Competing Auctions in a Monetary Economy*

Richard Dutu[†]
University of Waikato

Benoit Julien[‡]
University of New South Wales

Ian King[§]
University of Melbourne

August 2007

Abstract

We construct a monetary economy in which the goods are allocated via competing auctions as in McAfee (1993) and Peters and Severinov (1997). Sellers compete by offering auctions in which they decide for the quantity to be traded. Buyers observe the posted quantities, decide for a quantity of money to bring to the auctions and direct their search to the most attractive alternatives. We establish the existence of a unique symmetric monetary equilibrium and characterize buyers' and sellers' payoffs. We show that inflation cannot be too low nor too high for an equilibrium to exist, and that increasing inflation decreases the equilibrium posted quantity as for the number of buyers participating to the economy. Symmetric efficiency is attained for the lowest inflation rate compatible with existence of an equilibrium.

Keywords: Competing Auctions, Money, Directed Search, Inflation.
JEL Classification: D44; E40

*We thank James Chapman and Randy Wright for helpfull discussions, and the seminar participants at the Canadian Economics Assocation conference, the Society for Advanced Economic Theory meeting, Georgetown University and the Bank of Canada. All errors are ours.

[†] Department of Economics, University of Waikato, Knighton Road, Hamilton 3240, New Zealand. Tel: +64 7 858 5164. Email: rdutu@waikato.ac.nz

[‡] Australian Graduate School of Management, School of Economics and CAER, University of New South Wales, Australia. Tel: +61 2 9931 9324. Email: benoitj@agsm.edu.au

[§] Department of Economics, The University of Melbourne, 5th Floor, Economics and Commerce Building, Victoria 3010, Australia. Tel: +61 3 8344 3833. Email: ipking@unimelb.edu.au

1 Introduction

We construct a monetary economy in which goods are allocated by competing auctions. The economy is monetary: at the beginning of each period buyers decide how much money to bring to the auction market trading off the cost of carrying money against the expected gain from participating to an auction. The economy is competitive: sellers compete in attracting buyers trading off the production costs of the advertised quantity against the expected number of potential buyers. Finally search is directed: buyers allocate themselves trading off the posted quantity against the probability to win the good in the auction.

The main contribution of the paper is to turn the competing auction framework developed by McAfee (1993) and Peters and Severinov (1997) into a fully monetary economy. In the standard competing auction model, whether the finite case (Burguet and Sákovics 1999, Schmitz 2003, Hernando-Veciana 2005) or the infinite case (McAfee 1993, Peters and Severinov 1997), the resources available to buyers are exogenous as for the quantity produced and advertised by sellers. In contrast, in this paper we let buyers decide for their cash holdings and we let sellers decide how much of their production good will be auctioned. To conduct this exercise we embed the competing auctions framework into the Lagos and Wright (2005) model of monetary exchanges. In this model agents have a periodic access to a centralized market in which they can rebalance their money holdings after they have traded in an anonymous decentralized market. Assuming quasi-linearity of the production costs in this centralized market, Lagos and Wright (2005) show that agents' trading history are irrelevant for their choice of cash holdings. As a consequence, unless some for of heterogeneity is introduced, the distribution of money holdings is degenerate into one mass point: all buyers bring the same amount of money to the decentralized market. This results holds whether terms of trade are formed via bilateral bargaining, price posting or competitive search (Rocheteau and Wright 2005). In the context of competing auctions that we examine here, this means that any non-degenerate distribution of money holdings we may have will not be the result of different trading histories, but the product of our pricing mechanism.

We show the existence of a unique symmetric monetary equilibrium. To do that we extend the limit equilibrium concept developed by Peters and Severinov (1997) to the context of competing auctions with money. The idea consists in building the model, value and payoffs functions with finite numbers of buyers and sellers so that we can use the revelation principle. Then we turn this economy into a *competitive* economy by taking the limit of these numbers so that the buyer's utility is determined by the market and is insensitive to deviations in one single auction. This is the *market utility property* by which a deviation by one seller has no impact on the buyer's payoff in large markets (Peters 2000). Finally we assume rational expectations so that sellers believe their payoff function satisfy the market utility property. Another advantage of dealing with the limit case is that, since buyers know their cash holdings before they decide which auction to participate, sellers do not compete in reserve prices since they are equal to their production costs (McAfee 1993, Peters and Severinov 1997, Hernando-Veciana 2005). This means that in our environment sellers will compete for bidders by means of quantity announcements, and that we can conveniently represent the equilibrium allocation in a 2-dimension diagram with the posted quantity and the buyer-seller ratio on the axes.

In this equilibrium, all sellers post the same quantity of their production good. On the basis of this information, buyers make their entry decision and pick a quantity of money from a distribution of cash holdings that is determined endogenously. Because of the incentive to bring more money to win the auction, the distribution of money holdings is generally not degenerate in an auction environment (Galenianos and Kircher 2006). Finally buyers allocate themselves randomly among sellers by means of symmetry. The equilibrium posted quantity and buyer-seller ratio are derived from a representative seller's maximization program in which he maximizes his expected payoff from the auction subject to competition from the other sellers in attracting buyers.

We use the model to analyze the impact of monetary policy on the equilibrium allocation. We show that increasing inflation translates into a lower quantity posted by sellers and into a lower number of buyers participating to the auctions. We also show that inflation cannot be too low nor too high. If it is too low, holding money is not very costly so that buyers bring cash amounts very close to their private valuations, raising prices at auctions close to the buyers' reservation value. This means the seller extracts almost the entire surplus from trade leaving very little surplus to buyers. For some low values of the inflation rate the expected gains from trade are simply too low compared to the cost of holding money and the outside option, and no buyers would take part into the economy. This also means that the Friedman rule can never be an equilibrium as long as the buyer's outside option yields a positive utility. When inflation is too high, by contrast, the cost of holding cash is too high so that too few buyers participate to guarantee a positive net profit for sellers. Sellers simply drop out.

Competing auctions have already been studied in the context of a monetary economy by Julien Kennes and King (2006). The analysis is limited by the indivisibility of money in their model however. Here money is fully divisible as for the goods produced and consumed. Second-price auctions with divisible money have been examined by Galenianos and Kircher (2006), but the auctions in their paper are not competitive in the sense that both the quantity traded and the matching function are exogenous. Here we let sellers decide for the quantity they advertise and we let buyers decide which auction to participate into. That is, there is really competition between sellers who seek to maximize their expected revenue taking into account competition from other sellers; and there is directed search from buyers who allocate themselves to the sellers that offer the most attractive auctions in terms of expected net gains from trade. We also extend both papers by considering the mirror case in which sellers post a price and buyers compete in quantity.[1]

The paper also contributes to the literature on money with micro-foundations following Kiyotaki and Wright (1989,1991,1993) by considering a new competitive pricing mechanism. First the use of competing auction creates a new friction compared to the other pricing mechanisms that have been studied. In standard directed search models, buyers visit sellers on the basis of advertised fixed terms of trade and the corresponding probability to get the good. Frictions

[1] Search directed by posted prices has received a lot of attention in the labour literature. See for instance Montgomery (1991), Moen (1997), Acemoglu and Shimer (1999a,b) and the corresponding sections in the surveys by King (2003) and Rogerson, Shimer and Wright (2005). It is used in monetary models by Rocheteau and Wright (2005), Faig and Jerez (2005, 2006) and Berentsen Menzio and Wright (2006). Competition in auction has been applied to the job market by Julien Kennes and King (2000).

come from a lack of coordination among buyers who may decide to visit the same seller so that several buyers may not trade while several other sellers have no customer. When sellers compete in auctions, as they do here, the lack of coordination among buyers not only impacts on the probability to trade but also on terms of trade as prices increase with the number of buyers showing up. When competition is fierce, buyers' gains from trade may even exhaust. Combined with the divisibility of goods, the new friction generates interesting trade-offs for both sellers and buyers that have not been previously studied. Second, the use of auctions generates terms of trade dispersion. Most monetary models with micro-foundations do not have terms of trade dispersion, unless some form of heterogeneity is introduced (Faig and Jerez, 2005a, 2005b). Finally auctions are usually efficient mechanisms since the goods are allocated to those that value them most.

In the last section of the article, we examine the mirror case in which sellers post a price and let buyers compete in quantity. That is buyers bid in quantity rather than in price. We show the existence of an equilibrium in which sellers post a unique price and ex post opportunism in quantity leads to terms of trade dispersion: the quantity traded when several buyers show up is smaller than when the buyer is alone. Since sellers are bound by posted prices, the distribution of money holdings is degenerate and equal to the posted price. Increasing the money supply in this economy reduces the real value of money holdings and therefore the produced quantities whether in pairwise or multilateral meetings. Whether it increases terms of trade dispersion depends on the ratio of buyers to sellers. When this ratio is small, the value of real balances is low so that increasing the money supply decreases terms of trade dispersion by reducing the ability to distance oneself from average terms of trade using one's market power. For the same reason, when the ratio is high dispersion increases first then decreases.

The article is organized as follows. In section 2 we present the environment. Section 3 examines competition in price. Section 4 examines competition in quantity. Section 5 concludes.

2 The environment

The monetary side of the model borrows from Lagos and Wright (2005). Time is discrete and goes on forever. Each period is divided into two trading subperiods. In the first subperiod agents enter a frictional market, to be described shortly, where they can produce a first homogenous good and trade it for money. Then they meet on a Walrasian centralized market where they can produce a different good, called general good, and trade it for money. There is a continuum of anonymous infinitely lived agents who differ in terms of when they produce and consume. A first group of agents, called sellers and whose number is s, can produce and consume the general good but can only produce the first good in any quantity q. This set of seller is fixed and sellers are homogenous. The second group of agents, called buyers and whose number is b, can also produce and consume the general good but can only consume the first good traded in the frictional market. Thus each buyer wishes to trade the first good with a seller. It is assumed that neither good can be used as a medium of exchange in the other market. Since agents are anonymous and trade must be *quid pro quo*, money is essential in this environment. Money, whose quantity is M_t at time t, can be stored in any non negative quantity m_t by an agent. New money is injected by the Reserve Bank via lump-sum transfers at the beginning of centralized market at rate τ such that $M_{t+1} = (1 + \tau) M_t$.

At the end of the centralized market, which is used by agents to rebalance their money holdings, buyers and sellers enter an auction market where buyers direct their search towards sellers who advertise a certain quantity q to go via an ascending-bid auction. There are various type of ascending-bid auctions. We will use second-price auctions because they imply a unique optimal bidding strategy for buyers (Riley and Samuelson, 1981) and are therefore easier to work with. In second-price auctions the seller sells the good to the buyer who makes the final and highest bid and the winner pays a price that corresponds to second highest bid. When sellers post this auction, they take into account competition from other sellers. Observing all posted auctions, buyers decide which seller to visit. In equilibrium buyers will be indifferent between sellers.

Noting x the net consumption of the general good and β the discount factor, the instantaneous inter-period utility function for a representative agent is given by

$$x + \beta [u(q) - c(q)] \tag{1}$$

Note that production costs are linear in the general good. This greatly simplifies things by making agents' payment history irrelevant (Lagos and Wright, 2005) so that if the distribution of money holdings happens to be non-degenerate, this cannot come from wealth effects. We make standard concavity and convexity assumptions for u and c, and note $\tilde{q}$ the quantity that maximizes the trade surplus in a frictionless market, that is $u'(\tilde{q}) = c'(\tilde{q})$, and $\hat{q}$ the quantity such that $u(\hat{q}) = c(\hat{q})$. Also $c'(0) = 0$ and $u'(0) > 0$. We will consider large market so that both b and s are infinite, yet the ratio b/s is a real number. Finally we assume free entry on the buyers' side by posing that buyers that do not participate to the economy earn a sure surplus of k.

3 Competition in price

In this section sellers post a quantity and buyers bid in prices for it. A first question to address is the determination of buyers' cash holdings. Even though private valuations are identical among buyers–buyers are homogenous and have the same utility function $u(q)$–what matters is how much money buyers bring to the auctions. Assume all buyers decide to bring the same amount of money. If one agent deviates and bring an additional dollar, he is certain to win the auction at a negligible marginal cost. Because each buyer thinks in the same way, there is no focal point for buyers when it comes to deciding for their cash holdings. Because of this trade-off between the negligible additional cost and the discrete increase in the probability to win the auction, the distribution of money holdings across buyers is generally not degenerate in the context of second-price auctions (Galenianos and Kircher, 2006).

A second question is the set of variables under the seller's control when a seller designs an optimal auction. In competing auction models, the variable that can be used by sellers to extract surplus from buyers is the reserve price. Increasing the reserve price rules out low bids at the cost of decreasing the expected number of buyers. In the context of infinitely many buyers and sellers, McAfee (1993) and Peters and Severinov (1993) have proven that the reserve price is equal to the production cost. This means that in our context sellers will compete by ways of quantity announcements, and that's all. As will be shown, this is enough information for buyers to derive the distribution of money holdings from which they draw. Since sellers perfectly forecast this distribution and how it impacts on prices and the allocation, they can derive the price of money on the money market and therefore their reserve prices as the production costs expressed in nominal terms.

The sequence of events will be the following: the trading process begins when sellers publicly announce at the beginning of the Walrasian market a quantity to be auctioned in the coming auction market, quantity to be sold via ascending-bid auctions. On the basis of this information, buyers make their entry decision and then pick a quantity of money from the distribution of cash holdings derived endogenously. Then buyers enter the auction market and choose an auction to participate into. Finally they submit their bids and the good goes to the buyer that bids most. At the end of the auction market and once all goods have been sold, buyers and sellers proceed to the next Walrasian market. Note that, except for the monetary side, this sequence is close to that of the second case examined by Peters and Severinov (1997) in which buyers learn their valuations before they choose among sellers. The differences are that here buyers choose their cash holdings rather than valuations, and that sellers chose for the quantity traded.

A strategy for a seller is a posted q. A strategy for a buyer is rule that specifies the probability with which he will choose a particular seller as a function of the posted q. As is usual in competitive environments with large numbers of buyers and sellers, we will focus on symmetric equilibria: sellers post the same q and buyers follow the same decision rule. That is if two sellers offer the same q, they will be selected with the same probability by buyers. Symmetric equilibria where sellers receive a random number of buyers are indeed more realistic in large economies and more compatible with a decentralized economy.

3.1 The value functions

Let $W^b(m)$ and $V^b(m)$ be the value functions for a buyer holding m units of money in the centralized and auction market respectively. We have

$$\begin{aligned} W^b(m) &= \max_{x,m_{+1}} \left\{ x + \beta V^b(m_{+1}) \right\} \\ \text{s.t.} \quad \phi m_{+1} + x &= \phi\,(m+T) \end{aligned}$$

where m_{+1} corresponds to the money carried from the centralized market to the auction market and x is the net consumption of the good in the centralized market. In this program ϕ corresponds to the value of money in terms of the general good (1 unit of money buys ϕ units of the general good) and T corresponds to how many units of money per buyer are injected by the Reserve Bank each period. It says that when choosing for a quantity of general good to consume and produce and a quantity of money to bring to the auction market, buyers take into account that the real value of what they net consume and bring to the auction market must be equal to what they brought to this market and received from the Reserve Bank. Substituting for x, the program for a buyer in the day market can be rewritten

$$W^b(m) = \phi\,(m+T) + \max_{m_{+1}} \left\{ -\phi m_{+1} + \beta V^b(m_{+1}) \right\} \tag{2}$$

Since sellers have no reason to carry money in this economy, and assuming that only buyers receive a transfer of money, their program is

$$\begin{aligned} W^s(m) &= \max_{x,q,\theta} \left\{ x + \beta V^s(m_{+1}) \right\} \\ \text{s.t.} \quad x &= \phi m \end{aligned}$$

The seller's problem is to choose for a quantity of general good in the Walrasian market x, a posted quantity q and a queue length θ for the auction market that maximize his payoff. This simplifies into

$$W^s(m) = \phi m + \max_{q,\theta} \left\{ \beta V^s(m_{+1}) \right\}. \tag{3}$$

In the auction market we note $V^b(m)$ the value function of a buyer holding m units of money each worth ϕ_{+1} units of the general good in the next period Walrasian market, bidding for q units of goods in an economy where the buyer-seller ratio is θ. Noting $V_n^b(m)$ the same value function when the buyer faces exactly $n$competitors, the value function for the buyer at the beginning of the frictional market is then given by

$$V^b(m) = \sum_{n\in\mathbb{N}} P\left[X=n\right] V_n^b\left(m\right) \tag{4}$$

where X is the random variable equal to the number of competing buyers showing up at the seller's shop and $P\left[X=n\right]$ is the probability measure associated with the event $X=n$. The variable X takes values into $\mathbb{N}$ and follows a Poisson process of parameter $\theta = b/s$ so that

$$P\left[X=n\right] = \frac{\theta^n}{n!}e^{-\theta}$$

and

$$\sum_{n\in\mathbb{N}} P\left[X=n\right] = 1.$$

We note μ the random variable equal to how many units of money are held by one competitor and $F(m) = P\left[\mu \leq m\right]$ the probability measure associated with the event $\mu \leq m$. We note $f(m)$ its density so that

$$\int_{m\in S'} f(m)dm = 1.$$

in which the support of x is noted $S' = [\underline{\mathrm{m}}, \bar{m}]$ which we define shortly. Finally we note $S = [\underline{\mathrm{m}}, m] \subseteq S'$ and assume that F is continuous (this will appear clearly later).

A buyer facing n competitors wins the auction if he holds the highest money holding, which is distributed according to $F^n(m)$ with density $nF^{n-1}\left(m\right)f(m)$; with probability $1-F^n(m)$ he does not win the auction. We can now compute $V_n^b(m)$ the value function of a buyer holding m units of money, bidding for q units of goods and meeting n competitors. Noting z the number of units of money spent if he wins the auctions, this value function is given by

$$V_n^b\left(m\right) = \int_{z\in S} \left\{u(q) + W_{+1}^b\left(m-z\right)\right\} dF^n(z) + \left[1-F^n(m)\right] W_{+1}^b(m). \tag{5}$$

The first term corresponds to expected payoff to winning the auction. It is equal to the probability that all n competitors have less money than he has, multiplied by the payoff to consuming q units of the good and moving to the centralized market with $m - z$ units of money; we then sum over the quantity of money spent by the winning bidder, z, which takes value from the lowest money holding $\underline{m}$ and a quantity of money infinitely smaller than the buyer's own money holding m, noted $m - \varepsilon$. Since F is continuous by assumption, it is then continuous to the left with $\lim_{\varepsilon \to 0} F(m - \varepsilon) = F(m)$ so that z takes value into S. The second term corresponds to the probability of *not* winning the auction multiplied by the value of entering the centralized market with an unchanged amount of money m. Note that q depends neither on m the quantity of money held by the buyer nor on z the quantity of money spent by the buyer since the quantity traded is decided before the opening of the auction market and is therefore not influenced by local market conditions. The probability to trade does depend on m however.

Using $\phi = (1 + \tau)\, \phi_{+1}$, noting $i = (1 - \beta + \tau)/\beta$ the nominal interest rate, inserting (4) and (5) into (2) and getting rid of constant terms, the buyer's program becomes

$$\max_{m} \chi(m) = -i\phi m + \sum_{n \in \mathbb{N}} P[X = n] \left\{ u(q) F^n(m) - \phi \int_{z \in S} z dF^n(z) \right\} \tag{6}$$

Equation (6) says that the buyer chooses her money holdings in order to maximize the difference between the opportunity cost of carrying this money and the expected net gain of winning the auction in utility terms. This expected net gain is composed of the utility of consuming q multiplied by the probability of winning minus the expected payment associated with holding m units of money. Note that the second part of the above expression corresponds to the expected return of making a bid in auction theory: it is equal to the reservation value multiplied by the probability of winning, minus the expected payment (Riley and Samuelson, 1981).

As for sellers, we need first to characterize the distribution of the second highest money holding among the n buyers showing up since the winner pays the amount of the second richest bidder. Noting y_k the k^{th} order statistic, its density is given by

$$f_{x_{(k)}}(m) = n C_{n-1}^{k-1} F^{k-1}(m) \left[1 - F(m)\right]^{n-k} f(m)$$

where C_{n-1}^{k-1} corresponds to the number of $(k-1)$-combinations among $n-1$ elements, that is how many times $k-1$ different combinations of buyers can be chosen among $n-1$ buyers. Setting $k=n-1$ in the above formula and remembering that sellers do not hold any money, the value function for the seller posting q and taking the value of money ϕ as given is

$$V^s(q,\theta) = \sum_{n\in\mathbb{N}} P\left[X=n\right] \int\limits_{z\in S'} \left\{-c(q) + W_{+1}^s(z)\right\} f_{x_{(n-1)}}(z)dz. \tag{7}$$

Inserting (7) into (3) and getting rid of constant terms, the seller's program can be rewritten

$$\max_{q,\theta} -c(q) + \phi \sum_{n\in\mathbb{N}} P\left[X=n\right] \int\limits_{z\in S'} z f_{x_{(n-1)}}(z)dz. \tag{8}$$

The above equation corresponds to the seller's objective. It is composed of the difference between the cost of producing the posted q and the expected return of selling this q via a second-price auction. Note that since the quantity q chosen by sellers impacts on the distribution of money holdings by buyers F, the choice of q impacts on the right-hand side of (8) via $f_{x_{(n-1)}}(z)$.

To solve for how much money buyers bring to the auction market, let us take the first order condition of the buyer's program in equation (6). Getting rid of subscript and taking q and ϕ as given, we obtain

$$(1-\beta+\tau)\,\phi = \beta\left[u(q) - \phi m\right] f(m)\theta e^{-\theta[1-F(m)]} \tag{9}$$

Rearranging and integrating this expression over S gives the distribution of money holdings among buyers as a function of the price of money ϕ, the quantity q posted by sellers and the lower support of the distribution $\underline{m}$:

$$F(m) = \frac{1}{\theta}\ln\left\{1 - ie^{\theta}\ln\left[\frac{u(q)-\phi m}{u(q)-\underline{m}}\right]\right\}.$$

To find $\underline{m}$ note that the seller is indifferent between producing q for $\underline{m}$ and doing nothing such that $-c(q) + W(\underline{m}) = W(0)$ from which we extract $\underline{m} = \frac{c(q)}{\phi}$ using the linearity of W. One can check that $F(\underline{m}) = 0$ and that $F(\bar{m}) = 1$ implies

$$\bar{m} = \frac{u(q) - e^{-\frac{1-e^{-\theta}}{i}}\left[u(q) - c(q)\right]}{\phi}$$

so that

$$S' = \frac{1}{\phi}\left[c(q), u(q) - e^{-\frac{1-e^{-\theta}}{i}}\left[u(q) - c(q)\right]\right]$$

and

$$F(m) = \frac{1}{\theta}\ln\left\{1 - ie^{\theta}\ln\left[\frac{u(q) - \phi m}{u(q) - c(q)}\right]\right\} \tag{10}$$

which is continuous over S'.

3.2 Equilibrium

The equilibrium concept we will be working with is the *competing auction monetary equilibrium*. It differs from Peters and Severinov (1997) in that sellers do not compete in reserve prices but in the quantity they post. Also buyers are allowed to choose the cash they bring to the auctions. But we follow them by considering the limit of the equilibrium with finite buyers and sellers so that there is no profitable deviation for the seller when posting a quantity. It is important to note that these two choice variable, m for buyers and q for sellers, go together: the higher the posted quantity, the higher the cost of carrying the money that can make an agent win the auction for that quantity. As soon as buyers are allowed to choose their cash holdings, the posted quantity becomes a relevant choice variable.[2]

Our equilibrium concept will also differ from the competitive search monetary equilibrium in Rocheteau and Wright (2005) in that search is indeed directed in both equilibria, but we use auctions to allocate the good while they use fixed prices and rationing. This has an important consequence: since the distribution of money holdings is non degenerate, one cannot use the representative buyer's budget in the seller's constrained maximization program since it is a random variable. We go around this by taking the expected value of the buyer's net gain from trade and force it to be at least equal to the buyers' outside option utility, k.

[2] In auction theory it is usually assumed that *one* unit of the commodity is auctioned. Assuming zero production costs, noting x the buyer's private valuation and p the price paid, the seller's surplus is p and the buyer's surplus is $x - p$. The quantity produced does not matter as it is mainly a problem of sharing a surplus so if 2 units are to be auctioned, we simply rescale the surplus. Once money is introduced, things are different since bringing the real value of 2 units of goods costs more than bringing the real value of one unit. The quantity posted is no longer neutral and chosing m for buyers and choosing q for sellers go hand in hand.

Let us write the seller's program. A seller maximizes his expected payoff subject to the buyers' expected payoff in utility terms being at least equal to k. The buyer's payoff before deciding to enter the economy is just the expectation of the payoff that the buyer will earn conditional on his cash after he enters. Recalling from (6) that $\chi(m)$ is the net surplus of a representative buyer, the seller's problem becomes

$$\max_{q,\theta} -c(q) + \phi \sum_{n \in \mathbb{N}} P[X = n] \int_{z \in S'} z f_{x_{(n-1)}}(z) dz \tag{11}$$

$$\text{s.t.} \quad E[\chi(m)] = \int_{m \in S'} \chi(m) dF(m) \geq k. \tag{12}$$

Definition 1 *Let us note $Q \subset [0, \hat{q}]$ and $\Theta \subseteq \mathbb{R}_+$. When buyers compete in prices, a competing auction monetary equilibrium is a list $(q, \theta, \phi) \in Q \times \Theta \times \mathbb{R}_+$ and a distribution of money holdings F such that: (i) Agents have rational expectations; (ii) :Sellers maximize (11) subject to (12); (iii) :buyers are indifferent between all sellers; (iv) : the price of money is given by the clearing condition on the money market*

$$\int_{c(q)/\phi}^{\bar{m}} z f(z) dz = \bar{M}^S. \tag{13}$$

.

To give an intuition of how things work, consider a seller who contemplates changing his posted quantity. When doing so he correctly forecasts the impact of q on the distribution of money holdings given by (10) from which buyers pick from, then the impact it has on the value of money in the Walrasian market via (13) and therefore on buyers' and seller's expected payoffs appearing in (11) and (12). He also understands from (12) how the buyer-seller ratio will adjust to this new q and therefore its impact on the distribution of money holdings via (10). Note that money holdings are private information to buyers but that the distribution of money holdings is not by the rational expectations hypothesis.

Let us simplify the seller's program. Noting $v(z) = \frac{1-F(z)}{f(z)}$ the difference between the first order statistic and the second order statistic, we have the following lemma

Lemma 1 *The seller's program (11)-(12) simplifies into*

$$\max_{q,\theta} -c(q) + \phi \int_{z \in S'} [z - v(z)]\, \theta f(z) e^{-\theta[1-F(z)]} dz \tag{14}$$

$$s.t. \quad -i\phi \bar{M}^S + \frac{u(q)}{\theta} - \phi \int_{z \in S'} [z - v(z)]\, f(z) e^{-\theta[1-F(z)]} dz \geq k/\beta, \tag{15}$$

Proof. See Appendix. ■

The seller maximizes the sum of the desutility of producing the posted quantity q and the expected payment in real terms coming from the auction, subject to the buyer's incentive constraint. The first term inside the integral $z - v(z)$ corresponds to Myerson's (1981) virtual valuation of a buyer holding z units of money, that is the difference between what he actually holds z and the buyer's rent which is the difference between the first order statistic and the second order statistic $v(z)$. The term $\theta f(z) e^{-\theta[1-F(z)]}$ corresponds to the probability of a pairwise match for a seller in which the buyer holds more than z. The last thing to do is to sum over all possible z to get the expected value. The second term in the constraint corresponds to the expected gain in utility, $u(q)$, multiplied by how many sellers are present s and divided by the number of competitors b. If, from the expectation operator, how much money is held by a particular buyer does not play any role in allocating the good, the expected gross return to a buyer is simply equal to the expected gain at each seller's post $\frac{u(q)}{b}$, multiplied by the number of posts s. This results from the expectation operator. The last term corresponds to the expected payment by the buyer, which is equal to the seller's expected revenue multiplied by the number of sellers s and divided by the number of buyers b for exactly the same reason as above.

Lemma 2 *The seller's program is independent of the price of money ϕ.*

Proof. See Appendix. ■

Despite the complexity of the auction environment, Lemma 2 shows that terms of trade are formed in real terms as in the other pricing mechanisms studied in Rocheteau and Wright (2005). That is the price of money adjusts in the end via the clearing condition on the money

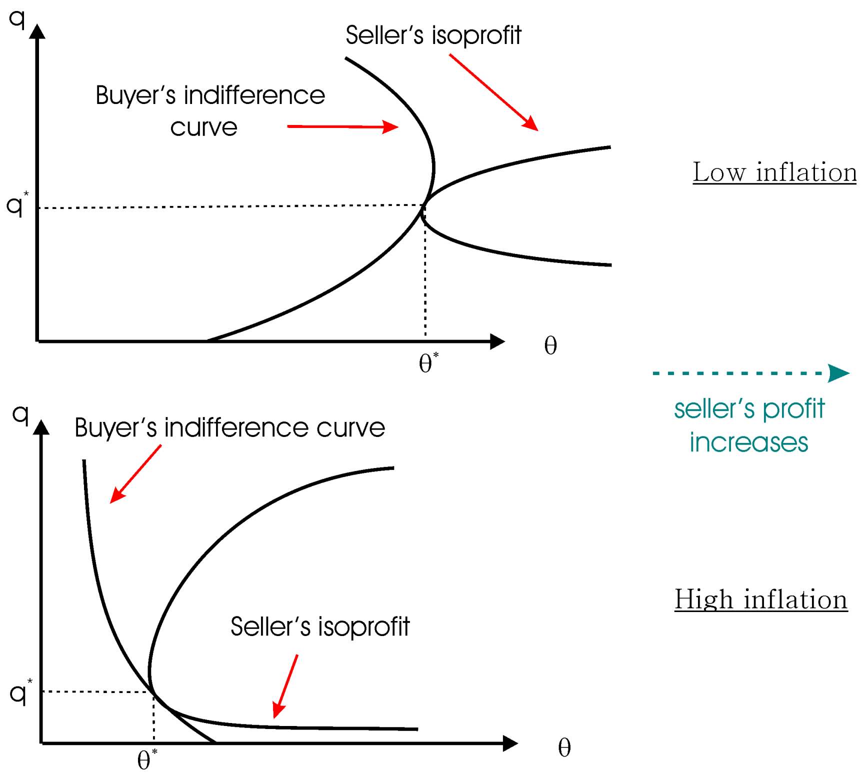

Figure 1: The impact of inflation on the posted quantity and the buyer-seller ratio.

market (13). This is a by-product of the Lagos Wright model (2005). We use this lemma to establish to the following proposition.

Proposition 1 *There exists an equilibrium and it is unique.*

Proof. See Appendix. ■

The buyers' and sellers' payoffs are nicely shaped so that uniqueness is guaranteed. Figure 1 illustrates how monetary policy impacts on the buyer's indifference curve and on the seller's isoprofit curve, and on the equilibrium (q^*, θ^*). Sellers' profit rise moving to the right, while buyers' profit decrease. An equilibrium is a couple (q, θ) where the seller's isoprofit curve and the buyer's indifference curve are tangent.

The buyer's and seller's isoprofit curves have the following interpretation. Let us start with the seller, in case inflation is low, and assume the posted q increases from a low value. Because of convexity the impact on production costs is small. But now buyers bring more money since there is more goods to buy and this contributes to increase seller's profit. The only way to keep profit constant is then to decrease competition among buyers by decreasing the buyer-seller ratio. However, when q is big, the increase in production costs in case q increases more than cancel the gains coming from the fact that buyers bring more money. In order to stay on the same isoprofit curve, the buyer-seller ratio must now increase. Let us consider a representative buyer now and assume that the posted q increases from a low value. The impact on utility is important due to concavity. If inflation is low the impact on the cost of carrying money is small so that profit is kept constant by increasing the buyer-seller ratio in order to increase competition and therefore decrease the probability to win the auction. When q is high, however, the gains in utility are small and offset by the cost of holding a large amount of cash. So the only way to keep profit constant for the buyer is to decrease the buyer-seller ratio.

The interesting point is that when inflation is high (last diagram on Figure 1), gains in utility following an increase in q are always more than cancelled by the increase in the cost of holding money so that the only way to keep profit constant following an increase in q is to *decrease* the buyer-seller ratio. That is why the buyer's isoprofit curve now is only downward sloping rather than "inverse C" shaped.

From numerical simulation and Figure 2, it can be shown that increasing the money supply tend to decrease the posted quantity and the number of buyers participating to the auctions. This is due to the buyer's indifference curve shifting to the left. And when inflation becomes very high, the cost of holding cash is too high so that too few buyers participate to guarantee a positive net profit for sellers. Sellers drop out and there is no equilibrium. As inflation decreases, by contrast, the traded quantity gets closer to the optimal quantity $\tilde{q}$ such that $u'(\tilde{q}) = c'(\tilde{q})$. This quantity can never be reached however, since the Friedman rule does not sustain an equilibrium for a strictly positive k. To see this, note that if inflation is too low, holding money is not very costly so that buyers bring cash amounts very close to their private valuations, raising prices at auctions close to the buyers' reservation value. This means that the seller extracts almost the entire surplus from trade leaving very little surplus to buyers. For

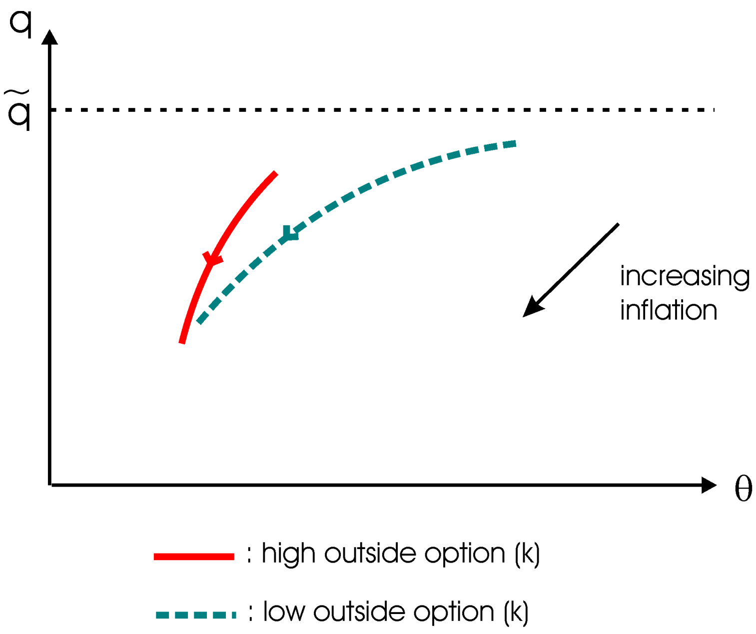

Figure 2: Equilibrium paths, low and high k.

some low values of the inflation rate the expected gains from trade are simply too low compared to the cost of holding money and the outside option, and no buyers would take part into the economy. At the Friedman rule, all agents bring the same amount of cash equal to $u(q)$ in utility terms so that gains from trade are $u(q) - u(q)$ which is necessarily smaller than k. There can never be an equilibrium at the Friedman rule as long as the buyer's outside option yields a positive utility. Finally the range of equilibrium (q, θ) widens as the buyer's outside option decreases.

Finally it can be seen from Fig. 2 that the equilibrium path when the outside option is low features a higher equilibrium posted q and a higher equilibrium buyer-seller ratio.

3.3 Optimal monetary policy

Due to the free entry condition on the buyer's side, buyers' expected payoff is unaffected by changes in monetary policy. In low inflation economies the quantity of goods auctioned is higher yet there are more buyers around. In high inflation economies there is less good at each auction but also less buyers on average. Regardless of monetary policy, buyers always get k in terms of utility. In the end a reasonable test for optimality in monetary policy in this environment is whether it maximizes the sellers' profit. This is known as symmetric efficiency.

Since the seller's profit increases as his isoprofit curve shifts to the right, we have the following proposition.

Proposition 2 *The optimal monetary policy when goods are allocated via competing auctions is the lowest possible inflation rate compatible with existence of a monetary equilibrium.*

4 Competition in quantity

Since we have divisibility on both sides, goods and money, we now address the case in which sellers post a price and buyers compete in quantity. That is, a bid from a buyer is a proposition of a quantity required in exchange for the posted amount of money (or price). By contrast to competition in price, buyers are not constrained in their bidding strategy and this makes things simpler to characterize. If two or more buyers show up at a seller's place, they compete until terms of trade leave the winning buyer, chosen at random, indifferent between trading or not. In case only one buyer shows up, he has enough bargaining power to impose terms of trade that leave the seller indifferent between trading or not. We note q_m and q_p the quantities of the good traded on the auction market in multilateral and pairwise meetings respectively.

When the number of sellers is small, a change in the seller's price implies a change in the probability that a buyer visits him. Here, by considering large markets (as in the previous section) in which the number of buyers and sellers are infinite, sellers take into account competition by assuming that they must provide buyers with a minimum of utility, k. As is usual in competitive environments, this utility is not affected by a deviant seller in the limit case.[3] An interpretation is that sellers do not compete against each other but against the market. Since there is no profitable deviation in the limit case, all sellers advertise the same price d. In the end, if two or more buyers compete, a buyer chosen at random gets q_m for d units of money. If

[3] This is not true when b and s are small. See Burdett Shi and Wright (2001).

the buyer is alone he gets $q_p > q_m$ with probability 1 against d.

The timing of events is the following. At the beginning of the centralized market sellers advertise a price without a quantity for the auction market so as to maximize their expected surplus taking into account competition from the market and the expected number of buyers. Then sellers and buyers, who know the posted price and the probability for a buyer to win an auction, trade the general good for money. Then they enter the auction market where buyers direct their search to the most attractive alternatives and where sellers are bound by the posted price.

Recalling that $\theta = b/s$ is the buyer-seller ratio (or the queue length) and using standard convergence properties of Binomial distributions, the probability of a pairwise match for a seller is $\xi_p = \theta e^{-\theta}$, the probability of a multilateral match (at least two buyers are present) is $\xi_m = 1 - e^{-\theta} - \theta e^{-\theta}$ and the probability that no buyer shows up is $1 - \xi_p - \xi_m$. Similarly, for a buyer the probability of a pairwise match is $\psi_p = e^{-\theta}$, the probability of winning the auction in a multilateral match is $\psi_m = \frac{1-e^{-\theta}}{\theta}$ and the probability of not winning the auction is $1 - \psi_p - \psi_m$.[4]

The value function of the buyer in the centralized market is identical to the competition in price case, which we simply recall

$$W^b(m) = \phi(m+T) + \max_{m_{+1}} \left\{ -\phi m_{+1} + \beta V^b(m_{+1}) \right\} \tag{16}$$

Let us turn now to the auction market. If a buyer trades d units of money against q units of the special good, then Bellman equation for a buyer in this market is given by

$$\begin{aligned} V^b(m) &= \psi_p \left\{ u(q_p) + W^b_{+1}(m-d) \right\} + \psi_m \left\{ u(q_m) + W^b_{+1}(m-d) \right\} \\ &\quad + (1 - \psi_p - \psi_m) W^b_{+1}(m). \end{aligned} \tag{17}$$

With probability ψ_p a buyer is alone and trades with a seller, in which case he consumes q_p units of the good and enters tomorrow's day market with $m - d$ units of money. With probability

[4] The derivation of these probabilities can be found in Burdett Shi Wright (2001) and Julien Kennes King (2007).

ψ_m the buyer meets several other buyers but ends up winning the auction, consuming q_m and carrying on to the centralized market with $m-d$ units of money. In all other cases he enters the centralized market with the same amount of money.

As for sellers, their program on the centralized market differs with that of competition in price in terms of which variables the sellers maximizes in. Here the seller will chose a price– rather than a quantity– and a queue length. We have

$$W^s(m) = \phi m + \max_{d,\theta} \{\beta V^s(m_{+1})\}. \tag{18}$$

Noting that sellers have no incentive to hold money, that is $m_{+1} = 0$, Bellman's equation for a seller in the decentralized market is given by

$$\begin{aligned} V^s(m) &= \xi_p \left\{-c(q_p) + W^s_{+1}(d)\right\} + \xi_m \left\{-c(q_m) + W^s_{+1}(d)\right\} \\ &\quad + \left(1-\xi_p-\xi_m\right) W^s_{+1}(0) \end{aligned} \tag{19}$$

with similar interpretation as (17).

Consider a menu of different terms of trade and let k denote the buyer's outside option utility. Competition implies that a seller must choose d and θ to make sure he provides buyers with at least k if he is to get any customer. Now consider a buyer. Using the linearity of W^b, inserting (17) into (16) and focussing on steady-state equilibria where real balances are constant, that is $\phi_{+1}(1+\tau) = \phi$, a buyer is willing to apply to a particular seller if he can get

$$-(1-\beta+\tau)\phi_{+1}d + \psi_p\beta\left[u(q_p) - \phi_{+1}d\right] + \psi_m\beta\left[u(q_m) - \phi_{+1}d\right] \geq k. \tag{20}$$

Inserting (19) into (18), using the linearity of W^s and getting rid of constant terms, in the end the seller chooses a price and a queue length such that

$$\max_{d,\theta} \xi_p\left[-c(q_p) + \phi_{+1}d\right] + \xi_m\left[-c(q_m) + \phi_{+1}d\right] \tag{21}$$

subject to (20).

Bertrand competition in case a seller faces only one buyer implies that the seller is indifferent between producing and trading q_p for d units of money and doing nothing. That is, real balances

z must be such that $z = \phi_{+1}d = c(q_p)$. Similarly Bertrand competition among two or more buyers leads to $z = \phi_{+1}d = u(q_m)$. Inserting these two values into (20) and (21) implies the following program for sellers

$$\max_{d,\theta} \xi_m \left[-c(q_m) + \phi_{+1}d\right] \tag{22}$$

$$\text{s.t.} \quad -(1-\beta+\tau)\phi_{+1}d + \psi_p\beta\left[u(q_p) - \phi_{+1}d\right] \geq k, \tag{23}$$

$$\text{s.t.} \quad \phi_{+1}d = c(q_p) = u(q_m). \tag{24}$$

Solving the buyer's constraint at equality for $\phi_{+1}d$ and substituting it into (22), maximizing over θ yields a first order condition from which k, once substituted into this first order condition using (23), enables to extract

$$\phi_{+1}d = z\left(q_m, q_p\right) = \frac{\left[1-\beta+\tau+\beta\psi_p\right]\xi_m' c(q_m) - \beta\psi_p'\xi_m u(q_p)}{\left[1-\beta+\tau+\beta\psi_p\right]\xi_m' - \beta\psi_p'\xi_m}. \tag{25}$$

To solve for the quantities, note from (24) that $q_m = u^{-1}\left[c(q_p)\right]$ which once inserted into $z\left(q_m, q_p\right)$ yields one equation in one unknown q_p of the form

$$z\left(u^{-1}\left[c(q_p)\right], q_p\right) = c(q_p). \tag{26}$$

Definition 2 *When buyers compete in quantities, a competing auction monetary equilibrium is a list* $\left\{z \in \mathbb{R}^+, (q_m, q_p) \in [0, \hat{q}]^2\right\}$ *such that* q_p *verifies (26),* $q_m = u^{-1}\left[c(q_p)\right]$ *and* z *verifies (25).*

Proposition 3 *There exists an equilibrium and it is unique. The posted price is unique so that the distribution of money holdings is degenerate. The quantities traded* (q_p, q_m) *decrease with the growth rate of the money supply. The dispersion in terms of trade* $(q_p - q_m)$ *decrease with the growth rate of the money supply* τ *for low* θ *and increases first then decreases for high* θ. *See Fig. 2.*

Proof. See Appendix. ■

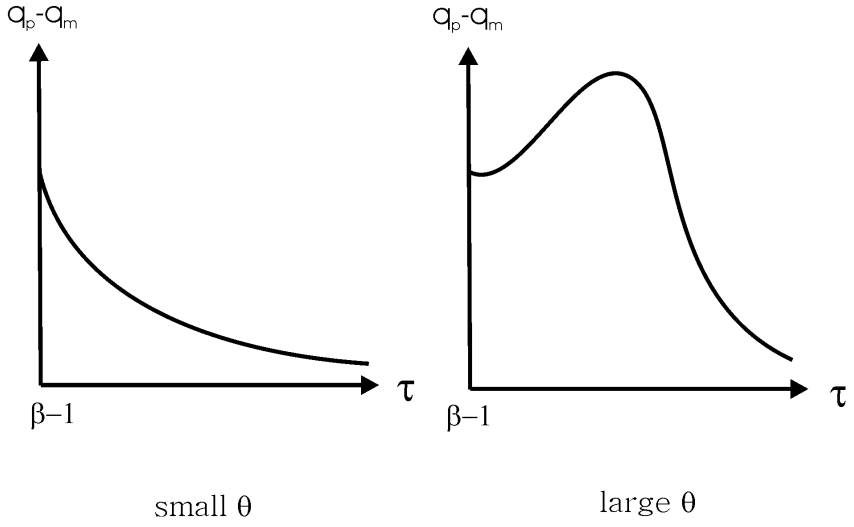

Figure 3: Monetary policy and terms of trade dispersion when buyers compete in quantity.

Regardless of the quantity that is finally produced and consumed, the price is the same across all shops so that the distribution of money holdings is degenerate and equal to the posted price. When a buyer is alone he has sufficient bargaining power to impose (d, q_p) that leave the seller indifferent between producing or not. When two or more buyers show up, because they are not constrained by their resources in their bidding strategy, competition leads to (d, q_m) that leave the winning buyer indifferent between trading or not.

Increasing the money supply reduces the real value of money holdings and then the produced quantities whether in pairwise or multilateral meetings. Whether it increases terms of trade dispersion actually depends on the ratio of buyers to sellers and then on k (there will be less buyers if k is high). When this ratio is small, the value of real balances is low so that increasing the money supply decreases terms of trade dispersion by reducing the ability to distance oneself from average terms of trade using one's market power. For the same reason, when the ratio is high dispersion increases first then decreases as is represented on Figure 3.

5 Conclusion

The main contribution of this paper is to turn the competing auctions environment into a fully monetary economy in which it is now possible to study the influence of monetary variables on the equilibrium allocation. The environment is competitive: sellers post terms of trade taking into account competition from the market. It is monetary since buyers must decide how much money to bring to the auction market. Finally search is directed since buyers allocate themselves on the basis of expected net gains from trade and the probability to win the auction. We have shown the existence of a unique symmetric monetary equilibrium and that increasing inflation tends to decrease the quantity traded as for the number of buyers taking part into the auctions. The optimal monetary policy is given by the lowest inflation rate compatible with existence of an equilibrium.

We are not able to obtain closed form solutions for the posted quantity and the buyer-seller ratio however. This is due to the complex auction environment that we have been working with and is not related to the introduction money. Actually the introduction of money has an interesting–and somehow simplifying–effect on the structure of the auction model: it replaces an unknown distribution of private valuations with a known distribution of cash holdings, which is what matters in the end for the allocation of the goods. But this is true only because we have been working with homogenous agents that have the same private valuations. An immediate extension would be to allow for heterogeneity among buyers' private valuations and see how the distribution of private valuations and the distributions of cash holdings interact on the equilibrium allocation.

Finally, in the conclusion of their paper, Peters and Severinov (1997) acknowledge that "it is difficult to think of markets in which sellers compete in reserve prices" in the manner they describe, but hope that "the relatively simple limit equilibrium [they introduce] will make it possible to discover the transaction costs that are missing from [their story]". What our research has shown is that 1) these missing transaction costs may well be the costs of holding and using money, and that 2) as soon as money is introduced in auction theory, since its cost is proportional to the quantity auctioned, the quantity posted by sellers becomes the key variable over which sellers compete rather than the reserve price.

Appendix

A1. Proof of Lemma 1

The seller's objective and the buyer's constraint are given by

$$\max_{q,\theta} -c(q) + \phi_{+1} \sum_{n\in\mathbb{N}} P\left[X = n\right] \int\limits_{z\in S'} z f_{x_{(n-1)}}(z)dz$$

$$\text{s.t.} \quad E\left[\chi(m)\right] = -\left(1-\beta+\tau\right)\phi_{+1} \int\limits_{m\in S'} mdF(m) +$$

$$\sum_{n\in\mathbb{N}} P\left[X=n\right] \left\{ \beta u(q) \int\limits_{m\in S'} F^n(m)dF(m) - \beta\phi_{+1} \int\limits_{\underline{m}}^{\bar{m}} \int\limits_{\underline{m}}^{m} zdF^n(z)dF(m) \right\} \geq k,$$

The integral in the second term of the seller's objective corresponds to the expected gross revenue for the seller. Using the definition of $f_{x_{(n-1)}}$ it is equal to

$$\begin{aligned} & \int\limits_{z\in S'} zn\left(n-1\right)F^{n-2}(z)\left[1-F(z)\right]f(z)dz \\ = \; & n \int\limits_{z\in S'} z\left[1-F(z)\right]dF^{n-1}(z) = n\left[z\left[1-F(z)\right]F^{n-1}(z)\right]_{\underline{m}}^{\bar{m}} - n \int\limits_{z\in S'} \left[1-zf(z)-F(z)\right]F^{n-1}(z)dz \\ = \; & n \int\limits_{z\in S'} \left[zf(z)+F(z)-1\right]F^{n-1}(z)dz. \end{aligned}$$

Taking the sum over n of the above expression multiplied by $P\left[X=n\right] = \frac{\theta^n}{n!}e^{-\theta}$ we obtain

$$\begin{aligned} & \int\limits_{z\in S'} \left[zf(z)+F(z)-1\right]\sum_{n\in\mathbb{N}} \frac{\theta^n}{n!}e^{-\theta}nF^{n-1}(z)dz \\ = \; & e^{-\theta}\theta \int\limits_{z\in S'} \left[zf(z)+F(z)-1\right] \sum_{n\in\mathbb{N}^{*+}} \frac{\theta^{n-1}}{(n-1)!}F^{n-1}(z)dz \\ = \; & \theta \int\limits_{z\in S'} \left[zf(z)+F(z)-1\right]e^{-\theta\left[1-F(z)\right]}dz. \end{aligned}$$

Factorizing by $f(z)$ yields the expression in (14).

Since $\int\limits_{m\in S'} mdF(m)$ is nothing but the money demand, the first term of the constraint corresponds to the cost to the economy of holding the money supply. Using similar techniques as above the second term simplifies into $\frac{\beta u(q)}{\theta}$. Finally, carefully reversing the order of integration, the double integral is the last term can be rewritten

$$\begin{aligned}
&\int_{\underline{m}}^{\bar{m}}\int_{z}^{\bar{m}} zdF^n(z)dF(m) \\
= &\int_{\underline{m}}^{\bar{m}} zdF^n(z)\left[1-F(z)\right] \\
= &\int_{z\in S'} \left[zf(z)+F(z)-1\right]F^n(z)dz.
\end{aligned}$$

It is straightforward to see that the sum over n of the above expression multiplied by $P\left[X=n\right]$ is equal to

$$\int_{s\in S'} \left[zf(z)+F(z)-1\right]e^{-\theta[1-F(z)]}dz$$

which is the seller's expected gross revenue divided by θ. Factorizing by $f(z)$ yields the expression in (15).

A.2. Proof of Lemma 2

The seller's objective and the buyer's constraint can be written

$$-c(q)+\phi\theta e^{-\theta}\int_{z\in S'} \left[zf(z)+F(z)-1\right]e^{\theta F(z)}dz \tag{27}$$

and

$$-i\phi\int_{z\in S'} zf(z)dz+\frac{u(q)}{\theta}-\phi e^{-\theta}\int_{z\in S'} \left[zf(z)+F(z)-1\right]e^{\theta F(z)}dz\geq k/\beta \tag{28}$$

respectively. From (10), the density $f(z)$ is given by

$$\frac{\partial F(z)}{\partial z}=\frac{ie^{\theta}\theta}{\theta\left[u(q)-\phi z\right]\left[1-ie^{\theta}\ln\left(\frac{u(q)-\phi z}{u(q)-c(q)}\right)\right]}.$$

We make the following change in variable, $v=1-ie^{\theta}\ln\left(\frac{u(q)-\phi m}{u(q)-c(q)}\right)$ so that $dv=\frac{ie^{\theta}\theta}{u(q)-\phi z}dz$ and

$$z=\frac{u(q)-e^{\frac{1-v}{ie^{\theta}}}}{\phi}.$$

Introducing these values into (27)-(28), the first part of the integral in the seller's objective $\phi\theta e^{-\theta} \int\limits_{z\in S'} zf(z)e^{\theta F(z)}dz$, for instance, transforms into

$$e^{-\theta} \int\limits_{1}^{e^{\theta}} \left(u(q) - e^{\frac{1-v}{i.e^{\theta}}} \left[u(q) - c(q) \right] \right) dv$$

which has no ϕ in it. Using the same change in variable in (27)-(28) yields

$$\max_{q,\theta} \Pi_S(q,\theta) = [u(q) - c(q)] \left[1 + e^{-\theta} \int\limits_{1}^{e^{\theta}} e^{\frac{1-v}{i.e^{\theta}}} \left(\frac{v \ln v - \theta v}{ie^{\theta}} - 1 \right) dv \right] - e^{-\theta} u(q) \quad (29)$$

$$\text{s.t.} \quad \frac{-i}{\theta} \int\limits_{1}^{e^{\theta}} \left(u(q) - e^{\frac{1-v}{i.e^{\theta}}} \left[u(q) - c(q) \right] \right) \frac{dv}{v} + \frac{u(q)}{\theta} - \frac{\Pi_S(q,\theta) + c(q)}{\theta} \geq k/\beta, \quad (30)$$

which do not depend on ϕ.

A3. Proof of Proposition 1

From Lemma 2, ϕ is not a choice variable for the seller so we can focus on q and θ.

(i): **Existence.** First let us note $x = (q, \theta)$ and $X = Q \times \Theta$ with $Q = [0, \hat{q}]$ and Θ is a closed set of $\mathbb{R}_+$. Also let $k \in K \subset \mathbb{R}_+$.

Let $f : X \times K \to \mathbb{R}$ be the objective function with $f(x; k) = -c(q) + \phi\Psi(q, \theta)$ where $\Psi(q, \theta)$ is the gross expected payment to the seller and is given by

$$\Psi(q, \theta) = \theta \int\limits_{z\in S'} \left[zf(z) + F(z) - 1 \right] e^{-\theta[1-F(z)]} dz.$$

Let $\Gamma : K \to X$ be the correspondence (point-to-set mapping) defined by

$$\Gamma(k) = \left\{ x : -i\phi\bar{M}^S + \frac{u(q)}{\theta} - \frac{\phi}{\theta}\Psi(q, \theta) \geq k/\beta \right\}$$

with $Graph(k) = \{(k, x) \in K \times X : x \in \Gamma(k)\}$.

Finally let

$$\begin{aligned} v^*(k) &= \max_x f(x; k) \\ \text{s.t.} \quad x &\in \Gamma(k) \end{aligned}$$

and

$$\begin{aligned} w^*(k) &= \arg\max_x f(x;k) \\ \text{s.t.} \quad x &\in \Gamma(k). \end{aligned}$$

with $w^*(k) = \{q^*(k), \theta^*(k)\}$.

The function $f(x;h)$ is continuous in both x and k over the compact set $X \times K$. Since X is compact and $Graph(k)$ is closed, Γ is upper hemicontinuous. It is also lower hemicontinuous and therefore continuous so that the theorem of the maximum applies (Stockey and Lucas, 1987, Theorem 3.6 p. 62) : $v^*(k)$ is continuous and $w^*(k)$ is non empty, compact valued and upper hemicontinuous, which guaranties the existence of an equilibrium.

(ii): Uniqueness:

A.4. Proof of Proposition 3

Let us note $L(q_p) = z\left(u^{-1}\left[c(q_p)\right], q_p\right) - c(q_p)$ which is given by

$$\frac{\left[1-\beta+\tau+\beta\psi_p\right]\xi_m' c(u^{-1}\left[c(q_p)\right]) - \beta\psi_p'\xi_m u(q_p)}{\left[1-\beta+\tau+\beta\psi_p\right]\xi_m' - \beta\psi_p'\xi_m} - c(q_p).$$

One can see that $L(0) = 0$ so that a non monetary equilibrium in which money is not valued always exists. Similarly one verifies that $L(\hat{q}) = 0$, yet $\hat{q}$ cannot be an equilibrium otherwise from (23) the buyer's net surplus would be negative.

Substituting ψ_p and ξ_m for their values in θ, one obtains

$$L'(0) = \frac{\left(1+\theta-e^{\theta}\right)u'(0)}{1-e^{\theta}\left(1+i\theta\right)}$$

with $i = \frac{1-\beta+\tau}{\beta} \geq 0$. Since both $1+\theta-e^{\theta}$ and $1-e^{\theta}\left(1+i\theta\right)$ are negative for any $\theta \in \mathbb{R}_+$ then $L'(0) > 0$. By the intermediate value theorem there exists at least one $q_p^* \in [0,\hat{q}]$ such that $L(q_p) = 0$ and we choose u and c so that there is a unique solution (which is the case for instance with $u(q) = \sqrt{q}$ and $c(q) = q$).

In terms of limits we have $\lim_{\theta\to 0} L(q_p) = \lim_{\tau\to\infty^+} L(q_p) = c(u^{-1}\left[c(q_p)\right]) - c(q_p)$, the solutions to which are 0 and $\hat{q}$. Since $\hat{q}$ cannot be an equilibrium we are left with $q_p^* \to 0$ when either $\theta \to 0$ or $\tau \to \infty$. Similarly one can show that $q \to \hat{q}$ as $\theta \to \infty^+$.

References

[1] Acemoglu, Daron and Robert Shimer (1999a). "Holdups and Efficiency with Search Frictions". International Economic Review 40(4), p. 827-849.

[2] Acemoglu, Daron and Robert Shimer (1999a). "Efficient Unemployment Insurance". Journal of Political Economy 107(5), p. 893-928.

[3] Berentsen Aleksander, Guido Menzio and Randall Wright (2006). "Inflation and Unemployment: Lagos-Wright meets Mortensen-Pissarides", mimeo.

[4] Burdett, Kenneth, Shouyong Shi and Randall Wright (2001). "Pricing and Matching with Frictions", Journal of Political Economy 109 (5), 1060-1085.

[5] Burguet, Roberto and József Sákovics (1999). "Imperfect Competition in Auction Designs", International Economic Review 40(1), 231-247.

[6] Faig Miquel and Belén Jerez (2005a). "Inflation, Prices and Information in Competitive Search", Advances in Macroeconomics 6(1).

[7] Faig Miquel and Belén Jerez (2005b). "Precautionary Balances and the Velocity of Circulation of Money", miméo.

[8] Galenianos, Manolis and Philipp Kircher, 2006. "Directed Multilateral Matching in a Monetary Economy", manuscript.

[9] Hernando-Veciana, Ángel (2005). "Competition among auctioneers in large markets", Journal of Economic Theory, p. 107-127.

[10] Julien, Benoit, John Kennes and Ian King (2000). "Bidding for Labour", Review of Economic Dynamics 3(4), p. 619-649.

[11] Julien, Benoit, John Kennes and Ian King (2006). "Bidding for Money", Journal of Economic Theory (forthcoming)

[12] King, Ian (2003). "A Directed Tour of Search-Theoretic Explanations for Unemployment". New Zealand Economics Papers 37, p. 245-267.

[13] Kiyotaki, N. and Wright, R. (1989). 'On money as a medium of exchange', *Journal of Political Economy*, vol. 97(4), pp. 927-954.

[14] Kiyotaki, N. and Wright, R. (1991). 'A contribution to the pure theory of money', *Journal of Economic Theory*, vol. 53(2), pp. 215-235.

[15] Kiyotaki, N. and Wright, R. (1993). 'A search-theoretic approach to monetary economics', *American Economic Review*, vol. 83(1), pp. 63-77.

[16] Lagos, Ricardo and Randall Wright (2005). "A Unified Framework for Monetary Theory and Policy Analysis", Journal of Political Economy 113 (3), pp. 463-484.

[17] McAfee, Preston R. (1993). "Mechanism Design by Competing Sellers". Econometrica 61(6), 1281-1312.

[18] Moen, Espen R. (1997). "Competitive Search Equilibrium". Journal of Political Economy 105(2), p.385-411.

[19] Montgomery, James (1991). "Equilibrium Wage Dispersion and Interindustry Wage Differentials", The Quarterly Journal of Economics 106, 163-79.

[20] Myerson, Roger B. (1981). "Optimal Auction Design", Mathematics of Operations Research 6(1), 58-73.

[21] Peters, Michael (1991). "Ex Ante Price Offers in Matching Games Non-steady States." Econometrica 59, 1425-1454.

[22] Peters, Michael (1997). "A Competitive Distribution of Auctions". Review of Economic Studies 64, 97-123.

[23] Peters, Michael (2000). "Limits of Exact Equilibria for Capacity Constrained Sellers with Costly Search". Journal of Economic Theory 95, 139-168.

[24] Peters Michael and Sergei Severinov (1997). "Competition among Sellers Who Offer Auctions Instead of Prices." Journal of Economic Theory 75, 141-179.

[25] Riley, J. and W. Samuelson (1981). "Optimal Auctions". American Economic Review 71, p. 381-392.

[26] Rocheteau, Guillaume and Randall Wright (2005). "Money in Search Equilibrium, in Competitive Equilibrium, and in Competitive Search Equilibrium", Econometrica 73, 175-202.

[27] Rogerson, Richard, Robert Shimer and Randall Wright (2005). "Search-Theoretic Models of the Labour Market: A Survey". Journal of Economic Literature 63, p. 959-988.

[28] Schmitz, Patrick W. (2003), "On second-price auctions and imperfect competition", Journal of Mathematica Economics 39, p.901-909.

[29] Stockey, Nancy L. and Robert E. Lucas, with Edward C. Prescott, 1989, *Recursive Methods in Economic Dynamics*. Cambridge, MA: Harvard University Press.