Quantile Combination of Density Forecasts: An Application to US GDP forecasts

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Introduction

Forecasting lower tail events

- Uncertainty and downside risk plays a prominent role for economic forecasts and policy decisions
    - It has become popular for economic forecasters' and policy institutions (e.g. central banks) to provide probabilistic (density) forecasts (BoE, Norges Bank, the Fed)

- In recent years, policymakers' have shifted their focus from forecasts uncertainty in general to being particularly interested in quantifying macroeconomic downside tail risk, often referred to as GDP-at-risk.
    - Adrian et al. (2019) argue that financial conditions are particularly informative above future downside macroeconomic risk.
    - Led to a surge of interest in growth-at-risk (e.g. Coe and Vahey, 2020; Reichlin et al., 2020; Carriero et al., 2020; Clark et al., 2021; Brownlees and Souza, 2021; Amburgey and McCracken, 2022).

- A large literature that find a variety of economic and financial variables contain predictive information about future economic recessions and downturns see e.g. Marcellino (2006) and Liu and Moench (2016) for an overview.

Combination of Density Forecasts

- To produce accurate and useful forecasts, forecasters and policy makers routinely rely on multiple sources to produce forecasts.
  - This has spurred a recent resurgence in interest in combination of density forecasts in macroeconomics and econometrics.
- Combining predictive densities using weighted linear combinations of prediction models, evaluated using various scoring rules
  - Hall and Mitchell (2007), Jore et. al (2010), Geweke and Amisano (2011), Gneiting and Ranjan (2011, 2012) and Aastveit et al. (2014)
- Complex combination approaches that allows for time-varying weights with possibly both learning and model set incompleteness
  - Billio et al. (2013), Casarin et al. (2015), Pettenuzzo and Ravazzolo (2016), Del Negro et al. (2016), Aastveit et al. (2018), McAlinn and West 2019 and McAlinn et al. (2020).

Combination of Density Forecasts

Suppose that a set of $k = 1, \ldots, K$ predictive distributions $f_{t+h,k}$ for the same variable of interest y_t for horizon h are available.

- Standard Combination methods apply a combination weight to the entire predictive distribution, i.e.:

$$\underset{1\times Q}{y_{t+h}} = \underset{1\times K}{w_k} \underset{K\times Q}{f_{t+h,k}}$$

where $q = 1, \ldots, Q$ indicates the quantiles or bins elements of the density distribution.

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However, this approach implicitly overlooks superior forecast accuracy of some $f_{t+h,k}$ over a specific region of the distribution.

Quantile Combination of Density Forecasts

Suppose that a subset of this set is indeed more accurate in predicting the mean (tails) of the distribution, while they perform poorly in the tails (mean);

- It would be desirable then to consider this heterogeneity in accuracy across regions of the distribution in constructing the combined density i.e.:

$$\underset{1\times Q}{y_{t+h}} = diag(\underset{Q\times K}{\boldsymbol{w_{q,k}}} \underset{K\times Q}{f_{t+h,k}})$$

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However: how can we estimate w_q?

We need a new measure of forecast accuracy which is quantile-specific.

This paper

1. Contribute to the literature on **combination of density forecasts** by proposing an alternative combination approach that assigns a different set of combination weights to the various quantiles of the predictive distribution.
   - Bayesian quantile regression models as in Kozumi and Kobayashi (2011) are used to build the individual forecasts $f_{t+h,k}$
   - Construct quantile-specific weights for each model using the quantile score by Gneiting and Ranjan (2011)
   - Quantile combination is compared to traditional combination approaches.

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    - Applying the quantile combination to forecast US GDP growth.

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   - Comparing the informativeness of various leading indicators for various parts of the GDP distribution
   - Applying the quantile combination to forecast US GDP growth.
3. Main findings:
   - Forecasts from our quantile combination approach outperforms forecasts from commonly used combination approaches
   - We find that also other variables than the NFCI are informative about future downside macroeconomic risk

Outline

1. Introduction
2. Econometric Framework
3. Empirical Exercise
4. Simulation
5. Conclusion

Econometric Framework

Quantile regression

$$y_{t+h,q} = \sum_{i=1}^{r+p} \boldsymbol{x}_t' \boldsymbol{\beta}_q + \varepsilon_{t+1} \qquad (1)$$

- $q = 1, \ldots, 5$ denotes the respective quantile.
- $\boldsymbol{x}_t'$ is the vector of lagged values of y_t (with maximum lag r) and of lagged values one of the N predictor (with maximum lag p).
- $\varepsilon_{t+1} = \sigma\theta z_{t+h} + \sigma\tau\sqrt{z_{t+h}} u_{t+h}$ following Kozumi and Kobayashi (2011)
- $z_{t+h} \sim \mathit{Exponential}(1)$, $\sigma \sim IG(n_0/2, s_0/2)$ and $u_{t+h} \sim \mathcal{N}(0, 1)$
- $\theta = (1 - 2q)/q(1 - q)$ and $\tau^2 = 2/1(1 - q)$

Bayesian Inference

Evaluation of quantiles' forecast accuracy: quantile scores

Continuous Ranked Probability Score (CRPS)

According to a loss function, the density forecast is evaluated by computing the distance at each point of the distribution to the realization. It is defined by:

$$CRPS(f_{t+h,k}, y_{t+h}) = -\int_{-\infty}^{\infty} (F_{t+h,k} - \mathbb{I}(F_{t+h,k} \geq y_{t+h}))^2 dy \qquad (2)$$

where $F_{t+h,k}$ represents the cdf of forecast $f_{t+h,k}$ and y_{t+h} the corresponding realization.

Gneiting and Ranjan (2011) proposes a quantile decomposition of the CRPS represented by:

$$CRPS_{t+h,k} = \int_{0}^{1} QS_{t+h,k}(q)dq$$

$$QS_{t+h,k}(q) = \frac{1}{n-h+1} \sum_{t=m}^{m+n-h} QS_q(F^{-1}_{t+h,k}(q), y_{t+h}) \qquad (3)$$

$$QS_q(F^{-1}_{t+h,k}(q), y_{t+h}) = 2\left(\mathbb{I}\{y_{t+h} \leq F^{-1}_{t+h,k}(q)\} - q\right)(F^{-1}_{t+h,k}(q) - y_{t+h})$$

Quantile Combination Weights

Quantile-specific combination weights:

$$\underset{K\times Q}{w_{t+h}(k,q)} = \frac{\sum_{t=m}^{m+n-h} 1/QS_{t,k,q}}{\sum_{k=1}^{K}\sum_{t=m}^{m+n-h} 1/QS_{t,k,q}} \tag{4}$$

We need to impose that $w(t,k,q) \geq 0$ and that:

$$\sum_{k=1}^{K} w(t,k,q) = 1$$

The combined density forecast y_{t+h}^{c} is obtained by multiplying the matrix of

combination weights computed according to (4) with the matrix of quantile forecasts:

$$y_{t+h}^{c} = \text{diag}(w_{t+h,k,q} \times f_{t+h,q,k}) \tag{5}$$

Alternative combination approaches

$$f(y_{t+h}) = \sum_{k=1}^{K} \omega_k f_{t+h,k}$$

- Equal Weights $\omega_k = 1/K$
- Optimal Weights by Hall and Mitchell (2007) and Geweke and Amisano (2011)
  $w_k = \frac{1}{T-h} \sum_{t=1}^{T-h} ln(f_{t+1,k}) \quad s.t. \quad w_k > 0, \quad \sum_{k=1}^{K} w_k = 1$
  the inference algorithm for w_k in Conflitti et al. (2015) is used.
- Log score weights by Jore et al. (2010)
  $w_{t+h}(k) = \frac{\sum_{t=m}^{m+n-h} LS_{t,k}}{\sum_{k=1}^{K} \sum_{t=m}^{m+n-h} LS_{t,k}}$
- CRPS weights $w_{t+h}(k) = \frac{\sum_{t=m}^{m+n-h} 1/CRPS_{t,k}}{\sum_{k=1}^{K} \sum_{t=m}^{m+n-h} 1/CRPS_{t,k}}$
- Bayesian Model Averaging
  $p(y_{t+h}|I_K) = \sum_{k=1}^{K} P(M_k)\, p(\widetilde{y}_{t+h}|k)$ where $P(M_k)$ is the posterior probability of model k, based on the predictive likelihood for model k.

Forecast evaluation

Gneiting and Ranjan (2011) uses versions of the continuous ranked probability score that emphasize regions of interest and retain propriety.

$$\text{emphCRPS}_{t+h,k} = \int_0^1 QS_q(F^{-1}_{t+h,k}(q), y_{t+h})\nu(q)dq$$

where ν is a nonnegative weight function on the unit interval.

- We focus on:
  - Uniform: $\nu(q)\ \nu_0(q) = 1$
  - Centre: $\nu_0(q) = 1$
  - Tails: $\nu_1(q) = q(1-q)$
  - Right Tail: $\nu_2(q) = (2q-1)^2 \nu_3(q) = q^2$
  - Left Tail: $\nu_4(q) = (1-q)^2$
  - Heavy Tails: $\nu_5(q) = (2q-1)^4$

Empirical Exercise

Forecasting US GDP growth in real time

We use the following set of predictors:

Label	Trans	Real Time	Description	Source
RGDP	Δln	73:Q1-19:Q4	Real GDP growth, sa	AL
NFCI	level	11:Q2-19:Q4	National Financial Conditions Index	Chicago Fed
ICS	level-100	98:Q3-19:Q4	Consumer Sentiment Index	AL
CreSpread	Level	none:Q1	Credit Spread: BAA corporate bond yield - 10-year treasury	F
U	Δ log	65:Q4-19:Q4	Unemployment rate	AL
ResInv	Δ%	65:Q4-19:Q4	Real Gross Private Domestic Investment: Fixed Investment: Residential	AL

- Quarterly real-time data
- Period: 1971Q2-2019Q4
- Out-of-sample forecasting sample: 1993Q1-2019Q4
- Expanding window forecasts
- Forecast Horizons: H={1,4}

Evaluation of single predictive models for one-quarter ahead forecasts.

Emphasis	Uniform	Centre	Tails	Right Tail	Left Tail	Heavy Tails
$\nu(q)$	$\nu_0(q) = 1$	$\nu_1(q) = q(1-q)$	$\nu_2(q) = (2q-1)^2$	$\nu_3(q) = q^2$	$\nu_4(q) = (1-q)^2$	$\nu_5(q) = (2q-1)^4$
GDP	0.342	0.070	0.063	0.098	0.104	0.020
NFCI	0.361	0.073	0.067	0.099	0.115	0.022
ICS	0.334	0.068	0.063	0.095	0.103	0.020
U	0.367	0.075	0.069	0.104	0.114	0.022
CR Spread	**0.317**	**0.065**	**0.059**	**0.093**	**0.095**	**0.019**
ResInv	0.343	0.070	0.064	0.103	0.100	0.021

Average CRPS values with emphasis on specific regions of the distribution.

Evaluation of single predictive models for one-year ahead forecasts.

Emphasis $\nu(q)$	Uniform $\nu_0(q) = 1$	Centre $\nu_1(q) = q(1-q)$	Tails $\nu_2(q) = (2q-1)^2$	Right Tail $\nu_3(q) = q^2$	Left Tail $\nu_4(q) = (1-q)^2$	Heavy Tails $\nu_5(q) = (2q-1)^4$
GDP	0.393	0.079	0.076	0.117	0.118	0.025
NFCI	0.350	0.071	0.066	0.095	0.113	0.021
ICS	0.344	0.070	0.065	0.096	0.108	0.021
U	0.393	0.08	0.074	0.112	0.121	0.024
CR Spread	0.352	0.072	0.066	0.099	0.109	0.021
ResInv	**0.329**	**0.067**	**0.062**	**0.097**	**0.099**	**0.020**

Average CRPS values with emphasis on specific regions of the distribution.

Comparison between quantile combination and linear combination: one-quarter ahead

Emphasis	$\nu(q)$	EQ	OPT	BMA	Log Score	CRPS	Q-comb
Uniform	$\nu_0(q) = 1$	0.604***	0.923	0.903	0.952	0.988	**0.336**
Centre	$\nu_1(q) = q(1-q)$	0.602***	0.919	0.906	0.957	**1**	**0.068**
Tails	$\nu_2(q) = (2q-1)^2$	0.604***	0.927	0.888	0.941	0.955	**0.064**
Right Tail	$\nu_3(q) = q^2$	0.489***	0.834	0.873	0.881	0.923	**0.096**
Left Tail	$\nu_4(q) = (1-q)^2$	0.776	1.029	0.937	1.019	**1.04**	0.104
Heavy Tails	$\nu_5(q) = (2q-1)^4$	0.617	0.913	0.913	0.954	0.954	**0.021**

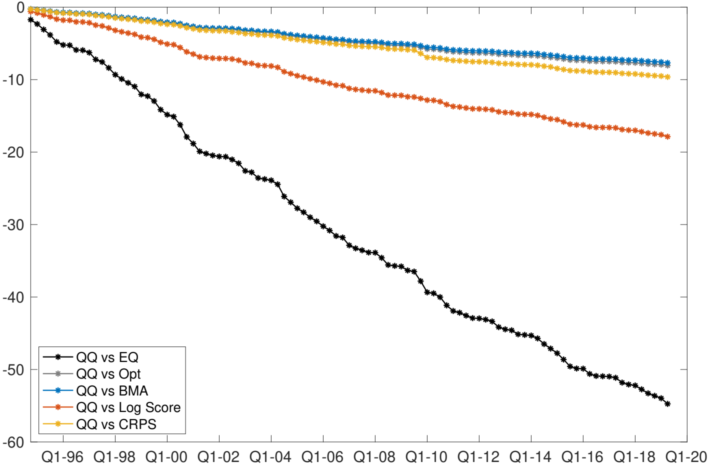
Average CRPS values with emphasis on specific regions of the distribution. Values > 1 denotes higher forecast accuracy than our quantile combination.

Comparison between quantile combination and linear combination: one-year ahead

Emphasis	$\nu(q)$	EQ	OPT	BMA	Log Score	CRPS	Q-comb
Uniform	$\nu_0(q) = 1$	0.561***	0.793***	1.000	0.996	0.846	**0.319**
Centre	$\nu_1(q) = q(1-q)$	0.579***	0.795***	1.000	1.000	0.833	**0.066**
Tails	$\nu_2(q) = (2q-1)^2$	0.504***	0.778***	**1.018**	0.833	0.857	0.056
Right Tail	$\nu_3(q) = q^2$	0.452***	0.833***	**1.021**	0.989	0.842	0.095
Left Tail	$\nu_4(q) = (1-q)^2$	0.707***	0.748***	0.978	0.989	0.858	**0.092**
Heavy Tails	$\nu_5(q) = (2q-1)^4$	0.500***	0.783***	**1.200**	1.000	0.833	0.018

Average CRPS values with emphasis on specific regions of the distribution. Values > 1 denotes higher forecast accuracy than our quantile combination.

Cumulative CRPS for alternative approaches relative to our combination for one-quarter ahead forecasts



Simulation

Simulation: Increasing sample size

Generate Simulated data with sample size $T = 1000$ for GDP and NFCI

- Draw GDP_t from $p(GDP_t | \boldsymbol{GDP_{t-r}}, \boldsymbol{NFCI_{t-s}}, \vartheta)$.
- Draw $NFCI_t$ from $p(NFCI_t | \boldsymbol{NFCI_{t-p}}, \vartheta)$

Where ϑ are estimated from quantile regression using "real data" in the application exercise.

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- Draw $NFCI_t$ from $p(NFCI_t|\boldsymbol{NFCI_{t-p}}, \vartheta)$

Where ϑ are estimated from quantile regression using "real data" in the application exercise.
The draws for GDP and NFCI are added to the "real data" dataset. The process is iterated 1000 times.

Simulation: Increasing sample size

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Where ϑ are estimated from quantile regression using "real data" in the application exercise.
The draws for GDP and NFCI are added to the "real data" dataset. The process is iterated 1000 times.

Predictive forecasts using simulated data for GDP and NFCI are then estimated and combined as in the previous application.

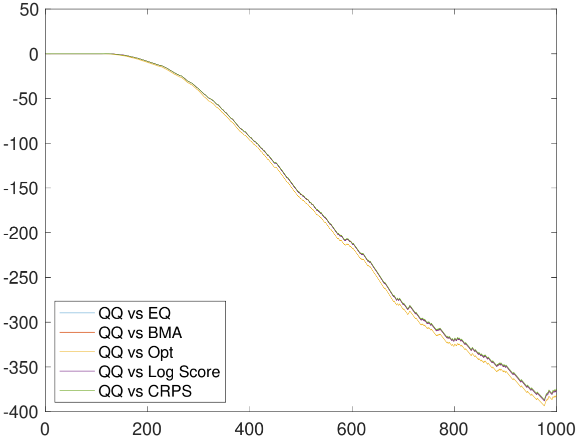
Simulation 1: $T = 1000$, $Q = 5$, $K = 2$

Table: Simulation 1 with T=1000 forecast origins. CRPS scores for Q-comb and relative performance of alternative models compared to Q-comb.

	EQ	Opt	BMA	Log Score	CRPS	Q-comb
Uniform	0.71617	0.71304	0.71617	0.71572	0.71661	**1.143**
Centre	0.77187	0.76947	0.77187	0.77187	0.77187	**0.247**
Tails	0.49211	0.49057	0.49211	0.49211	0.49211	**0.156**
Right Tail	0.93865	0.93578	0.93865	0.93865	0.94154	**0.306**
Left Tail	0.54358	0.54101	0.54358	0.54358	0.54444	**0.343**
Heavy Tails	0.44231	0.44231	0.44231	0.44231	0.44231	**0.046**

Values > 1 denotes higher forecast accuracy than our quantile combination.

Cumulative CRPS for alternative approaches relative to our combination for one-year ahead forecasts



Simulation 2: Increasing sample size and number of quantiles

Generate Simulated data with sample size $T = 1000$ for GDP and NFCI

- Draw GDP_t from $p(GDP_t|\boldsymbol{GDP_{t-r}}, \boldsymbol{NFCI_{t-s}}, \vartheta)$.
- Draw $NFCI_t$ from $p(NFCI_t|\boldsymbol{NFCI_{t-p}}, \vartheta)$

Where ϑ are estimated from quantile regression using "real data" in the application exercise.
The draws for GDP and NFCI are added to the "real data" dataset. The process is iterated 1000 times and quantiles estimated are increased from $Q = 5$ to $Q = 10$.

Predictive forecasts using simulated data for GDP and NFCI are then estimated and combined as in the previous application.

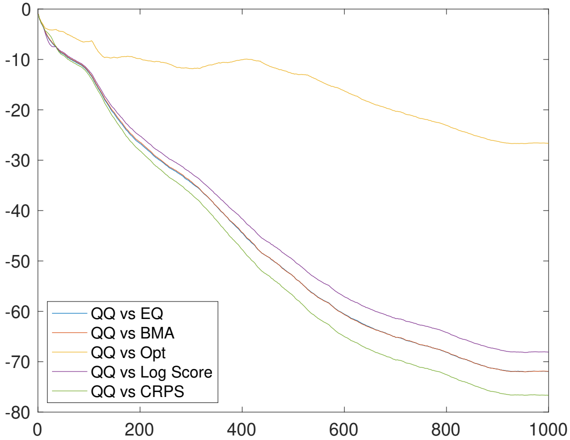
Simulation 2: $T = 1000$, $Q = 10$, $K = 2$

Table: Simulation 2 with T=1000 forecast origins and $Q = 10$ quantiles. CRPS scores for Q-comb and relative performance of alternative models compared to Q-comb.

	EQ	Opt	BMA	Log Score	CRPS	Q-comb
Uniform	0.96286	0.98788	0.96286	0.96503	0.96016	**1.711**
Centre	0.9633	0.98746	0.9633	0.96626	0.96037	**0.315**
Tails	0.96581	0.9869	0.96581	0.96788	0.96375	**0.452**
Right Tail	0.96578	0.98833	0.96578	0.96823	0.96334	**0.762**
Left Tail	0.95808	0.98765	0.95808	0.96096	0.95522	**0.320**
Heavy Tails	0.96364	0.98605	0.96364	0.96364	0.96364	**0.212**

Values > 1 denotes higher forecast accuracy than our quantile combination.

Simulation 2: $T = 1000$, $Q = 10$, $K = 2$ Cumulative CRPS



Simulation 3: $T = 1000$, $Q = 10$, $K = 10$

Generate Simulated data for GDP, using 10 predictors.

- Draw GDP_t from $p(GDP_t | GDP_{t-r}, X_{1,t-s}, \ldots, X_{K,t-s}, \vartheta)$.
- Draw $X_{k,t}$ from $p(X_{k,t} | X_{k,t-p}, \vartheta_k)$

Where $X_{k,t}$ denotes one of the predictor variables used in the Application: NFCI, ICS, U, CR Spread, ResInv, plus 5 other: CFNA, S&P500, OIL, PERMSA, BANKCRg. ϑ are estimated from quantile regression using "real data" in the application exercise.

The draws are added to the "real data" dataset. The process is iterated 1000 times.

Predictive forecasts using simulated data are then estimated and combined as in the previous application.

Simulation 3: $T = 1000$, $Q = 10$, $K = 10$

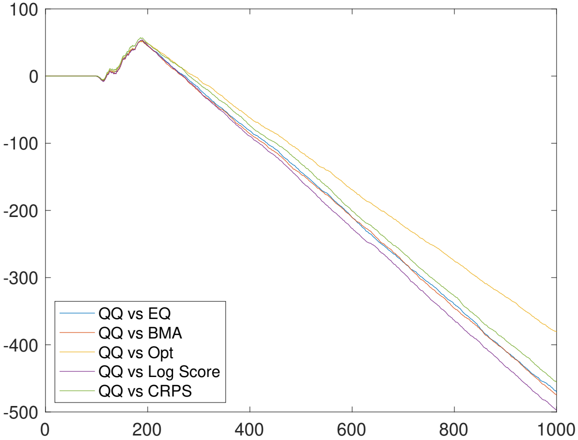
Table: Simulation (3): T=1000 forecast origins, Q=10 quantiles and K=10 models. CRPS scores for Quantile combination (Q-comb) and relative performance of alternative models compared to Q-comb.

	EQ	Opt	BMA	Log Score	CRPS	Q-comb
Uniform	0.103	0.539	0.443	0.317	0.354	**0.055**
Centre	0.083	0.415	0.419	0.334	0.237	**0.088**
Tails	0.083	0.415	0.419	0.334	0.237	**0.088**
Right Tail	0.087	0.443	0.423	0.356	0.246	**0.083**
Left Tail	0.082	0.408	0.415	0.328	0.234	**0.089**
Heavy Tails	0.083	0.419	0.417	0.337	0.237	**0.086**

Values $>$ 1 denotes higher forecast accuracy than our quantile combination.

Simulation 3: $T = 1000$, $Q = 10$, $K = 10$ Cumulative CRPS

Figure: Simulation (3) with T=1000 forecast origins Q=10 quantiles and K=10 models: Cumulative Scores for Linear and Quantile Combinations



Conclusion

Conclusion

- We propose a new forecast combination approach, that assigns a set of combination weights to the various quantities of the individual density forecasts
  - Bayesian quantile regression models as in Kozumi and Kobayashi (2011) are used to build the individual forecasts $f_{t+h,k}$
  - Construct quantile-specific weights for each model using the quantile score by Gneiting and Ranjan (2011)
- We apply it to real GDP growth rate for the Unites States for the period 1993Q1-2020Q1. Finding:
  - Forecasts from our quantile combination approach outperforms forecasts from commonly used combination approaches
- In simulation we relax some of the assumptions made due to dataset feasibility; where quantile combination resulted to be the most accurate approach.

Bayesian Inference

Re-parametrisation:

$$y_{t+h,q} = \sum_{i=1}^{r+p} \boldsymbol{x}_t' \boldsymbol{\beta}_q + \upsilon_t \theta z_{t+h} + \tau \sqrt{z_{t+h} \upsilon_i} u_{t+h} \tag{6}$$

Each unknown parameter is sampled from the following posterior distributions using a Gibbs sampler:

- $\boldsymbol{\beta}_q | \boldsymbol{y}, \boldsymbol{v}, \boldsymbol{x}, \sigma \sim \mathcal{N}(\bar{\beta}_q, \bar{V}_\beta)$

$$\bar{V}_\beta^{-1} = \left(\sum_{t=1}^{T} \frac{x_t' x_t}{\tau^2 \sigma \upsilon_t} + \boldsymbol{B}_{q_0}^{-1} \right) \quad \bar{\beta}_q = \bar{V}_\beta \left[\sum_{t=1}^{T} \frac{x_t (y_t - \theta \upsilon_t)}{\tau^2 \upsilon_t \sigma} + \boldsymbol{B}_{q0}^{-1} \boldsymbol{\beta}_{q_0} \right] \tag{7}$$

where $\boldsymbol{B}_{q0}^{-1}$ and $\boldsymbol{\beta}_{q_0}$ are priors:

$$\boldsymbol{B}_{q0}^{-1} = 100\boldsymbol{I} \quad \boldsymbol{\beta}_{q_0} = \boldsymbol{0} \tag{8}$$

Bayesian Inference

- $\boldsymbol{\beta}_q | \boldsymbol{y}, \boldsymbol{v}, \boldsymbol{x}, \sigma \sim \mathcal{N}(\bar{\beta}_q, \bar{V}_\beta)$
- $\upsilon_t | \boldsymbol{y}, \boldsymbol{\beta}_q \boldsymbol{x} \sigma \sim GIG(1/2, \delta_t, \gamma_t)$

$$\delta_t = \frac{(y_t - \boldsymbol{x}_t' \boldsymbol{\beta})^2}{\tau^2 \sigma} \quad \gamma_t^2 = 2\sigma + \theta^2/(\tau^2 \sigma) \tag{9}$$

- $\sigma | \boldsymbol{y}, \boldsymbol{\beta}_q \boldsymbol{x} \upsilon_t \sim IG(\frac{n}{2}, \frac{s}{2})$ where:

$$n = n_0 + 3T \quad s = s_0 + 2 \sum_{t=1}^{T} \upsilon_t + (y_t - \boldsymbol{x}_t' \boldsymbol{\beta}_q - \theta \upsilon_t)^2 / \tau^2 \upsilon_t \tag{10}$$

and n_0 ans s_0 are priors set to be equal to 0.1.

Results refers to full sample of data, we have used 8 000 Monte Carlo iterations after discarding the first 4 000. back