bmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 & \Gamma_6 \\ I_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & I_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & I_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & I_{2\times2} & 0_{2\times2} \end{bmatrix}, \quad J = \begin{bmatrix} I_{2\times2} \\ 0_{2\times2} \\ 0_{2\times2} \\ 0_{2\times2} \\ 0_{2\times2} \\ 0_{2\times2} \end{bmatrix}, \qquad \text{(A.7)}$$

where $I_{n\times n}$ is an $n \times n$ identity matrix. In addition, I use the notation $z_t = [x_t \quad u_t]'$. With these matrices and this notation, the VAR model in Equation (A.1) implies

$$\begin{bmatrix} x_{T+h} \\ u_{T+h} \end{bmatrix} = J'\left(\sum_{j=1}^{h} A_1^{j-1}\right)A_0 + J'A_1^h \begin{bmatrix} z_T \\ z_{T-1} \\ z_{T-2} \\ z_{T-3} \\ z_{T-4} \\ z_{T-5} \end{bmatrix} + J'\sum_{j=1}^{h} A_1^{h-j} J \begin{bmatrix} e_{x,T+j} \\ e_{u,T+j} \end{bmatrix}. \qquad \text{(A.8)}$$

Based on Equation (A.8), I can compute the h-step-ahead *unconditional* forecasts with

$$\begin{bmatrix} \hat{x}_{T+h|T} \\ \hat{u}_{T+h|T} \end{bmatrix} = J'\left(\sum_{j=1}^{h} \hat{A}_1^{j-1}\right)\hat{A}_0 + J'\hat{A}_1^h \begin{bmatrix} z_T \\ z_{T-1} \\ z_{T-2} \\ z_{T-3} \\ z_{T-4} \\ z_{T-5} \end{bmatrix}, \qquad \text{(A.9)}$$

where

$$\hat{A}_0 = \begin{bmatrix} \hat{\Gamma}_0 \\ 0_{2\times1} \\ 0_{2\times1} \\ 0_{2\times1} \\ 0_{2\times1} \\ 0_{2\times1} \end{bmatrix}, \quad \hat{A}_1 = \begin{bmatrix} \hat{\Gamma}_1 & \hat{\Gamma}_2 & \hat{\Gamma}_3 & \hat{\Gamma}_4 & \hat{\Gamma}_5 & \hat{\Gamma}_6 \\ I_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & I_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & I_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & I_{2\times2} & 0_{2\times2} \end{bmatrix}. \qquad \text{(A.10)}$$

In equation (A.9), the values of z_T, z_{T-1}, z_{T-2}, z_{T-3}, z_{T-4}, and z_{T-5} are the initial conditions or starting points that I use to compute the forecasts.

Next, I want to compute *conditional* forecasts of u_t, denoted with $\{\hat{u}^c_{T+1|T}, \dots, \hat{u}^c_{T+h|T}\}$. I condition these forecasts on a future path of x_t, denoted with $\{x^c_{T+1}, \dots, x^c_{T+h}\}$. In the *Commentary*, I use $x^c_{T+j} = 0$ for $j = 1, \dots, h$ where $h = 240$. I follow Doan, Litterman, and Sims (1983) to produce these conditional forecasts. I decompose the VAR innovations into a 2-dimensional process $\epsilon_t = [\epsilon_{1,t} \quad \epsilon_{2,t}]'$, in which $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are uncorrelated, with

$$\begin{bmatrix} e_{x,t} \\ e_{u,t} \end{bmatrix} = \begin{bmatrix} \sigma_{e,x} & 0 \\ \dfrac{\sigma_{e,ux}}{\sigma_{e,x}} & \sqrt{\sigma_{e,u}^2 - \dfrac{\sigma_{e,ux}^2}{\sigma_{e,x}^2}} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}. \tag{A.11}$$

Macroeconomic researchers sometimes interpret ϵ_t as structural or economically interpretable shocks. For example, Waggoner and Zha (1999) refer to ϵ_t as a vector of structural shocks. I am not using that interpretation here. Rather, I view $\epsilon_{1,t}$ or $\epsilon_{2,t}$ as alternative reduced-form innovations that will be convenient for conditional forecasting.

To see this interpretation more clearly, note that the first row of (A.11) implies that $\epsilon_{1,t} = \sigma_{e,x}^{-1} e_{x,t}$. Then, this equation and the second row of (A.11) implies $e_{u,t} = \frac{\sigma_{e,ux}}{\sigma_{e,x}^2} e_{x,t} + \sqrt{\sigma_{e,u}^2 - \frac{\sigma_{e,ux}^2}{\sigma_{e,x}^2}} \epsilon_{2,t}$, noting that $e_{x,t}$ and $\epsilon_{2,t}$ are uncorrelated. Then, the decomposition is akin to projecting or regressing $e_{u,t}$ on $e_{x,t}$ and having $\sqrt{\sigma_{e,u}^2 - \frac{\sigma_{e,ux}^2}{\sigma_{e,x}^2}} \epsilon_{2,t}$ be the residual that affects $e_{u,t}$ but not $e_{x,t}$. Hence, I view (A.11) as simply a way to orthogonalize the reduced-form innovations with linear projections – not as a way to uncover structural shocks. To ease notation, I use

$$L = \begin{bmatrix} \sigma_{e,x} & 0 \\ \dfrac{\sigma_{e,ux}}{\sigma_{e,x}} & \sqrt{\sigma_{e,u}^2 - \dfrac{\sigma_{e,ux}^2}{\sigma_{e,x}^2}} \end{bmatrix} \tag{A.12}$$

so that $[e_{x,t} \quad e_{u,t}]' = [\epsilon_{1,t} \quad \epsilon_{2,t}]' L'$.

I produce the conditional forecasts by choosing a collection of future shocks, denoted with $\left[\epsilon_{1,T+1|T}^c \quad \epsilon_{2,T+1|T}^c \quad \cdots \quad \epsilon_{1,T+h|T}^c \quad \epsilon_{2,T+h|T}^c\right]$, such that $x_{T+1} = x_{T+1}^c$, $x_{T+2} = x_{T+2}^c$, …, $x_{T+h} = x_{T+h}^c$. I define $\hat{X} = \left[\hat{x}_{T+1|T}, \dots, \hat{x}_{T+h|T}\right]'$ and $X^c = \left[x_{T+1}^c, \dots, x_{T+h}^c\right]'$ to be $h \times 1$ vectors of unconditional and conditional forecasts. Then using (A.8), (A.9), (A.10), (A.11), and (A.12), I write the constraints, $x_{T+1} = x_{T+1}^c$, $x_{T+2} = x_{T+2}^c$, …, $x_{T+h} = x_{T+h}^c$, as

$$X^c - \hat{X} = M \begin{bmatrix} \epsilon_{1,T+1|T}^c \\ \epsilon_{2,T+1|T}^c \\ \vdots \\ \epsilon_{1,T+h|T}^c \\ \epsilon_{2,T+h|T}^c \end{bmatrix}, \tag{A.13}$$

in which M is an $h \times 2h$ matrix given by

$$M = \begin{bmatrix} [1 \quad 0_{1\times 11}]JL & 0_{1\times 2} & \cdots & 0_{1\times 2} \\ [1 \quad 0_{1\times 11}]A_1 JL & [1 \quad 0_{1\times 11}]JL & \cdots & 0_{1\times 2} \\ \vdots & \vdots & \ddots & 0_{1\times 2} \\ [1 \quad 0_{1\times 11}]A_1^{h-1} JL & [1 \quad 0_{1\times 11}]A_1^{h-2} JL & \cdots & [1 \quad 0_{1\times 11}]JL \end{bmatrix}. \tag{A.14}$$

Then, I estimate the values of $\left[\epsilon_{1,T+1|T}^{c} \quad \epsilon_{2,T+1|T}^{c} \quad \cdots \quad \epsilon_{1,T+h|T}^{c} \quad \epsilon_{2,T+h|T}^{c}\right]$ by minimizing

$$\left(\epsilon_{1,T+1|T}^{c}\right)^{2} + \left(\epsilon_{2,T+1|T}^{c}\right)^{2} + \cdots + \left(\epsilon_{1,T+h|T}^{c}\right)^{2} + \left(\epsilon_{2,T+h|T}^{c}\right)^{2} \tag{A.15}$$

subject to (A.13). This minimization yields

$$\begin{bmatrix} \hat{\epsilon}_{1,T+1|T}^{c} \\ \hat{\epsilon}_{2,T+1|T}^{c} \\ \vdots \\ \hat{\epsilon}_{1,T+h|T}^{c} \\ \hat{\epsilon}_{2,T+h|T}^{c} \end{bmatrix} = \widehat{M}'\left(\widehat{M}\widehat{M}'\right)^{-1}\left(X^{c} - \hat{X}\right), \tag{A.16}$$

in which $\widehat{M}$ takes the same form and M, but with $\hat{A}_1$ in place of A_1 and

$$\hat{L} = \begin{bmatrix} \hat{\sigma}_{e,x} & 0 \\ \dfrac{\hat{\sigma}_{e,ux}}{\hat{\sigma}_{e,x}} & \sqrt{\hat{\sigma}_{e,u}^{2} - \dfrac{\hat{\sigma}_{e,ux}^{2}}{\hat{\sigma}_{e,x}^{2}}} \end{bmatrix}$$

in place of L. I then feed $\left[\hat{\epsilon}_{1,T+1|T}^{c} \quad \hat{\epsilon}_{2,T+1|T}^{c} \quad \cdots \quad \hat{\epsilon}_{1,T+h|T}^{c} \quad \hat{\epsilon}_{2,T+h|T}^{c}\right]$ into (A.8) using the estimates in (A.10) to compute the conditional unemployment rate forecasts, $\left\{\hat{u}_{T+1|T}^{c}, \dots, \hat{u}_{T+h|T}^{c}\right\}$.

References

Doan, Thomas, Robert Litterman, and Christopher A. Sims. 1983. "Forecasting and Conditional Projection Using Realistic Prior Distributions." NBER Working Paper, No. 1202. DOI: 10.3386/w1202.

Waggoner, Daniel F., and Tao Zha. 1999. "Conditional Forecasts in Dynamic Multivariate Models." *Review of Economics and Statistics*, 81(4): 639-651. DOI: 10.1162/003465399558508.