

Expectations formation, sticky prices, and the ZLB*

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Abstract

This paper studies boundedly rational expectations formation in the context of three key policy questions for which communication about the future path of monetary or fiscal policy is crucial. Optimal monetary policy under commitment at the ZLB, the effects of a delayed liftoff from the ZLB, and size of the fiscal multiplier at constant interest rates. We contrast the findings under rational expectations with those under level- k -thinking, a form of bounded rationality introduced by [Farhi and Werning \(2017\)](#) that is consistent with the micro evidence on expectations formation. The boundedly rational expectation formation process does not lead to a number of puzzling features from rational expectations models, such as the forward guidance and the reversal puzzle, Neo-Fisherian features of optimal monetary policy at the ZLB or implausible large fiscal multipliers. Our results show which qualitative features from rational expectations models survive under level- k thinking, and how those features change quantitatively.

Keywords: fiscal multiplier, fixed interest rates, new Keynesian model, zero lower bound, forward guidance puzzle, reversal puzzle

JEL Classification: E32

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1 Introduction

Most model-based analysis of monetary policy assumes that agents have rational expectations. However, the empirical literature documents that the expectation formation of firms and households is at a odds with that assumption, see for instance [Coibion and Gorodnichenko \(2015\)](#) or [Coibion, Gorodnichenko, and Kamdar \(2018\)](#). Motivated by this discrepancy, this paper studies macroeconomic dynamics in a model of bounded rationality. Our focus is on the zero lower bound (ZLB) in sticky price models. At the ZLB, expectations about outcomes in the distant future can be crucial for economics dynamics along the entire path. Furthermore, in the aftermath of the financial crisis central banks have widely used communication about the future path of monetary policy as an instrument to compensate for the inability to move the policy rate in the near term. Any assessment of the effects of such communication should take into account how expectations are actually formed.

In this paper, we assume that expectations are formed according to level- k thinking. This concept was introduced by [Farhi and Werning \(2017\)](#) and is closely related to the ideas proposed earlier in [Garcia-Schmidt and Woodford \(2015\)](#). When assessing the effect of a policy intervention, agents iteratively update their beliefs starting out from a baseline belief. At the first level, they calculate the direct effect of the policy tool holding fixed beliefs about future variables at baseline. This is roughly equal to the partial equilibrium effect. At higher levels, those beliefs are then updated from the sequence of economic outcomes that would obtain period by period given the direct effect of the intervention and the previous level of beliefs about the future. For any given set of beliefs, outcomes are a temporary equilibrium in these sense of [Grandmont \(1977\)](#). When this updating process is stable, the equilibrium converges to rational expectations. In this paper, we focus on low levels of belief revisions, in line with the empirical evidence, see for example [Mauersberger and Nagel \(2018\)](#) or [Coibion, Gorodnichenko, Kumar, and Ryngaert \(2018\)](#).

We focus on three key policy questions for which communication about the future path of monetary or fiscal policy and hence expectations formation is crucial.¹ First, how should monetary policy optimally be conducted under commitment when an adverse demand shock drives the economy to the ZLB for an extended period? Second, what are the macro effects of delaying the lift-off from the ZLB as in a time-dependent forward guidance policy? Third, how large are fiscal multipliers when monetary policy is at the lower bound? Our main findings are summarized below.

1. Optimal monetary policy under commitment at the ZLB:

A few authors, such as [Schmitt-Grohé and Uribe \(2014\)](#) and [Schmitt-Grohé and Uribe \(2017\)](#) have argued that in order to exit from a liquidity trap, central banks should raise nom-

¹These questions have received a large amount of attention in the context of rational expectation models. For related analysis under rational expectations see [Eggertsson and Woodford \(2003\)](#) regarding optimal monetary policy at the ZLB under commitment, regarding the effects of a delayed liftoff (including the so-called reversal puzzle) see [Carlstrom, Fuerst, and Paustian \(2015\)](#) or [Del Negro, Giannoni, and Patterson \(2012\)](#) and for the fiscal multiplier at the ZLB see [Christiano, Eichenbaum, and Rebelo \(2011\)](#), for example.

inal interest rates rather than keep them lower for longer. While the detailed logic depends on the specific model, the prescription rests on the assumption that the higher path for nominal interest rates goes hand in hand with an increase in inflation expectations. We show that such a policy prescription can indeed be the outcome of an optimal commitment policy under rational expectations in the context of a demand shock that brings the economy to the ZLB. In particular, the central bank optimally raises the nominal interest rate off the ZLB earlier than under a simple Taylor rule and prescribes a higher path thereafter. However, when we compute optimal monetary policy at the ZLB under commitment with level- k thinking, we show that no such Neo-Fisherian features arise. The path for the nominal interest rate that is optimal if the private sectors forms expectations according to level- k delays liftoff relative to the Taylor rule and prescribes a lower path for interest rates thereafter. In addition, we show that the results are robust when extending the model with an endogenous propagation mechanism through inflation persistence.

For the region of the parameter space where the rational expectations solution does not have Neo-Fisherian features, the optimal policy under level- k thinking looks qualitatively similar to the rational expectations model. However, the central bank must extend the duration of the stay at the ZLB relative to what is implied by optimal policy under rational expectations and hence promise even more accommodative policy. The central bank nevertheless achieves less beneficial macroeconomic stabilization outcomes. Hence, the key qualitative prescription from optimal monetary policy under commitment at the ZLB derived in rational expectations models for instance by [Eggertsson and Woodford \(2003\)](#) and [Jung, Teranishi, and Watanabe \(2005\)](#) continue to hold with expectations formation according to level- k thinking albeit with a reduced macroeconomic benefit.

2. The effects of time dependent forward guidance at the ZLB:

We examine the effects of time dependent forward guidance. We model such as a policy as delaying the liftoff from the ZLB relative to the Taylor rule by an exogenous number of quarters before returning to the rule. As shown in [Carlstrom, Fuerst, and Paustian \(2015\)](#), such a policy can be extremely powerful in rational expectation models and curiously, in models with strong enough endogenous propagation mechanisms the reversal puzzle can arise. The reversal puzzle is the implication that extending the length of an interest rate peg beyond some critical duration results in a fall in inflation and activity instead of an increase.

We find that the effects of a delayed liftoff policy under level- k thinking is qualitatively similar to rational expectations in parts of the parameter space. However, the macroeconomic effects are substantially smaller as expectations adjust slowly. However, in other parts of the parameter space where the rational expectations model displays the so-called reversal puzzle, the level- k framework does not. Furthermore, in that region of the parameter space, level- k thinking does not converge to the counter-intuitive rational expectations solutions as the level of thinking k grows. Hence, our analysis provides a basis for discounting the reversal puzzle based on implausible expectations formation.

3. The fiscal multiplier under constant interest rates:

The literature has pointed out that the fiscal multiplier under constant interest rates (at the ZLB) can be large.² For instance, in the well-known paper of [Christiano, Eichenbaum, and Rebelo \(2011\)](#) the fiscal multiplier on impact for an expansion with an expected duration of 5 quarters is 3.7 in the context of a simple purely forward looking model! We show that for those parameterizations where the rational expectations multiplier is extremely large, the level- k multiplier is only modestly larger than unity. Furthermore, convergence to the rational expectations multipliers is extremely slow. For instance, with level-5 thinking the fiscal multiplier is only 1.2. Even with level-100 thinking, the fiscal multiplier is only about 85 percent of its rational expectations counterpart and full convergence requires level-500 thinking. However, for those parameterization where the fiscal multiplier under rational expectations is more modest, convergence of level- k thinking to rational expectations is fairly fast. For instance, when the expected duration of the fiscal expansion is only a little less than 5 quarters, the fiscal multiplier under rational expectations drops to 1.3 and the level-5 thinking multiplier is already 1.13 with full convergence achieved at level-20 thinking.

Furthermore, recent work by [Mertens and Ravn \(2014\)](#) has shown that the effect of government spending is different in a non-fundamental liquidity trap than in a fundamentals driven one as distinguished by the expected duration of the fiscal expansion. In the non-fundamentals-driven ZLB episode government spending multipliers are always small than unity and inflation falls rather than rises with additional spending. That prediction from rational expectations model is certainly puzzling, as it implies that agents sharply change even the sign of their inflation expectations in response to small changes in their beliefs about the duration of the fiscal expansion. We compute fiscal multipliers under level- k thinking in the full parameter space covering the determinacy and indeterminacy region from the rational expectations model and show that there is no discontinuity in the behavior of inflation expectations and hence actual inflation as under rational expectations. Fiscal multipliers under constant interest rates are always larger than unity.

Related literature:

The contribution of this paper compared the closely related work of [Farhi and Werning \(2017\)](#) and [Garcia-Schmidt and Woodford \(2015\)](#) is as follows. We see the work of these authors as laying out the general framework and theoretical foundations of two similar forms of boundedly rational belief formation. [Garcia-Schmidt and Woodford \(2015\)](#) are concerned with the question whether low interest rates result in deflation in the context of a linear sticky price model. To this end they analyze the effects of an exogenous interest rate peg under an iterative process of belief revision. Our paper analyzes a wider set of very specific and practical monetary and fiscal policy questions. In particular, we characterize optimal monetary policy under commitment in the presence of the zero lower bound when households and firms form boundedly rational beliefs. In that setting a spell at the ZLB arises endogenously

²See [Christiano, Eichenbaum, and Rebelo \(2011\)](#), [Woodford \(2011\)](#), or [Carlstrom, Fuerst, and Paustian \(2014\)](#)

rather than exogenously as in their framework and we show how optimal monetary policy differs from rational expectations in the presence of an adverse demand shock. Differently from [Garcia-Schmidt and Woodford \(2015\)](#), we also extend our sticky price model to include inflation indexation as an endogenous state variable, which can give rise to so-called reversal puzzle as discussed above. [Garcia-Schmidt and Woodford \(2015\)](#) do not assume that the expectations formation process is identical across households, an important and clearly realistic assumption. For simplicity, we abstract from heterogeneity of belief formation by the private sector. Our work follows the earlier lead of [Woodford \(2010\)](#) and [Woodford \(2013\)](#) in deriving the implications of bounded rationality for monetary policy and in particular optimal monetary policy. For a recent contribution, see also [Angeletos and Sastry \(2018\)](#).

Compared to [Farhi and Werning \(2017\)](#), we focus on the effects of boundedly rational beliefs alone. These authors study the combination of level- k thinking, incomplete markets and occasionally binding borrowing constraints and find that all features together provide a solution to the forward guidance puzzle. Our paper shows that we can resolve several puzzles within a much simpler sticky price and complete markets frameworks.

This paper is also related to the literature that overcomes the forward guidance puzzle based on alternative approaches to expectations formation. [Angeletos and Lian \(2016\)](#) propose incomplete information along with higher-order beliefs. [Wiederhold \(2015\)](#) assumes dispersed information among households and shows that forward guidance is less powerful compared to full information. Differently from those authors, our work studies the implications for optimal policy under commitment at the ZLB under bounded rationality.

The rest of the paper is organized as follows. Section 2 defines the concept of bounded rationality. Section 3 examines optimal monetary at the zero lower bound under full information and bounded rationality. Section 4 discusses the implications of bounded rationality on the forward guidance and the reversal puzzle. Section 5 revisits fiscal multipliers at the zero lower bound with bounded rationality. Section 6 concludes.

2 Bounded Rationality

Throughout this paper we will contrast outcomes under rational expectations with those under level- k thinking, a form of bounded rationality that builds on the concept of temporary equilibrium defined below.

Definition 1 (*Temporary Equilibrium*) *A temporary equilibrium is a collection of choices for households and firms such that*

1. *given beliefs, household and firms optimize at all t*
2. *goods, labor and asset markets clear for all t*
3. *budget constraints for all agents are satisfied for all t*

Formally, a temporary equilibrium in period t is a mapping from beliefs $\{B_{t+j}\}_{j=1}^{\infty}$ into equilibrium values X_t that satisfies the assumptions above. We denote this mapping by Φ , which

may depend on predetermined variables X_{t-1}

$$X_t = \Phi \left(\{B_{t+j}\}_{j=1}^{\infty}, X_{t-1} \right) \quad (1)$$

Level- k thinking specifies how agents form beliefs. We assume agents know the correct structure of the economy and furthermore that they have perfect foresight about the setting of exogenous government policies (such as the monetary policy rate or government spending). However, they cannot necessarily make fully model consistent forecasts of future endogenous variables. Instead, agents are assumed to form beliefs by going through a (small) number of iterations of belief formations, called levels of thinking. We begin with an initial level of beliefs, which we often assume to coincide with the rational expectations equilibrium prior to a particular policy intervention. Given those beliefs, the temporary equilibrium mapping Φ then generates a sequence of temporary equilibria under level-1 thinking. That sequence is then used to update beliefs and the mapping (1) is used again to generate the temporary equilibrium under level-2 thinking and the iterations repeat k times.³ A key parameter for quantitative results is the level- k . We find it reasonable that boundedly rational agents will perform at most a small number of iterations when forming their beliefs. That assumption seems to be confirmed in micro-economic experimental evidence, see [Mauersberger and Nagel \(2018\)](#) who put a typical level of reasoning no higher than 3. One of the core questions in this paper then is whether the rational expectations outcome is quantitatively or even qualitatively similar to level- k thinking for reasonably small levels of k .

We believe this type of bounded rationality analysis is likely to be more appropriate for policy advice in a temporary ZLB environment than rational expectations, because one cannot merely assume that agents' expectations will approximate rational expectations. The typical justification that rational expectations equilibria are learnable need not apply when agents have spent little time in a liquidity trap.

3 Optimal monetary policy under commitment

Our analysis begins with optimal monetary policy at the ZLB in the purely forward looking three equation new Keynesian model. In the second part we allow for an endogenous state variable via inflation indexation.

3.1 A purely forward looking model

We consider an economy where output is produced with labor as the only input. Firms set price in a staggered fashion as in the model of [Calvo \(1983\)](#) and wages are flexible. A representative household supplies labor, consumes and demands bonds in zero net supply. For arbitrary expectations, the optimality conditions for households and for firms can be

³In infinite horizon models, updating beliefs about infinite sequence is infeasible in practice and we need to truncate. Add more detail.

expressed in log-linear form as follows, see [Preston \(2005\)](#):

$$y_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s [(1 - \beta)y_{t+1+s} - \frac{1}{\sigma}(i_{t+s} - nr_{t+s} - \pi_{t+1+s})], \quad (2)$$

$$\pi_t = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s [\beta(1 - \varphi)\pi_{t+1+s} + \kappa y_{t+s}], \quad (3)$$

$$nr_t = \rho nr_{t-1} + \epsilon_t. \quad (4)$$

Here, π_t , y_t , and i_t , denote inflation, the output gap, and the nominal rate, respectively, all measured as deviations from the steady state. The exogenous variable nr_t represents the natural rate of interest that acts as a demand shock in this framework. The first equation is the household Euler equation under the assumption that any initial wealth is zero, β is the discount factor, and σ is the inverse of the intertemporal elasticity of substitution. The second equation is the optimality condition for the firm's price setting problem, where φ is the Calvo probability that the firm will not have a chance to reset its price. The slope of the Phillips curve, κ is defined by

$$\kappa \equiv \frac{(1 - \beta\varphi)(1 - \varphi)}{\varphi} (\omega + \sigma^{-1}) \quad (5)$$

where ω is the Frisch elasticity of labor supply.

It will be useful to contrast the optimal monetary policy with outcomes under a simple interest rate rule:

$$i_t = \max(r_t + 1.5\pi_t + 0.5y_t, ZLB) \quad (6)$$

We refer to outcomes under this rule as the "baseline". At the ZLB and under rational expectations, this simple rule turns out to implement the optimal monetary policy under discretion: It prescribes to follow the natural rate one for one as long as permitted by the ZLB constraint. Once the ZLB is no longer binding, this results in zero inflation and zero output gap. During the ZLB spell, a gap between the natural rate and the real interest arises that depresses aggregate demand.

In the next subsection, we use this model to shed light on the role of rational expectations in the conduct of optimal monetary policy subject to the zero lower bound. We assume a standard calibration: $\sigma = 1$, $\omega = 0$, $\beta = 0.99$. The probability of keeping the price fixed is $\varphi = 0.8564$ such that the slope of the Phillips curve under rational expectations is $\kappa = 0.0255$. We assume a large natural rate shock that pushes the economy to zero lower bound under the Taylor rule for a number of periods. In particular, we assume that the natural rate in the initial period falls to -0.033 after which it follows its autoregressive process with persistence of $\rho = 0.9$. Under the Taylor rule, the ZLB is binding for 12 quarters.

3.1.1 Optimal commitment policy with rational expectations

Under rational expectations, the equilibrium conditions of the model reduce to the familiar IS and Phillips curves.

$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\sigma} (i_t - nr_t - \mathbb{E}_t \pi_{t+1}) \quad (7)$$

$$\pi_t = \kappa y_t + \beta \mathbb{E}_t \pi_{t+1} \quad (8)$$

We assume that the central bank has a loss function with equal weights on inflation and the output gap – an assumption that turns out to be important for our result as we will discuss later on. The discount factor of the central bank is equal to that of the private sector of 0.99. Figure 1 shows the resulting outcomes for optimal policy under rational expectations.

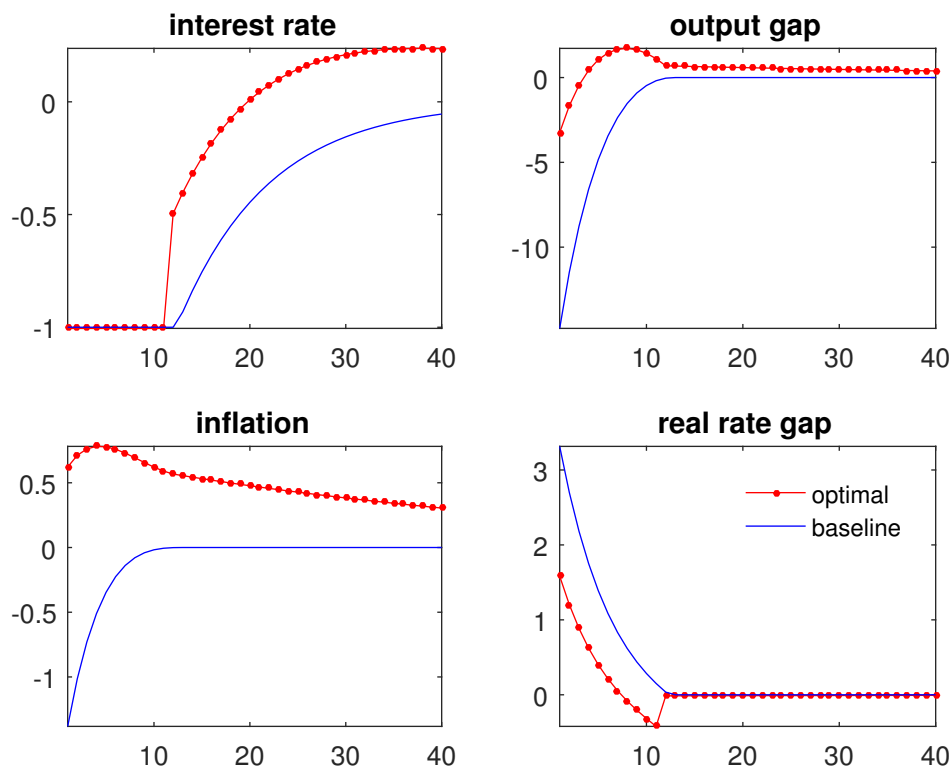


Figure 1: Optimal commitment policy with rational expectations (solid line denotes baseline evolution under the Taylor rule)

The blue line solid denotes the baseline outcome under the Taylor rule, while the red line (using dots as markers) shows the outcome under the optimal commitment policy. Remarkably, the optimal commitment policy lifts off from the zero lower bound earlier than the policy under Taylor rule and has a higher path for the nominal funds rate after liftoff. Despite this higher path for the policy instrument, macroeconomic outcomes are much better. This arises, because the real interest rate is lower under optimal policy due to the higher expected

inflation. This can be seen in the lower right panel, which plots the real rate gap - defined as the gap between the ex-ante real interest rate and the natural rate shock.

The optimal policy has a Neo-Fisherian flavor. It prescribes that nominal interest rates be raised relative to the Taylor rule (and not kept lower for longer as many central banks have done in practice) in order to improve macroeconomic stabilization in an ZLB environment, similar in spirit to the exit strategy advocated in a flexible price but sticky wage model by [Schmitt-Grohé and Uribe \(2017\)](#). Clearly, this prescription relies on being able to influence inflation expectations in line with the rational expectations path. It may be surprising that an optimal commitment policy at the ZLB is characterized by an earlier liftoff than under a simple Taylor rule (or equivalently compared to discretion). After all, previous work by [Jung, Teranishi, and Watanabe \(2005\)](#) or [Levin, Lopez-Salido, Nelson, and Yun \(2010\)](#) considers essentially the same model as we do and shows that optimal policy under commitment features staying at the ZLB for longer than under discretion. The difference to our work is in the weights in the loss function. When we use a micro-founded loss function which features relatively little weight on inflation stabilization as these authors do, we recover exactly their results under rational expectations. We will return to this calibration with a microfounded loss function later on.

3.1.2 Optimal policy with level- k thinking

The model under level- k thinking is described by the following recursions:

$$y_t^k = \sum_{s=0}^{\infty} \beta^s \left[(1 - \beta) y_{t+1+s}^{k-1} - \frac{1}{\sigma} (i_{t+s} - nr_{t+s} - \pi_{t+1+s}^{k-1}) \right] \quad (9)$$

$$\pi_t^k = \kappa y_t^k + \sum_{s=0}^{\infty} (\beta\varphi)^s \left[\beta(1 - \varphi) \pi_{t+1+s}^{k-1} + \kappa\beta\varphi y_{t+1+s}^{k-1} \right] \quad (10)$$

Here, we take the path of the interest rate and any natural rate shock as exogenous and known to the private sector. The equations (9) and (10) then determine the expectations mapping Φ defined earlier which give rise to the sequence of temporary equilibria under level- k thinking for any given sequence of beliefs of level $k - 1$. The sequence of temporary equilibria is then used to update beliefs in the next level.

We compute optimal commitment policy numerically by solving a quadratic programming problem subject to an inequality constraint on the equilibrium interest rate. The instrument in quadratic programming problem is the path of the nominal interest rate chosen via adding anticipated monetary policy shocks to an arbitrary interest rate rule as in [Laseen and Svensson \(2011\)](#), [Holden and Paetz \(2012\)](#) and others.⁴ Appendix A shows the impulse response to

⁴Under rational expectations, the particular rule chosen to calculate the impulse response to anticipated policy shocks is irrelevant as long as it generates a determinate equilibrium. Under level- k thinking, it is important that the rule does not include an endogenous feedback to equilibrium variables, but instead be purely exogenous. That is required so that both the actual and the expected path of the nominal interest rate obeys the ZLB when those expectations are non-rational. An interest rate rule that is purely exogenous does not generate issue of determinacy under level- k thinking as outlined in [Farhi and Werning \(2017\)](#).

anticipated policy shocks under level- k thinking that are at the core of our algorithm. The algorithm can be applied to rational expectation and bounded rationality models and is sketched in Appendix B. For numerical reasons, the solution method requires that the infinite horizon problem be approximated by an arbitrarily long but finite horizon problem.

We assume that the private sector forms expectations in line with level- k , the central bank knows the level k of belief formation and is choosing its instrument optimally to minimize the discounted sum of variances of realized inflation and realized output. In other words, only the private sector is subject to bounded rationality. The private sector does have, however, perfect foresight about the path for the policy rate itself which we justify with reference to forward guidance.

As in previous subsections and to provide a fair comparison, the initial beliefs in this exercise are set to the rational expectations solution under the Taylor rule subject to the ZLB.

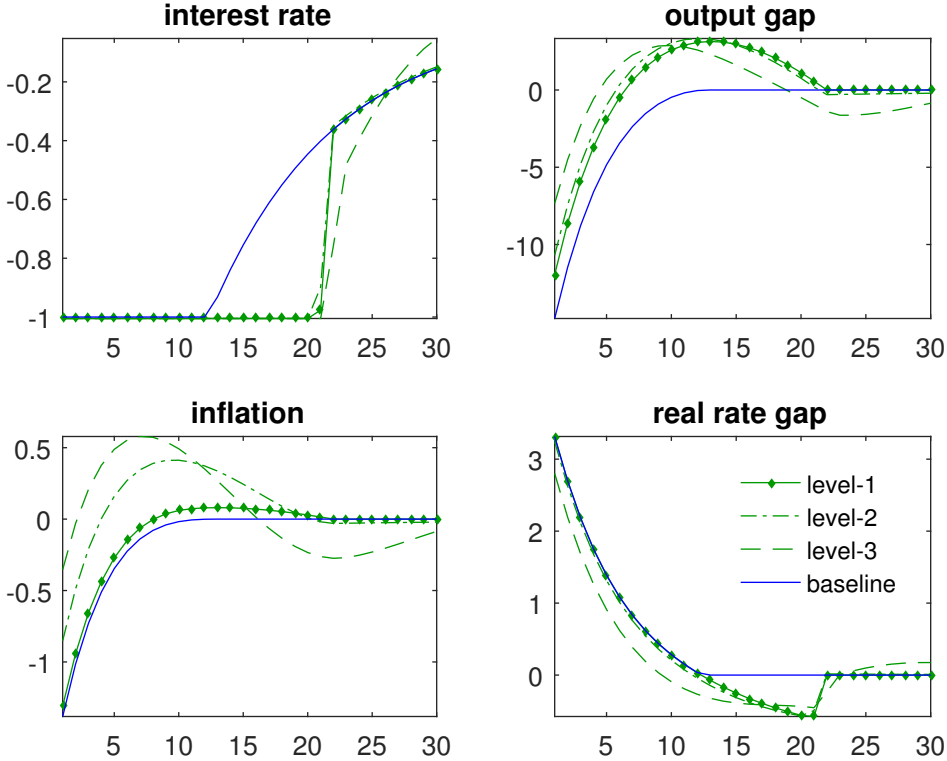


Figure 2: Optimal policy with level 1, 2, and 3 expectations

Figure 2 shows that with this process for expectations formation, rather than raising the nominal interest rate from the lower bound earlier than the Taylor rule prescribes as under rational expectations, the funds rate lifts off much later. In fact, liftoff is delayed to period 20, a full 2 years later than the Taylor rule. Macroeconomic outcomes improve, but only mildly so compared to the remarkable improvement under optimal commitment with rational expectations. We now discuss expectations formation and the associated macro outcomes for the different levels of belief revision k in more detail.

Under level-1 thinking, inflation and output gap expectations are at baseline and only the direction effect of the delayed liftoff from the ZLB affects macro outcomes. Since the path for the interest rate under optimal policy is assumed to be known and it enters the Euler equation directly (albeit with small discounting), the output gap is improved noticeably. Inflation barely moves with level-1 thinking, because expectations about future inflation and the future output gap are at baseline and the impact of current period output gap on inflation is small. Under level-2 thinking, the path for inflation is substantially improved as price setters update their expectations of the future output gap (and much less importantly also update their expectations of future inflation). The path for the output gap under level 2 thinking is little changed from that of level 1, because the level 1 path for inflation that is used in expectations formation for the real interest rate is little changed from baseline and the path for the output gap that *is* changed does not carry much weight in the Euler equation.⁵ Under level-3 thinking, the main change relative to level-2 thinking is in inflation expectations that now revise substantially and in line with the the actual paths for inflation under level 2 thinking. Hence, the ex-ante real interest rate gap (the gap between the ex-ante real interest rate computed using households subjective inflation expectation and the natural rate shock) is smaller than in the baseline and hence the output gap improves further compared to level-2 thinking.

Nevertheless, even under level-3 thinking, output falls about [6] percent below steady state on impact – almost twice as much as under rational expectations – and quarterly inflation remains [0.5] percentage points below baseline on impact against a [0.5] rise above baseline under rational expectations. Hence, the main finding for optimal policy under non-rational belief formation is that policy is required to stay at the ZLB longer than under rational expectations, but nevertheless that more accommodative policy is less successful at improving macro outcomes than under rational expectations.

3.1.3 Convergence to rational expectations?

The optimal path for the nominal interest rate under rational expectations will not deliver outcomes for inflation and output from the rational expectations model if the private sector forms expectations under any level of k . The reason is that the rational expectations path for the nominal interest rate is *uniformly* higher than under the Taylor rule at all horizons. It is trivial to see from equations (9) and (10) that a uniformly higher path for the nominal funds rate results in lower than baseline paths for output and inflation at any level of k .

Nevertheless, it is natural to ask whether the policy that a central bank would optimally choose knowing that the private sector forms expectations according to level- k would closely approximate those under rational expectations for high level of k .

⁵It is important to recognize that Figure 2 recomputes optimal monetary policy for any given level of thinking of the private sector. Hence, the path for inflation expectations under level-2 thinking that is used in the computations of optimal policy under level-2 thinking is not the one plotted in this figure for level 1 thinking, because the latter is based on a different optimal interest rate path. Nevertheless for the purpose of discussion and since the difference in optimal interest rate paths as k varies is small, one can gain intuition by reading beliefs under level 2 thinking for inflation (or output) off of the path under level-1 thinking shown in this figure for those variables.

Claim 1 (*Quasi-equivalence result*)

For sufficiently high levels of k , the path for the nominal interest rate chosen optimally by the policymaker brings about an equilibrium sequence for $\{y_t^k, \pi_t^k\}_{t=0}^T$ that is numerically within a neighborhood of $\epsilon < 10^{-6}$ of the rational expectations solution. The required path for the nominal federal funds rate is within a similarly sized neighborhood of its rational expectations counterpart in all periods but the very last period T .

Claim 2 states that the equilibrium for inflation and output is essentially the same as under rational expectations over the horizon of the experiment, but the policy path necessary to achieve this outcome under level- k differs from the rational expectation outcome in the very last period. In the example chosen in this subsection, policymaker operating in a level $k \sim 50$ environment sets an equilibrium funds rate in the final period $T = 100$ less than 1 basis point below steady state on an annualized basis! Despite this small magnitude, this final value is crucial for the dynamics along the entire path. In all periods prior to T the nominal funds rate is above baseline. It can easily be seen that for any given level of k a higher funds rate implies lower inflation and output along the entire path. Hence, without the small stimulus in the last period, inflation and output would have been massively below the solution under the baseline Taylor rule in all periods. This is merely another manifestation of the forward guidance puzzle under rational expectations and under the near rational expectations solution for high levels of k .⁶

The analysis makes clear that proponents of the Neo-Fisherian view for how to exit the zero lower bound (by *raising* nominal interest rates) rely heavily on rational expectations or in the context of this model a high level of k and the forward guidance puzzle. In practice, one may hold the view that inflation expectations will not instantaneously adjust perfectly in line with rational expectations and that small changes to interest rates at far distant horizons have little to no effect on current period outcomes. But if that is so, the prescriptions from the bounded rationality models for low levels of k examined here will likely give more suitable policy advice. These prescriptions turned out to be in line with actual forward guidance policies chosen during the crisis.

3.1.4 Optimal policy with a welfare-based loss function

We now examine the results of optimal policy with a welfare based loss function. That is, the relative weight on output gap stabilization is now $\frac{\kappa}{\epsilon} = \frac{.0255}{6} = 0.00425$. As pointed out earlier, under this calibration, the optimal policy under commitment does not display any Neo-Fisherian features.

⁶In a rational expectations model and without imposing the ZLB, the importance of small changes in the last period in a finite horizon model of optimal monetary policy has been pointed by [Campbell and Weber \(2018\)](#)

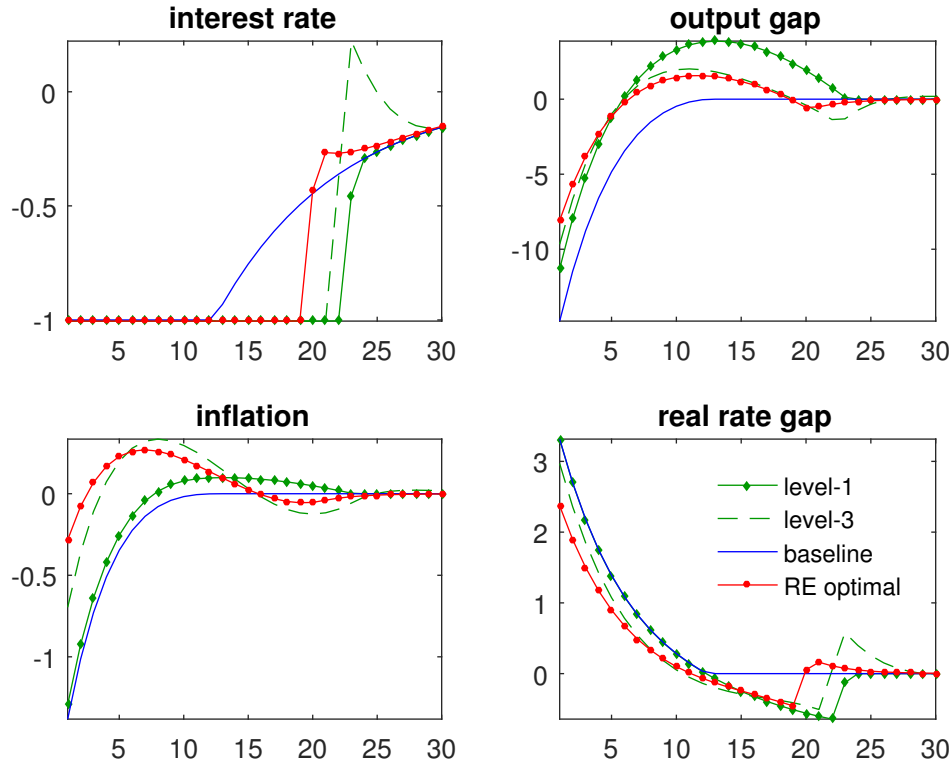


Figure 3: Optimal policy with with level 1, 2, and 3 expectations and a welfare based loss function.

Figure (3) shows the outcomes under the optimal commitment policies with rational expectations as well as with level- k thinking for levels 1 and 3 (level 2 is not shown in order to keep the Figure readable). In all of these cases, the optimal policy delays liftoff relative to the Taylor rule. In line with the results from the previous setting, monetary policy optimally stays at the ZLB for longer under level- k thinking than under rational expectations and achieves smaller improvements in inflation and output in the initial periods than under rational expectations.

3.2 Adding inflation persistence

The purely forward-looking model lacks an endogenous propagation mechanism that is needed to provide more realistic inflation dynamics. We therefore allow for indexation in price setting. In particular, we assume that those firms that do not receive a signal to update their their price fully index to last period's inflation rate. For arbitrary expectations, the equilibrium

conditions can now be expressed as :

$$y_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s [(1 - \beta)y_{t+1+s} - \frac{1}{\sigma}(i_{t+s} - nr_{t+s} - \pi_{t+1+s})] \quad (11)$$

$$p_t^* + \pi_t = (1 - \beta\varphi)\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s [\pi_{t+s} + (\omega + \sigma^{-1})y_{t+s}] \quad (12)$$

$$p_t^* = \frac{\varphi}{1 - \varphi}(\pi_t - \pi_{t-1}) \quad (13)$$

The second equation is the first-order condition for firms that optimally adjust their price where p_t^* denotes the price of adjusting firms relative to the aggregate price index. The third equation follows from the recursion for the aggregate price index. These equations can be reduced under rational expectations to the familiar system:

$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}(i_t - nr_t - \mathbb{E}_t \pi_{t+1}) \quad (14)$$

$$\pi_t = \frac{1}{1 + \beta}\pi_{t-1} + \frac{\beta}{1 + \beta}\mathbb{E}_t \hat{\pi}_{t+1} + \frac{1}{1 + \beta}\kappa y_t \quad (15)$$

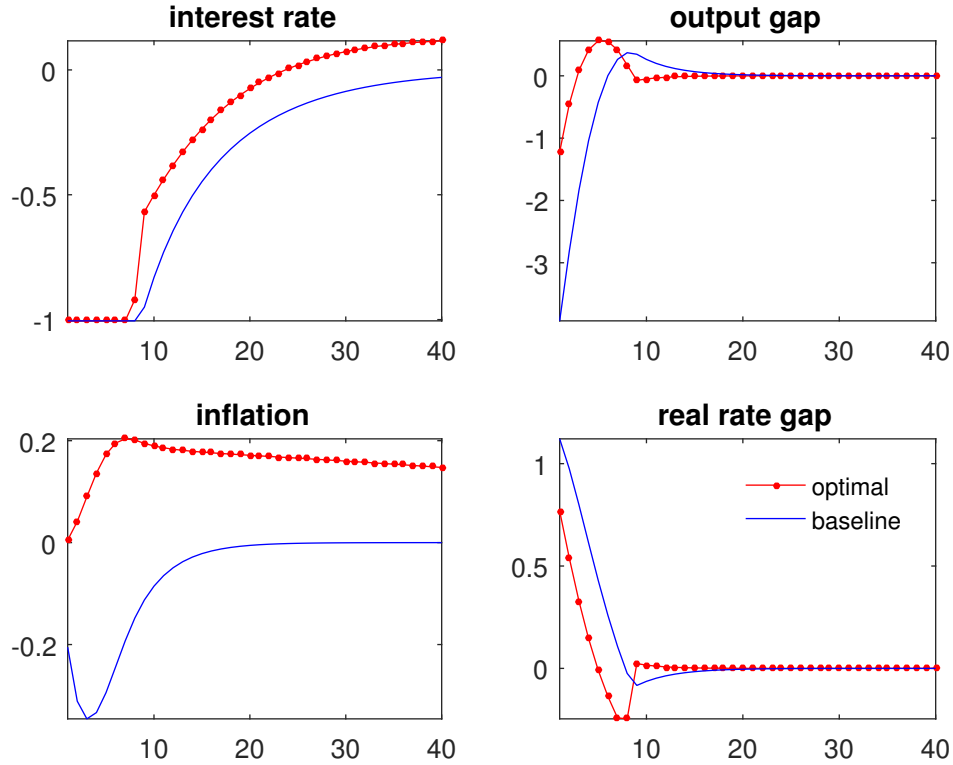


Figure 4: Optimal policy under rational expectations with inflation inertia.

As in the previous subsection, we consider optimal monetary policy when the economy is faced with a natural rate shock that causes the ZLB to bind under the baseline Taylor rule. We assume a smaller size of the natural rate shock so that the baseline outlook under the Taylor rule is similar to the one in the previous subsection.⁷ In particular, the initial innovation is only $\epsilon_1 = -0.018$.

The presence of inflation persistence alters the micro-founded loss function which penalizes the first difference of inflation. Consistent with this finding, we assume that the central bank now stabilizes the variances of the change in inflation and the output gap. Also consistent with the micro-founded loss function, we set the relative weight on output gap stabilization to $\frac{\kappa}{\epsilon}$

Figure 4 shows that under the optimal commitment policy with rational expectations, the interest rate is raised from the ZLB earlier than under the Taylor rule as in the purely forward looking model. And the nominal rate follows a higher path after lift-off. Outcomes for inflation and output are improved, because inflation expectations adjust upwards thereby improving the real interest rate gap relative to the Taylor rule.

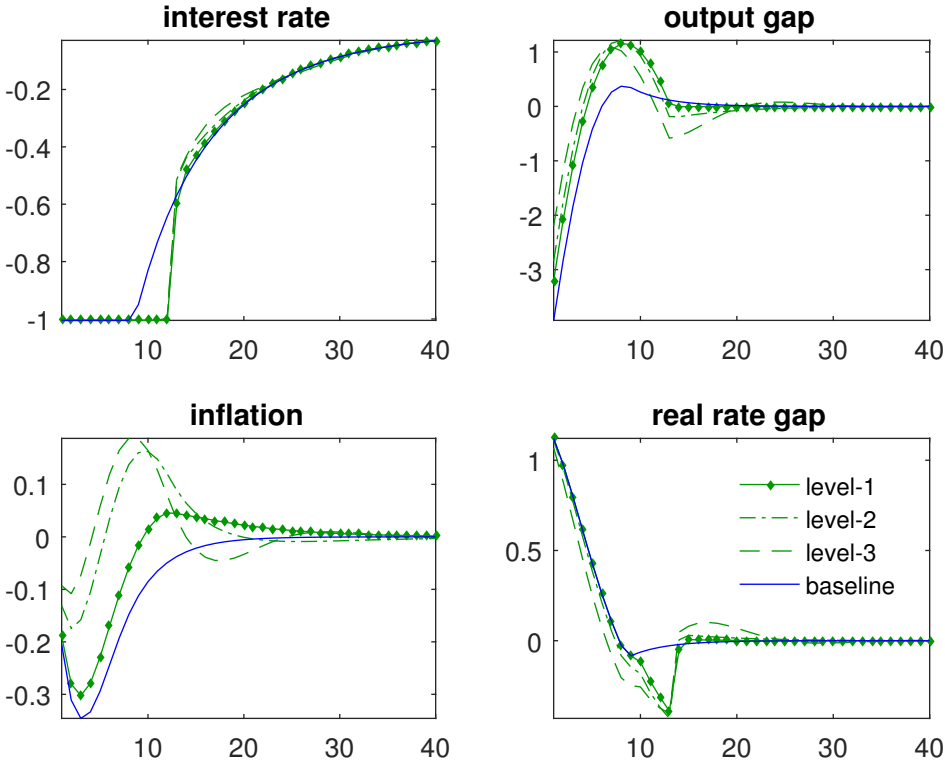


Figure 5: Optimal policy under levels 1, 2, and 3 with inflation inertia.

Again it is natural to compute optimal policy if the private sector forms beliefs according

⁷Given the inflation persistence, the economy may travel in and out of binding ZLB episodes if the shock is large which we want to avoid.

to level- k thinking. The equilibrium conditions are given by the following recursions:

$$y_t^k = \sum_{s=0}^{\infty} \beta^s [(1 - \beta)y_{t+1+s}^{k-1} - \frac{1}{\sigma}(\mathbb{E}_t i_{t+s} - \pi_{t+1+s}^{k-1})] \quad (16)$$

$$(p_t^*)^k + \pi_t^k = (1 - \beta\varphi) \left[\pi_t^k + (\omega + \sigma^{-1})y_t^k + \sum_{s=1}^{\infty} (\beta\varphi)^s [\pi_{t+s}^{k-1} + (\omega + \sigma^{-1})y_{t+s}^{k-1}] \right] \quad (17)$$

$$(p_t^*)^k = \frac{\varphi}{1 - \varphi} (\pi_t^k - \pi_{t-1}^k), \quad (18)$$

The latter two equations can be simplified to

$$\pi_t^k = \alpha_1 \pi_{t-1}^k + \alpha_2 \left[(\omega + \sigma^{-1})y_t^k + \sum_{s=1}^{\infty} (\beta\varphi)^s [\pi_{t+s}^{k-1} + (\omega + \sigma^{-1})y_{t+s}^{k-1}] \right] \quad (19)$$

with $0 < \alpha_1 = 1/(1 + \beta - \beta\varphi) < 1$ and $0 < \alpha_2 = (1 - \varphi)(1 - \beta\varphi)/(\varphi + \beta\varphi(1 - \varphi)) < 1$.

Figure 5 shows the results under level- k thinking for k of 1 through 3. Consistent with the findings from the purely forward looking model, optimal policy involves staying at the ZLB for longer than under the Taylor rule. The macroeconomic stabilization delivered by this policy is, however, more modest than under the optimal policy with rational expectations. The output gap falls to almost -3 percent vs a little over -1 percent under rational expectation.

4 Delayed liftoff and the reversal puzzle

The previous section has analyzed fully optimal monetary policy under commitment and shown that the process for expectations formation can be crucial for the design of optimal policy. In this section, we consider the effect of simpler policies, namely those that delay the liftoff from the ZLB by a fixed number of quarters relative to the liftoff date under a benchmark Taylor rule. Such “lower for longer policies” have been widely used in the aftermath of the financial crisis by central banks. A number of authors have shown that such a time-dependent forward guidance policy can have implausibly large effects on initial inflation and output, see for instance [Carlstrom, Fuerst, and Paustian \(2015\)](#) or [Del Negro, Giannoni, and Patterson \(2012\)](#).

We consider an environment where the central bank communicates that the nominal interest rate stays at the ZLB for an extended period before returning to an interest rate rule. This occurs against the background of an adverse demand shock that drives the economy endogenously to the zero lower bound. In particular, we assume the central bank announces an interest rate rule given by the following:

$$i_t = \begin{cases} ZLB & t = 1, 2, \dots, t^*, t^{*+1}, \dots, t^{*+k} \\ \max(ZLB, r_t + \phi_\pi \pi_t + \phi_y y_t) & t \geq t^{*+k+1} \end{cases}$$

Here, t^* is the period prior to lift-off under the baseline Taylor rule. This policy delays liftoff by k periods relative to the Taylor rule. Post-liftoff the baseline rule applies again, possibly

subject the ZLB (which never turns out to be binding). We use the standard calibration of $\phi_\pi = 1.5$ and $\phi_y = 0.5$. We use gain the model with indexation in price setting. The initial innovation into the natural rate is set to $\epsilon_1 = -0.015$ and we assume a persistence of 0.9 as before.

This policy experiment models in a very simple way key elements of forward guidance implemented by several central banks in the aftermath of the financial crisis. While the effects of such a policy are difficult to quantify empirically, most commentators agree that this policy provided macroeconomic stimulus that mitigated the adverse effects of the crisis.

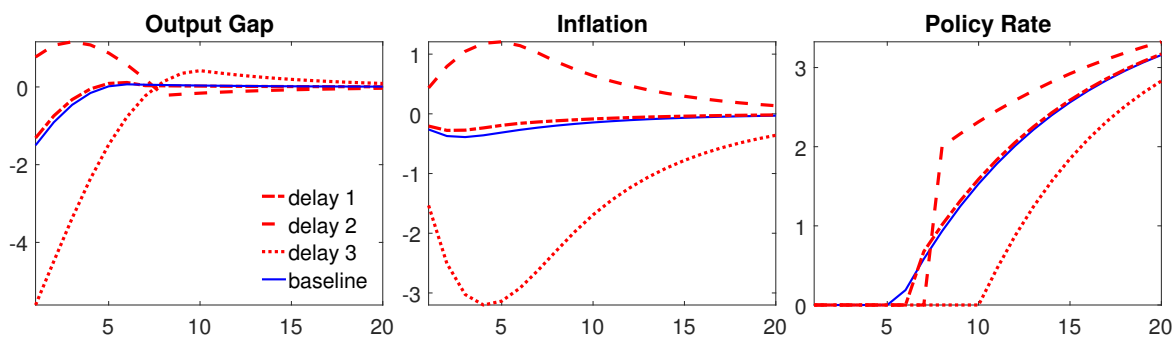


Figure 6: Delayed liftoff under rational expectations

Figure 6 shows the solution under rational expectations under the standard Taylor rule in blue and with a delayed liftoff by 1, 2, and 3 quarters. Note that a delay by one quarter improve outcomes very little. However, a delay by 2 quarters results in a dramatic improvement. Output rises above steady state instead of contracting by nearly 2 percent on impact under the baseline and inflation is persistently above steady state rather than below under the baseline rule. Under rational expectations, macro economic outcomes are extremely sensitive to very small variations in the liftoff date in this model.

One would suspect that a delay by 3 quarters would result in further increases in output and inflation. However, in the context of this calibrated model, a delayed liftoff policy under rational expectations can result in a so-called *reversal puzzle* as pointed out in [Carlstrom, Fuerst, and Paustian \(2015\)](#).⁸ Rather than further stimulating the economy, a delay in the liftoff date by 3 quarters results in a contraction in output and inflation. The weak economic outcomes then cause the ZLB to bind further and the ZLB binds endogenously for a total of 10 quarters. These perverse movements of inflation and output in response to a time-dependent delay in the liftoff have been discussed in detail in [Carlstrom, Fuerst, and Paustian \(2015\)](#) and they cast a doubt on the rational expectations assumptions. One may question why households and firms should be expected to revise their expectations in response to a delayed liftoff by 3 periods in a even qualitatively completely different way than for a delay by 2 periods.

⁸[Carlstrom, Fuerst, and Paustian \(2015\)](#) provide conditions on the degree of indexation in the Phillips curve for the reversal puzzle to occur.

We therefore examine the same delayed liftoff policy under level- k thinking. We again assume that the initial expectations are given by the baseline outlook under the Taylor rule. Figure 4 shows the solution to the model with a delayed liftoff by the same 1, 2 and 3 quarters under level 2 thinking. Similar plots under level-1 and level-3 thinking are contained in the Appendix C for reference.

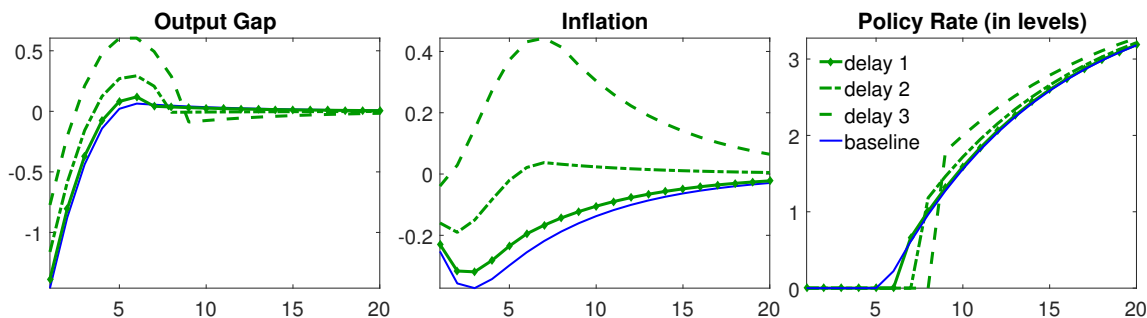


Figure 7: Level-2 solution with delayed liftoff

Figure shows that there is no reversal puzzle under level- k thinking. Keeping interest rates lower for longer improves outcomes by more the longer the interest rate stays at the ZLB. Numerical results not shown in here confirm that this is the case for further delays in the liftoff date beyond the 3 quarters that are plotted in the figure.

We close this section by discussing whether the iterative process of belief revision for large k converges to the rational expectations solution. We find numerically that it does so when the delay in the liftoff is small (a delay of 2 periods) such that the policy is expansionary for output and inflation under both rational expectations and level- k thinking. However, it does not converge to the rational expectations solution for a delay in the liftoff that results in a (counterintuitive) contraction in real activity and inflation under rational expectations. For instance, for a delay of 3 periods, as the level- k increases the effects of on initial inflation and output diverge to arbitrarily large values. In our judgment, non-convergence provides a formal basis for discounting this particular prescription from the rational expectations framework as not relevant in practice.

5 Fiscal multiplier

This section considers fiscal multipliers at the ZLB under level- k thinking. We begin with the case of an equilibrium at the ZLB driven by fundamentals and then proceed to an expectations-driven liquidity trap similar to [Mertens and Ravn \(2014\)](#).

5.1 A fundamentals-driven equilibrium

A large literature has pointed out that the fiscal multiplier under constant interest rates such as in a ZLB episode can be large. The mechanism is well understood: higher government

spending raises inflation expectations which reduces ex-ante real interest rates and hence crowds in private expenditures, thus raising the multiplier above unity. What is surprising is that the effect can be quantitatively very large. For instance, in the well-known paper of [Christiano, Eichenbaum, and Rebelo \(2011\)](#) the fiscal multiplier on impact for an expansion with an expected duration of 5 quarters is 3.7 in the context of a simple purely forward looking model. [Carlstrom, Fuerst, and Paustian \(2014\)](#) point out that the mechanism that model relies on rational expectations about small probability events in which the fiscal expansion lasts for a long time and the expected macro outcomes are huge.

It is thus natural to examine the sensitivity of the fiscal multiplier under constant interest rates to bounded rationality. For simplicity, we consider an environment similar to that of section 3 of [Christiano, Eichenbaum, and Rebelo \(2011\)](#). We consider a coordinated policy experiment in which (i) government spending is set above steady state $g_t = \bar{g} > 0$, and simultaneously (ii) the central bank announces an interest rate peg $i_t = 0$. Each period there is probability p that this policy will continue so that the expected duration of the expansion is $T = \frac{1}{1-p}$. With probability $(1 - p)$ the expansion ends, at which point fiscal policy returns to steady state, $g_t = 0$, and monetary policy reverts to a typical Taylor rule:

$$i_t = \phi_\pi \pi_t + \phi_y y_t. \quad (20)$$

Under standard assumptions on ϕ_π and ϕ_y , there is a unique equilibrium after the period of the peg. Since there are no state variables nor exogenous shocks during these subsequent periods, the unique equilibrium after the policy experiment is given by $\pi_t = y_t = 0$.

Under arbitrary expectations about the outcomes during a fiscal expansion, the equilibrium conditions are given by

$$c_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s p^{s+1} [(1 - \beta)c_{t+1+s} - \frac{1}{\sigma}(i_{t+s} - \pi_{t+1+s})] \quad (21)$$

$$\pi_t = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\varphi)^s [p^{s+1}\beta(1 - \varphi)\pi_{t+1+s} + p^s \kappa (\sigma c_{t+s} + \omega^{-1} y_{t+s})] \quad (22)$$

$$y_t = (1 - s)c_t + sg_t \quad (23)$$

and κ is defined by

$$\kappa = \frac{(1 - \beta\varphi)(1 - \varphi)}{\varphi} \quad (24)$$

Under rational expectations the model is given by:

$$c_t = \mathbb{E}_t c_{t+1} - \frac{1}{\sigma} (i_t - \mathbb{E}_t \pi_{t+1}) \quad (25)$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa m c_t \quad (26)$$

$$m c_t = \sigma c_t + \omega^{-1} y_t \quad (27)$$

$$y_t = (1 - s)c_t + sg_t \quad (28)$$

The constant $s = \frac{G}{Y_{ss}}$ is the share of government spending in the steady state. Under rational expectations, the fiscal multiplier during the interest rate peg is given by:

$$\frac{dY}{dG} \equiv \left(\frac{1}{s}\right) \frac{dy_t}{dg_t} = \left[\frac{\sigma [(1-p)(1-\beta p) - \kappa p]}{\Delta} \right] \quad (29)$$

where

$$\Delta \equiv \sigma (1-p)(1-\beta p) - \kappa [\sigma + \omega^{-1}(1-s)] p. \quad (30)$$

As shown for example in [Carlstrom, Fuerst, and Paustian \(2014\)](#), the model has unique stable equilibrium whenever $\Delta > 0$. We restrict attention to that case for now.

We use the same baseline parameter values as in [Carlstrom, Fuerst, and Paustian \(2014\)](#): $\beta = 0.99$, $\kappa = 0.028$, $\omega^{-1} = 0.5$, $\sigma = 2$, $s = 0.2$ and $p = 5/6$. Under this calibration, the rational expectations fiscal multiplier is 4.9. That is, one dollar of government spending crowds in almost a additional 4 dollars of private spending.

Our system of equations under level- k thinking is,

$$c_t^k = \sum_{s=0}^{\infty} p^{s+1} \beta^s [(1-\beta)c_{t+1+s}^{k-1} - \frac{1}{\sigma}(\hat{i}_{t+s} - \pi_{t+1+s}^{k-1})] \quad (31)$$

$$\pi_t^k = \kappa(\sigma c_t^k + \omega^{-1}y_t^k) + \sum_{s=0}^{\infty} (\beta\varphi)^s p^{s+1} [\beta(1-\varphi)\pi_{t+1+s}^{k-1} + \kappa\beta\varphi(\sigma c_{t+1+s}^{k-1} + \omega^{-1}y_{t+1+s}^{k-1})] \quad (32)$$

$$y_t^k = (1-s)c_t^k + sg_t^k \quad (33)$$

We assume that agents have boundedly rational beliefs about a an equilibrium where all future variables during the fiscal expansion take on constant values, that is $\{\pi_{t+1+s}^k\}_{s=0}^{\infty} = \bar{\pi}^k$, $\{c_{t+1+s}^k\}_{s=0}^{\infty} = \bar{c}^k$. The right hand side of equations (31)and (32) then provide updating formulas to revise these beliefs iteratively. The assumption of constant values during the expansion is consistent with rational expectations. [Table 1](#) contains the results under level- k thinking.

level-k	1	2	5	10	20	30	40	50	100	200	500
multiplier	1	1.03	1.23	1.53	2.05	2.5	2.87	3.19	4.16	4.76	4.9
(% of RE)	(20)	(21)	(25)	(31)	(42)	(51)	(59)	(65)	(85)	(97)	(100)

Table 1: Fiscal multiplier under level-k thinking

[Table 1](#) shows that to achieve beliefs consistent with the rational expectations fiscal multiplier of 4.9 agents are required to iteratively update their beliefs to an extraordinarily high level. For example, level 5 beliefs result in a fiscal multiplier that is only 1.23. And even level 100 beliefs only generate a multiplier that is 85 % of its rational expectations counterpart, while full convergence is only achieved at level 500.

Is this slow convergence to rational expectations under level- k thinking a generic feature of fiscal multipliers in this framework, or is convergence only slow when the multiplier is unusually large under rational expectations? To answer this, we reduce the probability of staying in the expansionary fiscal policy regime from $p = 5/6 \approx 0.83$ to $p = 0.8$. Under rational expectations, this lowers the fiscal multiplier from 4.9 to 1.3. Under level- k thinking, convergence to this smaller fiscal multiplier is relatively rapid. In particular, level 5 implies a multiplier of 1.13, level 10 a multiplier of 1.2 and at level 20 the model has almost converged to rational expectations producing a multiplier of 1.29.

We conclude that the huge fiscal multipliers which can occur in this very simple model under rational expectations require beliefs that can be approximated in this framework only by an unusually high level of k . When fiscal multipliers are smaller and arguably more reasonable, the approach of bounded rationality taken here approximates rational expectations closely for relatively low levels of k .

5.2 An expectations-driven liquidity trap

Recent work by [Mertens and Ravn \(2014\)](#) has shown that the effect of government spending is different in an expectations driven liquidity trap than in a fundamentals driven one. In particular, their analysis restricts attention to a minimum state variable solution (MSV) augmented with a sunspot shock that follows a Markov process. Whenever, the sunspot shock is persistent enough, the augmented MSV solution leads to self-fulfilling spells at the ZLB in which government spending multipliers are always small than one rather than bigger than unity and inflation falls rather than rises with additional spending.

In line with their approach, we can calculate fiscal multipliers under the minimum state variable solution even for $p > p^*$. Here, p^* is the critical value for which $\Delta = 0$ in equation (30), which is the boundary of the determinacy region. Another interpretation of this multiplier is that it is the fiscal multiplier in an expectations driven liquidity trap driven by a sunspot shock that persists with probability p . As discussed in [Farhi and Werning \(2017\)](#), with level- k thinking indeterminacy of equilibrium in a linear model never arises for a fixed set of initial beliefs, essentially because the updates of expectations are unique.

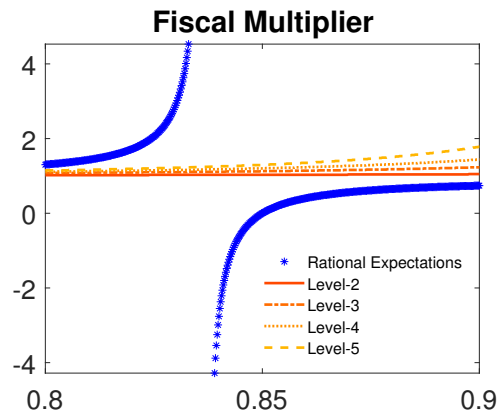


Figure 8: Fiscal multiplier as a function of the probability p

Figure 8 plots the fiscal multiplier across both regions of p under rational expectations and for level- k thinking. The multipliers are broadly similar for $p < 0.8$. Under rational expectations, the fiscal multiplier then grows rapidly for small increases of p . It asymptotes at the determinacy region to unboundedly large positive values, before collapsing to unboundedly negative values before rising again. In contrast, the fiscal multiplier under level- k thinking is a smooth function of p and only mildly increasing in p .

The sign flip in the fiscal multiplier under rational expectations reflects that inflation expectations rise with additional government spending for $p < p^*$, but fall for $p > p^*$. Under level- k thinking, none of these unusual switches in inflation expectations occur locally around p^* .

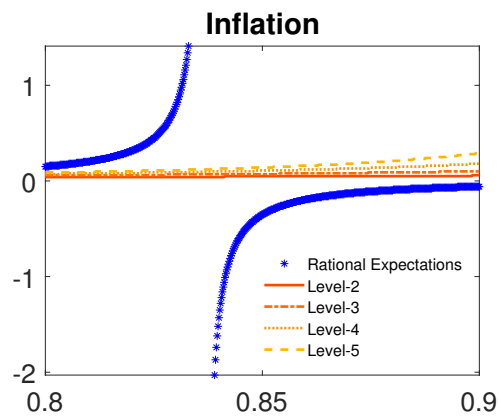


Figure 9: Inflation (annualized) as a function of the probability p

6 Conclusion

This paper has shown that the process of expectations formation can be crucial for the design of optimal monetary policy at the ZLB, for the quantitative and qualitative implications of time-dependent forward guidance and for the size of the fiscal multiplier at the ZLB. Many puzzling features of rational expectations model at the ZLB disappear once an empirically relevant process for expectations formation is embedded in an otherwise standard macro model. The process for expectations formation assumed here, namely level-k thinking, is clearly stylized. More research is needed to understand how households and business form their expectations in practice. Recent work by [Coibion, Gorodnichenko, and Weber \(2019\)](#) is a welcome contribution in that regard.

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Appendix A: IRFs to anticipated monetary policy shocks

A useful starting point for understanding the transmission of monetary policy is the impulse response to an anticipated loosening of monetary policy in the distant future. We assume further that the nominal interest rate is unchanged in all periods leading up to the future monetary loosening. This experiment is directly relevant for understanding optimal policy, because these impulse responses to anticipated monetary policy shocks are the key input into the computational algorithm we are using when solving for optimal monetary policy.

Figure 1 shows the responses to an anticipated shock 100 years in future. Because the model is purely forward looking one can also read that chart to show the response to any anticipated shock at any horizon earlier than 100 years in future. For instance, the effect of a policy loosening 25 years in future on current inflation and output can be read off the chart by considering quarter 300 (25 years prior to the loosening in quarter 400 assumed in the experiment plotted in the chart).

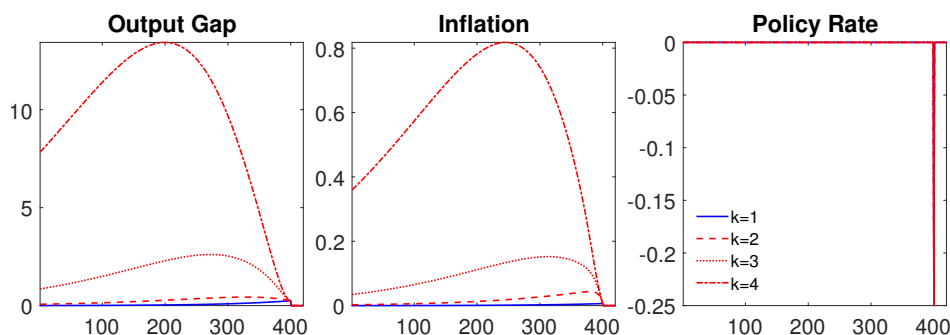


Figure 10: Interest rate pegged at steady state for 400 quarters

At level 1, all future endogenous variables are held at a baseline path equal to steady state. Since the interest rate is discounted in the Euler equation, the effect on inflation and output is smaller the further in the future the loosening occurs. [Farhi and Werning \(2017\)](#) call this the horizon effect. At higher levels, the response is hump shaped for the following reasons. Beliefs are now formed from the paths for output and inflation from previous levels which are above baseline. The higher income and inflation expectations cumulate for longer the farther in future the movement in the interest rate is and this creates an anti-horizon effect. But beliefs about future inflation and income are also discounted. With β close to unity, the discounting matters little for relatively modest horizons and hence monetary policy loosening is *more* powerful the farther in future it occurs. For longer horizon the effect of discounting becomes large, and the anti horizon effects dominates; monetary policy loosening is *less* powerful the farther in future it occurs. For comparison, under rational expectations the impact of the monetary policy loosening 100 years in future is incredibly large in this linear model ⁹⁾ for

⁹⁾The output response under rational expectation is on the order of 10^{25} percent above steady state.

reasons that have been pointed out in [Carlstrom, Fuerst, and Paustian \(2015\)](#), whereas it is only a couple of percentage points above steady state with low levels of k .

Appendix B: Computing Optimal Policy

We compute optimal monetary policy via choice of anticipated monetary policy shocks. The key idea behind the approach is that in a linear model, impulse responses are additive in the exogenous disturbances. So we can always write that a vector of endogenous variables is equal to the baseline projection plus the response to anticipated monetary policy deviations from the rule. Because these are anticipated, the approach assumes commitment. The deviations will be chosen optimally given a loss function. Let us stack the impulse responses of our n endogenous variables over the T^{**} periods in an nT^{**} vector Y .¹⁰

$$Y = B + DX \quad (34)$$

Here, B is the nT^{**} vector containing the baseline projection given the rule and some (non-policy) shocks. The matrix D is nT^{**} by T^{**} containing the response of observables to the anticipated deviations, which are denoted by X . Specifically, the first row column k collects the response of the first endogenous variable in period 1 of the planning horizon to an anticipated deviation k periods out. The second row contains the response of the first variable in period two of the planning problem to those deviations etc. The row $n + 1$ contains the response of the second variable in the first period of the planning problem to the anticipated deviations, etc.

The welfare weights are contained in a nT^{**} by nT^{**} matrix W . Note that we must include discounting here. Rows $1, \dots, T^{**}$ are multiplied by β^{t-1} for $t = 1, 2, \dots, T^{**}$ and the same for each block of rows $(k - 1)T^{**} + 1 : kT^{**}$ for $k = 1, 2, \dots, n$

The optimal policy problem is then simply to minimize the quadratic form

$$\min_X \frac{1}{2} Y' W Y = \min_X \frac{1}{2} X' D' W D X + B' W D X + t.i.p. \quad (35)$$

The optimal weights X^* are given by

$$(X^*)' = -B' W D (D' W D)^+ \quad (36)$$

Here, $+$ denotes the pseudo inverse reflecting that if the planning horizon T^{**} is chosen large enough D and W will contain many rows of zeros at far horizons creating singularities. Once the weights have been determined the path of all model variables under optimal policy is just the sum of the baseline responses plus the responses to anticipated deviations of size X^* , aka $Y^* = B + D X^*$

¹⁰For the purpose of constructing optimal policy, we do not need to include in Y the responses of all model variables, just those that feature the loss function. Once the optimal deviations X have been constructed, the implied response of those variables not in the loss function can be constructed ex-post.

Incorporating the zero lower bound in this algorithm amounts to adding a nonlinear constraint in the optimization problem.

$$\min_X \frac{1}{2} X' D' W D X + B' W D X \quad (37)$$

subject to:

$$-D_r X \leq B_r - c \quad (38)$$

Here D_r is the matrix containing the response of the nominal interest rates to the anticipated deviations and B_r is the vector containing the baseline projection for the nominal interest rate and c is a vector of constants containing the maximum loglinear deviation of the nominal rate from steady state admissible so that ZLB is just reached. One can use the MATLAB function `quadprog` to solve this problem.

Appendix C: The effects of a delayed liftoff

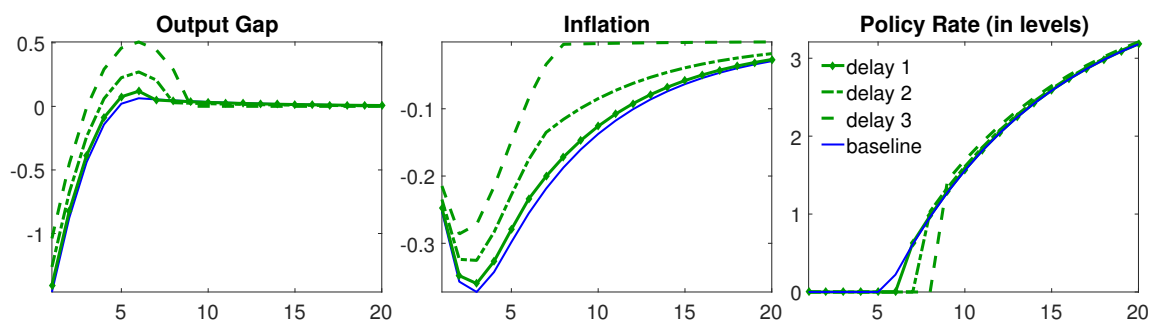


Figure 11: Level-1 solution with delayed liftoff

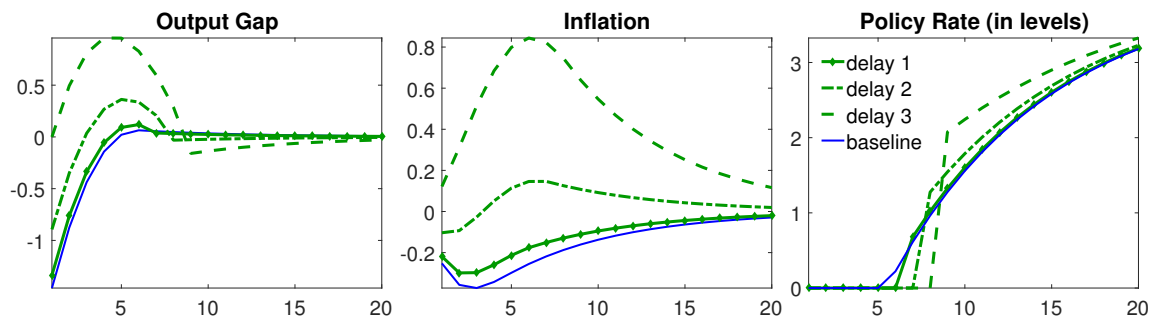


Figure 12: Level-3 solution with delayed liftoff