Equilibrium Concepts in the Large Household Model

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Abstract

This paper formulates two alternative equilibrium concepts in the large household model: one which allows individual household agents to make choices in their separate meetings, and the other which commits individual household agents to contingent actions prior to their meetings. In the first formulation, *large* converts a model with non-linear preferences for the household into one with quasi-linear preferences for the individual household's agents, which is critical to make degeneracy —all households experience a same distribution of meeting outcomes—as an equilibrium. In the second formulation, commitment instead of *large* is the critical factor.

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1 Introduction

Search models now play a dominant role in labor economics and a prominent role in monetary economics. In such models, meeting-specific shocks are obvious sources of heterogeneity. For example, in a money model with complete specialization in consumption and production and random pairwise meetings (e.g. Kyotaki and Wright [1]), two people who start with the same wealth end up with different wealth if one becomes a buyer and the other

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becomes a seller in the relevant meetings; or, in a labor model with random job destruction (e.g. Pissarides [7, Ch 1, 2]), two workers who start with the same wealth end up with different wealth if the relevant worker-firm pairs experience different separation or productivity shocks. Because heterogeneity precludes closed-form solutions, efforts have been made to create models in which equilibria have degenerate distributions of wealth.

One such model is the so-called large household model, where *large* means that each household consists of a non atomic measure of agents. In this model, each agent from a household meets someone from outside the household—a firm in Merz [3] or an agent from another household in Shi [5]. If all households start with the same wealth, then it is feasible that all households experience the same distribution of meeting outcomes, and, by a law of large numbers argument, end up with the same wealth. Of course, whether that happens depends not only on feasibility, but on whether the same distribution of meeting outcomes is an equilibrium. Whether it is is unclear because the literature contains neither clear definitions of equilibrium or existence proofs. Rauch [4] points out a defect in the formulation of [5] (which is also a defect of [3]), but Rauch's suggested alternative is itself not correct. More recent literature, initiated by Shi [6], avoids the problem pointed out by Rauch, but suffers from other deficiencies.

Here, in the context of a money model, I formulate two alternative concepts of search equilibrium. One completes and corrects Rauch's formulation. That formulation allows individual household agents to make choices in their separate meetings and, therefore, is called the *no-commitment* approach. The other completes the more recent approach of Shi [6]. It commits individual household agents to contingent actions prior to their meetings and, therefore, is called the *commitment* approach. In order to study the role of large in determining whether degeneracy (the absence of heterogeneity) is an equilibrium, I use a model in which degeneracy is feasible whether or not the household is large. In particular, I study a model in which each meeting is a single-coincidence meeting and I study a finite household version—a household that consists of n buyers and n sellers, and a large household version—a household that consists of equal non atomic measures of buyers and sellers.

In the no commitment approach, *large* converts a model with nonlinear preferences for the household into one with quasi-linear preferences for the individual household agents, and, this turns out to be critical to make degeneracy as an equilibrium. In the commitment approach, commitment itself, instead of *large*, is the critical factor. But since commitment eliminates subgame perfection,[1] it results in a weak equilibrium concept—in fact, there exists a continuum of commitment equilibria.

2 The environment

Time is discrete. There is a non atomic measure of each of $K \geq 3$ types of infinitely lived households. Each household consists of a set of buyers indexed by I and a set of sellers also indexed by I. I study two versions of I. In one version, $I = \{1, ..., n\}$, and the model is refered as the finite household model. In another version, I is a non atomic measure space, and the model is refered as the large household model.

There are $K \geq 3$ types of produced and perishable goods. For a type-k household, when its buyer $i \in I$ consumes q_{ib} units of type-k good (its buyers cannot consume other goods), and its seller $i \in I$ produces q_{is} units of type-$k+1$ good (its sellers cannot produce other goods), its period utility is $\int_{i \in I} u(q_{ib})di - \int_{i \in I} \phi(q_{is})di$.[2] The household maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. As is standard, u is strictly increasing, strictly concave, differentiable, and bounded, and with $u(0) = 0$ and $u'(0) = \infty$. And without loss of generality, $\phi(q) = q$.

There is another object called money which is durable and intrinsically useless. The stock of money is fixed and the per household holding is unity. Buyers can hold money while sellers cannot.[3]

Each date, agents from households are randomly matched in pairs, but in a way that makes each meeting a meeting between a seller who produces type-k good and a buyer who consumes type-k good. In each meeting, each agent's money holding and type is common knowledge, and, moreover, each

[1] As a simple way to see this, think about the well-known pie-splitting game. In the game, player 1 makes a take-it-or-leave-it offer to player 2. Before an offer is made, player 1 chooses a plan regarding what the offer is, and player 2 chooses a plan regarding how to respond to an offer. If players are committed to plans, then for each $x \leq 1$, it is an equilibrium outcome that player 1 gets x portion of the pie (the player 1's plan to offer x to himself, and the player 2's plan is to accept an offer that gives himself no less than x. If players are not committed to plans, the unique equilibrium outcome is that player 1 gets the whole pie.

[2] An alternative assumption is that buyers pool goods together after search, and the household's period utility by consuming q is $u(q)$. This does not change the logic to describe equilibrium or the results.

[3] This is equivalent to assuming that the household cannot hold an inventory of money. Such assumption simplifies the description of equilibrium and it does not affect the results.

household's start-of-date money holding is common knowledge.[4] These common knowledge assumptions permit me to avoid dealing with asymmetric information.

Throughout, I restrict attention to equilibrium that is *symmetric, stationary*, *degenerate*, and *monetary*, and with a value function v that is *non decreasing, continuous,* and *concave*. Symmetry is imposed across specialization types, and across buyers of a given household and sellers of a given household. Stationary means that the distribution of meeting outcomes obtained by the individual household agents only depends up the household's start-of-date money holding. Degeneracy means all households start with one unit of money at the start of each date. Monetary means that money is valued. Regarding v, it is needed by the household, and, in the no-commitment formulation, its agents, to deduce the effects of different actions that might be taken.

Throughout, I refer to a household with one unit of money as a *regular* household, an agent from a regular household as a regular agent, and a meeting between two regular agents as a regular meeting.

3 The no-commitment approach

In this section, I assume that households are not committed to pre-search plans; instead, each agent makes his own decision in meeting.

3.1 Equilibrium definition

To make a decision in meeting, an agent ought to evaluate each feasible trade. The payoff of a trade to the agent is defined as the additional contribution of the trade to the household's lifetime expected utility, taking as given other trade outcomes obtained by other agents from the same household when meeting regular agents. Regarding how a trade is determined in a meeting, following some of the literature (see Shi [5] and Rauch [4]), I assume generalized Nash bargaining,[5] and I denote the buyer's bargaining power by $\theta \in (0, 1]$.

[4] Imagine that each agent carries a sign that truly shows his household money holding at the start of the date, and this sign can be observed in meeting.

[5] Although generalized Nash bargaining does not explicitly describe the agent's decision making, as is well known, its solution may be interpreted as the limit of equilibrium outcomes of some game which explicitly describes the agent's decision making.

I begin by the finite household model. As indicated above, in order to describe the possibilities for a deviating household and its agents, I must describe the value to an arbitrary household starting with an arbitrary money holding x.

I first describe the *net* payoff (surplus-from-trade) of each feasible trade to each agent from the household with x. If v is the value function, then the net payoff to a buyer from the household with x who consumes q and transfers l units of money when each of other $n-1$ buyers from the household transfers $l_b(x)$ to a regular seller and each seller from the household acquires $l_s(x)$ from a regular buyer is

$$\begin{aligned} \Pi_b(q,l,x) &= u(q) + \beta v[x + nl_s(x) - (n-1)l_b(x) - l] \\ &\quad -\beta v[x + nl_s(x) - (n-1)l_b(x)], \end{aligned} \tag{1}$$

while the net payoff to a seller from that household who produces q and acquires l units of money when each buyer from the household transfers $l_b(x)$ to a regular seller and each of other $n-1$ sellers from the household acquires $l_s(x)$ from a regular buyer is

$$\begin{aligned} \Pi_s(q,l,x) &= -q + \beta v[x + (n-1)l_s(x) - nl_b(x) + l] \\ &\quad -\beta v[x + (n-1)l_s(x) - nl_b(x)]. \end{aligned} \tag{2}$$

Given those payoff functions defined by (1)-(2), the trade in a meeting between a buyer from the household with x and a regular seller is

$$(q_b(x), l_b(x)) \in \arg \max_{q \geq 0, 0 \leq l \leq x/n} [\Pi_b(q,l,x)]^{\theta} [\Pi_s(q,l,1)]^{1-\theta}, \tag{3}$$

while the trade in a meeting between a seller from that household and a regular buyer is

$$(q_s(x), l_s(x)) \in \arg \max_{q \geq 0, 0 \leq l \leq 1/n} [\Pi_b(q,l,1)]^{\theta} [\Pi_s(q,l,x)]^{1-\theta}. \tag{4}$$

In turn, the value function v must satisfy

$$v(x) = nu[q_b(x)] - nq_s(x) + \beta v[x - nl_b(x) + nl_s(x)]. \tag{5}$$

Finally, degeneracy requires

$$nl_b(1) = nl_s(1) = 1, \tag{6}$$

and I have the following definition.

Definition 1 *In the finite household model, a no-commitment equilibrium is a value function* v *on* $\mathbb{R}_+$*, and a collection of real-valued functions* (q_b, l_b, q_s, l_s) *on* $\mathbb{R}_+$*, that satisfy* (1)-(6)*.*

Notice that according to the definitions of Π_b and Π_s, the maximizations in (3) and (4) are carried out taking as given the actions of the other agents in the household. This sets the stage for quasi-linearity in the large household.

Now I turn to the large household model, and, without loss of generality, let the measure of I be unity. If the household has a value function v and ends up with x_+ at the end of the date, the agent's payoff from an extra unit of money is $\beta v'(x_+)$ (of course, v must be differentiable at x_+). Thus, the net payoff to a buyer from a household with x who consumes q and transfers l units of money when each of other buyers from the household transfers $l_b(x)$ to a regular seller and each seller from the household acquires $l_s(x)$ from a regular buyer is

$$\Pi_b(q, l, x) = u(q) - l\beta v'[x - l_b(x) + l_s(x)], \tag{7}$$

while the net payoff to a seller from that household who produces q and acquires l units of money when each buyer from the household transfers $l_b(x)$ to a regular seller and each of other sellers from the household acquires $l_s(x)$ from a regular buyer is

$$\Pi_s(q, l, x) = -q + l\beta v'[x - l_b(x) + l_s(x)]. \tag{8}$$

Given those payoffs, the trade in a meeting between a buyer from the household with x and a regular seller is

$$(q_b(x), l_b(x)) = \arg\max_{q \geq 0, 0 \leq l \leq x} [\Pi_b(q, l, x)]^\theta [\Pi_s(q, l, 1)]^{1-\theta}, \tag{9}$$

while the trade in a meeting between a seller from that household and a regular buyer is

$$(q_s(x), l_s(x)) = \arg\max_{q \geq 0, 0 \leq l \leq 1} [\Pi_b(q, l, 1)]^\theta [\Pi_s(q, l, x)]^{1-\theta}. \tag{10}$$

Thus

$$v(x) = u[q_b(x)] - q_s(x) + \beta v[x - l_b(x) + l_s(x)]. \tag{11}$$

Finally, degeneracy requires

$$l_b(1) = l_s(1) = 1, \tag{12}$$

and I have the following following definition.

Definition 2 *In the large household model, a no-commitment equilibrium is a value function* v *on* $\mathbb{R}_+$*, and a collection of real-valued functions* (q_b, l_b, q_s, l_s) *on* $\mathbb{R}_+$*, that satisfy* (7)-(12)*.*

For the large household model, I can prove existence only when there is a nonbinding upper bound Z on the household's money holdings, and the comparable definition including such a bound is the next.

Definition 3 *In the large household model, a no-commitment equilibrium is a number* $Z > 1$*, a value function* v *on* $[0, Z]$*, and a collection of real-valued functions* (q_b, l_b, q_s, l_s) *on* $[0, Z]$*, that satisfy* (7)-(12)*.*

When sellers can hold money, Definitions 1-3 can be revised as follows. Besides the household's money holding, the amount of money assigned to sellers is another argument of functions (q_b, l_b, q_s, l_s); also, the amount of money assigned to sellers is the choice variable of the household in the maximization problem in (3) or (11).

3.2 Comparison to the literature

Aside from details, Shi [5] and Rauch [4] share all the important assumptions of the environment, including the common knowledge assumptions. Shi [5], who initiated the use of the large household model for money applications, describes the household's problem in terms of sequences of the household's choices, and in his formulation, each household takes as given that the regular-meeting trade, denoted $(\hat{q}, \hat{l})$, is the trade that its buyers and sellers will make—independent of the household's start-of-date money holding. The household's problem can be recursively written as follows: When $x < 1$, $v(x)$ is not defined because $\hat{l} = 1$ (buyers from a household with $x < 1$ cannot transfer 1), and when $x \geq 1$, $v(x) = u(\hat{q}) - \hat{q} + \beta v(x - \hat{l} + \hat{l})$, and it follows that $v(x) = v(1)$ for $x \geq 1$, all of which is problematical. Rauch [4], in a lengthy comment on [5], points that out and proposes an alternative formulation.

Rauch deals with that the payoff of a trade to one agent depends on trade outcomes of other agents from the same household by describing an agent's action in a regular meeting as a function of its household's end-of-meeting money holdings. This treatment is awkward in general and impossible for the finite household; in the finite household, the household's end-of-meeting money holding depends on the agent's action. There is another problem in Rauch's formulation. Rauch describes the household problem in terms of sequences of the household's choices. But to define the payoff associated with an arbitrary sequence, he uses the Lagrangian multipliers that are associated with an optimal sequence, which makes the household's problem not well defined (this problem can be avoided by using a recursive approach and introducing a value function).

3.3 Existence results and the role of large

The main results here are non existence of a no-commitment equilibrium in the finite household model when $n = 1$, and existence of a no-commitment equilibrium in the large household model (recall that any equilibrium in consideration is symmetric, stationary, degenerate, and monetary, and with v that is non decreasing, continuous, and concave). As is discussed below, quasi linearity of the agent's payoff resulting from *large* makes the difference. This suggests that *large* be critical to make degeneracy as an equilibrium outcome.

Proposition 1 *There does not exist a Definition* 1 *equilibrium when* $n = 1$*.*

Proof. To begin with, notice that $v(1) > v(0)$ because money is valued. Then by continuity and concavity of v, v is strictly increasing over $[0, 1]$.

Now we consider $\theta = 1$. Suppose (v, q_b, l_b, q_s, l_s) is a Definition 1 equilibrium. By (3) and $l_b(1) = 1$,

$$l_b(x) \in \arg\max_{0 \le l \le x} u[\beta v(l) - \beta v(0)] + \beta v[x + l_s(x) - l]. \tag{13}$$

By (4) and $l_s(1) = 1$,

$$l_s(x) \in \arg\max_{l \le 1} u[\beta v(x - l_b(x) + l) - \beta v(x - l_b(x))] + \beta v(2 - l). \tag{14}$$

Letting $x = 1$ in (13) and using $l_b(1) = 1$, we have

$$1 \in \arg\max_{0 \le l \le 1} u[\beta v(l) - \beta v(0)] + \beta v(2 - l). \tag{15}$$

Fix $x < 1$. If $l_b(x) < x$, then because v is concave and strictly increasing over $[0, 1]$, and because u is strictly concave, (13) contradicts (15). So $l_b(x) = x$. Given this, if $l_s(x) < 1$, then because v is concave and strictly increasing over $[0, 1]$, and because u is strictly concave, (14) contradicts (15). So $l_s(x) = 1$. Therefore, we have

$$v(x) = u[\beta v(x) - \beta v(0)] - [\beta v(1) - \beta v(0)] + \beta v(1) \text{ for } x \leq 1. \tag{16}$$

It follows that $v(0) = 0$. This and (16) give

$$v(x) = u[\beta v(x)] \text{ for } x \leq 1. \tag{17}$$

But because u is strict concave, (17) cannot hold for more than one positive x. The proof for $\theta < 1$ is in the appendix. ■

Proposition 1 is similar to a result in Wallace and Zhu [8, section 2]. They study a model in which each household consists of one agent, each meeting is a single-coincidence meeting, and $\theta = 1$. They show that there does not exist any stationary and degenerate monetary equilibrium if β is close to unity. Proposition 1 is weaker than that result for it only rules out a class of stationary equilibria with concave value functions (but it is stronger for it deals with general θ and β); in the proof of Proposition 1, concavity over $[0, 2]$ is used to conclude that $l_b(x) = x$ and $l_s(x) = 1$ for $x \leq 1$.

Proposition 2 is partially converse to Proposition 1, partially because it concerns Definition 3 equilibrium, and because it requires additional assumptions on u when $\theta < 1$. For expositional convenience, I let $u(q) = \sqrt{q}$ for $q \leq Q$, where Q is large but finite (so u can still be bounded).

Proposition 2 *(i) If $\theta = 1$, then there exists a Definition 3 equilibrium. (ii) If $\theta < 1$ and if $u(q) = \sqrt{q}$ for $q \leq Q$, where Q is large but finite, then there exists a Definition 3 equilibrium.*

Proof. For part (i), we first construct $(Z, v, q_b, l_b, q_s, l_s)$. Let $\beta u'(\bar{q}) = 1$ and let Z satisfy $u'(Z\bar{q}) = 1$. Let (v, q_b, l_b, q_s, l_s) on $[0, Z]$ be defined by

$$\begin{aligned} v(x) &= u(x\bar{q}) - \bar{q} + \beta v(1), \\ (q_b(x), l_b(x)) &= (x\bar{q}, x), \\ (q_s(x), l_s(x)) &= (\bar{q}, 1). \end{aligned} \tag{18}$$

By $\beta u'(\bar{q}) = 1$, we have $\beta v'(1) = \bar{q}$. Next we verify that $(Z, v, q_b, l_b, q_s, l_s)$ is an equilibrium. First, functions (v, q_b, l_b, q_s, l_s) satisfy (11) and the degeneracy condition, and v is strictly increasing, strictly concave, and differentiable. Second, for $x \le Z$, by $\beta v'(1) = \bar{q}$ and $u'(Z\bar{q}) = 1$ and given functions (q_b, l_b, q_s, l_s) in (18), $(x\bar{q}, x)$ solves the problem in (9) because $u'(x\bar{q}) \ge 1$ (the first order condition of l), and $(\hat{q}, 1)$ solves the problem in (10) because $u'(\bar{q}) \ge 1$. The proof of part (ii) is in the appendix. ■

The quasi-linearity of agent payoffs (see (7) and (8)) and the bound Z play key roles in the last proof. The role of quasi-linearity can be seen by comparing (17) and (18). Regarding the bound Z, it is needed because in general v' in (7) and (8) depends on the household's money holding x. If it does, then quasi-linearity is not helpful. But the bound Z is constructed so that v' does not depend on x for $x \le Z$. This makes all buyers from a household with $x \le Z$ willing to spend x and makes all sellers from such households willing to receive 1.

A couple of other remarks are in order. Proposition 1 holds if Definition 1 is revised to allow a nonbinding upper bound on the household's money holdingsProposition 1 holds (notice that, to be nonbinding, any such upper bound should be no less than 2). Also, Propositions 1 and 2 hold if Definitions 1 and 3 are revised to allow sellers to hold money.

4 The commitment approach

In this section, I assume that households are committed to pre-search plans, that is, each household chooses a binding plan for all its buyers and sellers prior to meetings.

4.1 Equilibrium

Following some of the literature (see Shi [6]), I assume that buyers make take-it-or-leave-it offers in meetings.[6]

I start by describing an arbitrary plan made by the household with money holding x. Such a plan is a contingent plan and is denoted by $p_x \equiv (\sigma_x, \lambda_x)$,

[6]This assumption is not as restrictive as it appears. As it turns out, the equilibrium outcome under this assumption can be supported by a commitment equilibrium in which buyers and sellers make alternating offers.

where σ_x maps a buyer's contingency to an offer, and λ_x maps a seller's contingency to an acceptance-rejection response. For a buyer from the household, if he meets a seller whose household's money holding is y, then y is a contingency for him and so the offer is $\sigma_x(y)$. For a seller from the household, if he meets a buyer whose household's money holding is y and if the buyer offers σ, then (y, σ) is a contingency for him so the response is $\lambda_x(y, \sigma)$.

The payoff of p_x depends on the value function v and the plan chose by regular households, denoted $p_1^* \equiv (\sigma_1^*, \lambda_1^*)$. Depending on p_1^*, non convexity may arise if offers or responses are deterministic. To avoid non convexity and to restrict the set of equilibria, I allow stochastic offers and responses. Therefore, an offer σ is a probability measure over the set of feasible trades, where feasibility means the transfer of money in a meeting does not exceed the buyer's money holding; and an acceptance-rejection response is a probability to accept an offer, and, in particular, $\lambda_x(y, \sigma)$ is the probability to accept the offer σ made by a seller from a household with y.

For the household with x, p_x and p_1^* induce a distribution of realizations of $(q_{ib}, l_{ib}, q_{is}, l_{is})_{i \in I}$, where (q_{ib}, l_{ib}) is the trade in the meeting between buyer i and a regular seller, and (q_{ib}, l_{ib}) is the trade in the meeting between seller i and a regular buyer. Letting this distribution be denoted by $\pi(p_x, p_1^*)$, the payoff of $p(x)$ can be written as

$$f(p_x, p_1^*) = E_{\pi(p_x, p_1^*)}[\textstyle\int u(q_{ib})di - \int q_{is}di + \beta v(x - \int l_{ib}di + \int l_{is}di)], \quad (19)$$

where $E_{\pi(p_x, p_1^*)}$ stands for the expectation over the distribution $\pi(p_x, p_1^*)$. It follows that

$$v(x) = \max_{p_x} f(p_x^*, p_1^*). \quad (20)$$

Degeneracy requires

$$\pi(p_1^*, p_1^*)\{(q_{ib}, l_{ib}, q_{is}, l_{is})_{i \in I} : \textstyle\int l_{ib}di = \int l_{is}di = 1\} = 1. \quad (21)$$

By (20), for any $x \geq 0$, $p_x^* \equiv (\sigma_x^*, \lambda_x^*)$ is a best response to p_1^*, and this obviously imposes dependence of p_x^* on p_1^*. On the other hand, although p_1^* is a best response to p_x^* for any $x \neq 1$, this does not impose any dependence of p_1^* on p_x^* when $x \neq 1$; indeed, because the payoff of a plan is computed before matching and equilibrium is degenerate, any p_1 is a best response to p_x^* when $x \neq 1$. To strength the equilibrium concept and to restrict the set of equilibria, I introduce two types of constrains on p_1^*.

In the finite household model, the first constraint is

$$\sigma_1^*(x) \in \arg\max_{\sigma} \lambda_x^*(1,\sigma) E_\sigma[u(q) + \beta v(1-l)]. \tag{22}$$

That is, from a regular household's point of view, when one of its buyers meets a seller from a household with arbitrary x, the buyer's offer $\sigma_1^*(x)$ is a best response to the seller's response $\lambda_x^*(1,\sigma)$, by taking as given other agents from the household meet regular agents. The second constraint is

$$\lambda_1^*[x, \sigma_x^*(1)] \in \arg\max_{\lambda} \lambda E_{\sigma_x^*(1)}[-q + \beta v(1 - 1/n + l)]. \tag{23}$$

That is, from a regular household point of view, when one of its sellers meets a buyer from a household with arbitrary x, the seller's response $\lambda_1^*[x, \sigma_x^*(1)]$ is a best response to the buyer's offer $\sigma_x^*(1)$, by taking as given other agents from the household meet regular agents. In the large household model, constraints analogous to (22) and (23) are

$$\sigma_1^*(x) \in \arg\max_{\sigma} \lambda_x^*(1,\sigma) E_\sigma[u(q) - l\beta v'(1)], \tag{24}$$

and

$$\lambda_1^*[x, \sigma_x^*(1)] \in \arg\max_{\lambda} \lambda E_{\sigma_x^*(1)}[-q + l\beta v'(1)]. \tag{25}$$

Therefore, I have the following definitions.

Definition 4 *In the finite household model, a commitment equilibrium is a value function v on $\mathbb{R}_+$, and a collection of plans p_x^* for $x \geq 0$, that satisfy* (19)-(21), (22), *and* (23).

Definition 5 *In the large household model, a commitment equilibrium is a value function v on $\mathbb{R}_+$, and a collection of plans p_x^* for $x \geq 0$, that satisfy* (19)-(21), (24), *and* (25).

4.2 Comparison to the literature

The commitment approach is initiated by Shi [6]. In [6], there is no explicit definitions of contingencies or an explicit version of (24), while there is a special version of (25). By that version, in a meeting between a regular seller and a buyer from a household with x, the seller accepts any offer whose payoff to the seller's household is no worse than no trade, and so the regular-meeting output in any equilibrium must be equal to $\beta v'(1)$ (compare this with Proposition 3 below).

Aside from details, the purported equilibrium in [6] has the value function

$$v(x) = \max_{l \leq x, 0 \leq \rho \leq 1} u(l\bar{q}) - \rho\bar{q} + \beta v(x - l + \rho), \tag{26}$$

where $\beta u'(\bar{q}) = 1$. It follows that$\beta v'(1) = \bar{q}$. Letting $(l(x), \rho(x))$ be the optimal solution to the maximization problem in (26), the plans in the purported equilibrium are: (a) a regular seller accepts an offer (q, l) if and only if $q \leq l\bar{q}$, and a buyer from the household with x offers $(l(x)\bar{q}, l(x))$ to a regular seller; and (b) a regular buyer offers $(\bar{q}, 1)$ to a seller from the household with x, and a seller from the household with x accepts $(\bar{q}, 1)$ from a regular buyer with probability one.

This purported equilibrium has a defect, though. Given the response and offer of regular agents depicted in (a) and (b), the household with x shall let each of its sellers to accept $(\bar{q}, 1)$ with probability $\rho(x)$. This is also consistent with the way that the value function is described. It can be shown that $\rho(x) < 1$ as x is sufficiently large, and, hence, it is not optimal for the household with x to choose for its sellers the response depicted in (b).

4.3 Existence results and the role of large

The main results here are existence of a continuum of commitment equilibria in the finite household model when $n = 1$, and existence of a continuum of commitment equilibria in the large household model. Those results suggest that commitment instead of *large* be critical to make degenercy as an equilibrium. They suggest that commitment results in a weak equilibrium concept.

Proposition 3 *Let $\beta u'(\bar{q}) = 1$ and let $\hat{q} \in (0, \bar{q}]$. (i) There exists a Definition 4 equilibrium when $n = 1$ in which $\hat{q}$ is the regular-meeting output and $\hat{q} < \beta[v(1) - v(0)]$. (ii) There exists a Definition 5 equilibrium in which $\hat{q}$ is the regular-meeting output and $\hat{q} < \beta v'(1)$ when $\hat{q} < \bar{q}$.*

Proof. Here we prove part (ii). The proof of part (i) is in the appendix. First fix $\hat{q}$ and we define the value function v as

$$v(x) = \max_{0 \leq l \leq x, 0 \leq \rho \leq 1} u(l\hat{q}) - \rho\hat{q} + \beta v(x - l + \rho). \tag{27}$$

It is standard to show there exists a strictly increasing, strict concave, and differentiable v satisfying (27). Let $(l(x), \rho(x))$ denote the optimal solution to the maximization problem in (27), and by $\hat{q} \le \bar{q}$ and $\beta u'(\bar{q}) = 1$, $\rho(x) = 1$ and $l(x) = x$ for $x \le 1$.

Next we construct the plan p_1^*. Let

$$\sigma_1^*(x)\{(\rho(x)\hat{q}, \rho(x))\} = 1 \text{ all } x; \tag{28}$$

that is, the support of the measure $\sigma_1^*(x)$ is the singleton set $(\rho(x)\hat{q}, \rho(x))$, so a regular buyer offers $\rho(x)$ units of money for exchange of $\rho(x)\hat{q}$ units of good to a seller from a household with x. Also, for (x, σ) with $\sigma\{(q, l) : l \le x\} = 1$ (recall that σ should be feasible), let

$$\lambda_1^*(x, \sigma) = 1 \text{ if } \sigma\{(q, l) : q \le l\hat{q}\} = 1, \;\; \lambda_1^*(x, \sigma) = 0 \text{ otherwise;}$$

that is, a regular seller accepts an offer made by a buyer from a household with x if and only if the implied price of money is no greater than $\hat{q}$.

Next we construct the plan p_x^* for $x \ne 1$ and we only describe $\sigma_x^*(1)$ and $\lambda_x^*(1, \sigma)$ here. Let

$$\sigma_x^*(1)\{(l(x)\hat{q}, l(x))\} = 1; \tag{29}$$

that is, a buyer from the household with x offers $l(x)$ units of money for exchange of $l(x)\hat{q}$ units of good to a regular seller. Also, for $(1, \sigma)$ with $\sigma\{(q, l) : l \le 1\} = 1$, let

$$\lambda_x^*(1, \sigma) = 1 \text{ if } \sigma\{(q, l) : q \le l\hat{q}, \;\, l \le \rho(x)\} = 1, \;\; \lambda_x^*(1, \sigma) = 0 \text{ otherwise;}$$

that is, a seller from the household with x accepts an offer from a regular buyer if and only if the implied price of money is no greater than $\hat{q}$ and the transfer of money does not exceed $\rho(x)$.

Next we verify that such v, p_1^* and p_x^* constitute an equilibrium. Evidently, p_1^* satisfies (21). By (27) and $\beta u'(\hat{q}) \ge 1$, $v'(1) = u'(\hat{q})\hat{q}$. This and $u(\hat{q})/\hat{q} > u'(\hat{q})$ imply that σ_1^*, λ_x^* and v satisfy (24). By (25) and $\beta u'(\hat{q}) \ge 1$, σ_1^*, λ_x^* and v satisfy (25). To show p_x^*, p_1^* and v satisfy (20), first notice that $v(x) = f(p_x^*, p_1^*)$, and so it suffices to show $f(p_x^*, p_1^*) \ge f(p_x, p_1^*)$. But this follows from that $(l(x), \rho(x))$ is the optimal solution to the maximization problem in (27). ■

Because of commitment, buyer does not have all the bargaining power in a meeting even though he makes a take-it-or-leave-it offer. This is shown in

Proposition 3 by the relationship between $\hat{q}$ and $\beta v'(1)$ in the large household model, and the relationship between $\hat{q}$ and $\beta[v(1) - v(0)]$ in the finite household model.

Commitment also permits a variety of linear pricing. This can be seen from (28) and (29) in the large household model, and from (43) and (44) in the finite household model. When there is no commitment, linear pricing may occur in the large household model, but θ must be unity and the price of money must be $\bar{q}$.

It is straightforward to establish a version of Proposition 3 (i) in models where a household consists of one agent such as the one studied by Wallace and Zhu [8, section 2]. Also, Proposition 3 (i) seems applicable for $n > 1$, while the proof may be more complicate.

5 The concluding remarks

In the no-commitment formulation, the roles of quasi-linearity of the individual agent's payoff functions and the bound Z in the large household model are similar to their roles in the Lagos-Wright model [2]. In [2], agents trade in a centralized market after random matching, and preferences over the good in the centralized market are assumed to be quasi-linear; however, for an internal solution in the centralized market, the agent must enter the centralized market with money holdings that are not too large. In that case, the assumed quasi-linear preferences imply that the value function for the agent's end-of-match money holdings is affine, and that, in turn, implies that in a pairwise meeting, the buyer and seller payoff functions are quasi linear, linear in end-of-match money holdings. Moreover, those functions have the same linear coefficient, provided that the sum of the buyer and seller money holdings is consistent with an internal solution in the centralized market.

The way to describe the household's problem in the no-commitment approach is applicable to describe the household's problem in the labor search model of Merz [3], and it can also be adapted to describe the large firm's decision problem in the labor search literature, which has been dealt with under the assumption that the large firm takes the prevailing wage as given while it still bargains with each worker for each vacancy (see Pissarides [7, Ch 3.1]).

A natural way to refine the commitment equilibrium is by subgame perfection. After all, this is consistent with the standard adopted by most search models in which a household consists of one agent. Of course, doing so leads to the no-commitment equilibrium.

Appendix

Completion of the proof of Proposition 1

Proof. Here we proof part (ii). Let $\bar{\theta} \equiv (1-\theta)/\theta$. By (3) and $l_b(1) = 1$,

$$\begin{aligned} & u[q_b(x)] + \beta v[x - l_b(x) + l_s(x)] - \beta v[x + l_s(x)] \\ = \ & \bar{\theta} u'[q_b(x)]\{-q_b(x) + \beta v[l_b(x)] - \beta v(0)\}. \end{aligned} \tag{30}$$

By (4) and $l_s(1) = 1$,

$$\begin{aligned} & u[q_s(x)] + \beta v[2 - l_s(x)] - \beta v(2) \\ = \ & \bar{\theta} u'[q_s(x)]\{-q_s(x) + \beta v[x - l_b(x) + l_s(x)] - \beta v[x - l_b(x)]\}. \end{aligned} \tag{31}$$

Setting $x = 1$ in (30) and by $l_b(1) = 1$, we have

$$u(\hat{q}) + \beta v(1) - \beta v(2) = \bar{\theta} u'(\hat{q})\{-\hat{q} + \beta v(1) - \beta v(0)\}, \tag{32}$$

where $\hat{q} = q_b(1) = q_s(1)$. By (3) and $l_b(1) = l_s(1) = 1$,

$$u'(\hat{q}) v'_-(1) \geq v'_+(1), \tag{33}$$

where $v'_+(z)$ and $v'_-(z)$ are the right and left derivatives of v at z, respectively.

Now fix $x < 1$. Because v is concave and strictly increasing over $[0, 1]$, and because u is strictly concave, (30) and (32) imply $q_b(x) < \hat{q}$, and (31) and (32) imply $q_s(x) \leq \hat{q}$ and strict if $l_s(x) < 1$. Now we use those inequalities to show $l_b(x) = x$ and $l_s(x) = 1$. If $l_b(x) < x$, then by (3), we have

$$u'[q_b(x)] v'_+[l_b(x)] \leq v'_-[x - l_b(x) + l_s(x)]. \tag{34}$$

Because $q_b(x) < \hat{q}$, because v is concave and strictly increasing over $[0, 1]$, and because u is strictly concave, (34) and (33) imply $x - l_b(x) + l_s(x) < 1$, which in turn implies $l_s(x) < 1$, and then by (4), we have

$$u'[q_s(x)] v'_+[x - l_b(x) + l_s(x)] \leq v'_-[2 - l_s(x)]. \tag{35}$$

But because v is concave and strictly increasing over $[0, 1]$, and because u is strictly concave, (35) and (33) imply $q_s(x) > \hat{q}$, a contradiction. So $l_b(x) = x$. Now if $l_s(x) < 1$, then by (4), we have (35). But because $q_s(x) < \hat{q}$, because v

is concave and strictly increasing over $[0,1]$, and because u is strictly concave, (35) and (33) imply $x - l_b(x) + l_s(x) > 1$, which in turn implies $l_b(x) < x$, a contradiction. So $l_s(x) = 1$.

By $l_b(x) = x$ and $l_s(x) = 1$, (31) and (32) imply $q_s(x) = \hat{q}$. It follows that

$$v(x) = u[q_b(x)] - \hat{q} + \beta v(1) \text{ for } x \leq 1. \tag{36}$$

By $l_b(x) = x$ and $l_s(x) = 1$, (30) implies

$$u[q_b(x)] + \beta[v(1) - v(1+x)] = \bar{\theta} u'[q_b(x)]\{-q_b(x) + \beta[v(x) - v(0)]\}.$$

This and $v(1) \leq v(1+x)$ and (36) and $q_b(0) = 0$ imply

$$u[q_b(x)] \geq \bar{\theta} u'[q_b(x)]\{\beta u[q_b(x)] - q_b(x)\}, \tag{37}$$

By continuity of v and (3), $q_b(x) \to 0$ as $x \to 0$. But by $u'(0) = \infty$, (37) can not hold for sufficiently small x. ■

Completion of the proof of Proposition 2

Proof. Here we prove of part (ii). As in the proof of part (i), we first construct $(Z, v, q_b, l_b, q_s, l_s)$. Let $\bar{\theta} \equiv (1-\theta)/\theta$. Let ω satisfy

$$2\omega + \sqrt{4\omega^2 + 4\bar{\theta}(2+\bar{\theta})\omega} = (2+\bar{\theta})[\beta^{-1}\sqrt{4\omega^2 + 4\bar{\theta}(2+\bar{\theta})\omega} - \bar{\theta}]. \tag{38}$$

Then let $q(x)$ be the unique solution q to the equation

$$(\sqrt{q} - \omega x)2\sqrt{q} = \bar{\theta}(-q + \omega x), \tag{39}$$

so that

$$2(2+\bar{\theta})\sqrt{q(x)} = 2\omega x + \sqrt{(2\omega x)^2 + 4\bar{\theta}(2+\bar{\theta})\omega x}. \tag{40}$$

Then let Z satisfy $2\sqrt{q(Z)} = 1$. Then let (v, q_b, l_b, q_s, l_s) on $[0, Z]$ be defined by

$$\begin{aligned} v(x) &= \sqrt{q(x)} - q(1) + \beta v(1), \\ (q_b(x), l_b(x)) &= (q(x), x), \\ (q_s(x), l_s(x)) &= (q(1), 1). \end{aligned} \tag{41}$$

(So the requirement of Q is $Q \geq q(Z)$.) Notice that (38) and (39) imply $\beta v'(1) = \omega$.

Now we verify that $(Z, v, q_b, l_b, q_s, l_s)$ is an equilibrium. First, functions (v, q_b, l_b, q_s, l_s) as constructed satisfy (11) and the degeneracy condition, and the function v as constructed is strictly increasing, strictly concave and differentiable. Next, for any $x \leq Z$, by $\beta v'(1) = \omega$ and (40), and given functions (q_b, l_b, q_s, l_s) as constructed, $(q(x), x)$ is solves the problem in (9) because (39) holds with $q = q(x)$ (the first order condition of q) and $2\sqrt{q(x)} \leq 1$ (the first order condition of l), and $(q(1), 1)$ solves the problem in (10) because (39) holds with $q = q(1)$ and $2\sqrt{q(1)} \leq 1$. This completes the proof. ■

Completion of the proof of Proposition 3

Proof. Here we prove part (i). As in the proof of part (ii), fix $\hat{q}$ and we first define the value function v as

$$v(x) = u(x\hat{q}) - \hat{q} + \beta v(1) \text{ if } x \leq 1 \text{ and } v(x) = v(1) \text{ if } x > 1. \tag{42}$$

Unlike in the proof of part (ii), we use this satiated value function to deal with some technical issues in the finite model.

Next we construct the plan p_1^*. Let

$$\sigma_1^*(x)\{(\hat{q}, 1)\} = 1 \text{ all } x, \tag{43}$$

and for (x, σ) with $\sigma\{(q, l) : l \leq x\} = 1$, let

$$\lambda_1^*(x, \sigma) = 1 \text{ if } \sigma_x\{(q, l) : q \leq \dot{x}\hat{q},\ l = x\} = 1,\ \lambda_1^*(x, \sigma) = 0 \text{ otherwise},$$

where $\dot{x} \equiv \min\{x, 1\}$.

Next we construct the plan p_x^* for $x \neq 1$, and we only describe $\sigma_x^*(1)$ and $\lambda_x^*(1, \sigma)$. Let

$$\sigma_x^*(1)\{(\dot{x}\hat{q}, x)\} = 1, \tag{44}$$

and for $(1, \sigma)$ with $\sigma\{(q, l) : l \leq 1\} = 1$, let

$$\lambda_x^*(1, \sigma) = 1 \text{ if } \sigma\{(q, l) : q \leq l\hat{q}\} = 1,\ \lambda_x^*(1, \sigma) = 0 \text{ otherwise}.$$

Next we verify that the above v, p_1^* and p_x^* constitutes an equilibrium. Evidently, p_1^* satisfies (21). By (42),

$$v(1) - v(0) = u(\hat{q}). \tag{45}$$

By (45), v, p_1^* and p_x^* satisfy (22). Notice that $\beta u'(\hat{q}) \geq 1$ implies $\beta u(\hat{q}) > \hat{q}$. This and (45) imply v, p_1^* and p_x^* satisfy (23). By $v = f(p_x^*, p_1^*)$, to show v, p_1^* and p satisfy (20), it suffices to show $f(p_x^*, p_1^*) \geq f(p_x, p_1^*)$ for p_x such that

$$\sigma_x(1)\{(\dot{x}\hat{q}, x)\} = \mu \text{ and } \sigma_x(1)\{(0, 0)\} = 1 - \mu,$$

and $\lambda_x[1, \sigma_1^*(x)] = \lambda$. In turn, it suffices to show that

$$\begin{aligned}(1,1) \in \arg\max_{0\leq\mu,\lambda\leq1} & \mu\lambda[u(\dot{x}\hat{q}) - \hat{q} + \beta v(1)] + (1-\mu)\lambda[-\hat{q} + \beta v(1+x)] \\ & +\mu(1-\lambda)[u(\dot{x}\hat{q}) + \beta v(0)] + (1-\mu)(1-\lambda)\beta v(x).\end{aligned}$$

Clearly $\mu = 1$ is optimal when $x = 0$. When $x > 0$, if $\mu < 1$ is optimal, then $u(\dot{x}\hat{q}) \leq \beta[v(x) - v(0)]$, but this contradicts to (45), and so $\mu = 1$ is optimal. Now if $\lambda < 1$ is optimal, then $\beta[v(1) - v(0)] \leq \hat{q}$, but this and $\beta u(\hat{q}) > \hat{q}$ contradicts to (45), and so $\lambda = 1$ is optimal. ■

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