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Arbitrage: The Key to Pricing Options

by Ed Nosal and Tan Wang

Arbitrage is the act of simultaneously buying and selling assets or commodities in an attempt to exploit a profitable opportunity. Although the idea behind arbitrage is fairly simple, it is quite powerful because the ability to exploit such opportunities is needed for markets to operate efficiently. Arbitrage ensures, for example, that buyers and sellers of foreign exchange can be assured that they are getting the "correct" rates for the currencies they are buying and selling independent of the national foreign-exchange markets they happen to be using.

When markets are efficient, the prices of the objects being traded reflect their true value. And having prices reflect true values is important in decentralized economies, such as the United States, since it is the relative prices of various goods, services, and assets that determines how many will be produced, how they will be allocated, and how funds will be invested. If prices did not reflect true value, then the resulting allocation of goods, services, and investment would not be, in general, economically efficient.

This *Commentary* focuses on a particular episode in which the recognition of an arbitrage "opportunity" made financial markets more efficient. It wasn't a chance to make a profit that got noticed, it was the way the principles of arbitrage could be applied to the problem of correctly pricing options. Once financial economists figured it out, the solution enhanced the efficiency of financial markets because it made options useful as hedging instruments. Such instruments can be used to manage cash holdings only if they are correctly priced.

■ In Search of the Option Pricing Formula

Options are phenomenally popular these days, but when they were introduced, they didn't take off at first. No one knew how to price them. People were using methods based on familiar financial instruments, like equities and bonds, and these did not work well for pricing options.

An option is called a derivative security because it derives both its payoff structure and its value from some (other) underlying security. A call option on a stock, for example, gives the holder of the call option the right, but not obligation, to purchase a particular stock at some date in the future. The price at which the holder can purchase the stock is called the strike price and the latest possible date at which the holder can purchase the stock—or exercise the option—is called the expiration date. If the holder is only allowed to exercise on the expiration date, the option is called a European; if the holder can exercise any time between now and the expiration date, the option is called an American.

The ability to exercise—or not—gives options different characteristics from the underlying securities on which they're based, and that fact makes them difficult to price. Consider a European call option on ABC stock with a strike price equal to \$50 and an expiration date in one year. If you own this option, what will be your possible payoffs one year from now? Clearly, if ABC's stock price is below \$50 one year from now, you will not exercise because you would be paying \$50 for something that is worth less than that; in these circumstances your payoff will be zero. If the stock price is above \$50, you will exercise the option, and for every dollar that ABC's stock is above \$50, you will receive that amount.

Arbitrage has become associated in popular attitudes with the most ruthless and profit-driven of human impulses, but the opposite reputation might be more well-deserved. The ability to arbitrage is essential for the efficient operation of markets. An interesting application of the principle of arbitrage arose when it provided the breakthrough insight in economists' solution to a formerly intractable problem: how to properly price the emergent financial instruments known as options.

Determining the payoff structure for an option doesn't seem that difficult. How might one go about determining the value or price of this option? A standard way to think about pricing any financial asset is to first specify the payoffs that the asset is expected to generate in the future; then to appropriately discount these payoffs; and finally add up all of the discounted payoffs. The sum of the discounted payoffs represents the value of the asset, and the price of the asset should be equal to this value.

Future payoffs for financial assets are discounted for at least two reasons. First, a dollar in the future is worth less than a dollar today; so, future dollars must be discounted to make them comparable with dollars today. Second, if one asset's payoff stream is riskier than another's, then, holding all other things equal, the former asset is more valuable than the latter. Hence, riskier payoff streams

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should be discounted more heavily than less risky payoff streams.

It might seem as if calculating the value of the option should be fairly straightforward. We know when the payoffs to the option will be received and what possible values they may take. All we have to do is to discount these expected future payoffs to determine the option's value.

But here's the rub: What discount factor do we use? A discount factor should reflect the underlying risk of the asset. Since the option's value is related to movements in the stock price of the ABC company, it might seem reasonable to discount the option's future expected payoffs with ABC's discount factor. The problem here is although the movement in the option's payoff perfectly tracks the stock price movements when they are above $50, it does not at all track the movement of the stock price when the price is below $50.

This implies that the risk characteristics of the option are quite different from those of the underlying stock. And herein lies the obstacle that prevented generations of researchers from solving the option pricing formula: No appropriate discount factor could be found.

■ Arbitrage Basics

The big breakthrough came when two economists recognized that arbitrage was the secret to unlocking the pricing formula. The first step to grasping their discovery is to understand the implications arbitrage has for the pricing of anything in general.

To see these, consider a simple example. Imagine that you visit a street, Main Street, which is lined with apple vendors. On the south side of the street all the vendors sell apples for 10 cents a piece, and on the north side they sell them for 20 cents a piece. Whatever the reason apples are trading at two different prices, it is possible for you to profit from the apparent "mispricing."

Suppose you go to the south side of the street and ask a vendor if he would be willing to "lend" you an apple, which you would repay with an apple momentarily. You in turn sell the borrowed apple on the opposite side of the street for 20 cents. (If apples were stocks, what we have just described is "shorting" a stock.) With 20 cents in hand you run across the street, purchase an apple for 10 cents, and return this apple to the south-side vendor from whom you borrowed it. When all is said and done, you have earned 10 cents without using any of your own wealth: This corresponds to an infinite rate of return! But why stop at only 10 cents? If you borrowed 2 apples from the vendor, you could have made 20 cents; if you borrowed 100 apples you could have made $10. It appears that you will be able to generate an indefinite amount of money by simply selling apples on the one side of the street and buying them on the other.

Unfortunately (for you), the forces of supply and demand will take hold and limit how much you can make. By attempting to sell large amounts of apples on the north side, you will bid the north-side price down, and by attempting to buy large amounts on the south side, you will bid the south-side price up. In the end, the price of apples on both sides of the street will be the same, and your infinite-rate-of-return investment will disappear.

The apple parable helps explain why, for example, the U.S./Canadian dollar exchange rate in the New York foreign-exchange market will be the same as in the Tokyo foreign-exchange markets. If there was a discrepancy between these rates, financial institutions would have an opportunity to make an infinite return by selling the currency at the "overvalued exchange" and buying currency at the "undervalued exchange." The forces of supply and demand will ultimately equate the two exchange rates.

The key lesson here is that objects that are the same should trade at the same price; if they do not, then an arbitrage opportunity exists. Attempts to cash in on the opportunity will actually eliminate it and, in the end, prices will be the same.

■ Forget the Discount Factor, It's All About Arbitrage

Applying the lesson to options tells us that if there are two options written on the same stock, with the same strike price and with the same expiration date, then the prices of these two options must be the same. But it does not tell us what that price should be. Or does it?

In 1973, economists Fisher Black and Myron Scholes showed how the notion of arbitrage can be used to price an option. Their big insight was that the payoff structure of an option can be replicated by a portfolio of market-traded assets. Since the cash payoffs to the portfolio and the option are identical, it must be the case that the price of the option equals the value of the portfolio; otherwise, an arbitrage opportunity would exist. A simple example might help explain their approach.

Suppose that today's price for ABC stock is $40. In one period from now the price of ABC stock will either rise to $60 or fall to $20. There is a European option on the ABC stock that has an expiration date in one period, with a strike price at $50. The borrowing and lending interest rate is 25 percent per period.

This is all the information we require to price the option. Note that the payoff to the option in one period will be either $10 if the stock price rises or zero if it falls. The Black-Scholes insight was to construct a portfolio of existing assets that can replicate a payoff of $10 when the stock price goes up and zero when it goes down.

The portfolio in this example would consist of $^1/_4$ share of ABC stock and $4, which has been borrowed. The value of this portfolio today is $6—the stock is worth $10, and $4 must be repaid.

The payoff in one period matches that of the option: Your holdings of ABC stock, if the price rises, will be worth $^1/_4$ x $60, or $15. The $4 borrowed will be repaid with $5 since the interest rate is 25 percent, so the value of the entire portfolio will be $10. If the stock price falls, your holdings of ABC will be worth $^1/_4$ x $20 or $5. In this case, the portfolio's value one period from now is zero: Your holdings of ABC stock just offset what you owe. In summary, the portfolio pays $10 when the stock price is high, and zero when it is low.

Since the payoff to your portfolio is identical to the payoff of the option, it stands to reason that the price of the option should be $6, the value of your portfolio today. And it is an arbitrage argument that demonstrates that this must be so.

To see this, suppose that the option was selling at $7. Just as in the apple example, you can make an infinite rate of return by selling the "expensive good" (the option) and buying the "cheap good" (the portfolio). If you sell the option, you will receive $7 but you will be required to pay $10 in one period from now if the stock price turns out to be high. After selling an option, you can

FIGURE 1 STOCK PRICE TREE

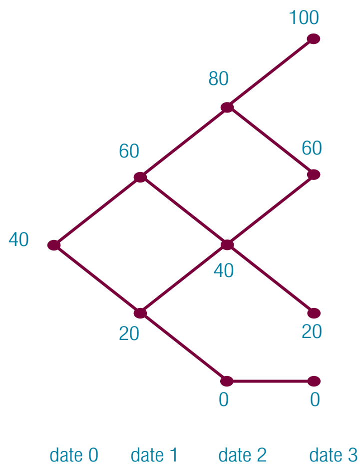
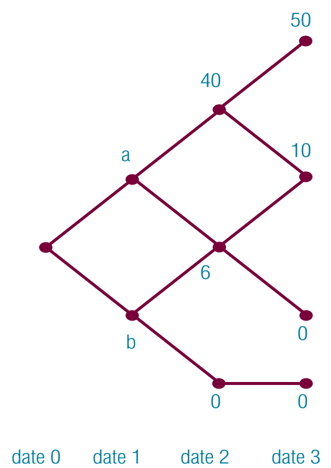


FIGURE 2 OPTION PRICE TREE



purchase the portfolio described above for $6. That leaves you with $1 in your pocket. The portfolio will pay off $10 one period from now when the stock price is high, which is exactly what you need to pay off your option obligation. Hence, you have made $1 for sure without using any of your wealth. But you will continue as long as the profit opportunity exists, and the forces of supply and demand will ultimately equate the value of the portfolio with the price of the option.

So far, all this seems pretty simple, but that is because the example is simple. In reality when someone purchases an option, the expiration date is many days into the future and, as a result, the stock price has the opportunity to move "up and down" many times. So the relevant question is how to price an option when the price of the underlying asset can move around many times before the expiration date. Black and Scholes realized that the arbitrage-portfolio replication argument described above needed to be used repeatedly. To illustrate, let's continue with our simple example.

Suppose that the option on the ABC stock, with strike price of $50, has an expiration date three periods from now. Today, date 0, the price of ABC is $40; at each date the price can either increase or decrease $20. If, however, the stock price ever is zero, it remains at zero forever. Figure 1, which is a "stock price tree," describes the set of possible stock prices for ABC over the next three periods. We will use it to calculate the price of the option today, at date 0.

Suppose that at date 2, the stock price is $40. In this situation, the option will either pay $10 or $0 at date 3. From the example above, we know that the price of the option will be $6, and that a portfolio consisting of $4 borrowed and $^1/_4$ stock of ABC can replicate the payoff of the option. Similarly, if the stock price at date 2 is $80, then the option will pay off either $50 ($100 – $50) or $10 ($60 – $50), depending upon whether the stock goes up or down at date 3. A portfolio consisting of borrowing $40 and buying one unit of ABC stock can replicate the payoffs of the option. This portfolio, and hence the option, has a value equal to $40. Finally, at date 2, if the stock price is zero, the payoff to the option will be zero at date 3, and a portfolio consisting of "nothing," which costs nothing, is able to replicate the option's payoff.

So, we have determined the price of the option at date 2 for all possible outcomes. By using the same kind of reasoning we will be able to determine the price of the option at period 1 when the stock price is $60 and when it is $20 and finally at period 0.

Figure 2, which is an "option price tree," describes the price of the option for the various realizations of the stock price. At date 1, if the stock price is $60, then a holder of the option is holding an asset that will be worth either $40 or $6 at date 2, depending upon whether the price of the stock goes up or down at date 2. A portfolio consisting of borrowing $22.40 and buying .85 of ABC stock can replicate these date-2 payoffs. This portfolio is worth $28.60; hence, the value of the option at date 1 when the stock price is $60, denoted by node a in figure 2, must be equal to $28.60. Similarly, if the stock price is $20 at date 1, a portfolio consisting of .15 of ABC stock will replicate the date-2 payoff of the option. This portfolio costs $3; hence, the value of the option at node b in figure 2 is $3. The price of the option at date 0 can be determined by working backward in the same fashion: It will be given by the value of a portfolio that pays off $28.60 if the stock price increases to $60 at date 1 and $3 if the stock price decreases to $20. A portfolio that consists of borrowing $7.84 and buying .64 of ABC stock will give such payoffs, and it costs $17.76 to buy such a portfolio. Therefore, the value of the option at date 0 is $17.76.

Black and Scholes used the method of repeated, or "dynamic," portfolio replication, along with an arbitrage argument, to determine the price of the option. At each date, the value of the portfolio is precisely equal to what is needed to buy a new portfolio that can replicate the subsequent period's payoffs. In the end, the date-3 payoffs to the option can be replicated. If, in our example, the price of the option is not equal to $17.76, say it is $20, an infinite rate of return can be made; you sell the option and buy the date-0 portfolio. The returns to the portfolio with dynamic replication will precisely pay off your option obligations, you pocket $2.24, and the desire to capture more of this "free money" will imply that the forces of supply and demand will equate the value of the replicating portfolio and the option.

■ Arbitrage and Economic Efficiency

Arbitrageurs help make markets efficient. When, for some reason, prices get out of line with one another, arbitrage will get the prices back in line. As a result, prices will reflect the "true values" of the traded objects. The notion of arbitrage has improved the efficiency of markets in another, and perhaps, unexpected way. The ideas behind arbitrage helped financial market participants price derivative assets, such as options. These derivative assets can be used to manage risk and, when correctly priced, will enhance the efficiency of markets and firms and, as a result, society.

Ed Nosal is a senior economic advisor at the Federal Reserve Bank of Cleveland, and Tan Wang is the Peter Lusztig Associate professor of Finance at the University of British Columbia.

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Federal Reserve Bank of Cleveland
Research Department
P.O. Box 6387
Cleveland, OH 44101