Endogenous Risk-Exposure and Systemic Instability *

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Abstract

Most research on financial systemic stability assumes an economy in which banks are subject to exogenous shocks, but in practice, banks choose their exposure to risk. This paper studies the determinants of this endogenous risk exposure when banks are connected in a financial network. I show that there exists a network risk-taking externality: connected banks' choices of risk exposure are strategically complementary. Banks in financial networks, particularly densely connected ones, endogenously expose to greater risks. Furthermore, they choose correlated risks, aggravating the systemic fragility. Banks, however, do have incentives to form networks to protect their charter values. The theory yields several novel perspectives on policy debates.

Keywords: systemic risks, financial networks, capital regulation, government bailout, CCP

JEL Classification: G21, G28, L14

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Introduction

Since the 2008 financial crisis, the relationship between financial networks and systemic stability has been an important subject of research (Glasserman and Young, 2016). Most of the existing literature assumes exogenous shocks and studies how these idiosyncratic shocks are propagated across a financial network.[1] However, banks' exposure to which particular shock is an endogenous choice variable. For example, a bank chooses between safe borrowers and subprime borrowers, or chooses its exposure on asset-backed securities.[2] This paper extends the theory of interbank networks and systemic stability by incorporating endogenous risk exposure. The introduction of a risk exposure choice changes the received intuition about financial stability in an important way and yields novel policy implications.

Pioneering works by Allen and Gale (2000) and Freixas et al. (2000) show that connected networks are more resilient to the contagion of exogenous shocks than unconnected ones due to a co-insurance mechanism. They conclude that a highly connected banking sector promotes financial stability. In contrast to the conclusions of the above papers, I show that although shocks are better co-insured in densely connected networks, banks in those networks initially choose greater risk exposure. Furthermore, they choose correlated risks. In other words, in densely connected networks, bank-specific endogenous losses are more likely, and they tend to happen simultaneously. As a result, the banking sector as a whole becomes more fragile.

The basic intuition for this result relies on a network risk-taking externality. Banks in networks, if solvent, partially reimburse failed banks through interbank payments, which I dub as cross-subsidy. The cross-subsidy reduces banks' upside payoffs (the payoffs when their own assets succeed). On the other hand, banks' downside payoffs are always zero due to limited liability. The asymmetric distortion disincentivizes banks from being prudent because they become less interested in increasing the probability of success when trading off risk and return. This risk-taking distortion is higher when each bank anticipates a higher likelihood of having to cross-subsidize other banks, that is when its counterparties take greater risks. As a result, banks' choices of risk exposure are strategically complementary.

Moreover, banks in greater connected networks will be more affected by such risk-taking externality. In particular, I show that banks in networks with a greater level of connections, in a maximum connected complete structure, or in networks with more counterparties will choose greater risk exposure. The model contributes to the debates on the relationship between a financial network's connectedness and systemic stability.[3] My result stands in contrast to the

[1] For example, Allen and Gale (2000), Freixas et al. (2000) and Gai et al. (2011) consider exogenous liquidity shocks. Shin (2009), Elliott et al. (2014) and Acemoglu et al. (2015) considers exogenous economic shocks.

[2] Mian and Sufi (2009) empirically documented an unprecedented growth of subprime credit right before the 2008 financial crisis. They also found a concurrent rapid increase in the securitization of subprime mortgages.

[3] For "connected-stability" view, Allen and Gale (2000) show that a complete network is more robust to the loss contagion due to a co-insurance mechanism. For "connected-fragility" view, Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) argue that the "complete-stability" relationship does not apply to larger shocks due to a propagation mechanism. Elliott, Golub, and Jackson (2014) find similar non-monotonic relationships for equity networks.

"connected-stability" view that argues for financial networks' co-insurance benefits. I show that the losses that are better co-insured, as in Allen and Gale (2000)'s complete network, will be more likely to endogenously evolve in the first place. Nevertheless, I also show that banks' choices of risk exposure are not monotonically increasing in the network's degree of connectedness. On the one hand, greater connectedness increases a bank's exposure to more counterparties' risk-taking externalities. On the other hand, the bank becomes less sensitive to particular other banks' failure. This nonmonotonicity result is similar to the observation of Elliott, Golub, and Jackson (2014), who use random networks to show that the ex-post contagion is not monotonic to a financial system's connectedness.

Notwithstanding the network distortion, banks do have incentives to form a financial network. With valuable expected present value of their future profits (charter values), banks do not want to risk defaulting on their deposits (Keeley, 1990; Hellmann et al., 2000). This implies that even though the interbank connection hurts banks' upside payoffs, being in a network can protect them from losing their valuable charter values as it provides co-insurance to their depositors. It is also worth noting that this paper's network risk-taking externality is distinct from the asset substitution problem as in Jensen and Meckling (1976). A conventional asset substitution model shows that the **level** of debt can encourage banks' risk-taking. Using the machinery of networks, my model shows that the **topology** of the financial system also matters for the risk-taking conditioning on the same level of debt.

This paper's model builds on a payment equilibrium model by Eisenberg and Noe (2001), which has later been utilized by Shin (2008, 2009) and Acemoglu et al. (2015). My innovation is to allow banks to choose their risk exposure endogenously after anticipating the payment equilibrium and their counterparties' risk exposure. One important contribution of this model is to show that the standard intuition about the stabilizing effect of financial networks reverses with endogenous risk-taking. The theory also yields several novel perspectives on policy debates:

- **Central Clearing Counterparty (CCP)**. According to LCH-Clearnet, the second-largest clearinghouse in the world, a CCP reduces risks by insuring members against counterparty losses.[4] In this paper, instead, I show that the risk-taking equilibrium with a CCP is equivalent to the outcome of a maximum connected complete network. This is because the CCP "forces" each member bank to be exposed to the risk-taking externalities of other banks. That implies, contrary to popular belief, a CCP may instead increase risk-taking incentives for banks in originally loosely connected networks. This result is echoed by the concern of a former SEC Chief Economist, stating "the clearinghouse is subject to considerable moral hazard and systemic risk".[5]

- **Network-adjusted Capital Regulation**: I show that each bank's equity buffer has a network

[4]See LCH-Clearnet's presentation to the New York Fed, https://www.newyorkfed.org/medialibrary/media/banking/international/11-LCH-Credit-Risk-2015-Lee.pdf

[5]See Chester Spatt's statement to the Senate Banking Committee, https://www.govinfo.gov/content/pkg/CHRG-112shrg71411/pdf/CHRG-112shrg71411.pdf

effect on systemic stability. It not only reduces a bank's risk-taking (Jensen and Meckling, 1976) but also reduces the risk-taking of other connected banks. A failed bank's equity first absorbs part of the loss, which may be otherwise propagated to other banks. That implies every bank in the financial network anticipates a smaller cross-subsidy to failed banks, and will ex-ante choose to expose to fewer risks. The result suggests that policymakers should consider banks' systemic footprint when deciding their regulatory capital. It provides a rationale for a recently proposed rule by FRB and OCC.[6]

- **Government Bailouts**. Conventional wisdom states that a government bailout, or simply anticipation of it, is harmful to the systemic stability since it encourages excessive risk-taking by reducing banks' "skin in the game". I show that a government bailout may instead reduce connected banks' risk-taking distortion. In presence of the possibility of a government bailout, every bank will anticipate a smaller cross-subsidy to its failed counterparties. Hence the network risk-taking distortion is reduced and so does every bank's choices of risk exposure.

In the final part of the paper, I endogenize banks' decisions to correlate their risk exposure. I show that in a financial network, banks will choose to expose to a single systemic risk. In anticipation of counterpart risks, a correlated portfolio reduces the possibility of a bank having to cross-subsidize others. Hence the correlated portfolios will increase each bank's expected profit. As a result of the correlation, a financial crisis (or simultaneous failure of several banks) will be more likely to evolve in a connected banking system endogenously. This observation explains the empirical findings of the Financial Crisis Inquiry Commission (2011) on the 2008 financial crisis, stating "some financial institutions failed because of a common shock: they made similar failed bets on housing."

The paper makes several contributions to the topic of systemic stability. In contrast to previous papers that study the ex-post contagion, this paper provides a tractable model to study banks' choices of risk exposure in financial networks. It reverses the previous intuition about the stabilizing effect of a highly connected financial system. The paper also explains the observation that connected banks tend to make similar bets, especially in the 2008 global financial crisis. Finally, the theory yields several novel perspectives on policy debates. It appeals to regulators to consider the financial system's topology when designing prudential policies.

Related Literature This paper is related to a recent and growing literature on the relationship between the interconnectedness of modern financial institutions and systemic stability. Most research focuses on the question do more connections tend to amplify or dampen systemic shocks. Glasserman and Young (2016) provide a survey of this literature, and here I will summarize a few related to the present paper. One branch of literature conforms to a "connected-stability"

[6]The proposed rule calibrates a bank's enhanced supplementary leverage ratio (eSLR) to its systemic importance rather than a fixed leverage standard. See `https://www.federalreserve.gov/newsevents/pressreleases/bcreg20180411a.htm`

view: a connected network provides better liquidity insurance against some exogenous shocks to one individual bank. The view is supported by Allen and Gale (2000), Freixas, Parigi, and Rochet (2000), Leitner (2005). Allen and Gale (2000) argues that the initial loss will be widely divided in a complete network. Therefore banks will be less likely to default in such a network. In Freixas et al. (2000), depositors face uncertainties about where they will consume. They also show that the interbank connections enhance the resiliency. Leitner (2005) argues that the interbank connection is optimal ex-ante due to the probability of private-sector bailouts.

On the other hand, the "connected-fragility" view is supported by Gai, Haldane, and Kapadia (2011), Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015), and Donaldson and Piacentino (2017). Using numerical simulations, Gai et al. (2011) demonstrate that a more complex and concentrated financial network may amplify the fragility. Acemoglu et al. (2015) use Eisenberg and Noe (2001)'s model to study the shock propagation. They conclude that a highly connected complete network becomes least stable under a large exogenous shock. Donaldson and Piacentino (2017) study the liquidity co-insurance benefits of long-term interbank debts. None of the above papers, nevertheless, studies how those initial shocks evolved in the first place.

Some recent papers study endogenous network formations and interbank liquidity. Acemoglu, Ozdaglar, and Tahbaz-Salehi (2014) study the network externalities of bilateral lending on other third parties in the same financial system. They show that although banks internalize the bilateral counterparty risks through the interest rate, they fail to internalize the externalities on the rest of the network. In this case, banks may "overlend" in equilibrium. The present paper utilizes the same framework to illustrate another financial network externalities: risk-taking externalities. Di Maggio and Tahbaz-Salehi (2014) study the interbank intermediation capacity with moral hazard. They show that the collateral's liquidity may have a huge effect on haircuts and intermediation capacity due to the moral hazard's cumulative nature.

There is sparse research on banks' portfolio choices when they are connected in financial networks. Brusco and Castiglionesi (2007) study banks' contracting behaviors in financial networks. They utilize the models of Diamond and Dybvig (1983) to study bankers' private benefit from gambling and their contracting behaviors with depositors. Contemporaneous papers such as Elliott et al. (2018) and Jackson and Pernoud (2020) also study banks' choices of correlation with each other in financial systems. Elliott et al. (2018) use German banks to show that banks are more likely to form connections with the ones with similar exposures to the real economy. Jackson and Pernoud (2020) argue that banks do not internalize the inefficiency resulting from their counterparties' bankruptcy cost. While the conclusions of their papers are complementary to mine, the structure of the underlying models is very different. The key innovation of this paper is that it provides the first micro-foundation showing how the financial system's risk-taking externality is the equilibrium outcome of the network structure of the banking system.

1 Model

The economy consists of $N \in \mathbb{N}^+$ risk-neutral banks that are interconnected through the cross-holdings of unsecured debt contracts $\bar{d}_{ij} > 0$, where $\bar{d}_{ij}$ is the face value of the interbank debt that bank j owes to bank i. Assume that all interbank liabilities have equal seniority. Denote $\bar{d}_j \equiv \sum_i \bar{d}_{ij}$ as bank j's total interbank liabilities. Following Acemoglu et al. (2015), I restrict most of the analysis to *regular* network structures in which the total interbank liabilities and claims are equal for all banks (i.e., $\sum_j \bar{d}_{ij} = \sum_j \bar{d}_{ji} = \bar{d}$ for all i). In this way, we abstract away the effect of network asymmetry (e.g. the existence of a dominant player). Define $\theta_{ij} \equiv \bar{d}_{ij}/\bar{d}_j$ as bank i's share in j's total interbank liabilities. By the regularity assumption, we have $\sum_j \theta_{ij} = \sum_i \theta_{ij} = 1$. Denote $\mathbf{\Theta} \equiv [\theta_{ij}]$ as an $N \times N$ matrix, which determines the network connectedness and will be further discussed in section 3. A topology $\mathbf{\Theta}$ is path-connected if every two nodes in the network can be connected by some path (Jackson, 2010). It is symmetric if each row of $\mathbf{\Theta}$ has the same set of elements.

Besides the interbank liabilities, each bank also owes a more senior outside debt $v_i = v > 0$ that needs to be paid in full before the interbank debt. One example of such outside debt is banks' retail deposits. In summary, an economy is characterized by $(\bar{d}, \mathbf{\Theta}, N, v)$, which is publicly observable. In the initial date, each bank i simultaneously chooses one project Z_i among a set of available projects $[\underline{Z}, \overline{Z}]$. This project Z_i will produce a random return of $\tilde{e}_i(Z_i)$ with the following payoff distribution.[7]

$$\tilde{e}_i = \begin{cases} Z_i & \text{w.p} \quad P(Z_i) \\ 0 & \text{w.p} \quad 1 - P(Z_i) \end{cases} \tag{1}$$

$P(Z) \in (0,1)$ is some deterministic function that denotes the probability of project Z's success. In the benchmark model, I assume each bank's project is independent. This assumption is later relaxed in Section 5. It's worth noting that $P(Z_i)$ denotes the success probability of bank i's primitive asset rather than the probability of it being solvent (i.e., able to fully pay back its deposits). As we will see in section 4.1, the probability that a bank is solvent also depends on the primitive assets of other banks in the network. To avoid confusion, throughout the rest of the paper, I use the word "successful" to denote that the primitive asset pays off (i.e. $\tilde{e}_i = Z_i$) and the word "solvent" to denote that the bank can fully pay back its deposits. To guarantee a non-trivial banking sector, a bank will be able to pay off its total liabilities whenever its project succeeds. That implies $\underline{Z} \geqslant v + \bar{d}$, and suppose this condition holds throughout the rest of the paper.[8] Let's further impose the following assumption.

[7] The payoff function assumes that a failed project generates a 0 return. In the online Appendix, I show that the main results of the paper still hold if the downside payoff is positive.

[8] This condition describes reality well. For example, in Morgan Stanley's 2020 Q1 call report, the bank has an interest income of 966 million dollars, of which 51 million dollars is interbank interest revenue. The bank needs to pay 303 million dollars as its total interest expense. This implies that even if Morgan Stanley receives nothing from its counterparties, it can fulfill its total liabilities, confirming the assumption $\underline{Z} \geqslant v + \bar{d}$. The same observation applies to all current major banks and even Lehman Brothers before its 2008 crash.

ASSUMPTION 1. *$P(Z)$ is decreasing in Z, and $P(Z) \cdot Z$ is concave in Z.*

The first part captures the fact that high-return projects come with high risks. Each bank faces a trade-off between project payoff and project safety. A large Z denotes a project with a large return along with high risks. Therefore, we can interpret Z_i as bank i's choice of its risk exposure. The Pareto optimal risk exposure for each individual bank is when $\mathbb{E}[\tilde{e}]$ is maximized: $Z^* = \text{argmax}_Z P(Z)Z$. An economy's total surplus will be later formalized in definition 3. The second part of the assumption is to ensure a unique interior risk exposure. A sufficient condition is to let $P()$ be concave: the project risk increases at a growing rate in the project return.

After all banks choose their risk exposure $\mathbf{Z} = (Z_1, ..., Z_N)$, the state of nature $\boldsymbol{\omega} = (\omega_1, ..., \omega_N)$ will be independently drawn from the distribution according to equation (1).[9] For each bank, ω_i can take one of the two values: success ($\omega_i = s$) or fail ($\omega_i = f$). As a result, $\boldsymbol{\omega} \in \boldsymbol{\Omega} = 2^N$. After realization of the state of nature, interbank debts' reimbursement will be determined from a payment equilibrium. A bank's total payments depend on what it possesses, which depends on the interbank payments from other banks. As a result, the payment equilibrium is solved by a fixed point system. This notion of the payment equilibrium is introduced by Eisenberg and Noe (2001) and then utilized by Shin (2008, 2009) and Acemoglu et al. (2015). The current paper differs from theirs in that the payment vector of my model is now parametrized by a vector of risk exposure $\mathbf{Z}$ and a vector of states $\boldsymbol{\omega}$. Definition 1 formally defines the payment equilibrium.

DEFINITION 1. *For a network structure ($\bar{d}, \boldsymbol{\Theta}, N$) and given a risk vector $\mathbf{Z}$, the payment equilibrium is a vector of functions $\boldsymbol{d}^*(\boldsymbol{\omega}; \mathbf{Z}) = [d_1^*(\boldsymbol{\omega}; \mathbf{Z}), ..., d_N^*(\boldsymbol{\omega}; \mathbf{Z})]$ that solves*

$$d_i^*(\boldsymbol{\omega}; \mathbf{Z}) = \left\{ \min \left[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + e_i(\omega_i, Z_i) - v, \bar{d} \right] \right\}^+ \quad \forall i \in \mathcal{N} \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega} \tag{2}$$

$d_i^*(\boldsymbol{\omega}; \mathbf{Z})$ denotes bank i's total payments of its interbank liabilities in state $\boldsymbol{\omega}$ after banks choosing risk exposure $\mathbf{Z}$. On the right hand side, $\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + e_i(\mathbf{Z}, \boldsymbol{\omega})$ is bank i's available resources for payments to its total liabilities (deposits and interbank debts). The function $\min[., \bar{d}]$ captures banks' limited liabilities, so they pay either what they owe or what they have, whichever is smaller. $\{.\}^+ \equiv \max\{., 0\}$ denotes the fact that banks' interbank payments are non-negative. It binds when the bank is not solvent (i.e., cannot fulfill its deposits). A bank starts to pay its interbank liabilities only after it fully fulfills its deposits.

We observe that the payment $d_i^*(\boldsymbol{\omega}; \mathbf{Z})$ is a function of $\boldsymbol{\omega}$. For each state of nature $\boldsymbol{\omega}$, we will have a separate fixed-point system. Therefore, given a risk vector $\mathbf{Z}$, we need to solve 2^N fixed-point systems, one for each state of nature. Before we proceed, one immediate task is to show that the above payment equilibrium exists and is unique.

LEMMA 1. [Eisenberg-Noe] *For any risk vector $\mathbf{Z}$, the payment equilibrium exists and is generic unique.*

[9] For the remaining text, I refer a vector as in bold letters. For example, $\boldsymbol{x} = (x_1, ...x_N)$ and $\boldsymbol{x_{-i}} = (x_1, .., x_{i-1}, x_{i+1}, ..x_N)$

The proof is a simple utilization of the Brouwer fixed point theorem and is identical to Eisenberg and Noe (2000) and Acemoglu et al. (2015). Part of the proof is subsumed in the proof of proposition 2. Hence, it is omitted here to conserve space. Acemoglu et al. (2015) show that for each $\tilde{\boldsymbol{e}}$, the fixed point exists and is generic unique. It is identical to say that for every combination of $(\boldsymbol{\omega}, \boldsymbol{Z})$, the fixed point exists and is generic unique. Hence lemma 1 naturally follows.

After the realization of $\boldsymbol{\omega}$ and the interbank payments $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$, each bank's profit at the final date becomes

$$\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}) = \Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\boldsymbol{Z}, \boldsymbol{\omega}) - v_i - d_i^*(\boldsymbol{\omega}; \boldsymbol{Z}) \Big)^+ \tag{3}$$

The profit $\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})$ depends on the risk exposure of all other banks. In Equilibrium, each bank choose its own risk exposure Z_i to maximize the expected payoff $\mathbb{E}_{\omega}[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})]$. The following figure summarizes the timeline.

Figure 1: Timeline

From equation 3, we can derive each bank's expected profit as

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})\Big] = \sum_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} \Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}) \cdot \Pr(\boldsymbol{\omega}) = \sum_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} \Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z}) \cdot \prod_j \Pr(\omega_j)\Big]$$

The last equality is due to the assumption that each bank's project outcome is independent. Each bank chooses its risk exposure to maximize the expected profit. Therefore, the Nash Equilibrium for banks' risk exposure can be expressed as the solution of the following fixed-point system:

$$Z_i^* = \underset{Z_i}{\operatorname{argmax}} \sum_{\boldsymbol{\omega} \in \boldsymbol{\Omega}} \Big[\Pi_i(\boldsymbol{\omega}; Z_i, \boldsymbol{Z}_{-\boldsymbol{i}}^*) \cdot \prod_j \Pr(\omega_j)\Big] \quad \forall i \in \mathcal{N} \tag{4}$$

We observe that $\boldsymbol{Z}_{-\boldsymbol{i}}$ enters bank i's expected profit in two ways: first through the distribution of the state of nature, $Pr(\omega_j = s) = P(Z_j)$, and second through the payment equilibrium $\boldsymbol{d}^*(\boldsymbol{\omega}, \boldsymbol{Z})$. In the next section, I will show that the second channel has no effect and bank j's risk choice affects bank i's expected profit only through the distribution of $\boldsymbol{\omega}$.

2 Risk-Taking Equilibrium and Network Distortion

It's immediate that we can define a risk-taking equilibrium as every bank chooses its risk exposure simultaneously, anticipating other banks' optimal risk exposure and the resulting payment equilibrium.

DEFINITION 2. *The risk-taking equilibrium in a financial network* $(\bar{d}, \boldsymbol{\Theta}, N)$ *is a pair* $(\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z}), \boldsymbol{Z}^*)$ *consisting of a vector of payment functions* $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ *and a vector of risk exposure* $\boldsymbol{Z}^*$ *such that:*

1. *The vector of functions* $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ *is a payment equilibrium for any* $\boldsymbol{Z}$.

2. *For each* $i \in \mathcal{N}$, Z_i^* *is optimal and solves equation 4, given* $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ *and* $\boldsymbol{Z}^*_{-i}$.

We first observe that the above risk-taking equilibrium is the solution of two intertwined systems of equations (equation 2 and 4): when choosing the risk vector Z_i, each bank anticipates the payment equilibrium. When determining the interbank debt payment $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$, banks' chosen risk vector is a parameter.

At first glance, the fixed point solutions to the two intertwined systems look complicated to derive. Thanks to the following lemma 2 and proposition 1, the existence and analytical solutions for the risk-taking equilibrium can be obtained.

LEMMA 2. *The payment equilibrium* $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ *is constant in the risk exposure vector* $\boldsymbol{Z}$.

Proof. In the Appendix

As a result, we can rewrite $\boldsymbol{d}^*(\boldsymbol{\omega}) = \boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$. The idea is that when a bank's project succeeds, its total interbank payment is the face value $\bar{d}$, independent of any bank's chosen risk exposure. On the other hand, when a bank's project fails, its contribution to the payment system is 0, also independent of any bank's chosen risk exposure.[10] Therefore, the payment equilibrium is independent of the risk exposure vector $\boldsymbol{Z}$.

As a result of lemma 2, we can disentangle the two intertwined fixed-point systems. We first solve the fixed-point vectors for the payment equilibrium (equation 2), and then use them to derive the fixed-point solution for the risk-taking Nash Equilibrium (equation 4).

We also observe that a bank will earn a positive profit only if its project succeeds. Suppose a bank's project fails, at most its available resource will be $\max_{\omega} \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) = \bar{d}$, that is when its interbank claims get paid in full. That implies this bank will default on its interbank debts (i.e. $\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - v < \bar{d}$). Therefore the bank with a failed project will earn a zero profit at the final date. Hence, we can rewrite bank i's expected profit as:

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \boldsymbol{Z})\Big] = P(Z_i) \sum_{\boldsymbol{\omega}_{-i}} \Big[Z_i - v - \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{s}})\Big)\Big] \cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})$$

[10] Although a failed bank's contribution to the payment system is zero, its interbank payments may be positive.

where $\boldsymbol{\omega}_{-i} \in 2^{N-1}$ denotes the vector of states for all banks except bank i. With a slight abuse of notation, I denote $\boldsymbol{\omega}^{i=s} \equiv (\omega_1..., \omega_{i-1}, s, \omega_{i+1}, ...\omega_N)$ as the vector that appends bank i's success to other banks' states of nature $\boldsymbol{\omega}_{-i}$. Define the function $\mathcal{D}(\mathbf{Z}_{-i})$ as

$$\mathcal{D}(\mathbf{Z}_{-i}) \equiv \sum_{\boldsymbol{\omega}_{-i}} \left(\bar{d} - \sum_{j} \theta_{ij} d_j^*(\boldsymbol{\omega}^{i=s}) \right) \cdot \Pr(\boldsymbol{\omega}_{-i}) \tag{5}$$

Note that $\mathcal{D}(\mathbf{Z}_{-i})$ is non-negative and is parameterized by the network structure $(\bar{d}, \boldsymbol{\Theta}, N)$. Plugging $\mathcal{D}(\mathbf{Z}_{-i})$ into the bank's expected profit, we have

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z})\Big] = P(Z_i)(Z_i - v) - P(Z_i)\mathcal{D}(\mathbf{Z}_{-i}) \tag{6}$$

Equation 6 consists of two parts. The first term $P(Z_i)(Z_i - v)$ is the expected payoff of a stand-alone bank. The second term $\mathcal{D}(\mathbf{Z}_{-i})$ is bank i's expected net interbank payment (or "cross-subsidy") to other banks when its project succeeds. This cross-subsidy $\mathcal{D}(\mathbf{Z}_{-i})$ can be interpreted as a risk-taking distortion as it will become clear in the next proposition. Since $\mathbf{Z}_{-i}$ enters bank i's expected payoff through this distortion, we will be interested to know how it affects bank i's choice of risk exposure. Proposition 1 provides the answer.

PROPOSITION 1. *The choice of risk exposure* $\mathbf{Z}$ *is strategically complementary among all banks in the same financial network.*

Proof. In the Appendix

The proposition states that a bank's optimal risk exposure is increasing in the risk exposure of any other bank in the network. To see the intuition, suppose a counterparty bank, say bank m, increases its risk exposure. As a result, bank m's project becomes more likely to fail. When it does fail, bank i's cross-subsidies to other banks will increase. This will decrease bank i's upside payoff (the payoff when its project succeeds). As a result of this distortion, bank i will be less interested in increasing the probability of success when trading off risk and return. In other words, bank i will optimally choose a greater risk exposure in response to bank m's increased risk exposure. As a result, banks' choices of risk exposure are strategically complementary.

Proposition 1 conveys the first important message of this paper. It assigns a new meaning to the view of the "too connected to fail" in the sense that a bank not only affects other connected banks through an ex-post loss contagion, as in Allen and Gale (2000), Elliott et al. (2014), or Acemoglu et al. (2015). It also creates an ex-ante moral hazard problem due to a risk-taking externality.

With the supermodular property for banks' choices of risk exposure at hand, we are now able to establish the existence of the risk-taking equilibrium.

PROPOSITION 2. *In any network structure* $(\bar{d}, \boldsymbol{\Theta}, N)$*, the risk-taking equilibrium exists.*

Proof. In the Appendix

The proof is a simple application of the Tarski (1955) fixed point theorem to a supermodular game. In general, the equilibrium is not unique. For the remaining text, let's focus on the Pareto-dominant equilibrium when $\boldsymbol{Z}$ is the smallest among the set of fixed points.[11].

After establishing the existence of the risk-taking equilibrium, we can now compare connected banks' choices of risk exposure with that of a stand-alone bank. The following proposition shows that the interconnectedness indeed encourages banks to expose to more risks.

COROLLARY 1. *A bank in any network structure* $(\bar{d}, \boldsymbol{\Theta}, N)$ *will choose a greater exposure to risks than a stand-alone bank.*

Proof. In the Appendix.

In financial networks, a bank with a successful project pays a net positive amount of cross-subsidy to failed banks' depositors. This cross-subsidy is reflected in the network distortion $\mathcal{D}(\boldsymbol{Z}_{-i})$ of a bank's upside payoff. As argued by proposition 1, every bank in the financial network, anticipating this distortion, will increase its exposure to risks. This leads to an amplification mechanism for banks' risk exposure as the increased risk, in turn, increases the distortion. In equilibrium, no bank will internalize the effect of its risk exposure on other banks' payoffs. There exists a risk-taking externality, and connected banks will endogenously expose to greater risks than stand-alone banks.

It's worth noting that a bank's risk-shifting incentive in a financial network is distinct from the asset substitution problem as in Jensen and Meckling (1976), who argue that the level of debt can encourage risk-taking. To see this, let's first define the total social welfare.

DEFINITION 3. *The social welfare is the sum of the expected returns to all agents in the economy, namely banks and retail depositors. Formally,*

$$u = \mathbb{E}\left\{ \underbrace{\sum_i \Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\boldsymbol{Z}, \boldsymbol{\omega}) - v_i - d_i^*(\boldsymbol{\omega}; \boldsymbol{Z}) \Big)^+}_{\text{expected return to banks}} + \underbrace{\sum_i \min\Big[v_i, \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\boldsymbol{Z}, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}) \Big]}_{\text{expected return to depositors}} \right\}$$

The first part is the expected return to banks' shareholders, and the second part is the expected return to their depositors. We can rewrite u as

$$u = \mathbb{E}\left\{ \sum_i \Big(e_i(\boldsymbol{Z}, \boldsymbol{\omega}) + \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}; \boldsymbol{Z}) \Big) \right\} = \mathbb{E}\left\{ \sum_i e_i(\boldsymbol{Z}, \boldsymbol{\omega}) \right\} = \sum_i P(Z_i) \cdot Z_i$$

[11] Focusing on the least exposure equilibrium is to abstract away a self-fulfilling failure. See Elliott et al. (2014) for more details. They also consider the "best-case" equilibrium, in which as few organizations as possible fail. Furthermore, all of the following results are robust to any stable equilibrium.

Comparing the social welfare u with each individual bank's objective function (equation 6), we notice that there exist two risk-taking distortions in a financial network: (i) friction between banks and depositors, and (ii) a risk-taking externality among connected banks, which is the main focus of this paper. The first distortion is known as the asset substitution problem of Jensen and Meckling (1976), who show that the level of debt financing can encourage risk-taking. In the next section, I will show that the topology of the debt also matters for banks' risk-taking even with the same level of total debt.

3 Network Structures

3.1 Size of interbank liabilities

So far, we have seen that a connected bank will endogenously expose to greater risks due to a network risk-taking distortion. Let's now examine the extent of this network distortion for different network structures. To begin with, I study in this section the effect of the interbank liabilities' size $\bar{d}$ on the network risk-taking distortion $\mathcal{D}(\mathbf{Z}_{-i})$ and the subsequent equilibrium risk exposure $\mathbf{Z}^*$. I do so by fixing the network topology $\mathbf{\Theta}$. Lemma 3 shows the result.

LEMMA 3. *In any network structure $(\bar{d}, \mathbf{\Theta}, N)$, the network risk-taking distortion $\mathcal{D}(\mathbf{Z}_{-i}; \bar{d})$ is increasing and concave in the size of interbank liabilities $\bar{d}$.*

Proof. In the Appendix.

To understand the intuition behind lemma 3, it is helpful to first notice that there are three types of bank outcomes at the final date. The first type contains banks with successful projects. Denote them $\mathcal{S}_\omega \equiv \{i : \omega_i = s\}$. The second type contains banks that failed its project but are still "solvent" (can fully fulfill their deposits). Denote them $\mathcal{F}^+_\omega \equiv \{i : \omega_i = f, \sum_j \theta_{ij} d^*_j(\boldsymbol{\omega}) \geqslant v\}$. Since those banks can fulfill their deposits, they will contribute back to the interbank payment system. The third type contains banks that failed its project and cannot fully fulfill their deposits. Denote them $\mathcal{F}^-_\omega \equiv \{i : \omega_i = f, \sum_j \theta_{ij} d^*_j(\boldsymbol{\omega}) < v\}$ and call them "insolvent" failed banks. The depositors of those banks will incur losses.

In a network with larger interbank liabilities, successful banks $\mathcal{S}$ will expect larger net interbank payments (cross-subsidies) to failed banks ($\mathcal{F}^- \cup \mathcal{F}^+$). Those cross-subsidies are due to the difference between what a successful bank pays, $\bar{d}$, and what it receives, $\sum_j \theta_{ij} d^*_j(\boldsymbol{\omega})$. They are naturally increasing in the size of the interbank liabilities. As argued earlier, those cross-subsidies are the causes of the network risk-taking distortion. Therefore, the network risk-taking distortion is increasing in the size of interbank liabilities.

On the other hand, the larger cross-subsidies also increase the likelihood for a failed bank to be solvent ($\mathcal{F}^- \to \mathcal{F}^+$). A solvent failed bank will contribute back to the payment system, which in turn partially lowers the cross-subsidies that a successful bank needs to pay. As a result of the above two countervailing effects, the network risk-taking distortion is increasing (due to larger interbank payment) and concave (due to more solvent failed banks) in the size of interbank liabilities. We can then apply lemma 3 to obtain the following equilibrium result on banks' choices of risk exposure.

PROPOSITION 3. *In any network structure $(\bar{d}, \mathbf{\Theta}, N)$, each bank's choice of risk exposure Z_i^* is increasing in the size of interbank liabilities $\bar{d}$.*

Proof. In the Appendix.

Proposition 3 is an equilibrium result stating that banks will choose greater risk exposure if the network has larger interbank liabilities. The proof is a simple application of the monotone selection theorem. Lemma 3 shows that each bank will experience a larger risk-taking distortion resulting from a larger $\bar{d}$. This will directly increase each bank's choice of risk exposure. From the strategic complementarity result, every bank's counterparties will also take greater risks, which in turn feedback to its risk-taking incentives. In equilibrium, a larger size of interbank liabilities will induce every bank in the network to expose to greater risks.

From the concavity result of lemma 3, we know that the size of interbank liabilities has a diminishing marginal effect on connected banks' risk-taking distortion. This implies that $\bar{d}$ will eventually cease to have an additional effect on $\mathcal{D}(\mathbf{Z}_{-i}; \bar{d})$ after a certain threshold. The following corollary formalizes this fact.

COROLLARY 2. *In a network with N banks,*

(a) For any network topology $\mathbf{\Theta}$, the network distortion $\mathcal{D}(\mathbf{Z}_{-i})$ is bounded from above.

(b) If the network $\mathbf{\Theta}$ is path-connected and symmetric, the upper bound is

$$\mathcal{D}^{max}(\mathbf{Z}_{-i}) = \sum_{f=1}^{N-1} \frac{f}{N-f} \cdot v \cdot \binom{N-1}{f} \Big[P(Z_{-i})\Big]^{N-1-f} \Big[1 - P(Z_{-i})\Big]^{f} \tag{7}$$

Proof. In the Appendix.

Part (a) states that there exists an upper bound for the network risk-taking distortion. We have shown that the network risk-taking distortion is the result of a bank's expected "cross-subsidy" to failed banks' depositors. That implies the distortion will stop increasing when the "cross-subsidy" can cover every connected bank's deposits in every state of nature.

Part (b) gives the analytical solution for this upper bound when the network is path-connected and symmetric. The maximum distortion in equation 7 has a clean interpretation. Suppose in some state of nature $\boldsymbol{\omega}$, there are f number(s) of banks with failed projects and $(N - f)$ number(s) of banks with successful projects. The maximum amount of money that needs to be bailed out is $f \cdot v$, the total amount of deposits from failed banks. Because of the symmetry, a successful bank is expected to cross-subsidize an amount of $f \cdot v/(N - f)$. The probability with which f

banks fail is $\binom{N-1}{f}[P(Z_{-i})]^{N-1-f}[1-P(Z_{-i})]^{f}$. Taking the expectation, we will have equation 7. It's worth mentioning that $\mathcal{D}^{max}(\boldsymbol{Z_{-i}})$ is independent of the network topology $\boldsymbol{\Theta}$ if the network is symmetric (e.g. ring or complete networks).

3.2 Complete and Ring Networks

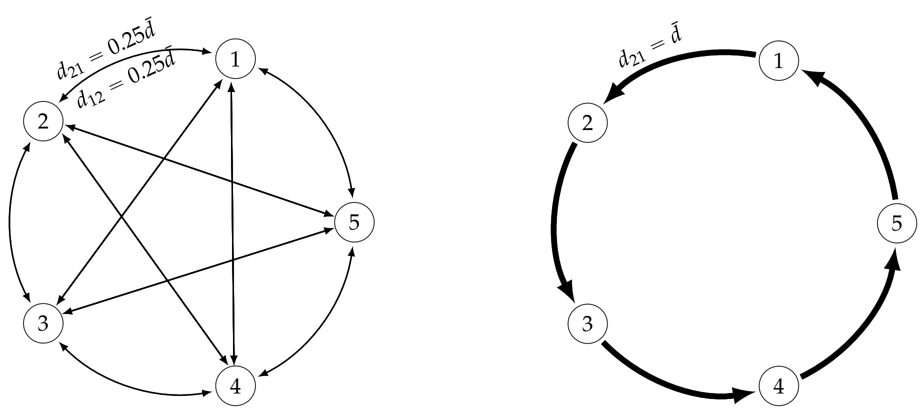
Let's now turn our attention to two particular network structures: the complete network and the ring network. The ex-post contagion of those two networks has been studied by Allen and Gale (2000) and Acemoglu et al. (2015) among others. Here we will study their effects on banks' ex-ante risk-taking incentives. In a ring network, every bank is connected only to its direct neighbors. In a complete network, every bank is connected to every other bank. Definition 4 formalizes the above description.

DEFINITION 4. *In a financial network with N banks, a ring network and a complete network are defined as*

$$\boldsymbol{\Theta}^{R} = \begin{bmatrix} \boldsymbol{0}'_{N-1} & 1 \\ \boldsymbol{I}_{N-1} & \boldsymbol{0}_{N-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Theta}^{C} = \frac{1}{N-1}(\mathbb{1}_{N,N} - \boldsymbol{I}_{N})$$

where $\boldsymbol{0}_{N-1}$ is a vector of $N-1$ zeros, $\mathbb{1}_{N,N}$ is a matrix of ones with a dimension (N, N), and $\boldsymbol{I}$ is an identity matrix. Figure 2 illustrates a complete and a ring network with 5 banks.

Figure 2: a complete and a ring network with 5 banks



(a) complete network (b) ring network

We observe that the total debt levels of banks in a complete and a ring network are identical: $\bar{d} + v$. This implies that the conventional asset substitution model, as in Jensen and Meckling (1976), is not suited to study connected banks' risk-taking incentives. With the help of my model, the following proposition compares banks' equilibrium risk exposure in a complete and a ring network.

PROPOSITION 4. *In any network structure $(\bar{d}, N)$, each bank's choice of risk exposure Z_i^* is larger in a complete network than in a ring network.*

Proof. In the Appendix.

The above proposition states that banks in a complete network choose greater risk exposure than banks in a ring network. The result stands in sharp contrast to the view of Allen and Gale (2000). They argue that a complete network is better at co-insurance and hence more resilient. Instead, I show that such co-insurance also creates an ex-ante risk-taking distortion. Banks with successful projects will anticipate a greater amount of "cross-subsidy" to failed banks' depositors. As argued earlier, due to such distortion, every bank will have an ex-ante incentive to expose to greater risks. As a result, in equilibrium, every bank in a complete network chooses a greater risk exposure.

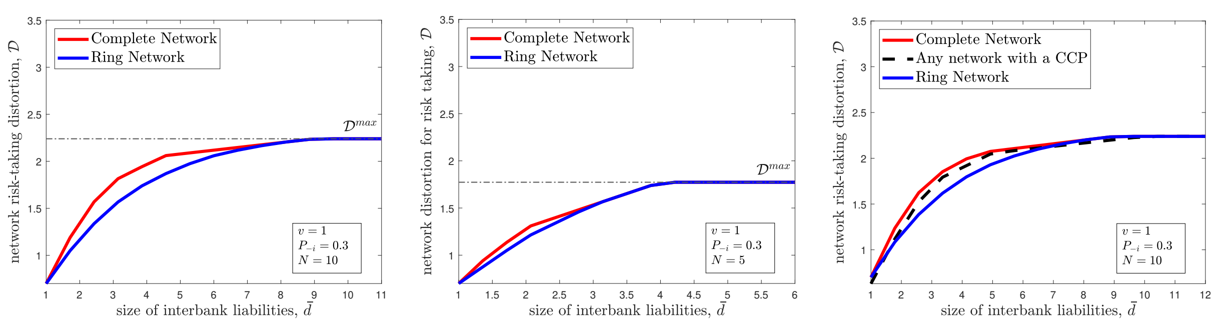
The same intuition can also be applied to networks with greater numbers of banks. Because the dimension of $\mathbf{\Theta}_N$ varies with N, the topology $\mathbf{\Theta}_{N+1}$ may not be well defined from an arbitrary $\mathbf{\Theta}_N$. I hence focus on the maximum risk-exposure of symmetric networks, which, according to corollary 2.(b), is independent of the network topology.

PROPOSITION 5. *In any symmetric financial network, the upper bound for each bank's risk exposure Z_i^* is increasing in the number of banks, N, in the network.*

Proof. In the Appendix.

Figure 3: Numerical Analysis

(a) Benchmark (b) Decrease N (c) With a CCP

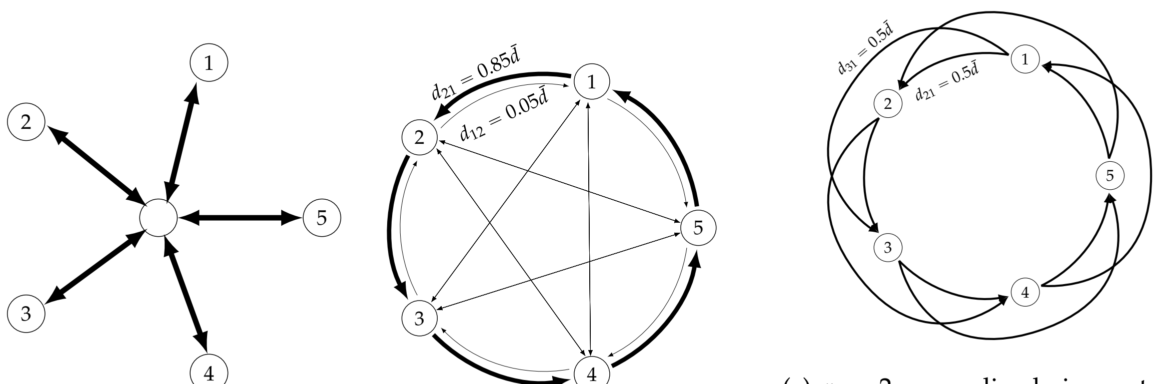


Proposition 5 confirms our conjecture. Figure 3 illustrates the numerical analysis summarizing the effects of network topologies we have studied so far. Figure 3.(a) displays the benchmark case where $N = 10$, $v = 1$, and $P(Z_{-i}) = 0.3$. It plots the network risk-taking distortion against the size of interbank liabilities for a complete and a ring network. We observe that the network distortion is increasing and concave in the size of interbank liabilities, confirming proposition 3. We also see that the distortion is larger in a complete network (red) than a ring network (blue), confirming proposition 4. In figure 3.(b), I decrease the number of banks from 10 to 5, and we see that the maximum risk-taking distortion decreases for both the ring and the complete network, confirming proposition 5.

3.3 Other Regular Networks

The machinery of networks also allows us to study the financial structures that are widely observed in the current financial systems, for example, central clearing counterparties (CCP). According to LCH-Clearnet, the world's leading multinational clearinghouse, a CCP nets down payment obligations across all the cleared contracts to one payment obligation to the CCP per member. In other words, it acts as interbank debts' buyer to all sellers and seller to all buyers.[12] This is equivalent to a core-periphery structure where the core acts as the clearing party with no asset and no outside liability. In this network, every bank has interbank claims and liabilities of size $\bar{d}$ to the core. Figure 4.(a) illustrates such a structure. The next proposition studies the risk-taking incentives for banks in networks with a CCP.

Figure 4: Other Regular Networks



(a) central-clearing (b) $\lambda = 0.2$ network (c) $r = 2$ generalized ring network

PROPOSITION 6. *In any network structure $(\bar{d}, \boldsymbol{\Theta}, N)$ with a central clearing counterparty, the risk-taking equilibrium is equivalent to that of a complete network with $(\frac{N-1}{N}\bar{d}, \boldsymbol{\Theta}^{\boldsymbol{C}}, N)$.*

Proof. In the Appendix.

From proposition 6, we observe that a CCP has two opposite effects on member banks' risk-taking incentives. First, a CCP increases banks' risk-taking incentives by increasing the network's completeness. Through central clearing, each bank is "forced" to connect to every other bank and become exposed to their risk-taking externalities. This "CCP-riskier" effect is greater for a loosely connected ring network than a complete network, on which a CCP has no effect. Second, a CCP reduces banks' risk-taking incentives by netting out some ex-post payment of interbank debts; it reduces the size of the connection from $\bar{d}$ to $\frac{N-1}{N}\bar{d}$.[13] This effect is equivalent the netting

[12]See LCH-Clearnet's presentation to the Federal Reserve Bank of New York, `https://www.newyorkfed.org/medialibrary/media/banking/international/11-LCH-Credit-Risk-2015-Lee.pdf`

[13]To illustrate this point, suppose there are four banks. In one state of nature, three succeed, and one fails. Suppose

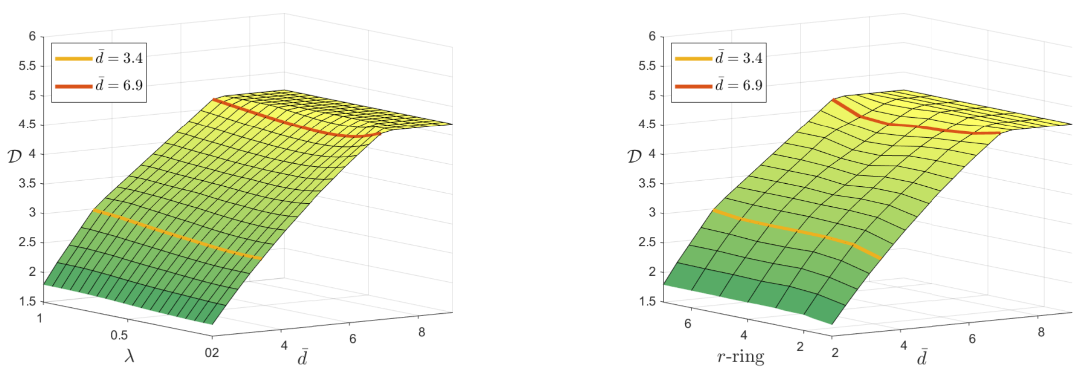
efficiency considered by Duffie and Zhu (2011).[14] Summing up the two forces, the effect of a CCP on banks' risk exposure depends on the banking system's original network topology.

Figure 3.c illustrates the effects of a CCP on a complete and a ring network. We obverse that the effect depends on the original network's topology. For a complete network, a CCP can decrease the network risk-taking distortion. This is because banks in a complete network enjoy more of CCP's netting efficiency. However, for a ring network with a modest $\bar{d}$, a CCP increases the network risk-taking distortion. This is because the CCP "forces" each member bank to be exposed to every other bank's risk-taking externalities. This implies, for loosely connected networks, the "CCP-riskier" dominates. In those cases, a CCP can create systemic instability, in contrast to conventional wisdom.

Notwithstanding greater risk exposure, banks still have incentives to join a CCP if they care sufficiently about their charter values. That is because a CCP can increase the likelihood of their depositors being paid in full. Section 4.1 will discuss this point in greater detail.

Figure 5: Intermediately Connected Networks

(a) λ networks (b) generalized ring networks



We have studied the risk-taking externalities for banks in two extreme networks: a fully-connected complete network and a loosely-connected ring network. Proposition 4 shows that banks in a complete network will choose greater risks than banks in a ring network. One may think that banks in an intermediately-connected network will choose a risk-exposure somewhere between the risk exposure of a complete and a ring network. The basis for this conjecture relies on the fact that the payment equilibrium of an intermediately-connected network is between that of a complete and a ring network, as shown by Eisenberg and Noe (2001) and Acemoglu et al. (2015).[15] However, this is not true for the network risk-taking externalities.

the failed bank is "insolvent". In this case, the distortion for a successful bank in a complete network is $\bar{d} - 2 \cdot \frac{1}{3}\bar{d} = \frac{1}{3}\bar{d}$. However, the distortion for a successful bank in a network with a CCP is $\bar{d} - 3 \cdot \frac{1}{4}\bar{d} = \frac{1}{4}\bar{d}$

[14]Duffie and Zhu (2011) study the CCP's ex-post netting efficiency by treating banks' defaults as unrelated events. The netting efficiency in my model is a (generalized) version of theirs after considering the joint determination of defaults using the technique of Eisenberg and Noe (2001).

[15]See Eisenberg and Noe (2001) lemma 6 and Acemoglu et al. (2015) proposition 4. To see why, suppose an intermediately-connected network has a $\mathbf{\Theta}$ that is the λ-convex combination of a ring and complete network. Because

There are two ways to define an intermediately-connected network. A λ network is the convex combination of a ring and a complete network: $\mathbf{\Theta}^{\lambda} = (1-\lambda)\mathbf{\Theta}^{R} + \lambda\mathbf{\Theta}^{C}$. According to Elliott et al. (2014), λ can be interpreted as a financial network's degree of diversification. From this definiteness, $\lambda = 0$ is a ring network and $\lambda = 1$ is a complete network. Another way to define an intermediately-connected structure is from the generalized ring network: each bank connects to r number of adjacent neighbors. From this definiteness, $r = 1$ is a ring network and $r = N-1$ is a complete network. Figure 4.(b) and (c) displays a $\lambda = 0.2$ network and a $r = 2$ generalized ring network. To illustrate the relationship between a network's degree of connectedness and the risk-taking distortion it induces, Figure 5 displays the distortion for different parameter values of λ, r, and $\bar{d}$ for an 8-bank network. Online Appendix provides a numerical example.

We notice that when $\bar{d}$ is low, the network distortion is increasing in both λ and r. This is because a higher degree of connectedness increases a bank's exposure to more counterparties' risk-taking externalities. This is consistent with the findings of Eisenberg and Noe (2001) and Acemoglu et al. (2015), who show that the payment equilibrium is increasing in λ. However, when $\bar{d}$ is large, the network distortion is not monotone in either λ or r. In this case, a λ network or a r generalized ring network may have a lower network distortion than a ring network. This is because as λ or r increases, banks become less sensitive to particular other banks' risk-taking externalities. This non-monotonicity result is consistent with the observation of Elliott et al. (2014), who show that contagion is most likely to occur when integration (similar to $\bar{d}$) and diversification (similar to λ) are in the middle range. Finally, if $\bar{d}$ is large enough such that all failed banks' depositors can be repaid, the degree of connectedness λ or r is irrelevant (corollary 2.b).

3.4 Non-regular Network: European Debt Cross-Holding Example

So far, the analysis has focused on regular networks where all banks' total interbank liabilities and claims are equal. Nevertheless, the model's tractability allows us in addition to study other types of networks, including those observed in the real world. In this section, I use the European debt cross-holding of Elliott et al. (2014) as an example to illustrate the risk-taking equilibrium when countries are interconnected. The example serves to give conceptual insight and is based on simplified estimates. The objective of this section is to show how systemic risks can endogenously evolve in a financial network.

The financial network consists of six European countries' banking systems: France, Germany, Greece, Italy, Portugal, and Spain. The data on the countries' cross-holdings of debt is directly taken from Elliott et al. (2014), who collected the information from the BIS Quarterly Review. I also use their estimate that a country's debt held internally is two-thirds of its total debt. To normalize the scale of the economy, I use 20 years of each country's GDP as the denominator.[16]

the RHS of equation 2 is monotone in λ, the fixed point solution is monotone in λ.

[16]The GDP is measured in 2011, to be consistent with the cross-holding data. They are \$2.861 trillion, \$3.744 trillion, \$287.8 billion, \$2.292 trillion, \$244.8 billion, and \$1.479 trillion, respectively.

The idea is to let each country choose a safe or risky economy with a 0.95 discount factor. The resulting network structure is given by

$$\Theta = \begin{bmatrix} 0 & 0.41 & 0.47 & 0.66 & 0.16 & 0.37 \\ 0.70 & 0 & 0.39 & 0.27 & 0.23 & 0.47 \\ 0.01 & 0.01 & 0 & 0.00 & 0.00 & 0.00 \\ 0.16 & 0.47 & 0.03 & 0 & 0.02 & 0.09 \\ 0.03 & 0.00 & 0.10 & 0.00 & 0 & 0.07 \\ 0.11 & 0.11 & 0.01 & 0.06 & 0.59 & 0 \end{bmatrix} \quad \bar{d} = \begin{bmatrix} 0.004 \\ 0.007 \\ 0.015 \\ 0.011 \\ 0.027 \\ 0.011 \end{bmatrix} \quad v = \begin{bmatrix} 0.009 \\ 0.013 \\ 0.029 \\ 0.022 \\ 0.054 \\ 0.021 \end{bmatrix}$$

To interpret the above matrices, $\Theta_{21} = 0.70$ means that France owes Germany 70% of France's total interbank debt, which is $\bar{d}_1 = 0.438\%$ of France's 20-year GDP. This means that France owes Germany \$175 trillion (\$2.861 trillion $\times 20 \times 0.438\% \times 70\%$). It's worth noting that the network is non-regular in that a country's total inter-country debt does not equal to its inter-country liability. To study the risk-taking incentives of each banking system, let each country choose a risk structure consisting of one of the two following choices.

$$\text{safe economy} = \begin{cases} 1 & \text{w.p} \quad 1 \\ 0 & \text{w.p} \quad 0 \end{cases} \qquad \text{risky economy} = \begin{cases} 1.1 & \text{w.p} \quad P_{risky} \\ 0 & \text{w.p} \quad 1 - P_{risky} \end{cases}$$

There are two choices for each country's banking system. Choosing a safe economy guarantees the country no economic shock. Choosing a risky economy will increase the country's output by 10% but reduces the certainty to P_{risky}. By construction, if a country is debt-free, it will choose a safe economy if $P_{risky} < 1/1.1$. For different values of P_{risky}, I will explore each country's choice of the economy on scenarios when they are (i) interconnected; (ii) stand-alone, or (iii) debt-free. To construct the counterfactual scenario where each country is stand-alone, I net-out each country's inter-country debt and add this to its internal debt v. The following table displays the identity of countries that choose the safe economy for different values of P_{risky}, ranging from 90.2% to 91%.

Countries that Choose Safe Economies

	P_{risky}			
	90.20%	90.60%	90.90%	91.00%
(i) interconnected	All	All except Greece and Portugal	None	None
(ii) stand-alone	All	All except Greece and Portugal	France	None
(iii) debt-free	All	All	All	None

We first observe that, as P_{risky} grows, it becomes less attractive for countries to choose safe economies for every scenario. This is because the risky choice's fundamental becomes better. We also see that countries choose safer economies when they are debt-free. This reflects the canonical asset substitution problem that arises from debt financing (Jensen and Meckling, 1976). Greece and Portugal are more subject to this risk-shifting due to their large amount of debt.

More interestingly, countries choose riskier economies if they are connected compared with if stand-alone, confirming corollary 1. If $P_{risky} = 90.90\%$, France chooses a safe economy when it is stand-alone, but a risky economy when it is connected. Intuitively, France anticipates its cross-subsidies to Greece and Portugal and optimally increases its own risk exposure. This illustrates the concept of endogenous systemic instability resulting from the network risk-taking distortion.

4 Extension and Policy Implications

In this section, I will first illustrate why banks have incentives to form a network in the first place. In other words, I will show why banks do not want to net out of their interbank claims and liabilities.[17] Then I will extend the benchmark model to study several widely adopted prudential policies that aim at stabilizing the financial system. Understanding the network effects of those government policies is particularly important in the current growingly connected financial systems.

4.1 Banks' Incentives to Form Networks

A natural question is why banks have incentives to form a network in the first place. In fact, in the benchmark model, a successful bank pays $\mathcal{D}(\mathbf{Z}_{-i})$ as cross-subsidies to its counterparties. It does not benefit from those cross-subsidies when it fails. In this section, I will illustrate that banks, possessing valuable expected present value of their future profits (charter values), do have incentives to form interbank connections, notwithstanding the risk-taking distortion. The introduction of banks' charter values is relevant to their risk-taking incentives.[18] It also describes reality well: in financial systems with deposit insurance, regulators (e.g., the FDIC in the U.S.) seize insolvent banks and put them into receivership.[19] As a result, banks do not want to risk defaulting on their deposits to protect their continuation values.

To model this, let $c_i \in \mathbb{R}^+$ denote bank i's charter value.[20] The bank can preserve this charter value if and only if its depositors get paid in full, either through its own project or other banks'

[17] The literature has proposed several reasons. Acemoglu et al. (2014) argue that banks may form interbank claims and liabilities because they have heterogeneous investment opportunities. Donaldson and Piacentino (2017) argues that the interbank debts embed the option to dilute with new debt to a third party. In this section, I will show that banks have incentives to form networks for co-insurance purposes.

[18] For example, Hellmann, Murdock, and Stiglitz (2000) show that reducing banks' charter values can create instability.

[19] During the global financial crisis of 2008, FDIC seized over 500 banks. For example, Washington Mutual, the sixth-largest bank in the United States at the time, ceased to exist after FDIC placed it into receivership.

[20] For expository purpose, c_i is assumed to be exogenous. One can micro-found c_i as bank i's discounted future payoff streams: $c_i = \beta/(1-\beta) \cdot \mathbb{E}[\Pi_i]$. The result is not driven by this abstraction.

cross-subsidies. From here, we can rewrite bank i's expected payoff as

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\mathbf{Z})\Big] = P(Z_i)\Big[Z_i - v - \mathcal{D}(\mathbf{Z}_{-i})\Big] + \underbrace{c_i - \Big[1 - P(Z_i)\Big]\mathrm{Pr}\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}}|\omega_i = f\Big)\cdot c_i}_{\text{expected charter value}} \tag{8}$$

$$= P(Z_i)\Big[Z_i - v - \mathcal{D}(\mathbf{Z}_{-i}) + \mathrm{Pr}\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}}|\omega_i = f\Big)\cdot c_i\Big] + c_i - \mathrm{Pr}\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}}|\omega_i = f\Big)\cdot c_i$$

where $\mathcal{F}^-_{\boldsymbol{\omega}} \equiv \{i : \omega_i = f, \sum_j \theta_{ij} d^*_j(\boldsymbol{\omega}) < v\}$ denotes the set of insolvent banks – the ones that cannot adequately reimburse their depositors. Thus $\mathrm{Pr}(i \in \mathcal{F}^-_{\boldsymbol{\omega}}|\omega_i = f)$ is the probability that bank i is insolvent given that its project fails. For example, if it is in a "fully" connected financial network as in corollary 2.(b), $\mathrm{Pr}(i \in \mathcal{F}^-_{\boldsymbol{\omega}}|\omega_i = f) = \prod_{j\neq i} 1 - P(Z_j)$. It means that bank i will become insolvent and lose its charter value only when all of its counterparties fail in addition to its own failure. In contrast, if bank i is stand-alone, it will lose its charter value simply when its own project fails. This implies that a stand-alone bank has an expected payoff of $\mathbb{E}\Big[\Pi^{SL}_i(\boldsymbol{\omega};\mathbf{Z})\Big] = P(Z_i)(Z_i - v) + P(Z_i)\cdot c_i$. Comparing this with equation 8, we observe that being in a financial network can increase the probability of a bank being solvent, hence protecting its charter value.

Because $-\mathcal{D}(\mathbf{Z}_{-i}) + \mathrm{Pr}(i \in \mathcal{F}^-_{\boldsymbol{\omega}}|\omega_i = f)\cdot c_i < c_i$, we can verify that corollary 1 still holds: connected banks choose greater risk exposure than stand-alone banks. Intuitively, there are two forces that make a connected bank choose greater risks: (i) a network risk-taking distortion as in the benchmark model, and (ii) a downside protection from the financial network's co-insurance. The second force is new here due to the introduction of banks' charter values. Both forces induce banks to become less interested in increasing the probability of success, hence creating systemic instability.

Let's now examine whether banks have incentives to form interbank connections in the face of the network risk-taking distortion. From bank i's expected payoff, it will prefer to form the connection (over stand-alone) if

$$P(Z^*_i)\Big[Z^*_i - v - \mathcal{D}(\mathbf{Z}^*_{-i})\Big] + c_i - \Big[1 - P(Z^*_i)\Big]\mathrm{Pr}\Big(i \in \mathcal{F}^-_{\boldsymbol{\omega}}|\omega_i = f\Big)\cdot c_i > P(Z^{**}_i)\Big[Z^{**}_i - v + c_i\Big] \tag{9}$$

where Z^*_i is equilibrium risk-taking of a bank in the network: $Z^*_i = \mathrm{argmax}\,\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega};\mathbf{Z}^*)\Big]$, and Z^{**}_i is the optimal risk-taking of a stand-alone bank: $Z^{**}_i = \mathrm{argmax}\,P(Z_i)(Z_i - v + c_i)$. The next proposition shows that condition 9 is possible if banks care sufficiently about their charter values.

PROPOSITION 7. *There exists $\bar{c} \in \mathbb{R}^+$, such that if $\min\{c_i\} > \bar{c}$, banks have incentives to form a network.*

Proof. In the Appendix.

To decide whether to form a network, banks face three considerations : (i) protection of their charter values, (ii) cross-subsidy $\mathcal{D}(\mathbf{Z}_{-i})$, and (iii) distorted investment Z^*. On the one hand, being in a network can protect banks from losing their valuable charter values, as it provides co-insurance to their depositors. On the other hand, due to this co-insurance, banks expect to cross-subsidize other banks, decreasing their upside payoffs. This also distorts investment and results in systemic instability. Proposition 7 states that banks will form an interbank network if they care sufficiently about their charter values.

The proposition is silent on the optimal topology that banks want to connect. A natural direction for further research is to fully endogenize the network formation while taking into account the risk-taking externalities.

4.2 Capital Requirement

So far, we have been studying banks' risk-taking equilibrium in financial networks where banks do not hold any equity. Since the 1980s, regulators began using capital adequacy requirements to ensure that banks do not take excessive risks (Gorton, 2012). With more "skin in the game", banks are less willing to gamble with their equity (Jensen and Meckling, 1976). In this section, I will extend the benchmark model to study the network effects of banks' capital when connected in financial networks.

Now suppose each bank is required to hold equity of size $r_i = r$. The amount of deposits that a bank needs to borrow decreases to $v - r$. Let's assume equity is junior to both deposits and interbank liabilities. That implies when a bank's total cash flow is smaller than its total liabilities, equity holders will be the first to incur a loss. As a result, the payment equilibrium becomes

$$d_i^*(\boldsymbol{\omega}; r) = \left\{ \min \Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; r) + e_i(\mathbf{Z}, \boldsymbol{\omega}) - v + r, \bar{d} \Big] \right\}^+ \quad \forall i \tag{10}$$

The expected profit becomes

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z})\Big] = P(Z_i)(Z_i - v + r) - P(Z_i)\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}; r) \tag{11}$$

where

$$\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}; r) \equiv \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; r) \Big) \cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})$$

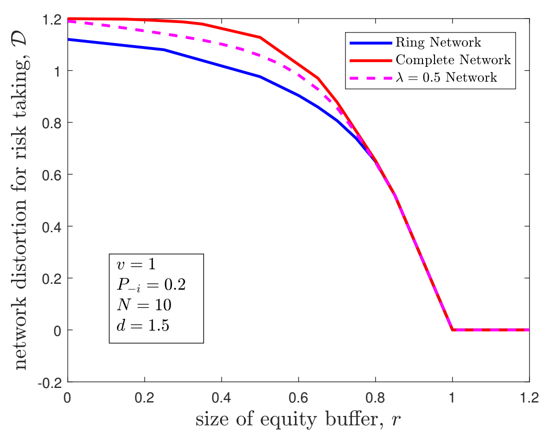
We notice that equity enters a bank's expected payoff in two ways: the upside payoff $(Z_i - v + r)$ and the network risk-taking distortion $\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}; r)$. The next proposition studies how an equity buffer will affect the network risk-taking distortion.

LEMMA 4. *In any network structure $(\bar{d}, \boldsymbol{\Theta}, N; r)$, the network risk-taking distortion $\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}; r)$ is decreasing and concave in the size of equity buffers r.*

Proof. In the Appendix.

If a bank fails, its equity holders will first incur the loss before its depositors or interbank counterparties. The loss that may be otherwise propagated to other banks will now be absorbed by this equity buffer. In other words, the equity buffer decreases the cross-subsidy that successful banks pay to failed counterparties. The network risk-taking distortion is hence reduced. Moreover, with greater equity, failed banks are more likely to become solvent, hence contributing back to the payment system. This further reduces successful banks' cross-subsidy to other failed banks. As a result, the network risk-taking distortion is decreasing at a growing rate in the size of an equity buffer. Figure 6 plots the network risk-taking distortion against the size of the equity buffer. Lemma 4 immediately implies that banks' equilibrium risk exposure will be reduced by an equity buffer.

Figure 6: Equity Buffer



PROPOSITION 8. *In any network structure* $(\bar{d}, \mathbf{\Theta}, N; r)$*, each bank's choice of risk exposure* Z_i^* *is decreasing in the size of equity* r*.*

Proof. In the Appendix.

There are two effects of an equity buffer on banks' choices of risk exposure. First, an equity buffer has a direct impact on a bank's risk-taking. A bank will choose to expose to fewer risks if it has a higher equity ratio: it is unwilling to gamble if there is more "skin in the game" (Jensen and Meckling, 1976). More interestingly, lemma 4 shows that equity buffers have a network effect on reducing systemic risks. An equity buffer curbs failed banks' loss at the origin, hence mitigating the network risk-taking distortion. This implies that one bank's equity can reduce the risk-taking incentives of its counterparties. Moreover, the strategic complementarity implies that reducing a bank's risk-taking will, in return, reduce other banks' risk-taking.

The finding suggests that when deciding banks' capital requirement, policymakers should consider not only its effect on a bank's own "skin in the game" but also the network effect on the bank's counterparties. A recently proposed rule by FRB and OCC steps in this direction by tailoring leverage ratio requirements to banks' systemic footprint. The result is also related to Erol and Ordoñez (2017), who also study the network response of capital regulation. They show that a capital requirement can discontinuously discourage interbank connections, hence reducing the ex-post co-insurance benefits. Combining our results, a tighter regulation can, on the one hand, decreases the interbank network's risk-taking externalities. On the other hand, it can also break the interbank connections if beyond a tipping point. This implies that the effect of capital requirement on systemic stability may exhibit a phase transition.

4.3 Government Bailout

The 2008 bailout of Bear Stearns and the subsequent Troubled Asset Relief Program (TARP) have sparked continuing debates among both policy-makers and academics. Government bailouts have been widely argued to incentivizes harmful ex-ante behaviors (Gale and Vives, 2002; Farhi and Tirole, 2012; Erol, 2019). In this section, I will study the effect of a government bailout on banks' ex-ante risk-taking incentives when connected in financial networks.

Similar to Erol (2019), I assume government bailouts only occur in crisis when a large number of banks have failed.[21] I define a government bailout (n, t) as a transfer t from the government to each failed bank if the number of failed banks exceeds n. With the government bailout in place, the payment equilibrium becomes

$$d_i^*(\boldsymbol{\omega}; \mathbf{Z}) = \left\{ \min \Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + e_i(\omega_i, Z_i) + t_i(\boldsymbol{\omega}) - v, \bar{d}, \Big] \right\}^+ \quad \forall i \in \mathcal{N} \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega} \tag{12}$$

where the transfer is state-contingent and is defined as

$$t_i(\boldsymbol{\omega}) \equiv t \cdot \mathbb{1}(\omega_i = f) \cdot \mathbb{1}(\#\text{ failed banks} \geqslant n)$$

The rest of the definition for the network risk-taking equilibrium remains unchanged from equation 4. The following proposition shows that a government bailout can contribute to systemic stability by reducing the network risk-taking distortion.

PROPOSITION 9. *In any financial network $(\bar{d}, \boldsymbol{\Theta}, N)$, each bank's network risk-taking distortion and equilibrium risk exposure is reduced if there exists a government bailout.*

Proof. In the Appendix.

In contrast to the conventional wisdom, proposition 9 states that a credible government bailout can instead discourage ex-ante risk-taking. During Crises, a government bailout can curb the loss before spreading to successful banks. The network risk-taking distortion resulting from cross-subsidy is hence reduced. This will encourage connected banks to reduce their choices of risk exposure.

[21] This definition is consistent with section 101 of the 2008 Emergency Economic Stabilization Act (EESA). It states "TARP was only part of the government's response to the crisis" and is "to restore liquidity and stability to the financial system". See https://www.treasury.gov/initiatives/financial-stability/about-tarp.

Proposition 9 stands in contrast to Erol (2019), who show that a government bailout can create systemic instability by encouraging excessive network formation. In his model, banks will not worry about contagion during network formation if there exists a government bailout. In contrast, this paper shows that because banks do not worry about the cross-subsidy, they will be subject to less network risk-taking externalities. While both effects (excessive network formation and less risk-taking externalities) are reasonable, the net effect of a government bailout is an empirical question.

5 Correlated Risk Exposure

In previous sections, we assumed that banks' project outcomes are independent. While this is a reasonable assumption for local banks serving mortgages in different regions, large national banks' portfolios may be well correlated. In this section, I model each bank's decision whether to expose to correlated risks and explain why a systemic crisis can endogenously evolve due to interbank connectedness.

Suppose each bank, besides choosing its project outcome's marginal distribution $P(Z_i)$, also chooses its conditional distribution $\boldsymbol{\lambda}_i = [\lambda_1, ..., \lambda_N]$ on the project outcomes of other connected banks in the network. Define the matrix $\boldsymbol{\Lambda} \equiv [\lambda_{ij}]$ as

$$\lambda_{ij} \equiv Pr(\omega_i = s | \omega_j = s)$$

where $\mathbf{0} \leqslant \boldsymbol{\lambda}_i \leqslant \mathbf{1}$. We can interpret λ_{ij} as bank i's choices of correlation with bank j. This notion of pairwise conditional probabilities matrix was proposed in the IMF's *Global Financial Stability Review* (2009) and later utilized by Bisias et al. (2012).

From the above definition, the pairwise correlation between bank i and j's projects is

$$\rho_{ij} \equiv \frac{\lambda_{ij} P(Z_j) - P(Z_i) P(Z_j)}{P(Z_i)^{1/2} P(Z_j)^{1/2} [1 - P(Z_i)]^{1/2} [1 - P(Z_j)]^{1/2}}$$

In contrast to the benchmark model, each banks' project outcomes are no longer independent. Bank i's expected profit becomes

$$\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; Z_i, \lambda_i)\Big] = P(Z_i)(Z_i - v) - \sum_{\boldsymbol{\omega}_{-i}} \Big(\bar{d} - \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{s}})\Big) \cdot \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}}) \cdot \Pr(\omega_i = s | \boldsymbol{\omega}_{-\boldsymbol{i}})$$

The above equation uses the property $\Pr(\boldsymbol{\omega}^{\boldsymbol{i}=\boldsymbol{s}}) = \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}}) \cdot \Pr(\omega_i = s | \boldsymbol{\omega}_{-\boldsymbol{i}})$. The dependence vector $\boldsymbol{\lambda}_{\boldsymbol{i}}$ enters the last term.

DEFINITION 5. *The correlated risk-taking equilibrium in a financial network $(\bar{d}, \boldsymbol{\Theta}, N)$ is a triplet $(\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z}), \boldsymbol{Z}^*, \boldsymbol{\lambda}^*)$ consisting of a vector of payment functions $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$, a vector of risk exposure $\boldsymbol{Z}^*$, and a matrix of conditional distribution $\boldsymbol{\Lambda}^* \equiv [\lambda_{ij}^*]$ such that:*

1. *The vector of functions $\boldsymbol{d}^*(\boldsymbol{\omega}; \mathbf{Z})$ is a payment equilibrium for any $\mathbf{Z}$.*

$$d_i^*(\boldsymbol{\omega}; \mathbf{Z}) = \left\{ \min \left[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + e_i(\omega_i, Z_i) - v, \bar{d}, \right] \right\}^+ \quad \forall i \in \mathcal{N} \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega}$$

2. *For each bank $i \in \mathcal{N}$, $(Z_i^*, \boldsymbol{\lambda}_i^*)$ is optimal and solves the following equation, given $\boldsymbol{d}^*(\boldsymbol{\omega}; \mathbf{Z})$, $\mathbf{Z}_{-i}^*$ and $\boldsymbol{\Lambda}_{-i}^*$*

$$(Z_i^*, \boldsymbol{\lambda}_i^*) = \underset{\substack{\underline{Z} \leqslant Z_i \leqslant \overline{Z} \\ \mathbf{0} \leqslant \boldsymbol{\lambda}_i \leqslant \mathbf{1}}}{\operatorname{argmax}} \mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; \mathbf{Z}_{-i}^*, \boldsymbol{\lambda}_{-i}^*)\Big] \quad \forall i \in \mathcal{N}$$

3. *The pairwise correlations are compatible among all banks. i.e. $\boldsymbol{\rho} = [\rho_{ij}]$ is symmetric and positive semi-definite.*

Part 2 of the above definition implies that banks are unrestricted in choosing their conditional dependence with their counterparties. Any bank can choose a project that is arbitrarily correlated with any other bank: a notion similar to Denti (2018). However, part 3 of the above definition states that the conditional dependence has to be mutually and jointly compatible in equilibrium.[22] Part 3 also implies $\lambda_{ij}^*/\lambda_{ji}^* = P(Z_i^*)/P(Z_j^*)$ for all i, j. This shows a dependence between $\boldsymbol{\lambda}$ and $\mathbf{Z}$. In equilibrium, the marginal and conditional distribution should also be compatible.

PROPOSITION 10. *In any network structure $(\bar{d}, \boldsymbol{\Theta}, N)$, the correlated risk-taking equilibrium exists and every bank's risk exposure is perfectly correlated: $\lambda_{ij}^* = 1$ for all $i, j \in \mathcal{N}$.*

Proof. In the Appendix.

Proposition 10 states that connected banks will coordinate to expose to one single systemic risk. In anticipation of the interbank transfers (cross-subsidy) to failed banks, each bank will optimally align their project outcomes with other connected banks, for any chosen risk exposure. By doing so, there will be no downward distortion when the bank's project succeeds, and hence each bank will enjoy a higher expected profit. The perfect correlation, however, will be harmful to the economy as a whole. Since every bank chooses to exposure to one single systemic risk, there is no co-insurance among economic agents.

Proposition 10 predicts that a financial crisis will be more likely to *endogenously* evolve in a highly connected banking system. It confirms the empirical findings of International Monetary Fund (2009) and Bisias et al. (2012) that there existed a large distress dependence among major banks before the 2008 financial crisis when the banking system became unprecedentedly connected. The result is also consistent with the observation of Acharya (2009), who argues that

[22] For example, $\rho_{ij} = 1$ and $\rho_{ji} = 0$ is not compatible because $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not symmetric. For another example, $\rho_{ij} = 1$, $\rho_{jk} = 1$, $\rho_{ik} = 0$ is not compatible because $\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$ is not positive semi-definite.

banks choose correlated investments due to a pecuniary externality: a failed bank reduces counterparties' profitability through an increase in the market-clearing rate for deposits. The result is also related to recent papers such as Elliott et al. (2018) and Jackson and Pernoud (2020). Using data on German banks, Elliott et al. (2018) illustrate banks' incentive to form partners with similar portfolios. Jackson and Pernoud (2020) show that banks have incentives to minimize the set of states where they pay debts and have have their values diluted.

6 Discussion and Concluding Remarks

This paper studies banks' incentives to choose their risk exposure in financial networks, where banks are connected through cross-holdings of unsecured debts. In contrast to previous literature that focuses on the co-insurance mechanism for exogenous shocks, I show that connected banks ex-ante choose to expose to greater risks. In addition, they choose to expose to correlated risks, aggravating the systemic fragility. Nevertheless, banks do have incentives to form a network as it provides co-insurance to their charter values.

This paper brings about several testable empirical predictions. For example, the strategic complementarity result suggests that an individual bank's risk-taking is positively related to the risks of the entire financial system. This is exactly what Mink, Ramcharan, and van Lelyveld (2020) have found. Using granular bond portfolio of EU banks, they find that regulatory solvency shocks (proxied by the banking system's tier 1 capital ratio) can induce banks to shift into riskier assets (higher-yielding sovereign debt) and correlated assets (domestic bonds).

Another interesting real-world example of financial networks is the credit union industry. Individual natural person credit unions (NPCU), like community banks, make loans to and take deposits from local consumers. Geographically proximate NPCUs often form interbank networks through a corporate credit union (CCU), commonly referred to as "the credit union's credit union".[23] One empirical prediction of this paper is that the NPCUs in highly connected CCUs choose riskier loan portfolios.

By studying banking networks, this paper sheds new insights on several government policies. For example, the paper formalizes the conjecture that a CCP, although providing co-insurance to its member banks, can create moral hazard and systemic instability. The model also suggests that capital regulation should consider banks' systemic footprint. However, the paper does not aim to design actual government policies or provide a holistic study of each particular policy. A natural step for further research is to examine how interbank connectedness can affect different aspects of government policies.

[23] In 2005, there were around 7500 NPCUs and 26 CCUs. For more details about NPCUs and CCUs, see Ramcharan et al. (2016). They also document that geography proximity is an important factor in explaining the topology of CCUs, lending variations for empirical identification.

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PROOFS

PROOF OF LEMMA 2: From the assumption $\underline{Z} \geqslant v + \bar{d}$, a successful bank's interbank payment is $d_i = \bar{d}$, independent of its choice of risk exposure Z_i. A failed bank's cash flow that will contribute to the interbank payment system is $e_i = 0$, also independent of its choice of risk exposure. Reordering equation 2 gives us,

$$\begin{aligned} d_i^*(\boldsymbol{\omega}; \boldsymbol{Z}) &= \bar{d} && \forall \omega_i = s \\ d_i^*(\boldsymbol{\omega}; \boldsymbol{Z}) &= \left\{ \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \boldsymbol{Z}) - v \right\}^+ && \forall \omega_i = f \end{aligned} \tag{13}$$

We can see that the vector of risk exposure $\boldsymbol{Z}$ does not enter the system of equations. As a result, the fixed point $(d_1^*(\boldsymbol{\omega}), ...d_N^*(\boldsymbol{\omega}))$ is constant in $\boldsymbol{Z}$. □

Before proving proposition 1, it is useful to have the following auxiliary lemma.

AUXILIARY LEMMA: *the payment vector $\boldsymbol{d}^*$ is weakly increasing in any bank's cash flow $\tilde{e}_j$. In particular, $\boldsymbol{d}^*(\boldsymbol{\omega})$ is higher when any bank's project succeeds ($\omega_j = s$) compared with when it fails ($\omega_j = f$).*[24]

Proof. The above lemma is identical to Eisenberg and Noe (2000) Lemma 5. The payment equilibrium (equation 2) is a fixed point solution of a function $\boldsymbol{d}^* = \Phi(\boldsymbol{d}^*; \tilde{e}_j)$. Since both min and max operator preserve monotonicity, Φ is increasing in $\tilde{e}_j$. By monotone selection theorem (Milgrom and Roberts, 1990 Theorem 1), the fixed point $\boldsymbol{d}^*$ is increasing in $\tilde{e}_j$.

□

PROOF OF PROPOSITION 1: Taking the first- and second-order conditions of the equation 6, we have

$$\begin{aligned} F(Z_i; \boldsymbol{Z_{-i}}) &= P'(Z_i)(Z_i - v) + P(Z_i) - P(Z_i)'\mathcal{D}(\boldsymbol{Z_{-i}}) = 0 \\ S(Z_i; \boldsymbol{Z_{-i}}) &= P''(Z_i)(Z_i - v) + 2P'(Z_i) - P(Z_i)''\mathcal{D}(\boldsymbol{Z_{-i}}) < 0 \end{aligned}$$

From assumption 1, we obtain $S(Z_i; \boldsymbol{Z_{-i}}) < 0$. From the total derivative of the FOC, we have

$$\frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}\mathcal{D}(\boldsymbol{Z_{-i}})} = -\frac{\partial F(\widehat{Z}_i; \boldsymbol{Z_{-i}})/\partial \mathcal{D}(\boldsymbol{Z_{-i}})}{\partial F(\widehat{Z}_i; \boldsymbol{Z_{-i}})/\partial Z_i} = \frac{P'(\widehat{Z}_i)}{S(\widehat{Z}_i; \boldsymbol{Z_{-i}})} > 0 \tag{14}$$

The above inequality implies that whatever increases $\mathcal{D}(\boldsymbol{Z_{-i}})$ will increase bank i's optimal $\widehat{Z}_i$. Intuitively, the distortion $\mathcal{D}(\boldsymbol{Z_{-i}})$ decreases bank i's upside payoff (the payoff when its project succeeds). As a result, it will make bank i care less about the probability of success when trading off risk and return.

To see the effect from bank m's risk exposure Z_m on bank i's risk-taking distortion $\mathcal{D}(\boldsymbol{Z_{-i}})$, let's vary it from Z_m to Z'_m with $Z'_m > Z_m$. Let $\boldsymbol{Z'_{-i}}$ denote the new risk-exposure vector that differs from $\boldsymbol{Z_{-i}}$ only

[24]Throughout this paper, whenever the ordering of a vector is mentioned, it refers to a pointwise ordering. i.e $\boldsymbol{x} \geqslant \boldsymbol{y} \Leftrightarrow x_i \geqslant y_i$ for all i. For the following text, all orderings are weakly.

in Z_m. We have

$$
\begin{aligned}
&\mathcal{D}(\boldsymbol{Z'_{-i}}) - \mathcal{D}(\boldsymbol{Z_{-i}}) \\
&\quad = \sum_{\boldsymbol{\omega_{-i-m}}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[\Pr(Z'_m) \Big(\bar{d} - \sum_j \theta_{ij} d^*_j (\boldsymbol{\omega^{m=s}}) \Big) + \Big(1 - \Pr(Z'_m) \Big) \Big(\bar{d} - \sum_j \theta_{ij} d^*_j (\boldsymbol{\omega^{m=f}}) \Big) \Big] \\
&\quad\quad - \sum_{\boldsymbol{\omega_{-i-m}}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[\Pr(Z_m) \Big(\bar{d} - \sum_j \theta_{ij} d^*_j (\boldsymbol{\omega^{m=s}}) \Big) + \Big(1 - \Pr(Z_m) \Big) \Big(\bar{d} - \sum_j \theta_{ij} d^*_j (\boldsymbol{\omega^{m=f}}) \Big) \Big] \\
&\quad = \sum_{\boldsymbol{\omega_{-i-m}}} \Pr(\boldsymbol{\omega_{-i-m}}) \Big[\Big(\Pr(Z'_m) - \Pr(Z_m) \Big) \Big(\sum_j \theta_{ij} d^*_j (\boldsymbol{\omega^{m=f}}) - \sum_j \theta_{ij} d^*_j (\boldsymbol{\omega^{m=s}}) \Big) \Big] \geqslant 0
\end{aligned}
\tag{15}
$$

With slight abuse of notation, $\boldsymbol{\omega_{-i-m}}$ denotes a vector of $\boldsymbol{\omega}$ without the element i and m, $\boldsymbol{\omega^{m=s}}$ denotes a vector that appends $\boldsymbol{\omega_{-i-m}}$ with $\omega_m = s$ and $\omega_i = s$, and $\boldsymbol{\omega^{m=f}}$ denotes a vector that appends $\boldsymbol{\omega_{-i-m}}$ with $\omega_m = f$ and $\omega_i = s$. The last inequality is from the auxiliary lemma. The inequality states that bank i's risk-taking distortion is increasing in bank m's risk-exposure. To see the intuition, suppose bank m has a greater risk exposure Z_m, its project becomes more likely to fail. When bank m's project fails, bank i's net interbank payments to other banks will increase due to a greater amount of cross-subsidy. Finally, joining equation 14 and 15, we have

$$
\frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}Z_{-i}} = \frac{\mathrm{d}\widehat{Z}_i}{\mathrm{d}\mathcal{D}(\boldsymbol{Z_{-i}})} \frac{\mathrm{d}\mathcal{D}(\boldsymbol{Z_{-i}})}{\mathrm{d}Z_{-i}} > 0 \quad \forall i \quad \text{and} \quad -i
$$

□

PROOF OF PROPOSITION 2: The payment equilibrium in any state of nature is the fixed-point solution to a system of equations (equation 13). Denote the fixed point as $\boldsymbol{d}^* = \Phi(\boldsymbol{d}^*)$, where Φ a continuous mapping with a convex and compact domain $[0, \bar{d}]^N$. By the Brouwer fixed point theorem, the payment equilibrium $\boldsymbol{d}^*(\boldsymbol{\omega}; \boldsymbol{Z})$ exists for all $\boldsymbol{\omega}$ and $\boldsymbol{Z}$ (Eisenberg and Noe, 2001). This establishes the existence of the payment equilibrium for all $\boldsymbol{\omega}$ and $\boldsymbol{Z}$. From proposition 1, $\mathrm{d}\widehat{Z}_i / Z_{-i} \geqslant 0$ for all i and $-i$. It implies the Nash equilibrium is a supermodular game. The domain for the risk-exposure vector $[\underline{Z}, \bar{Z}]^N$ is a complete lattice. By Tarski's theorem, the fixed-point solution to the first order conditions $F(Z^*_i; \boldsymbol{Z^*_{-i}}) = 0$ exists. The equilibrium risk exposure $\boldsymbol{Z}^* = (Z^*_1, ..., Z^*_N)$ is this fixed point. □

PROOF OF CORROLARY 1: Denote Z^N and Z^S as the equilibrium risk exposure of a bank in a financial network and a stand-alone bank respectively. Formally, they are the solutions to their respective first order conditions, i.e.

$$
\begin{aligned}
&P'(Z^N)(Z^N - v) + P(Z^N) - P(Z^N)' \mathcal{D}(\boldsymbol{Z^N}) = 0 \\
&P'(Z^S)(Z^S - v) + P(Z^S) = 0
\end{aligned}
$$

By equation 14, $\mathrm{d}Z^N / \mathrm{d}\mathcal{D}(\boldsymbol{Z^N}) > 0$. We also know that the distortion $\mathcal{D}(\boldsymbol{Z^N})$ is positive because $P(Z_j) < 1$ for all Z_j. Therefore, $Z^N > Z^S$. □

PROOF OF LEMMA 3: For any state of nature $\boldsymbol{\omega}$, conjecture that there exists two vectors, $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$, such that $d^*_i(\boldsymbol{\omega}) = \{a_i(\boldsymbol{\omega})\bar{d} - b_i(\boldsymbol{\omega})v\}^+$. By definition, they should satisfy equation 13. After plugging $\boldsymbol{a}(\boldsymbol{\omega})$

and $\boldsymbol{b}(\boldsymbol{\omega})$ into equation 13, we have $(a_i, b_i) = (1, 0)\ \forall \omega_i = s$, and

$$
\begin{aligned}
d_i^*(\boldsymbol{\omega}) &= \left\{ \sum_{\omega_j = s} \theta_{ij}\bar{d} + \sum_{j \in \mathcal{F}_{\boldsymbol{\omega}}^+} \theta_{ij}\Big(a_j(\boldsymbol{\omega})\bar{d} - b_j(\boldsymbol{\omega})v\Big) - v \right\}^+ \\
&= \left\{ \Big(\sum_{j \in \mathcal{F}_{\boldsymbol{\omega}}^+} \theta_{ij} a_j(\boldsymbol{\omega}) + \sum_{\omega_j = s} \theta_{ij} \Big)\bar{d} - \Big(\sum_{j \in \mathcal{F}_{\boldsymbol{\omega}}^+} \theta_{ij} b_j(\boldsymbol{\omega}) + 1 \Big)v \right\}^+ \qquad \forall \omega_i = f
\end{aligned}
$$

where $\mathcal{F}_{\boldsymbol{\omega}}^+ \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v \geqslant 0\}$. We call it "solvent" failed banks. Similarly, define $\mathcal{F}_{\boldsymbol{\omega}}^- \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v < 0\}$ as the "insolvent" failed banks, and $\mathcal{S}_{\boldsymbol{\omega}} \equiv \{i : \omega_i = s)$ as successful banks. Per the conjecture, we need $\forall \omega_i \in f$,

$$
a_i(\boldsymbol{\omega}) = \sum_{j \in \mathcal{F}_{\boldsymbol{\omega}}^+} \theta_{ij} a_j(\boldsymbol{\omega}) + \sum_{\omega_j = s} \theta_{ij} \tag{16}
$$

$$
b_i(\boldsymbol{\omega}) = \sum_{j \in \mathcal{F}_{\boldsymbol{\omega}}^+} \theta_{ij} b_j(\boldsymbol{\omega}) + 1 \tag{17}
$$

Since the RHS of above equations are increasing in $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$ respectively, the fixed points exist by the Tarski's theorem. The conjecture is hence verified. Let's rewrite the above equations in a matrix form for banks in $\mathcal{F}_{\boldsymbol{\omega}}^+$.

$$
\boldsymbol{a}_+(\boldsymbol{\omega}) = \boldsymbol{\Theta}_{++}\boldsymbol{a}_+(\boldsymbol{\omega}) + \boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s \tag{18}
$$

$$
\boldsymbol{b}_+(\boldsymbol{\omega}) = \boldsymbol{\Theta}_{++}\boldsymbol{b}_+(\boldsymbol{\omega}) + \mathbb{1}_+ \tag{19}
$$

where $\boldsymbol{a}_+(\boldsymbol{\omega})$ and $\boldsymbol{b}_+(\boldsymbol{\omega})$ are truncated vectors of $\boldsymbol{a}(\boldsymbol{\omega})$ and $\boldsymbol{b}(\boldsymbol{\omega})$ with rows that belong to $\mathcal{F}_{\boldsymbol{\omega}}^+$. Similarly, $\boldsymbol{\Theta}_{++}$ is a truncated matrix of $\boldsymbol{\Theta}$ with rows and columns that belong to $\mathcal{F}_{\boldsymbol{\omega}}^+$, and $\boldsymbol{\Theta}_{+\boldsymbol{s}}$ is the truncated matrix of $\boldsymbol{\Theta}$ where each row belongs to $\mathcal{F}_{\boldsymbol{\omega}}^+$ and each column belongs to $\mathcal{S}$. $\mathbb{1}_+$ and $\mathbb{1}_s$ are column vectors of ones with appropriate dimension. Note that $\boldsymbol{\Theta}_{++}$, $\boldsymbol{\Theta}_{+\boldsymbol{s}}$, $\mathbb{1}_+$, and $\mathbb{1}_s$ are all state-contingent. To conserve space, I suppress their underscript $\boldsymbol{\omega}$.

By the Markovian property of $\boldsymbol{\Theta}$ (row-sum equals to one), we have $\boldsymbol{\Theta}_{++}\mathbb{1}_+ + \boldsymbol{\Theta}_{+-}\mathbb{1}_- + \boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s = \mathbb{1}_+$. By equation 18

$$
\boldsymbol{a}_+(\boldsymbol{\omega}) = (\boldsymbol{I}_+ - \boldsymbol{\Theta}_{++})^{-1}\boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s < \mathbb{1}_+ \tag{20}
$$

After plugging $(\boldsymbol{a}_+, \boldsymbol{b}_+)$ into the network risk-taking distortion, We can rewrite $\mathcal{D}(\boldsymbol{Z}_{-\boldsymbol{i}})$ in a matrix form as

$$
\begin{aligned}
\mathcal{D}(\boldsymbol{Z}_{-\boldsymbol{i}}) &= \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\bar{d} - \Big(\Theta_{is}\mathbb{1}_s\bar{d} + \Theta_{i+}(\boldsymbol{a}_+\bar{d} - \boldsymbol{b}_+ v) + \Theta_{i-}\cdot 0\Big)\Big] \\
&= \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{i+}\Big((\mathbb{1}_+ - \boldsymbol{a}_+)\bar{d} + \boldsymbol{b}_+ v\Big) + \boldsymbol{\Theta}_{i-}\mathbb{1}_-\bar{d} + \boldsymbol{\Theta}_{is}\mathbb{1}_s \cdot 0\Big]
\end{aligned} \tag{21}
$$

Each part of the above definition has a clean interpretation: $\boldsymbol{\Theta}_{i+}[(\mathbb{1}_+ - \boldsymbol{a}_+)\bar{d} + \boldsymbol{b}_+ v]$ is bank i's subsidy to "solvent" failed banks , $\boldsymbol{\Theta}_{i-}\mathbb{1}_-\bar{d}$ is bank i's subsidy to "insolvent" failed banks, and $\boldsymbol{\Theta}_{is}\mathbb{1}_s \cdot 0$ is bank i's subsidy to successful banks.

To prove lemma 3, compare three financial networks with same $\boldsymbol{\Theta}$ and N but different interbank liabilities $\bar{d}_1$, $\bar{d}_2$, and $\bar{d}_3$, with $\bar{d}_3 - \bar{d}_2 = \bar{d}_2 - \bar{d}_1 = \varepsilon$. To prove the monotonicity and concavity, it suffices to prove $\mathcal{D}^3(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^2(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^1(\mathbf{Z}_{-i})$ and $\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i})$ with inequality happens somewhere.

Observe that $\mathcal{F}^+_{\boldsymbol{\omega}} \equiv \{i : \omega_i = f, a_i\bar{d} - b_i v \geqslant 0\}$ is a function of $\bar{d}$. We hence denote $\mathcal{F}^+_1(\boldsymbol{\omega})$, $\mathcal{F}^+_2(\boldsymbol{\omega})$, and $\mathcal{F}^+_3(\boldsymbol{\omega})$ the set of "solvent" failed bank in state $\boldsymbol{\omega}$ for network $(\bar{d}_1, \boldsymbol{\Theta}, N)$, $(\bar{d}_2, \boldsymbol{\Theta}, N)$, and $(\bar{d}_3, \boldsymbol{\Theta}, N)$ respectively. By monotone selection theorem (see auxiliary lemma), $d^{3*}_i(\boldsymbol{\omega}) \geqslant d^{2*}_i(\boldsymbol{\omega}) \geqslant d^{1*}_i(\boldsymbol{\omega}), \forall i \in \mathcal{N}$ and $\boldsymbol{\omega} \in \Omega$. That implies $\mathcal{F}^+_1(\boldsymbol{\omega}) \subseteq \mathcal{F}^+_2(\boldsymbol{\omega}) \subseteq \mathcal{F}^+_3(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \Omega$. It means that increasing $\bar{d}$ can make more failed banks "solvent".

Let's consider the following four cases: (1) $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$. (2) $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. (3) $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$. (4) $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

Case I: $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$

From equation 18 and 19, it's easy to see that $\boldsymbol{a}^{\mathbf{1}}_+ = \boldsymbol{a}^{\mathbf{2}}_+ = \boldsymbol{a}^{\mathbf{3}}_+$ and $\boldsymbol{b}^{\mathbf{1}}_+ = \boldsymbol{b}^{\mathbf{2}}_+ = \boldsymbol{b}^{\mathbf{3}}_+$. We also have $\boldsymbol{\Theta}_{\boldsymbol{i}+}$, $\mathbb{1}_+$, $\boldsymbol{\Theta}_{\boldsymbol{i}-}$, and $\mathbb{1}_-$ unchanged for the three networks. Therefore,

$$\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\mathbb{1}_+ - \boldsymbol{a}_+)(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}_{\boldsymbol{i}-}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2) \Big] > 0$$
$$\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\mathbb{1}_+ - \boldsymbol{a}_+)(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}_{\boldsymbol{i}-}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1) \Big] > 0$$

The last inequality is due to equation 20. With $\bar{d}_3 - \bar{d}_2 = \bar{d}_2 - \bar{d}_1 = \varepsilon$, we have $\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) = \mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) > 0$. Intuitively, this case means that the network risk-taking is linearly increasing in $\bar{d}$, if the change of $\bar{d}$ does not make additional "insolvent" banks "solvenet".

Case II: $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

We first compare the interbank liabilities $\bar{d}_2$ with $\bar{d}_1$. In some state of nature $\boldsymbol{\omega}$, some otherwise "insolvent" failed banks for $(\bar{d}_1, \boldsymbol{\Theta}, N)$ become "solvent" for $(\bar{d}_2, \boldsymbol{\Theta}, N)$. Denote those banks $t_1, t_2, ..., t_T$, where $T \geqslant 1$. Due to continuity of the payment equilibrium in terms of $\bar{d}$ (equation 2). There exists $\bar{d}_1 < \tilde{d}_1 < \tilde{d}_2 ... < ... < \tilde{d}_S < \bar{d}_2$ (where $1 \leqslant S \leqslant T$), such that when the interbank liabilities $\bar{d} = \tilde{d}_s$, some bank t_t is exactly "solvent", or $\tilde{a}_t(\boldsymbol{\omega})\tilde{d}_s - \tilde{b}_t(\boldsymbol{\omega})v = 0$. In other words, this margin bank t is "solvent" when $\bar{d} \in [\tilde{d}_s, \tilde{d}_{s+1})$ and "insolvent" when $\bar{d} \in (\tilde{d}_{s-1}, \tilde{d}_s]$ respectively. Denote $\widetilde{\mathcal{D}}^s(\mathbf{Z}_{-i})$ the network risk-taking distortion at those cut-offs $\tilde{d}_s$. We have

$$\begin{aligned}
\mathcal{D}^2(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}^S(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}^{\mathbf{2}}_{\boldsymbol{i}+}(\mathbb{1}_+ - \boldsymbol{a}^{\mathbf{2}}_+)(\bar{d}_2 - \tilde{d}_S) + \boldsymbol{\Theta}^{\mathbf{2}}_{\boldsymbol{i}-}\mathbb{1}_-(\bar{d}_2 - \tilde{d}_S) \Big] > 0 \\
\widetilde{\mathcal{D}}^{s+1}(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}^s(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{\boldsymbol{i}+}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}^{\boldsymbol{s}}_+)(\tilde{d}_{s+1} - \tilde{d}_s) + \widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{\boldsymbol{i}-}\mathbb{1}_-(\tilde{d}_{s+1} - \tilde{d}_s) \Big] > 0 \\
\widetilde{\mathcal{D}}^1(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i}) \Big[\boldsymbol{\Theta}^{\mathbf{1}}_{\boldsymbol{i}+}(\mathbb{1}_+ - \boldsymbol{a}^{\mathbf{1}}_+)(\tilde{d}_1 - \bar{d}_1) + \boldsymbol{\Theta}^{\mathbf{1}}_{\boldsymbol{i}-}\mathbb{1}_-(\tilde{d}_1 - \bar{d}_1) \Big] > 0
\end{aligned} \tag{22}$$

where each column of $\boldsymbol{\Theta}^{\mathbf{1}}_{\boldsymbol{i}+}$ corresponds to an "solvent" failed bank at the state $\boldsymbol{\omega}$ in a network with

$\bar{d} \in [\bar{d}_1, \tilde{d}_1]$. Each column of $\widetilde{\boldsymbol{\Theta}}^s_{i+}$ corresponds to an "solvent" failed bank at the state $\boldsymbol{\omega}$ in a network with $\bar{d} \in [\tilde{d}_s, \tilde{d}_{s+1}]$. Each column of $\boldsymbol{\Theta}^2_{i+}$ corresponds to an "solvent" failed bank at the state $\boldsymbol{\omega}$ in a network with $\bar{d} \in [\tilde{d}_S, \bar{d}_2]$. The same notation applies to $\boldsymbol{a}_+$ and $\widetilde{\boldsymbol{\Theta}}_{i-}$ as well. They are state-contingent, and to conserve space we suppress the underscript.

The above inequalities show that $\mathcal{D}^2(\mathbf{Z}_{-i}) \geqslant \widetilde{\mathcal{D}}^S(\mathbf{Z}_{-i}) \geqslant \ldots \geqslant \widetilde{\mathcal{D}}^2(\mathbf{Z}_{-i}) \geqslant \widetilde{\mathcal{D}}^1(\mathbf{Z}_{-i}) \geqslant \mathcal{D}^1(\mathbf{Z}_{-i})$ and hence the monotonicity result follows. To prove the concavity, we observe that $\widetilde{\boldsymbol{\Theta}}^s_{i+}\mathbb{1}_+ + \widetilde{\boldsymbol{\Theta}}^s_{i-}\mathbb{1}_- = \boldsymbol{\Theta}_{if}\mathbb{1}_f$ for all s and $\boldsymbol{\omega}$. By definition, $\widetilde{\boldsymbol{\Theta}}^s_{i+}$ and $\tilde{\boldsymbol{a}}^s_+$ are sub-matrix of $\widetilde{\boldsymbol{\Theta}}^{s+1}_{i+}$ and $\tilde{\boldsymbol{a}}^{s+1}_+$ respectively. Hence we have

$$\widetilde{\boldsymbol{\Theta}}^s_{i+}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}^s_+) + \widetilde{\boldsymbol{\Theta}}^s_{i-}\mathbb{1}_- > \widetilde{\boldsymbol{\Theta}}^{s+1}_{i+}(\mathbb{1}_+ - \tilde{\boldsymbol{a}}^{s+1}_+) + \widetilde{\boldsymbol{\Theta}}^{s+1}_{i-}\mathbb{1}_- \qquad \forall s = 1, \ldots, S-1$$

After summing every difference in equation 22 and replacing all of RHS with the first line, i.e. the smallest, we have

$$\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) > \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^2_{i+}(\mathbb{1}_+ - \boldsymbol{a}^2_+)(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}^2_{i-}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

Since $\mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$, we have the following identity as in case I,

$$\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^2_{i+}(\mathbb{1}_+ - \boldsymbol{a}^2_+)(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}^2_{i-}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$

Hence $\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) < \mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i})$ and the concavity follows.

Intuitively, this case means that the network risk-taking distortion is increasing in $\bar{d}$, but at a slower rate. This is because the change of $\bar{d}$ (from $\bar{d}_1$ to $\bar{d}_2$) makes some "insolvent" banks "solvenet" in some state of nature.

Case III: $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

The proof is identical to case II with a slight twist. Instead of replacing all RHS of equation 22 with the first line, we replace it with the last line. Hence, we obtain,

$$\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) < \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^2_{i+}(\mathbb{1}_+ - \boldsymbol{a}^2_+)(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}^2_{i-}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$
$$\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^2_{i+}(\mathbb{1}_+ - \boldsymbol{a}^2_+)(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}^2_{i-}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

The monotonicity and concavity result follows.

Case IV: $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

The proof is a combination of case 2 and case 3:

$$\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) < \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^2_{i+}(\mathbb{1}_+ - \boldsymbol{a}^2_+)(\bar{d}_3 - \bar{d}_2) + \boldsymbol{\Theta}^2_{i-}\mathbb{1}_-(\bar{d}_3 - \bar{d}_2)\Big]$$
$$\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) > \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^2_{i+}(\mathbb{1}_+ - \boldsymbol{a}^2_+)(\bar{d}_2 - \bar{d}_1) + \boldsymbol{\Theta}^2_{i-}\mathbb{1}_-(\bar{d}_2 - \bar{d}_1)\Big]$$

The monotonicity and concavity result follows.

Because $\mathcal{F}_1^+(\omega) \subseteq \mathcal{F}_2^+(\omega) \subseteq \mathcal{F}_3^+(\omega)$ for all $\omega \in \Omega$, Case I-IV (or some combination of them) exhaust all the possibilities. Intuitively, the proof shows that the network risk-taking distortion is increasing in $\bar{d}$, but at a slower rate. This is because the change of $\bar{d}$ makes some "insolvent" banks "solvenet" and this decreases the marginal effect of $\bar{d}$. □

PROOF OF PROPOSITION 3: By lemma 1, the Nash Equilibrium for risk exposure $\mathbf{Z}^*$ is a supermodular game. By lemma 3, bank's expected profit exhibits an increasing difference in Z_i and $\bar{d}$. Then the Pareto-dominant equilibrium risk exposure is increasing in $\bar{d}$ (Milgrom and Roberts 1990, theorem 6). □

PROOF OF COROLLARY 2:
Part (a): Let $\widetilde{\mathbf{\Theta}}$ denote the largest path-connected sub-network of $\mathbf{\Theta}$ where bank i belongs to. Suppose when $\bar{d} = \bar{d}_1$, all failed banks in this sub-network are "solvent" in any state of nature. This means $\widetilde{\mathbf{\Theta}}_{++}\mathbb{1}_+ + \widetilde{\mathbf{\Theta}}_{+s}\mathbb{1}_s = \mathbb{1}_+$. As a result, equation 20 becomes $\boldsymbol{a}_+(\boldsymbol{\omega}) = (\boldsymbol{I}_+ - \widetilde{\mathbf{\Theta}}_{++})^{-1}\widetilde{\mathbf{\Theta}}_{+s}\mathbb{1}_s = \mathbb{1}_+$ for all $\boldsymbol{\omega}$. If this is the case, equation 21 implies $\mathcal{D}(\mathbf{Z}_{-i}; \bar{d}_2) - \mathcal{D}(\mathbf{Z}_{-i}; \bar{d}_1) = 0$ for all $\bar{d}_2 > \bar{d}_1$.

To show the upper bound exists, it remains to prove that $\bar{d}_1$ exits: i.e. there exists a $\bar{d}_1$ such that all failed banks in $\widetilde{\mathbf{\Theta}}$ are "solvent" in any state of nature. Because $\widetilde{\mathbf{\Theta}}$ is path-connected by construction, there is a chain $\{j, a, b, c, ..., i\}$ from any failed bank j to the successful bank i. Then consider

$$\bar{d}_j^{max} = \frac{1}{\widetilde{\theta}_{bc}}\left(\frac{1}{\widetilde{\theta}_{ab}}\left(\frac{1}{\widetilde{\theta}_{ja}} \cdot v_j + v_a\right) + v_b\right) + v_c + ...$$

Clearly, $\bar{d}_j^{max}$ is finite because the network is path-connected ($\widetilde{\theta}_{ja}$, $\widetilde{\theta}_{ab}$, $\widetilde{\theta}_{bc}$.. are all strictly positive). Suppose $\bar{d} = \bar{d}_j^{max}$, then even when any bank outside this chain failed and "insolvent" (i.e. unable to contribute to the chain), bank j can fulfill its deposits and become "solvent". Intuitively, that means $\bar{d}$ is so large that bank i can itself bail out bank j even though they may not be directly connected. Then let's define

$$\bar{d}^{max} = \max_j \bar{d}_j^{max}$$

When $\bar{d}_1 = \bar{d}^{max}$, then in any state of nature, all failed bank are "solvent". This completes the proof. □

Part (b): From path-connectedness, $\widetilde{\mathbf{\Theta}} = \mathbf{\Theta}$. From part (a), $\mathcal{D}(\mathbf{Z}_{-i})$ reaches the maximum when every failed banks are 'solvent" in all possible states of nature. In this case, we can rewrite failed banks' equilibrium payment (equation 13) as

$$\boldsymbol{d}_f^*(\boldsymbol{\omega}) = \mathbf{\Theta}_{ff}\boldsymbol{d}_f^*(\boldsymbol{\omega}) + \mathbf{\Theta}_{fs}\mathbb{1}_s\bar{d} - \mathbb{1}_f v \qquad \forall \boldsymbol{\omega}$$

It implies

$$\boldsymbol{d}_f^*(\boldsymbol{\omega}) = (\mathbf{I}_f - \mathbf{\Theta}_{ff})^{-1}(\mathbf{\Theta}_{fs}\mathbb{1}_s\bar{d} - \mathbb{1}_f v) = \mathbb{1}_f\bar{d} - (\mathbf{I}_f - \mathbf{\Theta}_{ff})^{-1}\mathbb{1}_f v \qquad \forall \boldsymbol{\omega}$$

The interbank payments received by the successful banks are

$$\mathbf{\Theta}_{sf}\boldsymbol{d}_f^*(\boldsymbol{\omega}) + \mathbf{\Theta}_{ss}\mathbb{1}_s\bar{d} = \mathbb{1}_s\bar{d} - \mathbf{\Theta}_{sf}(\mathbf{I}_f - \mathbf{\Theta}_{ff})^{-1}\mathbb{1}_f v \qquad \forall \boldsymbol{\omega}$$

That means successful banks' network distortion vector in state $\boldsymbol{\omega}$ is $\vec{\mathcal{D}}(\boldsymbol{\omega}) = \mathbf{\Theta}_{sf}(\mathbf{I}_f - \mathbf{\Theta}_{ff})^{-1}\mathbb{1}_f v$. By the network symmetry, the expected distortion conditional on the set f fails will be the ratio of column sum of $\vec{\mathcal{D}}(\boldsymbol{\omega})$ and the number of columns. That is

$$\mathbb{E}[\mathcal{D}^{max}|\text{set } f \text{ fails}] = \frac{\mathbb{1}_s'\boldsymbol{\Theta}_{sf}(\mathbf{I}_f - \boldsymbol{\Theta}_{ff})^{-1}\mathbb{1}_f v}{\mathbb{1}_s'\mathbb{1}_s} = \frac{\mathbb{1}_f'\mathbb{1}_f}{\mathbb{1}_s'\mathbb{1}_s} v$$

Then a bank's unconditional expected network distortion is $\sum_f \frac{\mathbb{1}_f'\mathbb{1}_f}{\mathbb{1}_s'\mathbb{1}_s} v \cdot \Pr(\mathcal{F} = f)$. Again due to the symmetry, the permutation among the failed banks is irrelevant. Therefore, the maximum network risk-taking distortion is

$$\mathcal{D}^{max}(\mathbf{Z}_{-i}) = \sum_{f=1}^{N-1} \frac{f}{N-f} \cdot v \cdot \binom{N-1}{f} \Big[P(Z_{-i})\Big]^{N-1-f} \Big[1 - P(Z_{-i})\Big]^{f}$$

It's worth mentioning that $\mathcal{D}^{max}(\mathbf{Z}_{-i})$ is independent of the network topology $\boldsymbol{\Theta}$ when it's symmetric (e.g. ring or complete networks). □

PROOF OF PROPOSITION 4: Let's separately analyze the two types of networks.

Complete Network

In A complete network, failed banks are either altogether "solvent" or "insolvent". That means we have either $\mathcal{F}^+(\boldsymbol{\omega}) = \mathcal{F}(\boldsymbol{\omega})$ or $\mathcal{F}^+(\boldsymbol{\omega}) = \varnothing$. Let's solve the payment equilibrium (equation 16 and 17) in those two types of states of nature.

1. For $\boldsymbol{\omega}$ where $\mathcal{F}^+(\boldsymbol{\omega}) = \mathcal{F}(\boldsymbol{\omega})$ (i.e. failed banks are "solvent"),
   If $\omega_i = f$, then $a_i(\boldsymbol{\omega}) = 1$ and $b_i(\boldsymbol{\omega}) = 1/(1 - \sum_{j\in\mathcal{F}_\omega}\theta_{ij})$.
2. For $\boldsymbol{\omega}$ where $\mathcal{F}^+(\boldsymbol{\omega}) = \varnothing$ (i.e. failed banks are "insolvent"),
   If $\omega_i = f$, then $a_i(\boldsymbol{\omega}) = \sum_{\omega_j = s}\theta_{ij}$ and $b_i(\boldsymbol{\omega}) = 1$.

By definition, a bank is "solvent" if $a_i\bar{d} - b_i v \geqslant 0$. Plugging the solution in case 1, we know $\mathcal{F}(\boldsymbol{\omega})^+ = \mathcal{F}(\boldsymbol{\omega})$ if and only if $\bar{d} \geqslant 1/(1 - \sum_{j\in\mathcal{F}_\omega}\theta_{ij}) \cdot v$. We can hence solve the payment equilibrium as

$$d_i^C(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \left(\bar{d} - \frac{1}{\sum_{\omega_j \in s}\theta_{ij}} v\right)^+ & \forall \quad \omega_i = f \end{cases}$$

where $1/\sum_{\omega_j=s}\theta_{ij} = (N-1)$ / # of successful banks. We observe that conditioning on m numbers of banks fail, $d_i^C(\boldsymbol{\omega})$ is independent of $\boldsymbol{\omega}$. We can rewrite the network risk-taking distortion as

$$\begin{aligned} \mathcal{D}^C(\mathbf{Z}_{-i}) &= \sum_{m=1}^{N-1} \Bigg(\bar{d} - \underbrace{\left(\bar{d} - \frac{N-1}{N-m} v\right)^+ \cdot \frac{m}{N-1}}_{\text{payment from failed banks}} - \underbrace{\bar{d} \cdot \frac{N-1-m}{N-1}}_{\substack{\text{payment from} \\ \text{successful banks}}}\Bigg) \cdot \Pr(m \text{ banks failed}) \\ &= \sum_{m=1}^{N-1} \min\left(\frac{m \cdot v}{N-m}, \frac{m \cdot \bar{d}}{N-1}\right) \cdot \Pr(m \text{ banks failed}) \end{aligned} \tag{23}$$

where

$$\Pr(m \text{ banks failed}) = \binom{N-1}{m}\Big(1-P(Z_{-i})\Big)^m\Big(P(Z_{-i})\Big)^{N-1-m}$$

Ring Network

For a failed bank, there are three scenarios: (1) its debtor succeeds, (2) its debtor failed but "solvent", and (3) its debtor failed and "insolvent". Let's solve the payment equilibrium (equation 16 and 17) in those three types of states of nature.

1. For $i \in \mathcal{F}$ with $\omega_{i-1} \in \mathcal{S}(\boldsymbol{\omega})$,
$$a_i(\boldsymbol{\omega}) = 1 \text{ and } b_i(\boldsymbol{\omega}) = 1.$$
2. For $i \in \mathcal{F}$ with $\omega_{i-1} \in \mathcal{F}^+(\boldsymbol{\omega})$,
$$a_i(\boldsymbol{\omega}) = a_{i-1}(\boldsymbol{\omega}) \text{ and } b_i(\boldsymbol{\omega}) = b_{i-1}(\boldsymbol{\omega}) + 1.$$
3. For $i \in \mathcal{F}$ with $\omega_{i-1} \in \mathcal{F}^-(\boldsymbol{\omega})$,
$$a_i(\boldsymbol{\omega}) = 0 \text{ and } b_i(\boldsymbol{\omega}) = 1.$$

By induction, we have

$$d_i^R(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \Big(\bar{d} - K_i(\boldsymbol{\omega})v\Big)^+ & \forall \quad \omega_i = f \end{cases}$$

where $K_i(\boldsymbol{\omega}) \equiv \min\{o : \omega_{i-o} = s\}$ is the number of failed debtors in the chain before reaching the first successful bank. Conditioning on m number(s) of banks failed, the total interbank payments received by bank i is

$$\sum_j \theta'_{ij} d_j^R(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \text{w.p.} \quad \binom{N-2}{N-2-m}\Big/\binom{N-1}{m} \\ \Big(\bar{d} - v\Big)^+ & \text{w.p.} \quad \binom{N-3}{N-2-m}\Big/\binom{N-1}{m} \\ \ldots\ldots & \\ \Big(\bar{d} - mv\Big)^+ & \text{w.p.} \quad \binom{N-2-m}{N-2-m}\Big/\binom{N-1}{m} \end{cases} \tag{24}$$

Equation 24 has a clean interpretation. The first line corresponds to the scenario where i's direct debtor succeeded. In this case, bank i will receive an interbank payment of $\bar{d}$. Conditioning on m number of bank failed, the probability of this scenario is $\binom{N-2}{N-2-m}\Big/\binom{N-1}{m}$. Similarly, the second line corresponds to the scenario where i's direct debtor failed but its debtor's debtor succeeded. In this case, bank i will receive an interbank payment of $(\bar{d}-v)^+$. The probability of this scenario is $\binom{N-3}{N-2-m}\Big/\binom{N-1}{m}$. The same logic applies till all m banks failed. It is easy to confirm by Hockey-stick identity (emma I.A) that the total probability in equation 24 is one. Taking the expectation, the network risk-taking distortion of a ring network is

$$\mathcal{D}^R(\mathbf{Z}_{-i}) = \sum_{m=1}^{N-1}\left[\bar{d} - \sum_{l=0}^{m}\Big(\bar{d} - lv\Big)^+\binom{N-2-l}{N-2-m}\Big/\binom{N-1}{m}\right]\cdot \Pr(m \text{ banks failed})$$

To compare it with the network distortion of a complete network,

$$\mathcal{D}^R(\mathbf{Z}_{-i}) \leqslant \sum_{m=1}^{N-1}\left[\bar{d} - \left(\sum_{l=0}^{m}\left(\bar{d}-lv\right)\binom{N-2-l}{N-2-m}\Big/\binom{N-1}{m} - \bar{d}\cdot\frac{N-1-m}{N-1}\right)^+ - \bar{d}\cdot\frac{N-1-m}{N-1}\right]$$
$$\cdot \Pr(m \text{ banks failed}) \qquad \textit{\textbf{(By lemma I.B)}}$$
$$= \sum_{m=1}^{N-1}\left(\bar{d} - \left(\bar{d} - \frac{N-1}{N-m}v\right)^+ \cdot \frac{m}{N-1} - \bar{d}\cdot\frac{N-1-m}{N-1}\right)\cdot \Pr(m \text{ banks failed}) \qquad \textit{\textbf{(By lemma I.A)}}$$
$$= \mathcal{D}^C(\mathbf{Z}_{-i})$$

It's worth noting that $\mathcal{D}^R(\mathbf{Z}_{-i}) = \mathcal{D}^C(\mathbf{Z}_{-i}) = \mathcal{D}^{max}(\mathbf{Z}_{-i})$ if $\bar{d} - mv \geqslant 0$ for all m. A necessary and sufficient condition is $\bar{d} \geqslant (N-1)v$. It confirms Corollary 2. Finally, by monotone selection theorem, the equilibrium risk exposure of banks in a complete network is larger than that of banks in a ring network. □

PROOF OF PROPOSITION 5: By binomial theorem, we can rewrite equation 7 as

$$\mathcal{D}^{max}(\mathbf{Z}_{-i}) = \frac{1 - P(Z_{-i}) - [1-P(Z_{-i})]^N}{P(Z_{-i})}\cdot v$$

It is immediate that $\mathrm{d}\mathcal{D}^{max}(\mathbf{Z}_{-i})/\,\mathrm{d}N > 0$. By monotone selection theorem, each bank's maximum risk exposure Z_i^* is increasing in the number of banks N in the network. □

PROOF OF PROPOSITION 6: Denote the central clearing counterparty (CCP) as bank 0. Because the CCP has no outside liability, it's always "solvent". Hence, the payment equilibrium when m banks fail can be solved by

$$\begin{aligned} d_s^* &= \bar{d} \\ d_f^* &= (d_0^*/N - v)^+ \\ d_0^* &= (N-m)\cdot d_s^* + m\cdot d_f^* \end{aligned}$$

The above fixed point system is solved as

$$d_i^{CCP}(\boldsymbol{\omega}) = \begin{cases} \bar{d} & \forall \quad \omega_i = s \\ \left(\bar{d} - \frac{N}{N-m}v\right)^+ & \forall \quad \omega_i = f \end{cases}$$

As a result, the risk-taking distortion of a successful bank is

$$\begin{aligned} \mathcal{D}^{CCP}(\mathbf{Z}_{-i}) &= \sum_{m=1}^{N-1}\Bigg(\bar{d} - \underbrace{\left(\bar{d} - \frac{N}{N-m}v\right)^+\cdot\frac{m}{N}}_{\text{payment from failed banks}} - \underbrace{\bar{d}\cdot\frac{N-m}{N}}_{\substack{\text{payment from}\\ \text{successful banks}}}\Bigg)\cdot \Pr(m \text{ banks failed}) \\ &= \sum_{m=1}^{N-1}\min\left(\frac{m\cdot v}{N-m}, \frac{m\cdot\bar{d}}{N}\right)\cdot \Pr(m \text{ banks failed}) \qquad (25) \end{aligned}$$

Compare equation 25 with 23, it's easy to see that $\mathcal{D}^{CCP}(\mathbf{Z}_{-i};\bar{d}) = \mathcal{D}^C(\mathbf{Z}_{-i};\frac{N-1}{N}\bar{d})$. □

PROOF OF PROPOSITION 7: Consider a network $(\bar{d}, \boldsymbol{\Theta}, N)$ where $\bar{d} > v$. From definition of the Nash equilibrium, the LHS of equation 9 is greater than

$$A \equiv P(Z_i^{**})\Big[Z_i^{**} - v - \mathcal{D}(\mathbf{Z}_{-i}^{*})\Big] + c_i - \Big[1 - P(Z_i^{**})\Big]\Pr\Big(i \in \mathcal{F}_{\boldsymbol{\omega}}^{-}|\omega_i = f\Big) \cdot c_i$$

Define the RHS of equation 9 as $B \equiv P(Z_i^{**})\Big[Z_i^{**} - v + c_i\Big]$

$$A - B = \Big[1 - P(Z_i^{**})\Big]\Big[1 - \Pr\Big(i \in \mathcal{F}_{\boldsymbol{\omega}}^{-}|\omega_i = f\Big)\Big] \cdot c_i - P(Z_i^{**}) \cdot \mathcal{D}(\mathbf{Z}_{-i}^{*})$$

The condition $\bar{d} > v$ implies $\Pr(i \in \mathcal{F}_{\boldsymbol{\omega}}^{-}|\omega_i = f) < 1$. This means that it is possible that bank i's deposits get fully fulfilled from counterparties' cross subsidies. Since Z^* and Z^{**} are bounded, there exists $\bar{c} \in \mathbb{R}^+$ such that if all $c_i > \bar{c}$, $A - B > 0$. □

PROOF OF LEMMA 4 The proof is similar to that of lemma 3. In any state of nature $\boldsymbol{\omega}$, the payment vector for "solvent" failed banks is $\boldsymbol{d}_+^* = \boldsymbol{\Theta}_{++}\boldsymbol{d}_+^* + \boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s\bar{d} + \mathbb{1}_+(r - v)$, or

$$\boldsymbol{d}_+^* = (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}(\boldsymbol{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s\bar{d} + \mathbb{1}_+(r - v))$$

To conserve space, I suppress the state $\boldsymbol{\omega}$ in $\boldsymbol{d}_+^*(\boldsymbol{\omega})$, $\boldsymbol{\Theta}_{++}(\boldsymbol{\omega})$, $\boldsymbol{\Theta}_{+\boldsymbol{s}}(\boldsymbol{\omega})$, $\mathbb{1}_s(\boldsymbol{\omega})$ and $\mathbb{1}_f(\boldsymbol{\omega})$. We can again write the risk-taking distortion in a matrix form as

$$\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\mathbb{1}_+\bar{d} - \boldsymbol{d}_+^*) + \boldsymbol{\Theta}_{\boldsymbol{i}-}\mathbb{1}_-\bar{d}\Big] \tag{26}$$

To prove the lemma 4, compare three financial systems with different sizes of equity buffers r_1, r_2, r_3 , with $\bar{r}_3 - \bar{r}_2 = \bar{r}_2 - \bar{r}_1 = \varepsilon$. Similar to the proof of lemma 3, we need to consider the following four cases.

Case I: $\mathcal{F}_1^+(\boldsymbol{\omega}) = \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$

For all $\boldsymbol{\omega}$, $\boldsymbol{d}_+^*$ is linearly increasing in r: $\boldsymbol{d}_+^{\mathbf{3}*} - \boldsymbol{d}_+^{\mathbf{2}*} = \boldsymbol{d}_+^{\mathbf{2}*} - \boldsymbol{d}_+^{\mathbf{1}*} = (\mathbf{I}_+ - \boldsymbol{\Theta}_{++})^{-1}\mathbb{1}_+\varepsilon > 0$. Then it is easy to see that the network risk-taking distortion is linearly decreasing in r.

$$\mathcal{D}^3(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}^2(\mathbf{Z}_{-\boldsymbol{i}}) = \mathcal{D}^2(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}^1(\mathbf{Z}_{-\boldsymbol{i}}) = \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\boldsymbol{\Theta}_{\boldsymbol{i}+}(\boldsymbol{d}_+^{\mathbf{1}*} - \boldsymbol{d}_+^{\mathbf{2}*})\Big] < 0$$

Case II: $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_3^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

We first compare the equity r_2 with r_1. In some state of nature $\boldsymbol{\omega}$, some otherwise "insolvent" failed banks for $(\bar{d}, \boldsymbol{\Theta}, N; r_1)$ become "solvent" for $(\bar{d}_2, \boldsymbol{\Theta}, N; r_2)$. Denote those banks $t_1, t_2, ..., t_T$, where $T \geqslant 1$. Due to the continuity of the payment equilibrium in terms of r (equation 10), there exists $r_1 < \tilde{r}_1 < \tilde{r}_2 ... < ... < \tilde{r}_S < r_2$ (where $1 \leqslant S \leqslant T$), such that when the equity buffer $r = \tilde{r}_s$, some banks t_t are exactly "solvent". As a result, those margin banks t_t are "solvent" when $r \in (\tilde{r}_s, \tilde{r}_{s+1})$ and "insolvent" when $r \in (\tilde{r}_{s-1}, \tilde{r}_s)$

respectively. Denote $\widetilde{\mathcal{D}}^s(\mathbf{Z}_{-i})$ the network risk-taking distortion when $r = \tilde{r}_s$. We have

$$\begin{aligned}
\mathcal{D}^2(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}^S(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\tilde{\boldsymbol{d}}^{S*}_{+} - \boldsymbol{d}^{2*}_{+})\Big] \leqslant 0 \\
\widetilde{\mathcal{D}}^{s+1}(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}^s(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}(\tilde{\boldsymbol{d}}^{s*}_{+} - \tilde{\boldsymbol{d}}^{s+1*}_{+})\Big] \leqslant 0 \qquad \forall s = 1, ..., S-1 \qquad (27) \\
\widetilde{\mathcal{D}}^1(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) &= \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{1}}_{i+}(\boldsymbol{d}^{1*}_{+} - \tilde{\boldsymbol{d}}^{1*}_{+})\Big] \leqslant 0
\end{aligned}$$

Summing above equations, it is easy to see that $\mathcal{D}^2(\mathbf{Z}_{-i}) - \widetilde{\mathcal{D}}^1(\mathbf{Z}_{-i}) \leqslant 0$. It then remains to prove the concavity. By construction, $\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}$ is a submatrix of $\widetilde{\boldsymbol{\Theta}}^{\boldsymbol{s+1}}_{i+}$. We also know $\tilde{\boldsymbol{b}}^s_+ = (\mathbf{I}^s_+ - \boldsymbol{\Theta}^{\boldsymbol{s}}_{++})^{-1}\mathbb{1}^s_+$ is a submatrix of $\tilde{\boldsymbol{b}}^{s+1}_+$. This is due to the construction that at the cutoff $r = \tilde{r}_s$, bank t can be treated either as solvent or insolvent. With those two facts, we have

$$\tilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{i+}(\mathbf{I}^s_+ - \tilde{\boldsymbol{\Theta}}^{\boldsymbol{s}}_{++})^{-1}\mathbb{1}^s_+ < \tilde{\boldsymbol{\Theta}}^{\boldsymbol{s+1}}_{i+}(\mathbf{I}^{s+1}_+ - \boldsymbol{\Theta}^{\boldsymbol{s+1}}_{++})^{-1}\mathbb{1}^{s+1}_+$$

After summing every difference in equation 27 and replacing all of RHS with the $\boldsymbol{\Theta}^2_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^2_{++})^{-1}\mathbb{1}^2_+$ (the largest), we have

$$\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) > \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^{\mathbf{2}}_{++})^{-1}\mathbb{1}^2_+(-\varepsilon)\Big]$$

Since $\mathcal{F}^+_2(\boldsymbol{\omega}) = \mathcal{F}^+_3(\boldsymbol{\omega})$, we have the following identity as in case I,

$$\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) = \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^{\mathbf{2}}_{++})^{-1}\mathbb{1}^2_+(-\varepsilon)\Big]$$

Hence $\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) < \mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i})$ and the concavity follows.

Case III: $\mathcal{F}^+_1(\boldsymbol{\omega}) = \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

The proof is identical to case II with a slight twist. When comparing r_3 with r_2. Again replacing all RHS of equation 27 with $\boldsymbol{\Theta}^2_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^2_{++})^{-1}\mathbb{1}^2_+$, the smallest, we obtain

$$\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) < \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^{\mathbf{2}}_{++})^{-1}\mathbb{1}^2_+(-\varepsilon)\Big] = \mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i})$$

Case IV: $\mathcal{F}^+_1(\boldsymbol{\omega}) \subset \mathcal{F}^+_2(\boldsymbol{\omega}) \subset \mathcal{F}^+_3(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

The proof is a combination of case 2 and case 3:

$$\begin{aligned}
\mathcal{D}^2(\mathbf{Z}_{-i}) - \mathcal{D}^1(\mathbf{Z}_{-i}) &> \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^{\mathbf{2}}_{++})^{-1}\mathbb{1}^2_+(-\varepsilon)\Big] \\
\mathcal{D}^3(\mathbf{Z}_{-i}) - \mathcal{D}^2(\mathbf{Z}_{-i}) &< \sum_{\boldsymbol{\omega}_{-i}} \Pr(\boldsymbol{\omega}_{-i})\Big[\boldsymbol{\Theta}^{\mathbf{2}}_{i+}(\mathbf{I}^2_+ - \boldsymbol{\Theta}^{\mathbf{2}}_{++})^{-1}\mathbb{1}^2_+(-\varepsilon)\Big]
\end{aligned}$$

The monotonicity and concavity result follows.

Because $\mathcal{F}_1^+(\boldsymbol{\omega}) \subseteq \mathcal{F}_2^+(\boldsymbol{\omega}) \subseteq \mathcal{F}_3^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in \Omega$, Case I-IV (or some combination of them) exhaust all the possibilities. □

PROOF OF PROPOSITION 8: The first and second order conditions of maximizing bank's expected profit (equation 11) over its choice of risk exposure Z_i:

$$F(Z_i; \mathbf{Z}_{-\boldsymbol{i}}, r) = P'(Z_i)(Z_i + r - v) + P(Z_i) - P(Z_i)'\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}; r)$$
$$S(Z_i; \mathbf{Z}_{-\boldsymbol{i}}, r) = P''(Z_i)(Z_i + r - v) + 2P'(Z_i) - P(Z_i)''\mathcal{D}(\mathbf{Z}_{-\boldsymbol{i}}; r)$$

Taking the total derivative of FOC, we have

$$\frac{\mathrm{d}Z_i^*}{\mathrm{d}r} = -\frac{\frac{\partial F}{\partial \mathcal{D}}\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}r} + \frac{\partial F}{\partial r}}{S(Z_i; \mathbf{Z}_{-\boldsymbol{i}}, r)} = \frac{1}{-S}\Big[- P'(Z_i)\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}r} + P'(Z_i)\Big] < 0 \quad \forall \mathbf{Z}_{-\boldsymbol{i}}$$

where $P'(Z_i) < 0$ is the direct effect of an equity buffer and $\mathrm{d}\mathcal{D}/\mathrm{d}r < 0$ is the network effect. □

PROPOSITION 9: The proof is similar to the proof of lemma 4. The payment vector for "solvent" failed banks is

$$\boldsymbol{d}_+^*(\boldsymbol{\omega}) = \begin{cases} (\mathbf{I}_+ - \mathbf{\Theta}_{++})^{-1}(\mathbf{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s\bar{d} + \mathbb{1}_+(t - v)) & \text{if} \quad \#\{l|\omega_l = f\} \geqslant n \\ (\mathbf{I}_+ - \mathbf{\Theta}_{++})^{-1}(\mathbf{\Theta}_{+\boldsymbol{s}}\mathbb{1}_s\bar{d} + \mathbb{1}_+(0 - v)) & \text{if} \quad \#\{l|\omega_l = f\} < n \end{cases}$$

The first line corresponds to the state of nature where a bailout occurs. The second line corresponds to the other cases. Compare two bailout amount t_1 and t_2 with $t_2 - t_1 = \varepsilon > 0$. We again have two cases: (1) $\mathcal{F}_2^+(\boldsymbol{\omega}) = \mathcal{F}_1^+(\boldsymbol{\omega})$ for all $\boldsymbol{\omega}$. (2) $\mathcal{F}_1^+(\boldsymbol{\omega}) \subset \mathcal{F}_2^+(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}$.

Denote the bailout event indicator $\mathbb{1}[\#\{l|\omega_l = f\} > n]$ as $\mathcal{B}(\boldsymbol{\omega})$. Since $n < N$, $\mathcal{B}(\boldsymbol{\omega}) = 1$ for some $\boldsymbol{\omega}$. For case 1,

$$\boldsymbol{d}_+^{\mathbf{2}*}(\boldsymbol{\omega}) - \boldsymbol{d}_+^{\mathbf{1}*}(\boldsymbol{\omega}) = \mathcal{B}(\boldsymbol{\omega})(\mathbf{I}_+ - \mathbf{\Theta}_{++})^{-1}\mathbb{1}_+\varepsilon \quad \forall \boldsymbol{\omega} \in \mathbf{\Omega}$$

From equation 26,

$$\begin{aligned} &\mathcal{D}^2(\mathbf{Z}_{-\boldsymbol{i}}) - \mathcal{D}^1(\mathbf{Z}_{-\boldsymbol{i}}) \\ &\quad = \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} \Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\mathbf{\Theta}_{\boldsymbol{i}+}(\boldsymbol{d}_+^{\mathbf{1}*} - \boldsymbol{d}_+^{\mathbf{2}*})\Big] = \sum_{\boldsymbol{\omega}_{-\boldsymbol{i}}} -\mathcal{B}(\boldsymbol{\omega}^{i=s})\Pr(\boldsymbol{\omega}_{-\boldsymbol{i}})\Big[\mathbf{\Theta}_{\boldsymbol{i}+}(\mathbf{I}_+ - \mathbf{\Theta}_{++})^{-1}\mathbb{1}_+\varepsilon\Big] < 0 \end{aligned}$$

The proof of case 2 is identical to case 2 of lemma 3 and 4. I omit here to avoid repetition. From monotone selection theorem, banks' equilibrium risk exposure is lower if $t = t_2$ compared with t_1. Intuitively, the proof shows that a government bailout decreases the cross-subsidy a successful bank pays during crises. □

PROOF OF PROPOSITION 10: From equation 13, the payment vector $\boldsymbol{d}^*$ is still independent of the risk vector $\boldsymbol{Z}$ or the correlation matrix $\boldsymbol{\lambda}$. Let's compare bank i's expected profit when it chooses between λ_{ij}

and $\widetilde{\lambda}_{ij}$ with $\widetilde{\lambda}_{ij} > \lambda_{ij}$

$$\begin{aligned}
\mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; Z_i, \widetilde{\boldsymbol{\lambda}}_{\boldsymbol{ij}})\Big] - \mathbb{E}\Big[\Pi_i(\boldsymbol{\omega}; Z_i, \boldsymbol{\lambda}_{\boldsymbol{ij}})\Big] = \\
&- \sum_{\boldsymbol{\omega}_{-i-j}} \Big(\bar{d} - \sum_l \theta_{il} d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,j=s}})\Big) \cdot \Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = s) \cdot P(Z_j) \cdot (\widetilde{\lambda}_{ij} - \lambda_{ij}) \\
&+ \sum_{\boldsymbol{\omega}_{-i-j}} \Big(\bar{d} - \sum_l \theta_{il} d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,j=f}})\Big) \cdot \Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = f) \cdot P(Z_j) \cdot (\widetilde{\lambda}_{ij} - \lambda_{ij})
\end{aligned}$$

Suppose $\lambda_{j,k}^* = 1$ for all $k \neq i$. That implies $\Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = s) = 1$ if and only if every element of $\boldsymbol{\omega}_{-i-j}$ is s. Similarly, $\Pr(\boldsymbol{\omega}_{-i-j} | \omega_i = s, \omega_j = f) = 1$ if and only if every element of $\boldsymbol{\omega}_{-i-j}$ is f.

By Auxiliary Lemma in the appendix above, $\sum_l \theta_{il} d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,-i=s}}) \geqslant \sum_l \theta_{il} d_l^*(\boldsymbol{\omega}^{\boldsymbol{i=s,-i=f}})$. This implies bank i's expected profit is increasing in its project's dependence λ_{ij} with other banks. Therefore, for all $\boldsymbol{Z}$, bank i's choices of conditional dependence with bank j won't deviate from $\lambda_{i,j}^* = 1$. With perfect correlation, the network risk-taking distortion disappears: $\mathcal{D}(\boldsymbol{Z}_{-\boldsymbol{i}}^*, \mathbb{1}) = 0$ for all $\boldsymbol{Z}_{-\boldsymbol{i}}^*$. Hence, the equilibrium is characterized by

$$\begin{aligned}
&\lambda_{ij}^* = 1 \quad \forall i, j \in \mathcal{N} \\
&P'(Z_i^*)(Z_i^* - v) + P(Z_i^*) = 0 \quad \forall i \in \mathcal{N}
\end{aligned}$$

And $\rho_{ij}^* = 1$ for all i, j. $\square$

Online Appendix

A. Omitted Proofs

LEMMA I.A [Hockey-stick Identity]
For all $n > r$, we have

$$\text{(i)} \quad \sum_{l=r}^{n} \binom{l}{r} = \binom{n+1}{r+1} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{l=r}^{n} \binom{l}{r}(n-l) = \binom{n+1}{r+1}\frac{n-r}{r+2}$$

PROOF
We proceed by induction. For an initial $n = r+1$

$$\text{(i)} \quad \binom{r}{r} + \binom{r+1}{r} = \binom{r+2}{r+1}$$
$$\text{(ii)} \quad \binom{r}{r} * 1 + \binom{r+1}{r} * 0 = \binom{r+2}{r+1} * \frac{1}{r+2} = 1$$

The above equations are to confirm the initial conditions hold. Now suppose that for $n = k$, the two equality holds. For $n = k+1$, we have

$$\text{(i)} \quad \sum_{l=r}^{k+1} \binom{l}{r} = \sum_{l=r}^{k} \binom{l}{r} + \binom{k+1}{r} = \binom{k+1}{r+1} + \binom{k+1}{r} = \binom{k+2}{r+1}$$
$$\text{(ii)} \quad \sum_{l=r}^{k+1} \binom{l}{r}(k+1-l) = \sum_{l=r}^{k} \binom{l}{r}(k+1-l) = \binom{k+1}{r+1}\frac{k-r}{r+2} + \binom{k+1}{r+1} = \binom{k+2}{r+1}\frac{k+1-r}{r+2}$$

Q.E.D by induction. □

LEMMA I.B [Triangle Inequality]
For any sequence $\{A_i\}$ and $B \in \mathbb{R}$ with $B < \max_i(A_i)$, we have

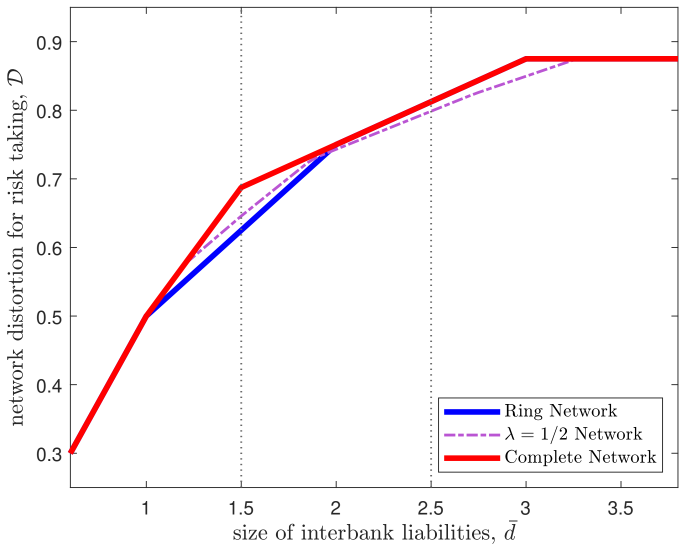
$$\sum_i \Big(A_i\Big)^+ \geqslant \Big(\sum_i A_i - B\Big)^+ + B$$

PROOF Without loss of generality, let $A_0 = \max_i(A_i)$

$$\sum_i \Big(A_i\Big)^+ - B = \sum_{i \neq 0} \Big(A_i\Big)^+ + (A_0 - B)^+ \geqslant \Big(\sum_i A_i - B\Big)^+$$

□

B. Numerical Example: intermediately-connected networks



We consider a ring, a $\lambda = 0.5$, and a complete network with four banks. Let the bank of interest be bank $i = 4$. The purpose of this section is to numerically solve the network risk-taking distortion for the three kinds of networks. Let $P(Z_j) = 0.5 \quad \forall j \neq i$.

$$, \boldsymbol{\Theta}^R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \boldsymbol{\Theta}^\lambda = \begin{bmatrix} 0 & 1/6 & 1/6 & 2/3 \\ 2/3 & 0 & 1/6 & 1/6 \\ 1/6 & 2/3 & 0 & 1/6 \\ 1/6 & 1/6 & 2/3 & 0 \end{bmatrix}$$

$$\boldsymbol{\Theta}^C = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$$

define $\boldsymbol{\theta}_4^R$, $\boldsymbol{\theta}_4^\lambda$, and $\boldsymbol{\theta}_4^C$ as vectors that represent the last row of each $\boldsymbol{\Theta}$.

(i) Small $\bar{d}$ ($\bar{d} = 1.5$)

$\boldsymbol{\omega} = (s, s, s, s)$:
$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 0$
$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 0$
$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 0$

$\boldsymbol{\omega} = (s, s, f, s)$:
$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 0.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 1$
$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 0.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 2/3$
$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 1.5 & 0.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1/3$

$\boldsymbol{\omega} = (s, f, s, s)$:
$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 0$
$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 1/6$
$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1/3$

$\boldsymbol{\omega} = (f, s, s, s)$:
$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 0$
$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 1/6$
$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 1.5 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1/3$

$\boldsymbol{\omega} = (s, f, f, s)$:
$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0.5 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 1.5$
$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0.25 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 29/24$
$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 1.5 & 0 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1$

$\boldsymbol{\omega} = (f, s, f, s)$:
$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 1.5 & 0.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 1$
$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 0.3 & 1.5 & 0.3 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 1$
$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 0 & 1.5 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1$

$\boldsymbol{\omega} = (f, f, s, s)$:
$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 0 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 0$
$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 0.25 & 0 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 11/24$
$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 0 & 0 & 1.5 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1$

$\boldsymbol{\omega} = (f, f, f, s)$:
$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix} 0.5 & 0 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 1.5 = 1.5$
$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix} 0 & 0 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 1.5 = 1.5$
$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix} 0 & 0 & 0 & 1.5 \end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 1.5 = 1.5$

Let m denotes the number failed banks. Conditioning on bank i succeeds, $\Pr(m = 0) = \frac{1}{8}$, $\Pr(m = 1) = \frac{3}{8}$, $\Pr(m = 2) = \frac{3}{8}$, and $\Pr(m = 3) = \frac{1}{8}$. From here we can calculate the network risk-taking distortion as

$\mathcal{D}^R = \Pr(m = 0) \cdot 0 + \Pr(m = 1) \cdot \frac{1}{3} + \Pr(m = 2) \cdot \frac{5}{6} + \Pr(m = 3) \cdot \frac{3}{2} = \frac{5}{8}$
$\mathcal{D}^\lambda = \Pr(m = 0) \cdot 0 + \Pr(m = 1) \cdot \frac{1}{3} + \Pr(m = 2) \cdot \frac{8}{9} + \Pr(m = 3) \cdot \frac{3}{2} = \frac{31}{48}$
$\mathcal{D}^C = \Pr(m = 0) \cdot 0 + \Pr(m = 1) \cdot \frac{1}{3} + \Pr(m = 2) \cdot 1 + \Pr(m = 3) \cdot \frac{3}{2} = \frac{11}{16}$

(ii) Large $\bar{d}$ ($\bar{d} = 2.5$)

$\boldsymbol{\omega} = (s, s, s, s)$:
$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 1.5 & 1.5 & 1.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 0$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 1.5 & 1.5 & 1.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 0$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 1.5 & 1.5 & 1.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 0$$

$\boldsymbol{\omega} = (s, s, f, s)$:
$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 2.5 & 1.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 1$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 2.5 & 1.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 2/3$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 2.5 & 1.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1/3$$

$\boldsymbol{\omega} = (s, f, s, s)$:
$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 1.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 0$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 1.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 1/6$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 1.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1/3$$

$\boldsymbol{\omega} = (f, s, s, s)$:
$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 2.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 0$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 2.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 1/6$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 2.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1/3$$

$\boldsymbol{\omega} = (s, f, f, s)$:
$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 1.5 & 0.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 2$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 19/16 & 5/8 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 47/32$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}2.5 & 1 & 1 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1$$

$\boldsymbol{\omega} = (f, s, f, s)$:
$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 2.5 & 1.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 1$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}1.3 & 2.5 & 1.3 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 1$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 2.5 & 1 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1$$

$\boldsymbol{\omega} = (f, f, s, s)$:
$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 0.5 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 0$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}19/16 & 5/8 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 17/32$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}1 & 1 & 2.5 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 1$$

$\boldsymbol{\omega} = (f, f, f, s)$:
$$\mathcal{D}^R(\boldsymbol{\omega}) = -\begin{pmatrix}1.5 & 0.5 & 0 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^R + 2.5 = 2.5$$
$$\mathcal{D}^\lambda(\boldsymbol{\omega}) = -\begin{pmatrix}2/3 & 0 & 0 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^\lambda + 2.5 = 43/18$$
$$\mathcal{D}^C(\boldsymbol{\omega}) = -\begin{pmatrix}0 & 0 & 0 & 2.5\end{pmatrix} \cdot \boldsymbol{\theta}_4^C + 2.5 = 2.5$$

$$\mathcal{D}^R = \Pr(m=0) \cdot 0 + \Pr(m=1) \cdot \tfrac{1}{3} + \Pr(m=2) \cdot 1 + \Pr(m=3) \cdot 2.5 = \tfrac{13}{16}$$
$$\mathcal{D}^\lambda = \Pr(m=0) \cdot 0 + \Pr(m=1) \cdot \tfrac{1}{3} + \Pr(m=2) \cdot 1 + \Pr(m=3) \cdot \tfrac{43}{18} = \tfrac{115}{144}$$
$$\mathcal{D}^C = \Pr(m=0) \cdot 0 + \Pr(m=1) \cdot \tfrac{1}{3} + \Pr(m=2) \cdot 1 + \Pr(m=3) \cdot 2.5 = \tfrac{13}{16}$$

As we see from this example, if $\bar{d} = 2.5$, bank 4's risk-taking distortion is not monotonic to the degree of connectedness λ: the distortion of a $\lambda = 0.5$ network is lower than that of a complete and a ring network.

C: Robustness

Relax the Assumption of zero downside payoff

In this section, I will show that the network risk-taking externality is not dependent on the assumption that a failed project generates 0 gross return. To do so, let's assume a project Z_i will produce a random return of $\tilde{e}_i(Z_i)$ with the following payoff distribution.

$$\tilde{e_i} = \begin{cases} Z_i & \text{w.p} \quad P(Z_i) \\ \varepsilon & \text{w.p} \quad 1 - P(Z_i) \end{cases}$$

where $\varepsilon < Z_i$ is a constant. Consider the following two parameter specifications.

(a) Suppose $\varepsilon < v$

The payment equilibrium (equation 13) becomes

$$d_i^*(\boldsymbol{\omega}; \mathbf{Z}) = \bar{d} \qquad \forall \omega_i = s$$
$$d_i^*(\boldsymbol{\omega}; \mathbf{Z}) = \left\{ \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + \varepsilon - v \right\}^+ \qquad \forall \omega_i = f$$

This implies Lemma 2 still holds: the payment equilibrium $\boldsymbol{d}^*(\boldsymbol{\omega}; \mathbf{Z})$ is constant in the risk exposure vector $\mathbf{Z}$. Equation 6 is the same as before because any failed bank will default on its interbank debts (i.e. $\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + \varepsilon - v < \bar{d}$). As a result, the proof of corollary 1 will be the same as before.

(b) Suppose $\varepsilon \geqslant v$

From equation 2, we know that the payment equilibrium will be

$$d_i^*(\boldsymbol{\omega}; \mathbf{Z}) = \bar{d} \quad \forall i \in \mathcal{N} \quad \forall \boldsymbol{\omega} \in \mathbf{\Omega}$$

All banks are solvent in all states of nature. Then there is no risk-taking externality resulting from banks' cross-subsidy. However, this also implies that financial networks will not exist in the first place. To see why, consider the setup in Section 4.1. The following equation displays each bank's payoff regardless of whether it is connected.

$$\max_Z P(Z)(Z - v) + \big[1 - P(Z)\big](\varepsilon - v) + c$$

As a result, this is no incentives for banks to form networks.

Endogenizing Deposit Rate

In the main text, the deposit rate v is assumed to be constant. In this section, I model depositors' rational decisions to lend to a bank while being aware of the interbank connections. In other words, I endogenize the deposit rates. To be more specific, each bank in the network $(\bar{d}, \mathbf{\Theta}, N)$ needs to borrow $M_i = 1$ (normalized to 1) from atomistic depositors to finance a productive project Z_i. The borrowing takes the form of a standard debt contract with a face value v_i, which will be determined in equilibrium. v_i can be interpreted as the gross interest rate. For expository purposes, I assume depositors are risk neutral and have time discount rate β. After each bank receives the deposits, they simultaneously choose their project choices. The subsequent timeline follows figure 1. A competitive market results in a zero profit for atomistic depositors. The equilibrium $(d_i^*(\boldsymbol{\omega}), \boldsymbol{v}^*, \boldsymbol{Z}^*)$ is hence characterized by

$$
\begin{aligned}
d_i^*(\boldsymbol{\omega}) &= \left\{ \min\Big[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}; \mathbf{Z}) + e_i(\omega_i, Z_i) - v_i^*, \bar{d}, \Big] \right\}^+ && \forall i \in \mathcal{N}, \quad \forall \boldsymbol{\omega} \in \boldsymbol{\Omega} \\
Z_i^* &= \underset{Z_i}{\operatorname{argmax}}\, \mathbb{E}\Big[\Pi_i^B(\boldsymbol{\omega}; \mathbf{Z}, v_i^*) \Big] && \forall i \in \mathcal{N} \\
0 &= -M_i + \beta \cdot \mathbb{E}\Big[\Pi_i^D(\boldsymbol{\omega}; v_i^*, \mathbf{Z}^*) \Big] && \forall i \in \mathcal{N}
\end{aligned}
$$

where $\Pi_i^B(\boldsymbol{\omega})$ is bank i's payoff in state $\boldsymbol{\omega}$, which is the same as equation 3, and $\Pi_i^D(\boldsymbol{\omega})$ is bank i's depositors' payoff in state $\boldsymbol{\omega}$. With deposit insurance, $\Pi_i^D(\boldsymbol{\omega}) = v_i^*$ for all $\boldsymbol{\omega}$. In this case, the return to depositors are guaranteed by the government. Without deposit insurance, $\Pi_i^D(\boldsymbol{\omega}) = \min[v^*, \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\mathbf{Z}^*, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega})]$ as a debt contract. It's worth noting that, in this case, $\Pi_i^D(\boldsymbol{\omega})$ is a function of $\boldsymbol{\Theta}$ and $\mathbf{Z}^*$: depositors perfectly observe the network structure and perfectly anticipate banks' optimal risk exposure. The following results show the equilibrium for the two cases.

(a) Without deposit insurance, (i) banks' equilibrium risk exposure is identical in any network structure: $Z^{C*} = Z^{R*} = Z^{S*}$ (ii) banks' equilibrium deposit rates are higher in a ring network than in a complete network: $v^{S*} > v^{R*} \geqslant v^{C*}$.

(b) With deposit insurance, (i) banks' equilibrium risk exposure is higher in a complete network than in a ring network: $Z^{C*} \geqslant Z^{R*} > Z^{S*}$. (ii) banks' equilibrium deposit rates are identical in any network structure: $v^{C*} = v^{R*} = v^{S*}$.

(where the superscript C denotes complete network, R denotes ring network, and S denotes stand-alone)

Part (a) states that without deposit insurance, banks' choices of risk exposure are identical in any network structure, in contrast to proposition 3 of the benchmark model. The benchmark model assumes fixed deposit rates and shows that banks in highly connected networks expose to greater risks due to a network risk-taking distortion. To understand the difference, let's first recall that this network risk-taking distortion is the result of "cross-subsidy" from successful banks to failed banks' depositors. Without deposit insurance, depositors in highly connected networks will feel more co-insured from the interbank connections and will demand lower interest rates. Both the lowered deposit rates and the "cross-subsidy" affect connected banks' upside payoffs. Their countervailing effects will equalize banks' choices of risk exposure in any network structure.

Part (b) considers financial systems where depositors are fully insured by the government. The result is identical to the benchmark model with a fixed deposit rate. With a government's guarantee, depositors are "informative insensitive" to banks' financial structures. As a result, the deposit rates are constant across all network structures and equal to depositors' time cost $(1/\beta)$. Without deposit rates' *price disciplining*, banks in highly connected networks will face greater network risk-taking distortion and choose greater exposure to risks (proposition 3 to 5).

Proof Suppose there is no deposit insurance. Bank i's depositors' the expected return is

$$
\begin{aligned}
&\mathbb{E}\Big[\Pi_i^D(\boldsymbol{\omega}; v_\Theta^*, \mathbf{Z}_\Theta^*) \Big] = \mathbb{E}\left\{ \min\Big[v_\Theta^*, \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\mathbf{Z}_\Theta^*, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}) \Big] \right\} \\
&= \underbrace{\mathbb{E}\left\{ e_i(\mathbf{Z}_\Theta^*, \boldsymbol{\omega}) \right\}}_{=P(Z_\Theta^*)\cdot Z_\Theta^*} + \underbrace{\mathbb{E}\left\{ \sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}) \right\}}_{=0} - \underbrace{\mathbb{E}\left\{ \Big(\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) + e_i(\mathbf{Z}_\Theta^*, \boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega}) - v_\Theta^* \Big)^+ \right\}}_{=P(Z_\Theta^*)(Z_\Theta^* - v_\Theta^*) - P(Z_\Theta^*)\mathcal{D}_\Theta(Z_\Theta^*)} \\
&= P(Z_\Theta^*) \cdot \Big(v_\Theta^* + \mathcal{D}_\Theta(Z_\Theta^*) \Big)
\end{aligned}
$$

With a slight abuse of notation, the subscript Θ represents the full network structure $(\bar{d}, \boldsymbol{\Theta}, N)$. The second line follows the first line because for all $x, y \in \mathbb{R}$, $\min(x, y) = y - (y - x)^+$. From the symmetry assumption, $\mathbb{E}[d_j^*(\boldsymbol{\omega})] = \mathbb{E}[d_i^*(\boldsymbol{\omega})]$,

$\forall i \neq j$. Hence $\mathbb{E}[\sum_j \theta_{ij} d_j^*(\boldsymbol{\omega}) - d_i^*(\boldsymbol{\omega})] = 0$. Plugging $\mathbb{E}[\Pi_i^D(\boldsymbol{\omega}; v_\Theta^*, \mathbf{Z}_\Theta^*)]$ to the equilibrium condition

$$\beta \cdot P(Z_\Theta^*) \cdot \Big(v_\Theta^* + \mathcal{D}_\Theta(Z_\Theta^*)\Big) - M = 0 \tag{28}$$

where bank's risk exposure Z_Θ^* is the result of a Nash equilibrium defined by equation 4. Explicitly,

$$P'(Z_\Theta^*) \cdot \Big(Z_\Theta^* - v_\Theta^* - \mathcal{D}_\Theta(Z_\Theta^*)\Big) + P(Z_\Theta^*) = 0 \tag{29}$$

Equation 28 and 29 jointly determine banks' equilibrium risk exposure as

$$P'(Z_\Theta^*) \cdot \Big(Z_\Theta^* - \frac{M}{\beta \cdot P(Z_\Theta^*)}\Big) + P(Z_\Theta^*) = 0 \tag{30}$$

It is easy to see that the equilibrium risk exposure $Z_\Theta^* = Z^*$ is independent of the network structure $(\bar{d}, \boldsymbol{\Theta}, N)$. From equation 28, the equilibrium deposit rates are determined by

$$v_\Theta^* = \frac{M}{\beta \cdot P(Z^*)} - \mathcal{D}_\Theta(Z^*)$$

From proposition 4, $\mathcal{D}_S < \mathcal{D}_R(Z) \leqslant \mathcal{D}_C(Z)$ for all Z. Finally, we have $v^{C*} \leqslant v^{R*} < v^{S*}$.

With deposit insurance, $\Pi_i^D(\boldsymbol{\omega}) = v_i^*$ for all $\boldsymbol{\omega}$. The equilibrium condition becomes:

$$\beta \cdot v_\Theta^* - M = 0$$

Or $v_\Theta^* = v^* = M/\beta$, independent of the network structure. Plugging into equation 29, we have

$$P'(Z_\Theta^*)\Big(Z_\Theta^* - \frac{M}{\beta} - \mathcal{D}_\Theta(Z_\Theta^*)\Big) + P(Z_\Theta^*) = 0$$

It's identical to the benchmark case with fixed deposit rates. Corollary 1 and proposition 4 implies $Z^{C*} \geqslant Z^{R*} > Z^{S*}$.

$\square$