Appendix to "Lingering Residual Seasonality in GDP Growth" by Kurt G. Lunsford

This appendix accompanies the Federal Reserve Bank of Cleveland *Economic Commentary* entitled "Lingering Residual Seasonality in GDP Growth," by Kurt G. Lunsford. This appendix provides the details for testing for residual seasonality in GDP growth.

The *Commentary* assumes that GDP growth follows

$$y_t = c_t + s_t + i_t,$$

where c_t denotes the business cycle component of the data, s_t denotes the seasonal component of the data, and i_t denotes the irregular component of the data. In the terminology of the Census Bureau's X-13 seasonal adjustment filter, the business cycle component may also be referred to as the "trend" component. As noted in the *Commentary*, the testing procedure follows four steps:

1. Estimate the business cycle component of GDP growth and subtract it from GDP growth from 1985 to 2015.

2. Collect the difference of GDP growth and its cyclical component by quarter of the year.

3. Estimate the average of the quarter-by-quarter difference between GDP growth and the cycle.

4. Use Müller and Watson's (2008, 2015) low-frequency econometrics to test if the averages from step 3 are statistically distinct from zero over the 1985 to 2015 sample.[1]

Each step is now covered in more detail

The first step estimates and removes the business cycle component of GDP growth, using the following linear regression

$$y_t = \alpha_o + \alpha_1 x_{1,t} + \cdots + \alpha_J x_{J,t} + e_t,$$

where $x_{1,t}, \ldots, x_{J,t}$ is a sequence of cosine waves. These cosine waves are given by $x_{j,t} = \sqrt{2}\cos(\pi j r_t)$ where $r_t = (t - 1/2)/T$, and T denotes the sample size of 124. This equation is estimated with least squares. The estimated parameters are denoted by $\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_J$, and the estimated business cycle is $\hat{c}_t = \hat{\alpha}_0 + \hat{\alpha}_1 x_{1,t} + \cdots + \hat{\alpha}_J x_{J,t}$.

The following figure shows the first four cosine waves used in this regression. The first wave completes half of a cycle from 1985 to 2015, giving it a period of 62 years; the second wave completes a whole cycle from 1985 to 2015, giving it a period of 31 years; and so on. Following this pattern, each cosine wave has a shorter period than the one before it. 31 cosine waves are used in the regression so that the wave with the shortest period has a period of 2 years. Thus, the linear regression and the corresponding estimate of the business cycle account for patterns in GDP growth that have periods of 2 years or longer. Using more cosine waves with periods down to 1.5 years does not change the primary results of the *Commentary*. Similarly, using fewer cosine waves with periods down to 3 years does not change the primary results of the *Commentary*.

[1] The methodology used in this *Commentary* is based on the assumption that the quarter-by-quarter differences between GDP growth and the business cycle are stationary. This assumption is supported by Müller and Watson's (2008, 2015) low-frequency stationarity (LFST) and low-frequency unit root (LFUR) tests, which are not reported here for brevity.

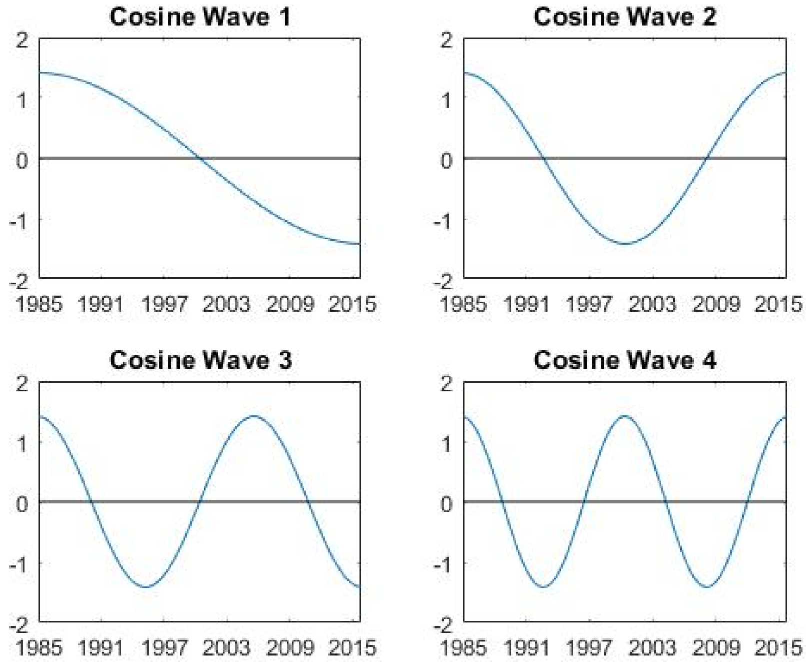

Figure 1: The first four cosine waves used in the linear regression.

I note here that the 31 cosine waves used in this step greatly exceed the number of cosine waves used in Müller and Watson (2008, 2015). However, the estimated coefficients on these cosine waves will not be used for inference as in Müller and Watson (2008, 2015). Inference will not be performed until step 4. Rather, the regression on the cosines in this step is simply meant to capture the patterns in GDP growth that have periods of 2 years or longer. In more technical terms, the estimated business cycle acts as a band-pass filter on GDP growth that includes frequencies of 2 years and longer. Other band pass filters may also be used in this step. However, I use the above regression because it a particularly simple way to capture the relevant frequencies. A brief comparison of the cosine regression to band-pass filters is given in Müller and Watson (2015).

The second step in the process for testing for residual seasonality is to collect the difference between GDP growth and its estimated business cycle (the data in the bottom panel of Figure 1 of the *Commentary*) by quarter of the year. Define $\tilde{y}_{q,n}$ to be the difference between GDP growth and the estimated cycle in quarter q of year n. Then, the third step takes the average of the quarter-by-quarter differences between GDP growth and its cycle, which is given by

$$\bar{s}_q = \sum_{n=1985}^{2015} \tilde{y}_{q,n},$$

for $q = 1, \ldots, 4$. Here, $\bar{s}_q$ denotes the average seasonal effect for quarter q.

The fourth step produces confidence intervals for $\bar{s}_q$. The methodology for producing these confidence intervals is as follows. First, estimate the regression

$$\tilde{y}_{q,n} = \beta_o + \beta_{1,q}\tilde{x}_{1,n} + \cdots + \beta_{K,q}\tilde{x}_{K,n} + \tilde{e}_{q,n}$$

by least squares where $\tilde{x}_{1,t}, \ldots, \tilde{x}_{K,t}$ is a sequence of cosine waves. This sequence is given by $\tilde{x}_{1,n} = \sqrt{2}\cos(\pi k r_n)$ and $r_n = (n - 1/2)/N$ for $n = 1, \ldots, N$ where N is the number of years in the sample.

These cosine waves have the same properties as those used in step 1. Unlike in step 1 however, the regression coefficients in this regression will be used for inference. Because of this, I choose a small number of cosine waves as in Müller and Watson (2008, 2015). Specifically, I choose $K = 6$ so that the shortest cosine wave in the regression has a period of about 10 years.

Next, Müller and Watson (2008, 2015) show that $\sqrt{N}$ times $\hat{\beta}_{1,q}, \ldots, \hat{\beta}_{K,q}$ behave as independent and identically distributed normal random variables with mean zero and variance σ_q^2. This variance is the long-run variance of $\tilde{y}_{q,n}$, which will be used to test the null hypotheses that $\bar{s}_q = 0$ for $q = 1, \ldots, 4$. To estimate the long-run variance, I use

$$\hat{\sigma}_q^2 = \frac{N}{K}\left(\sum_{k=1}^{K} \hat{\beta}_{k,q}^2\right).$$

Then, the standard error of $\bar{s}_q$ is given by

$$se_q = \sqrt{\frac{\hat{\sigma}_q^2}{N}}.$$

Finally, the 10% confidence interval is

$$CI_{10} = \left[\bar{y}_q - 1.94se_q, \bar{y}_q + 1.94se_q\right],$$

and the 5% confidence interval is

$$CI_5 = \left[\bar{y}_q - 2.45se_q, \bar{y}_q + 2.45se_q\right].$$

Note that the scaling on the confidence intervals, 1.94 and 2.45, are different than those for 10% and 5% confidence intervals from a standard normal distribution, which would be 1.64 and 1.96. That is because the confidence intervals in this *Commentary* follow a student's t distribution, and 1.94 and 2.45 are relevant critical values for a t distribution with 6 degrees of freedom.

References

Müller, Ulrich K. and Mark W. Watson. 2008. "Testing Models of Low-Frequency Variability." *Econometrica* 76(5): 979-1016.

Müller, Ulrich K. and Mark W. Watson. 2015. "Low-Frequency Econometrics." NBER Working Paper, No. 21564.