Optimal Stabilization Policy with Flexible Prices*

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Abstract

We construct a dynamic stochastic general equilibrium model to study optimal monetary stabilization policy. Unlike existing New Keynesian models we assume prices are fully flexible. The critical element for effective stabilization policy is the central bank's commitment to a price path in order to control inflation expectations. Policy is optimal because the central bank maximizes the lifetime expected utility of the representative agent subject to being a competitive equilibrium. We show that away from the Friedman rule there is welfare improving role for stabilization policy and it involves smoothing the nominal interest rate.

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1 Introduction

In the last decade there has been a tremendous amount of research focusing on the question of how to conduct optimal monetary policy. From the New Keynesian literature a key feature of "good" monetary policy is commitment to a price or inflation target in order to control inflation expectations. The choice of the inflation target is tied to the question of what is the optimal trend inflation rate while controlling inflation expectations allows the central bank to implement a more effective stabilization response to aggregate shocks.

A defining element of New Keynesian models is some form of nominal rigidity such as price or wage stickiness. Without this nominal rigidity there is no role for stabilization policy since money is both neutral and superneutral. From this literature one is tempted to conclude that price stickiness is a necessary element to generate a role for stabilization policy. In this paper we show that, in fact, the critical element for effective stabilization policy is the central bank's commitment to a price path.

To show the importance of commitment and the unimportance of price stickiness for effective stabilization policy, we construct a dynamic stochastic general equilibrium model where money is essential for trade and prices are fully flexible.[1] The basic framework is the Berentsen, Camera and Waller (2004) model of money and credit that builds on Lagos-Wright (2005). We introduce aggregate shocks to preferences and technology to examine the optimal stabilization response of a monetary authority. The existence of the credit sector generates a nominal interest rate that the monetary authority manipulates by changing the aggregate money stock. Policy is optimal because the monetary authority maximizes the lifetime expected utility of the representative agent subject to the allocation being a competitive equilibrium.

[1] By essential we mean that the use of money expands the set of allocations (Kocherlakota (1998) and Wallace (2001)).

Our main results are as follows. In the absence of any constraints on the long-run inflation rate, the optimal long-run policy is to implement the Friedman rule (a zero nominal interest rate). Under this policy the allocation is first-best for all shocks so there is no welfare improving role for stabilization policy. However, if the central bank is constrained to inflate at a rate above the Friedman rule, stabilization policy has important consequences for aggregate activity and welfare. In general, it involves smoothing the nominal interest rate. An interesting aspect of our result is that our model displays a liquidity effect similar to the ones in Lucas (1990) or in Fuerst (1992) and a standard inflation expectation effect. The working of the liquidity effect requires that the central bank controls inflation expectations. Without such a control injections in the first market simply change price expectations as predicted by the Fisher equation.

The paper proceeds as follows. In Section 2 we describe the environment. In Section 3 the agents optimization problems are presented. Section 4 examines the optimal monetary policy and Section 5 derives the equilibrium in the absence of central bank intervention to illustrate the gains from optimal stabilization. Section 6 concludes.

2 The Environment

The basic environment is that of Berentsen, Camera and Waller (2004). Time is discrete. In each period there are two perfectly competitive markets that open sequentially. There is a $[0, 1]$ continuum of infinitely-lived agents and one perishable good produced and consumed by all agents.

At the beginning of the first market agents receive a preference shock such that they either consume, produce or neither. With probability n an agent consumes, with probability s he produces and with probability $1 - n - s$ he is inactive in the goods market. We refer to consumers as buyers and producers as sellers.

Buyers get utility $\varepsilon u(q)$ from $q > 0$ consumption in the first market, where ε is a preference parameter and $u'(q) > 0$, $u''(q) < 0$, $u'(0) = +\infty$ and $u'(\infty) = 0$. Furthermore, we impose that the elasticity of utility $e(q) = \frac{qu'(q)}{u(q)}$ is bounded. Producers incur utility cost $c(q)/\alpha$ from producing q units of output where α is a measure of productivity. We assume that $c'(q) > 0$, $c''(q) \geq 0$ and $c'(0) = 0$.

Following Lagos and Wright (2005) we assume that in the second market all agents consume and produce, getting utility $U(x)$ from x consumption, with $U'(x) > 0$, $U'(0) = \infty$, $U'(+\infty) = 0$ and $U''(x) \leq 0$.[2] Agents can produce one unit of x with one unit of labor h. Production of x units of output generates disutility h. The discount factor across dates is $\beta \in (0, 1)$.

To motivate a role for fiat money, we assume that all goods trades are anonymous.[3] In particular, trading histories of agents are private information. Consequently, sellers require immediate compensation so buyers pay with money. There is also no public communication of individual trading outcomes (public memory), which eliminates the use of trigger strategies to support gift-giving equilibria.

2.1 Banking

At the beginning of a period, after the idiosyncratic shocks are realized, sellers and inactive agents hold idle money balances while buyers may have a desire for more money. Thus, we have two sides to a non-existent credit market where some agents would like to lend at positive interest and some would pay this premium to consume today. This missing credit market creates a role for financial intermediation between borrowers and lenders.

As in Berentsen, Camera and Waller (2004) we assume that a perfectly competitive banking sector creates this market. Banks accept nominal deposits and

[2] The difference in preferences over the good sold in the last market allows us to impose technical conditions such that the distribution of money holdings is degenerate at the beginning of a period. [3] As in Aliprantis, Camera and Puzzello, (2004) agents are not able to to lend or borrow from each other or to share information across periods. Consequently, all goods trades must be immediately settled with money.

pay the nominal interest rate i_d and make nominal loans at nominal rate i. Since the banking sector is perfectly competitive, banks take these rates as given. There are no operating costs so zero profits imply $i_d = i$. Banks have a record-keeping technology over financial transactions.

In the first market the banking sector opens and agents borrow and deposit after observing the shocks (see Figure 1). Then, they trade. In the second market all financial claims are settled. This essentially means that loans and deposits cannot be rolled over. Consequently, all financial contracts are one-period contracts. In all models with credit default is a serious issue. However, since we focus on stabilization issues, we assume that default is not feasible.[4]

2.2 Aggregate shocks

To study the optimal response to aggregate shocks we assume that n, s, α and ε are stochastic. The random variable n has support $[\underline{n}, \overline{n}] \in (0, 1/2]$, s has support $[\underline{s}, \overline{s}] \in (0, 1/2]$, α has support $[\underline{\alpha}, \overline{\alpha}]$, $0 < \underline{\alpha} < \overline{\alpha} < \infty$, and ε has support $[\underline{\varepsilon}, \overline{\varepsilon}]$, $0 < \underline{\varepsilon} < \overline{\varepsilon} < \infty$. Let $\omega = (n, s, \alpha, \varepsilon) \in \Omega$ be the aggregate state in market 1, where $\Omega = [\underline{n}, \overline{n}] \times [\underline{s}, \overline{s}] \times [\underline{\alpha}, \overline{\alpha}] \times [\underline{\varepsilon}, \overline{\varepsilon}]$ is a closed and compact subset on $\boldsymbol{R}_+^4$. The shocks are serially uncorrelated. Let $f(\omega)$ denote the density function of ω.

Shocks to n and ε are aggregate demand shocks, while shocks to s and α are aggregate supply shocks. We call shocks to ε and α intensive margin shocks since they change the desired consumption of each buyer and the productivity of each seller, respectively, without affecting the number of buyers or sellers. In contrast, shocks to n and s affect the number of buyers and sellers.

[4] In Berentsen, Camera and Waller (2004) we derive the equilibrium when the only punishment for default is exclusion from the banking system in all future periods.

2.3 Monetary Policy

We assume a central bank exists that controls the supply of fiat currency. The growth rate of the money stock is given by $M_t = \gamma M_{t-1}$ where M_t denotes the per capita money stock in market 2 in period t. The net change in the aggregate money stock is $\tau M_{t-1} = (\gamma - 1)M_{t-1}$ where these deterministic injections take place in the first market. If $\gamma > 1$, agents receive lump-sum transfers of money. If $\gamma < 1$, we assume the central bank has the authority to levy lump-sum taxes in the form of currency to extract cash from the economy. For notational ease variables corresponding to the next period are indexed by $+1$, and variables corresponding to the previous period are indexed by -1.

Monetary policy has a long and short-run component. The long-run component focuses on the trend inflation rate which in our model is captured by the choice of γ. The short-run component is concerned with the stabilization response to aggregate shocks. Let $\tau_1(\omega) M_{-1}$ and $\tau_2(\omega) M_{-1}$ denote state contingent transfers in market 1 and 2 respectively with $\tau_1(\omega) + \tau_2(\omega) = 0$. Since the growth rate of the money supply is still deterministic, i.e., τM_{-1} is not state dependent, changes in $\tau_1(\omega)$ allows us to affect the money stock for stabilization purposes without affecting the trend money growth rate.[5] With $\tau_2(\omega) = -\tau_1(\omega)$ we are implicitly assuming the central bank is committed to a given path of the money stock in market 2. As we show later, this commitment allows the central bank to control price expectations in market 2.

[5] Lucas (1990) employs a similar process for the money supply so that changes in nominal interest rates result purely from liquidity effects and not changes in expected inflation.

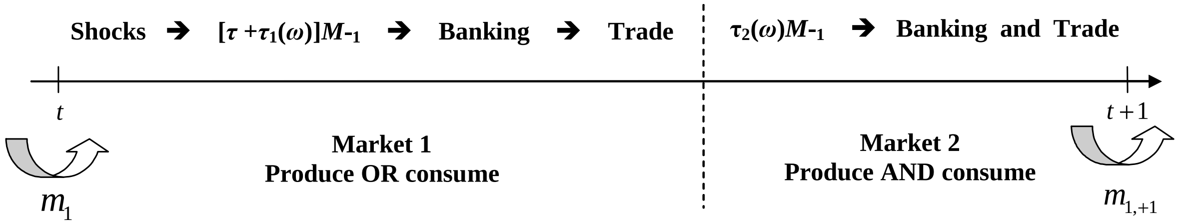

Figure 1: Sequence of events.

The precise sequence of action after the shocks are observed is as follows. First, the monetary injection $[\tau + \tau_1(\omega)]\, M_{-1}$ occurs. Second, sellers and inactive agents deposit their idle cash and buyers borrow money from the banking sector. Agents then move on to the goods market and trade. In the second market the goods market and banking sector open where all financial claims are settled. The central bank takes out respectively injects $\tau_2(\omega) = -\tau_1(\omega)\, M_{-1}$ units of money through lump-sum taxes respectively lump-sum subsidies.

3 Agents' Choices and Value functions

In period t, let ϕ be the real price of money in the second market. We study equilibria where end-of-period real money balances are time and state invariant

$$\phi M = \phi_{-1} M_{-1} \equiv z, \qquad \omega \in \Omega. \tag{1}$$

We refer to it as a stationary equilibrium. This implies that ϕ is not state dependent and so $\phi_{-1}/\phi = M/M_{-1} = \gamma$. This effectively means that the central bank commits to a price path $\phi = \frac{1}{\gamma}\phi_{-1}$ in market 2.

Consider a stationary equilibrium. Let $V(m_1)$ denote the expected value from trading in market 1 with m_1 money balances. Let $W(m_2, l, d)$ denote the expected value from entering the second market with m_2 units of money, l loans, and d deposits when the aggregate state is ω. Note that all quantities and prices are functions of the aggregate state ω, i.e., $m_2 = m_2(\omega)$, $l = l(\omega)$, and $d = d(\omega)$. We suppress this dependence for notational simplicity. In what follows, we look at a representative period t and work backwards from the second to the first market to examine the agents' choices.

3.1 The second market

In the second market agents consume x, produce h, and adjust their money balances taking into account cash payments or receipts from the bank. Loans are repaid by borrowers and bank redeem deposits. If an agent has borrowed l units of money, then he pays $(1+i)\,l$ units of money. If he has deposited d units of money, he receives $(1+i)\,d$. The representative agent's program is

$$W(m_2, l, d) = \max_{x,h,m_{1,+1}} \left[U(x) - h + \beta V(m_{1,+1})\right] \tag{2}$$

$$\text{s.t.} \quad x + \phi m_{1,+1} = h + \phi(m_2 + \tau_2 M_{-1}) + \phi(1+i)\,d - \phi(1+i)\,l$$

where $m_{1,+1}$ is the money taken into period $t+1$.

Rewriting the budget constraint in terms of h and substituting into (2) yields

$$\begin{aligned} W(m_2, l, d) &= \phi\left[m_2 + \tau_2 M_{-1} - (1+i)\,l + (1+i)\,d\right] \\ &+ \max_{x,m_{1,+1}} \left[U(x) - x - \phi m_{1,+1} + \beta V(m_{1,+1})\right]. \end{aligned}$$

The first-order conditions are $U'(x) = 1$ and

$$-\phi_{-1} + \beta V'(m_1) = 0 \tag{3}$$

where the first-order condition for money has been lagged one period. Thus, $V'(m_1)$ is the marginal value of taking an additional unit of money into the first market open in period t. Since the marginal disutility of working is one, $-\phi_{-1}$ is the utility cost of acquiring one unit of money in the second market of period $t-1$.

The envelope conditions are

$$W_m = \phi \tag{4}$$

$$W_d = -W_l = \phi(1+i)\,. \tag{5}$$

As in Lagos-Wright (2005) the value function is linear in wealth. The implication is that all agents enter the following period with the same amount of money.

3.2 The first market

Let q_b and q_s respectively denote the quantities consumed by a buyer and produced by a seller trading in market 1. Let p be the nominal price of goods in market 1. It is straightforward to show that buyers will never deposit funds in the bank and sellers and inactive agents will never take out loans. It is straightforward to show that it is optimal for sellers and inactive agents to deposit all their money balances if $i > 0$. If $i = 0$, they are indifferent since they earn no money. In what follows we assume that they also deposit their money when $i = 0$. Let l denote loans taken out by buyers and d deposits of sellers and inactive agents.

An agent who has m_1 money at the opening of the first market has expected lifetime utility

$$\begin{aligned} V(m_1) = \int_\Omega & \{n\left[\varepsilon u(q_b) + W(m_1 + (\tau+\tau_1)M_{-1} + l - pq_b, l, 0)\right] \\ & + s\left[-c(q_s)/\alpha + W(m_1 + (\tau+\tau_1)M_{-1} - d + pq_s, 0, d)\right] \\ & + (1-n-s)W(m_1 + (\tau+\tau_1)M_{-1} - d, 0, d)\} f(\omega)\, d\omega \end{aligned} \tag{6}$$

where pq_b is the amount of money spent as a buyer and pq_s the money received as a seller.

A seller's problem is $\max_{q_s}\left[-c(q_s)/\alpha + W(pq_s, 0, d)\right]$. Using (4), the first-order conditions are

$$c'(q_s) = \alpha p\phi, \qquad \omega \in \Omega. \tag{7}$$

Note that sellers cannot deposit receipts of cash obtained from selling output.[6]

[6] This is a form of limited participation in financial markets.

If an agent is a buyer in the first market, his problem is:

$$\max_{q_b,l} \left[\varepsilon u\left(q_b\right) + W\left(m_1 + \left(\tau + \tau_1\right) M_{-1} + l - pq_b, l, 0\right)\right]$$

$$\text{s.t.} \quad pq_b \leq m_1 + \left(\tau + \tau_1\right) M_{-1} + l$$

Notice that buyers can spend more cash than what they bring into the first market since they can borrow cash to supplement their money holdings at the cost of the nominal interest rate. Using (4) the buyer's first-order conditions can be written as

$$\varepsilon u'\left(q_b\right) - p\phi - p\lambda = 0, \qquad \omega \in \Omega, \tag{8}$$

$$-i\phi + \lambda = 0, \qquad \omega \in \Omega, \tag{9}$$

$$\lambda\left[m_1 + \left(\tau + \tau_1\right) M_{-1} + l - pq_b\right] = 0, \qquad \omega \in \Omega, \tag{10}$$

where $\lambda = \lambda\left(\omega\right)$ are the multipliers of the buyer's budget constraints.

Equations (7), (8) and (9) imply that

$$\alpha\varepsilon u'\left(q_b\right) = c'\left(q_s\right)\left(1 + i\right), \qquad \omega \in \Omega \tag{11}$$

Define the set of states where the constraint is nonbinding as $\Omega_0 = \{\omega \in \Omega \,|\, \lambda\left(\omega\right) = 0\}$. Accordingly, define $\Omega_1 = \{\omega \in \Omega \,|\, \lambda\left(\omega\right) > 0\}$. If the constraint is not binding, (9) implies $i = 0$ and so (11) reduces to

$$\alpha\varepsilon u'\left(q_b\right) = c'\left(q_s\right), \qquad \omega \in \Omega_0.$$

Hence trades are efficient.[7]

If the constraint is binding, then (9) implies $i > 0$ so we have $\alpha\varepsilon u'\left(q_b\right) > c'\left(q_s\right)$ which means trades are inefficient. The buyer spends all of his money, $pq_b = m_1 + \left(\tau + \tau_1\right) M_{-1} + l$, and consumes $q_b = \frac{m_1 + (\tau + \tau_1) M_{-1} + l}{p}$.

[7] With n buyers and s sellers, the first-best quantities are obtained by maximizing $n\varepsilon u\left(q_b\right) - \left(s/\alpha\right) c\left(q_s\right)$ s.t. $nq_b = sq_s$ for all $\omega \in \Omega$. The first-order conditions for q_b are $\alpha\varepsilon u'\left(q_b\right) = c'\left(q_s\right)$ for all $\omega \in \Omega$.

We show in the proof of Lemma 1 in the appendix that the marginal value of money is given by

$$V'(m_1) = \int_{\Omega} \phi \left\{ n\alpha\varepsilon u'(q_b)/c'(q_s) + (1-n)(1+i) \right\} f(\omega)\, d\omega. \tag{12}$$

Note that banks increase the marginal value of money because agents can earn interest on idle money as opposed to the non-bank case where $i = 0$.

Then, using (11) we can write $V'(m_1)$ as follows

$$V'(m_1) = \int_{\Omega} \left[\phi\alpha\varepsilon u'(q_b)/c'(q_s) \right] f(\omega)\, d\omega. \tag{13}$$

From this equation one can see that the value function is concave in m_1.

3.3 Equilibrium

We now derive the symmetric monetary steady-state equilibrium. Since in a symmetric equilibrium all sellers produce the same quantity, market clearing in the good market implies

$$q(\omega) \equiv q_b(\omega) = (s/n)\, q_s(\omega), \qquad \omega \in \Omega. \tag{14}$$

Since in a symmetric equilibrium all borrowers take out the same loan, $l(\omega)$, and depositors deposit the same amount $d(\omega)$ market clearing in the credit market implies

$$l(\omega) = \frac{1-n}{n} d(\omega), \qquad \omega \in \Omega. \tag{15}$$

Since in equilibrium $m_1 = M_{-1}$ and $d(\omega) = [1 + \tau + \tau_1(\omega)]\, M_{-1}$ the budget constraint of the buyer $pq(\omega) \leq [1 + \tau + \tau_1(\omega)]\, M_{-1} + l(\omega)$ has to satisfy

$$q(\omega) \leq \frac{[1 + \tau + \tau_1(\omega)]\, M_{-1}}{np}$$

Then from (7), (14) and (15) we get

$$(n/\alpha)\, q(\omega)\, c'\left[(n/s)\, q(\omega)\right] \leq [1 + \tau + \tau_1(\omega)]\, \phi M_{-1}. \tag{16}$$

Since the central bank has committed to a price path for ϕ, changes in $\tau_1(\omega)$ do not affect ϕ. Hence, ϕM_{-1} is constant. It then follows that increases in $\tau_1(\omega)$ raise real balances in market 1. Consequently, state contingent injections are not neutral.

In any monetary equilibrium (16) holds with equality in at least one state. In these states we have

$$(n/\alpha)\, q(\omega)\, c'[(n/s)\, q(\omega)] = \upsilon(\omega)\, z. \tag{17}$$

where $z = \phi M$ is the real stock of money and $\upsilon(\omega) = [1 + \tau + \tau_1(\omega)] / (1 + \tau)$. For any given state ω, $q(\omega)$ is an increasing function of z. Consequently, we have

$$\begin{aligned} q(\omega, z) &< q^*(\omega) \text{ if } \omega \in \Omega_1 \\ &\text{and} \\ q(\omega) &= q^*(\omega) \text{ if } \omega \in \Omega_0 \end{aligned} \tag{18}$$

where the efficient quantity $q^*(\omega)$ solves $\alpha\varepsilon u'[q^*(\omega)] = c'[(n/s)\, q^*(\omega)]$.

Use (3) to eliminate $V'(m_1)$ and (14) to eliminate q_s from (13). Then, multiply the resulting expression by M_{-1} to get

$$\frac{\gamma - \beta}{\beta} = \int_\Omega \left\{ \frac{\alpha\varepsilon u'[q(\omega, z)]}{c'[(n/s)\, q(\omega, z)]} - 1 \right\} f(\omega)\, d\omega. \tag{19}$$

Definition 1 *A symmetric monetary steady-state equilibrium is a z that satisfies (19).*

We define the equilibrium as the value of z that solves (19). The reason is that once the equilibrium stock of money is derived all other endogenous variables can be derived recursively. For example, equations (17) and (18) yield $q(\omega)$ for all $\omega \in \Omega$.

4 Optimal stabilization

We now derive the optimal stabilization policy. We assume that the central bank's objective is to maximize the welfare of the representative agent. It does so by choosing the quantities consumed and produced in each state subject to the constraint that the chosen quantities satisfy the conditions of a competitive equilibrium. The policy is implemented by choosing state contingent injections $\tau_1(\omega)$ and $\tau_2(\omega)$ accordingly.

It is straightforward to show that the expected lifetime utility of the representative agent at the beginning of period t is given by

$$(1-\beta) V(M_{-1}) = U(x) - x + \int_{\Omega} \{n\varepsilon u[q(\omega)] - (s/\alpha) c[(n/s) q(\omega)]\} f(\omega) d\omega$$

It is obvious that $x = x^*$ so all that remains is to choose $q(\omega)$.

The Ramsey problem facing the central bank is

$$\begin{aligned} \underset{q(\omega)}{M\alpha x} \; & \int_{\Omega} \{n\varepsilon u[q(\omega)] - (s/\alpha) c[(n/s) q(\omega)]\} f(\omega) d\omega \qquad (20) \\ \text{s.t. } \frac{\gamma-\beta}{\beta} &= \int_{\Omega} \left\{ \frac{\alpha\varepsilon u'[q(\omega)]}{c'[(n/s) q(\omega)]} - 1 \right\} f(\omega) d\omega \end{aligned}$$

where the constraint facing the central bank is that the quantities choosen must be compatible with a competitive equilibrium.

Proposition 1 *The Friedman rule generates the first-best allocation for all states.*

According to Proposition 1 the Friedman rule $\gamma = \beta$ implements the first-best allocation $q = q^*(\omega)$ in all states. The reason is that at the Friedman rule holding money is costless so agents can perfectly self-insure against consumption risks. Consequently, there are no welfare gains from stabilization policies.[8] The only friction in our model is the cost of holding money across periods. The Friedman rule eliminates this friction.

[8] Ireland (1996) derives a similar result in a model with nominal price stickiness. He finds that at the Friedman rule there is no gain from stabilizing aggregate demand shocks.

The message of Proposition 1 is that the central bank should follow a policy that is as close as possible to the Friedman rule. Suppose, however, that for some reason it is constrained to set $\gamma = \bar{\gamma} > \beta$.[9] In this case we have the following result.

Proposition 2 *Away from the Friedman rule, the central bank chooses $q < q^*(\omega)$ in all states.*

Surprisingly, the central bank never chooses $q = q^*(\omega)$. The reason is that the central bank wants to smooth consumption across states. Intuitively, consider two states $\omega, \omega' \in \Omega$ with $q = q^*(\omega)$ and $q' < q^*(\omega')$. Then, the first-order loss from lowering q is zero while there is a first-order gain from increasing q'. Note that under the optimal policy buyers are cash constrained in all states.

The interest rate associated with the optimal policy is

$$i(\omega) = \frac{\alpha\varepsilon u'[q(\omega)]}{c'[(n/s)\,q(\omega)]} - 1 > 0, \qquad \omega \in \Omega. \tag{21}$$

Thus, although $i = 0$ is the first-best policy for all states, unless it can be done for all states, it is optimal to never set $i = 0$. Hence, zero nominal interest rates should be an all-or-nothing policy. An interesting implication of the optimal policy is that the central bank is essentially providing an elastic supply of currency – when demand for liquidity is high, it provides additional currency and withdraws it when the demand for liquidity is low.

Suppose that the central bank cannot commit to undo the state contingent injections of the first market. Suppose in particular that there are no injections in the second market, i.e., $\tau_2(\omega) = 0$. Nevertheless the central bank reacts to the aggregate shocks in the first market by choosing $\tau_1(\omega)$ optimally.

Proposition 3 *Assume that $\tau_2(\omega) = 0$ for all $\omega \in \Omega$. Then, changes in $\tau_1(\omega)$ have no real effects and any stabilization policy is ineffective.*

[9] One reason why a central bank might be constrained from implementing the Friedman rule are seigniorage needs which require $\bar{\gamma} > 1$. Since our focus is not government financing we don't model seigniorage explicitly.

According to Proposition 3, if the central bank cannot commit to undo the state contingent injections of the first market stabilization policy is ineffective. The price of goods in market 1 changes proportionately with changes in $\tau_1(\omega)$. The real money holdings of the buyers are unaffected and so consumption in market 1 does not react to changes in $\tau_1(\omega)$.

An interesting aspect of this result is that our model displays a liquidity effect similar to the ones in Lucas (1990) or in Fuerst (1992) and a standard inflation expectation effect. The working of the liquidity effect requires that the central bank controls inflation expectations. Without such a control injections in the first market simply change price expectations as predicted by the Fisher equation.

Liquidity effect To see the importance of controlling price expectations for the liquidity effect, assume for simplicity that marginal cost is constant, i.e., $c'[(n/s)q(\omega)] = 1$. Then rewrite (17) and (21) to get

$$\begin{aligned} q(\omega) &= (\alpha/n)[1+\tau+\tau_1(\omega)]\phi M_{-1} \text{ and} \\ i(\omega) &= \alpha\varepsilon u'[q(\omega)] - 1 > 0, \qquad \omega \in \Omega. \end{aligned}$$

Since the central bank has committed to a price path for ϕ, changes in $\tau_1(\omega)$ do not affect ϕ in the first equation. Hence, ϕM_{-1} is constant. It then follows that increasing $\tau_1(\omega)$ raises $q(\omega)$. From the second equation it is clear that when $q(\omega)$ increases $i(\omega)$ decreases. The intuition is that increasing $q(\omega)$ reduces the marginal utility of consumption so the demand for loans decreases which combined with the increase in the supply of deposits pushes the nominal interest rate down.

Inflation expectation effect To see the inflation expectation effect, substitute (21) into the constraint of the central bank problem to get

$$\frac{\gamma-\beta}{\beta} = \int_{\Omega} i(\omega)\, dF(\omega).$$

Evidently, an increase in γ increases the expected nominal interest rate. From the proof of Proposition 3 we know that changing $\tau_1(\omega)$ with $\tau_2(\omega) = 0$ increases the expected inflation rate and has therefore no real effects. It simply increases the expected price in market 2 $p_2 = 1/\phi$ proportional to the increase in $\tau_1(\omega)$ such that $[1 + \tau + \tau_1(\omega)]\,\phi M_{-1}$ remains constant. Such a policy therefore has no real effects.

5 The inefficiency of a passive policy

What are the inefficiencies arising from a passive policy? In order to study this question we now derive the allocation when the central bank follows a policy where the injections are not state dependent, i.e., $\tau_1(\omega) = \tau_2(\omega) = 0$, and compare it to the central bank's optimal allocation. We also analyze each shock separately to understand their individual effects on the equilibrium allocation.

5.1 Extensive margin demand shocks

For the analysis of shocks to n, we assume that α, ε and s are constant. Note that the optimal quantities solve $\alpha\varepsilon u'[q^*(n)] = c'[(n/s)q^*(n)]$ for all $n \in [\underline{n}, \overline{n}]$ where $q^*(n)$ is strictly decreasing in n.

Proposition 4 *For $\gamma \geq \beta$, a monetary equilibrium exists with $q = q^*(n)$ for all n at $\gamma = \beta$. For $\gamma > \beta$, the equilibrium is unique with $q = q^*(n)$ if $n \leq \tilde{n}$ and $q < q^*(n)$ if $n > \tilde{n}$, where $\tilde{n} \in [0, \overline{n}]$. Moreover, $d\tilde{n}/d\gamma < 0$.*

Not surprisingly, the Friedman rule replicates the first-best allocation since it eliminates the cost of holding money. Away from the Friedman rule, buyers are constrained when there are many borrowers (high n) and are unconstrained when there are many depositors (low n). Since $d\tilde{n}/d\gamma < 0$, the higher is the inflation rate, the larger is the range of shocks where the quantity traded is inefficiently low. Note that for large γ we can have $\tilde{n} \leq \underline{n}$ which implies that $q < q^* (n)$ in all states.

How does this allocation differ from the one obtained by following an active policy? We illustrate the differences in figure 2 for a linear cost function. The thick curve represents equilibrium consumption with a passive policy and the thin curve consumption when the central bank is active.

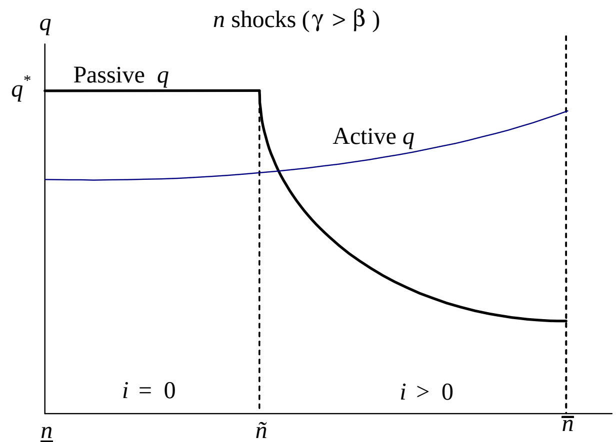

Figure 2: Active vs passive policy.

As shown earlier, with an active policy buyers never consume q^* and equilibrium consumption q is increasing in n. This is just the opposite from what happens when the central bank is passive. With a passive policy, buyers consume q^* in low n states and $q < q^*$ in high n states. Moreover, q is strictly decreasing in n for $n > \tilde{n}$. These differences are also reflected in the nominal interest rates. With an active policy the nominal interest rate is strictly positive in all states and decreasing in n. In contrast, with a passive policy the nominal interest rate is $i = 0$ for $n \leq \tilde{n}$ and $i = \varepsilon \alpha u' (q) - 1 \geq 0$ for $n > \tilde{n}$, and increasing in n.

What is the role of banking sector? With a linear cost function, in the no-banking equilibrium, the quantities consumed are the same across all states since buyers can only spend the cash they bring into market 1, which is independent of the state that is realized. In contrast, when banks exist, idle cash from sellers is deposited and lent back out to buyers. Note that individual consumption is high in low demand states and low in high demand states. The reason is that when n is high demand for loans is high and the supply of deposits is low. This pushes up the nominal interest rate and decreases individual consumption. The interesting aspect of this result is that while financial intermediation raises average consumption across states, it also causes individual consumption to fluctuate.

5.2 Extensive margin supply shocks

For the analysis of shocks to s, we assume that α, ε and n are constant. It then follows that $\omega = s$. Note that the optimal quantities solve $\alpha\varepsilon u' [q^* (s)] = c' [(n/s)q^* (s)]$ for all $s \in [\underline{s}, \overline{s}]$ where $q^* (s)$ is strictly increasing in s.

Proposition 5 *For $\gamma \geq \beta$, a monetary equilibrium exists with $q = q^* (s)$ for all s at $\gamma = \beta$. For $\gamma > \beta$, the equilibrium is unique with $q = q^* (s)$ if $s \geq \tilde{s}$ and $q < q^* (s)$ if $s > \tilde{s}$, where $\tilde{s} \in [0, \overline{s}]$. Moreover, $d\tilde{s}/d\gamma > 0$.*

Again, the Friedman rule replicates the first-best allocation since it eliminates the cost of holding money. Away from the Friedman rule, buyers are constrained when there are few producers and are unconstrained when there are many. Since $d\tilde{s}/d\gamma > 0$, the higher is the inflation rate, the larger is the range of shocks where the quantity traded is inefficiently low. For large γ we can have $\tilde{s} > \overline{s}$ which implies that $q < q^* (s)$ for all s.

Figure 2b illustrates how the allocation under a passive policy differs from the one obtained by following an active policy. The thick curve represents equilibrium consumption with a passive policy and the thin curve consumption when the central bank is active.

Make figure

As shown earlier, with an active policy buyers never consume $q^*(s)$ and equilibrium consumption q is decreasing in s. This is just the opposite from what happens when the central bank is passive. With a passive policy, buyers consume $q^*(s)$ in high s states and $q < q^*(s)$ in low s states. Moreover, q is strictly increasing in s for $s < \tilde{s}$. These differences are also reflected in the nominal interest rates. With an active policy the nominal interest rate is strictly positive in all states and increasing in s. In contrast, with a passive policy the nominal interest rate is $i = 0$ for $s \geq \tilde{s}$ and $i = \varepsilon\alpha u'(q) - 1 \geq 0$ for $s < \tilde{s}$, and decreasing in s.

5.3 Intensive margin demand shocks

To study ε shocks we assume that α, n and s are constant. It then follows that $\omega = \varepsilon$. Note that the optimal quantities solve $\alpha\varepsilon u'[q^*(\varepsilon)] = c'[(n/s)q^*(\varepsilon)]$where $q^*(\varepsilon)$ is strictly increasing in ε.

Proposition 6 *For $\gamma \geq \beta$, a monetary equilibrium exists with $q = q^*(\varepsilon)$ for all ε at $\gamma = \beta$. For $\gamma > \beta$ the equilibrium is unique with $q < q^*(\varepsilon)$ for $\varepsilon > \tilde{\varepsilon}$ and $q = q^*(\varepsilon)$ for $\varepsilon < \tilde{\varepsilon}$, where $\tilde{\varepsilon} \in [0, \bar{\varepsilon}]$. Moreover, $d\tilde{\varepsilon}/d\gamma < 0$.*

Once again the Friedman rule replicates the first-best allocation where there is no role for stabilization. Away from the Friedman rule, buyers are constrained in high marginal utility states but not in low states. If γ is sufficiently high, buyers are constrained in all states.

Note that with a passive policy $dq/d\varepsilon > 0$ for $\varepsilon \leq \tilde{\varepsilon}$ and $dq/d\varepsilon = 0$ for $\varepsilon > \tilde{\varepsilon}$. For $\varepsilon \leq \tilde{\varepsilon}$, buyers have more than enough real balances to buy the efficient quantity. So when ε increases, they simply spend more of their money balances. For $\varepsilon > \tilde{\varepsilon}$, buyers are constrained. So when ε increases, the demand for loans increases but the supply of deposits is unchanged so no additional loans can be made. Thus, the interest rate simply increases to clear the loan market.

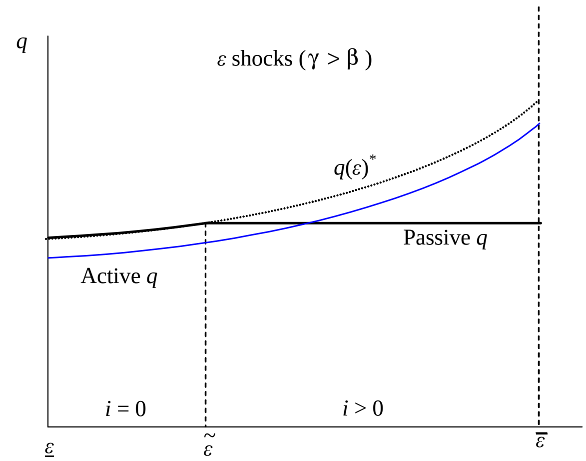

Figure 4: Marginal utility shocks.

Figure 4 illustrates how the allocation resulting from a passive policy differs from the one obtained under an active policy. The dashed curve represents the first-best quantities $q^*(\varepsilon)$.

5.4 Intensive margin supply shocks

To study aggregate productivity shocks, we assume that ε, n and s are constant. It then follows that $\omega = \alpha$. Note that in this case the optimal quantities are $\alpha\varepsilon u'[q^*(\alpha)] = c'[(n/s)q^*(\alpha)]$.

Proposition 7 *For $\gamma \geq \beta$, a monetary equilibrium exists with $q = q^*(\alpha)$ for all α at $\gamma = \beta$. For $\gamma > \beta$ the equilibrium is unique with $q = q^*(\alpha)$ for $\alpha \geq \tilde{\alpha}$ and $q < q^*(\alpha)$ for $\alpha < \tilde{\alpha}$.*

As productivity increases, the marginal cost of producing falls which increases the efficient quantity. It also implies that prices fall. The issue is whether prices fall sufficiently far to raise real balances enough to buy the higher efficient quantity. This in turn depends on preferences – for sufficiently concave preferences it is sufficient, while for preferences sufficiently close to linear, prices do not decrease enough to buy the efficient quantity. Finally note that passive policy, $dq/d\alpha > 0$ for all α.

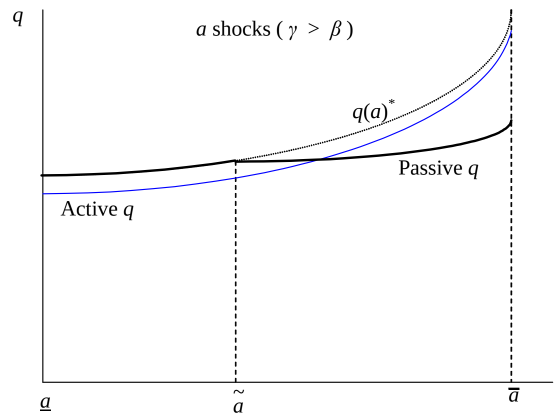

Figure 5. Productivity shocks.

Figure 5 illustrates how the allocation resulting from a passive policy differs from the one obtained under an active policy for productivity shocks. The dashed curve represents the first-best quantities $q^* (\alpha)$.

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Appendix

Lemma 1 *The marginal value of money satisfies (12).*

Proof of Lemma 1. Differentiate (6) with respect to m_1 to get

$$\begin{aligned} V'(m_1) = & \int_\Omega \Big\{ n \left[\varepsilon u'(q_b) \tfrac{\partial q_b}{\partial m_1} + W_m \left(1 - p\tfrac{\partial q_b}{\partial m_1} + \tfrac{\partial l}{\partial m_1}\right) + W_l \tfrac{\partial l}{\partial m_1} \right] \\ & + s\left[-\tfrac{c'(q_s)}{\alpha} \tfrac{\partial q_s}{\partial m_1} + W_m \left(1 + p\tfrac{\partial q_s}{\partial m_1} - \tfrac{\partial d}{\partial m_1}\right) + W_d \tfrac{\partial d}{\partial m_1} \right] \\ & (1-n-s) \left[W_m \left(1 - \tfrac{\partial d}{\partial m_1}\right) + W_d \tfrac{\partial d}{\partial m_1} \right] \Big\} f(\omega)\, d\omega. \end{aligned}$$

Recall from (4) and (5) that $W_m = \phi$ and $W_d = -W_l = \phi(1+i)\ \forall m_2$. Furthermore, $\frac{\partial q_s}{\partial m_1} = 0$ because the quantities sellers produce are independent of their money holdings and $\frac{\partial d}{\partial m_1} = 1$ since sellers and inactive agents deposit all their money holdings when $i \geq 0$. Hence,

$$\begin{aligned} V'(m_1) = & \int_\Omega \Big\{ n \left[\varepsilon u'(q_b) \tfrac{\partial q_b}{\partial m_1} + \phi \left(1 - p\tfrac{\partial q_b}{\partial m_1} + \tfrac{\partial l}{\partial m_1}\right) - \phi(1+i) \tfrac{\partial l}{\partial m_1} \right] \\ & + (1-n)\phi(1+i) \Big\} f(\omega)\, d\omega \end{aligned}$$

If $\omega \in \Omega_1$ we have $pq_b = m_1 + (\tau + \tau_1) M_{-1} + l$ and so $\left(1 - p\frac{\partial q_b}{\partial m_1} + \frac{\partial l}{\partial m_1}\right) = 0$. Furthermore,

$$\begin{aligned} & \varepsilon u'(q_b) \tfrac{\partial q_b}{\partial m_1} - \phi(1+i) \tfrac{\partial l}{\partial m_1} = \varepsilon u'(q_b) \tfrac{\partial q_b}{\partial m_1} - \phi(1+i) \left[p\tfrac{\partial q_b}{\partial m_1} - 1 \right] \\ = & \tfrac{\partial q_b}{\partial m_1} \left[\varepsilon u'(q_b) - \phi(1+i) p \right] + \phi(1+i) = \phi(1+i) = \phi\alpha\varepsilon u'(q_b) / c'(q_s). \end{aligned}$$

If $\omega \in \Omega_0$, we have $\frac{\partial q_b}{\partial m_1} = \frac{\partial l}{\partial m_1} = 0$ since buyers are unconstrained. Hence, we get (12). ■

Proof of Proposition 1. From (20) the unconstrained optimum corresponds to $q = q^*(\omega)$ for all $\omega = \Omega$. From the constraint of the central bank problem, since $\gamma < \beta$ is not compatible with the existence of a monetary equilibrium, the only value that is consistent with the unconstrained optimum is $\gamma = \beta$. ■

Proof of Proposition 2. The Ramsey problem facing the central bank is

$$\underset{q(\omega)}{M\alpha x} \int_\Omega \{ n\varepsilon u[q(\omega)] - (s/\alpha) c[(n/s) q(\omega)] \} f(\omega)\, d\omega$$

$$\text{s.t. } \frac{\gamma - \beta}{\beta} = \int_\Omega \left\{ \frac{\alpha\varepsilon u'[q(\omega)]}{c'[(n/s) q(\omega)]} - 1 \right\} f(\omega)\, d\omega$$

Note that λ is independent of ω. The first-order conditions are

$$n\varepsilon u'\left[q\left(\omega\right)\right]-\left(n/\alpha\right)c'\left[\left(n/s\right)q\left(\omega\right)\right]+\lambda\Psi\left(\omega\right)=0 \qquad \omega\in\Omega, \tag{22}$$

where

$$\Psi\left(\omega\right)=\alpha\varepsilon\left\{\frac{u''\left[q\left(\omega\right)\right]c'\left[\left(n/s\right)q\left(\omega\right)\right]-\left(n/s\right)c''\left[\left(n/s\right)q\left(\omega\right)\right]u'\left[q\left(\omega\right)\right]}{c'\left[\left(n/s\right)q\left(\omega\right)\right]^{2}}\right\}<0$$

Sufficient conditions for a maximum are $u'''\left[q\left(\omega\right)\right]\geq 0\geq c'''\left[\left(n/s\right)q\left(\omega\right)\right]$ for all $\omega\in\Omega$. The proof immediately follows from inspecting the first-order conditions (22). ■

Proof of Proposition 3. In any equilibrium buyers' real money holdings are

$$\phi\left(\omega\right)\left\{M_{-1}\left[1+\tau+\tau_{1}\left(\omega\right)\right]+l\left(\omega\right)\right\}=\frac{\phi\left(\omega\right)M_{-1}\left[1+\tau+\tau_{1}\left(\omega\right)\right]}{n}=\frac{\phi\left(\omega\right)M\left(\omega\right)}{n}$$

since the end-of-period nominal money stock is $M\left(\omega\right)=M_{-1}\left[1+\tau+\tau_{1}\left(\omega\right)\right]$. Thus, in any equilibrium we must have

$$\phi\left(\omega\right)p\left(\omega\right)q\left(\omega\right)\leq\frac{\phi\left(\omega\right)M\left(\omega\right)}{n} \qquad \text{for all } \omega\in\Omega.$$

From the first-order conditions of the sellers (7) we have

$$\alpha^{-1}c'\left[\left(n/s\right)q\left(\omega\right)\right]q\left(\omega\right)\leq\frac{\phi\left(\omega\right)M\left(\omega\right)}{n} \qquad \text{for all } \omega\in\Omega.$$

In a steady-state equilibrium $\phi\left(\omega\right)M\left(\omega\right)=\phi_{-1}\left(\omega\right)M_{-1}\left(\omega\right)=z\left(\omega\right)$ for all $\omega\in\Omega$. Hence,

$$\alpha^{-1}c'\left[\left(n/s\right)q\left(\omega\right)\right]q\left(\omega\right)\leq\frac{z\left(\omega\right)}{n} \qquad \text{for all } \omega\in\Omega. \tag{23}$$

We now show that in any steady state equilibrium $z\left(\omega\right)=z$ is a constant. To do so rewrite (12) as follows

$$V'(m_{1})=\int_{\Omega}\left[\phi\left(\omega\right)\alpha\varepsilon u'\left[q_{b}\left(\omega\right)\right]/c'\left[q_{s}\left(\omega\right)\right]\right]f\left(\omega\right)d\omega.$$

Use (3) to eliminate $V'(m_{1})$ and (14) to eliminate q_{s} to get

$$\phi_{-1}\left(\omega_{-1}\right)/\beta=\int_{\Omega}\left[\phi\left(\omega\right)\alpha\varepsilon u'\left[q\left(\omega\right)\right]/c'\left[\left(n/s\right)q\left(\omega\right)\right]\right]f\left(\omega\right)d\omega.$$

Multiply this expression by $M_{-1}(\omega_{-1})$ to get

$$M_{-1}(\omega_{-1})\,\phi_{-1}(\omega_{-1})/\beta = \int_{\Omega}\left\{\frac{M(\omega)\,\phi(\omega)}{\gamma(\omega)}\frac{\alpha\varepsilon u'[q(\omega)]}{c'[(n/s)\,q(\omega)]}\right\} f(\omega)\,d\omega.$$

since $M(\omega) = [1+\tau+\tau_1(\omega)]\,M_{-1}(\omega_{-1}) = \gamma(\omega)\,M_{-1}(\omega_{-1})$. Note that in any steady-state equilibrium the right-hand side is independent of ω_{-1} and therefore a constant. This immediately implies that

$$M_{-1}(\omega_{-1})\,\phi_{-1}(\omega_{-1}) = z_{-1} = constant \qquad \text{for all } \omega_{-1}\in\Omega$$

Since in a steady state equilibrium we have $z_{-1} = z$ we can rewrite this equation as follows

$$1/\beta = \int_{\Omega}\left\{\frac{\alpha\varepsilon u'[q(\omega)]}{\gamma(\omega)\,c'[(n/s)\,q(\omega)]}\right\} f(\omega)\,d\omega.$$

Finally from (23) we have

$$\alpha^{-1}c'[(n/s)\,q(\omega)]\,q(\omega) \le \frac{z}{n} \qquad \text{for all } \omega\in\Omega.$$

Since the right-hand side is independent of $\gamma(\omega)$, changes is $\gamma(\omega)$ are neutral. Hence, stabilization policy is ineffctive. ■

Lemma 2 *Under efficient trading, real aggregate spending $n\phi p(\omega)\,q^*(\omega)$ is increasing in ε. It is increasing in n and decreasing in s and α if*

$$\Phi = 1 + \frac{q^* u''(q^*)}{u'(q^*)} - \frac{q^* c''[(n/s)q^*]\,(n/s)}{c'[(n/s)\,q^*]} < 0$$

Proof of Lemma 2. In equilibrium buyer's real money holdings are $(\upsilon/n)\,z$. Thus, in any equilibrium $n\phi pq \le \upsilon z$. The right-hand side is the aggregate real money stock in market 1 which is independent of ω. The left-hand side is real aggregate spending which is a function of ω. For a given state ω, trades are efficient if $n\phi p(\omega)\,q^*(\omega) \le \upsilon z$ and inefficient if $n\phi p(\omega)\,q^*(\omega) > \upsilon z$ where $p = p(\omega)$ is a function of ω but ϕ is not. We would like to know how real aggregate spending $g(\omega) = n\phi p(\omega)\,q^*(\omega)$ changes in ω when trades are efficient:

$$dg(\omega) = \phi p(\omega)\,q^*(\omega)\,dn + n\phi q^*(\omega)\,dp + n\phi p(\omega)\,dq^*$$

The first term reflects the change in real liquidity that is intermediated in the economy. This effect only occurs if n changes. The second term reflects changes in the relative price ϕp of goods and the third term changes in the efficient quantity. Rewrite it as follows

$$dg(\omega) = n\phi p q^* \left[\frac{dn}{n} + \frac{dp}{p} + \frac{dq^*}{q^*} \right]$$

The term $\frac{dp}{p}$ can be derived from (7) as follows

$$\frac{dp}{p} = \frac{c''[(n/s)q^*]}{c'[(n/s)q^*]} \frac{q^* n}{s} \left[\frac{dn}{n} - \frac{ds}{s} \right] - \frac{d\alpha}{\alpha}$$

and the term $\frac{dq^*}{q^*}$ can be derived from $\varepsilon \alpha u'(q^*) = c'[(n/s)q^*]$ as follows

$$\frac{dq^*}{q^*} = \frac{c''[(n/s)q^*](n/s)}{\alpha\varepsilon u''(q^*) - c''[(n/s)q^*](n/s)} \left[\frac{dn}{n} - \frac{ds}{s} \right] - \frac{\varepsilon\alpha u'(q^*)/q^*}{\alpha\varepsilon u''(q^*) - c''[(n/s)q^*](n/s)} \left[\frac{d\alpha}{\alpha} + \frac{d\varepsilon}{\varepsilon} \right]$$

Investigating each shock separately we get

$$\begin{aligned}
\frac{\partial g(n)}{\partial n} &= c'[(n/s)q^*]\, q^*(n/\alpha s) \left\{ 1 + \frac{c''[(n/s)q^*]\,\Phi}{\alpha\varepsilon u''(q^*) - c''[(n/s)q^*](n/s)} \right\} \geq 0 \\
\frac{\partial g(n)}{\partial s} &= -\frac{c'[(n/s)q^*]\, q^*(n/s)^2 c''[(n/s)q^*]\,\Phi}{\alpha[\alpha\varepsilon u''(q^*) - c''[(n/s)q^*](n/s)]} \leq 0 \\
\frac{\partial g(n)}{\partial \alpha} &= \frac{-c'[(n/s)q^*]\, n\varepsilon u'(q^*)\,\Phi}{\alpha[\alpha\varepsilon u''(q^*) - c''[(n/s)q^*](n/s)]} < 0 \\
\frac{\partial g(n)}{\partial \varepsilon} &= -\frac{c'[(n/s)q^*]\, n u'(q^*)}{\alpha\varepsilon u''(q^*) - c''[(n/s)q^*](n/s)} > 0
\end{aligned}$$

■

Proof of Proposition 4. Here $\omega = n$. From (19) we have

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{n}}^{\overline{n}} \left[\frac{\alpha\varepsilon u'(q(n,z))}{c'[(n/s)q(n,z)]} - 1 \right] f(n)\, dn. \tag{24}$$

From Lemma 2 $\frac{\partial g(n)}{\partial n} \geq 0$. If $g(\underline{n}) > \upsilon z$, then agents are constrained in all states. If $g(\overline{n}) < \upsilon z$, then agents are never constrained. If $g(\overline{n}) \geq \upsilon z \geq g(\underline{n})$, for a given value of z there is a unique critical value $\tilde{n}$ such that

$$g(\tilde{n}) = \upsilon z \tag{25}$$

This implies that $q = q^*(n)$ for $n \leq \tilde{n}$ and $q < q^*(n)$ for $n > \tilde{n}$. Note that $\frac{\partial \tilde{n}}{\partial z} \geq 0$.

Existence: Using (25) we can write (24) as follows

$$\frac{\gamma - \beta}{\beta} = \int_{\check{n}}^{\overline{n}} \left\{ \frac{\alpha \varepsilon u'[q(n,z)]}{c'[(n/s)\,q(n,z)]} - 1 \right\} f(n)\,dn \equiv RHS \tag{26}$$

where $\check{n} = \max\{\tilde{n}, \underline{n}\}$. Only the right-hand side is a function of z. Note that $\lim_{z \to 0} RHS = \infty$. For $v\overline{z} = g(\overline{n})$ we have $\tilde{n} = \overline{n}$ and therefore $RHS\,|_{z=\overline{z}} = 0 \leq \frac{\gamma - \beta}{\beta}$. Since RHS is continuous in z an equilibrium exists.

Uniqueness: The right-hand side of (26) is monotonically decreasing in z. To see this use Leibnitz's rule to get

$$\frac{\partial RHS}{\partial z} = \int_{\check{n}}^{\overline{n}} \frac{\alpha \varepsilon \left[u'' c' - (n/s)\,c'' u'\right]}{(c')^2} \frac{\partial q(n,z)}{\partial z} f(n)\,dn - \left\{ \frac{\alpha \varepsilon u'[q(\check{n},z)]}{c'[(\check{n}/s)\,q(\check{n},z)]} - 1 \right\} f(\check{n}) \frac{\partial \check{n}}{\partial z}$$

Since $q(\check{n}, z) = q^*(\check{n})$ by construction we have

$$\frac{\partial RHS}{\partial z} = \int_{\check{n}}^{\overline{n}} \frac{\alpha \varepsilon \left[u'' c' - (n/s)\,c'' u'\right]}{(c')^2} \frac{\partial q(n,z)}{\partial z} f(n)\,dn < 0.$$

Since the right-hand side is decreasing in z, we have a unique z that solves (26). Consequently, we have

$$q = q^*(n) \text{ if } n \leq \check{n} \text{ and } q < q^*(n) \text{ otherwise.}$$

Finally, if buyers have been constrained in market 1 money holdings at the opening of the second market are $m_2 = 0$ for buyers and inactive agents and $m_2 = pq_s$ for sellers. Solving for equilibrium consumption and production in the second market, with $x^* = U'^{-1}(1)$, gives

$$\begin{aligned} h_b &= x^* + n e_c(q)\,c[(n/s)\,q] + (1-n)\,e_u(q)\,u(q) \\ h_s &= x^* - n e_c(q)\,c[(n/s)\,q]\,(1-s)\,s^{-1} - n e_u(q)\,u(q) \\ h_{in} &= x^* + n e_c(q)\,c[(n/s)\,q] - n e_u(q)\,u(q) \end{aligned}$$

Notice that $nh_b + sh_s + (1-n-s)\,h_{in} = x^*$. Moreover, we have $h_b \geq h_{in} \geq h_s$. For existence we need that all agents work a positive amount in the second market. This, it is sufficient to show that $h_s > 0$.

Given $\underline{s} > 0$, n/s is bounded and since the elasticities $e_c(q)$ and $e_u(q)$ are bounded, we can scale $U(x)$ such that there is a value $x^* = U'^{-1}(1)$ greater than the last term for all $q \in [0, q^*]$. Hence, h_s is positive for for all $q \in [0, q^*]$ ensuring that the equilibrium exists. Note that the states where the buyers are constrained are the ones where the sellers have all the money after trading. Therefore, if h_s is positive in constrained states it is positive in all unconstrained states. ■

Proof of Proposition 5. Here $\omega = s$. From (19) we have

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{s}}^{\overline{s}} \left[\frac{\alpha\varepsilon u'[q(s,z)]}{c'[(n/s)\, q(s,z)]} - 1 \right] f(s)\, ds. \tag{27}$$

From Lemma 1 $\frac{\partial g(s)}{\partial s} < 0$. If $g(\overline{s}) > \upsilon z$, then agents are constrained in all states. If $g(\underline{s}) < \upsilon z$, then agents are never constrained. If $g(\underline{s}) \geq \upsilon z \geq g(\overline{s})$, for a given value of z there is a unique critical value $\tilde{s}$ such that

$$g(\tilde{s}) = \upsilon z \tag{28}$$

This implies that $q = q^*(s)$ for $s \geq \tilde{s}$ and $q < q^*(s)$ for $s < \tilde{s}$. Note that $\frac{\partial \tilde{s}}{\partial z} < 0$.

Existence: Using (28) we can write (27) as follows

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{s}}^{\check{s}} \left\{ \frac{\alpha\varepsilon u'[q(s,z)]}{c'[(n/s)\, q(s,z)]} - 1 \right\} f(s)\, ds \equiv RHS \tag{29}$$

where $\check{s} = \min\{\tilde{s}, \overline{s}\}$. Only the right-hand side is a function of z. Note that $\lim_{z \to 0} RHS = \infty$. For $\upsilon \overline{z} = g(\overline{s})$ we have $\tilde{s} = \overline{s}$ and therefore $RHS\,|_{z=\overline{z}} = 0 \leq \frac{\gamma-\beta}{\beta}$. Since RHS is continuous in z an equilibrium exists.

Uniqueness: The right-hand side of (29) is monotonically decreasing in z. To see this use Leibnitz's rule to get

$$\frac{\partial RHS}{\partial z} = \int_{\underline{s}}^{\check{s}} \frac{\alpha\varepsilon\,[u''c' - (n/s)\, c''u']}{(c')^2} \frac{\partial q(s,z)}{\partial z} f(s)\, ds + \left\{ \frac{\alpha\varepsilon u'[q(\check{s},z)]}{c'[(n/\check{s})\, q(\check{s},z)]} - 1 \right\} f(\check{s}) \frac{\partial \check{s}}{\partial z}$$

Since $q(\check{s}, z) = q^*(\check{s})$ by construction we have

$$\frac{\partial RHS}{\partial z} = \int_{\underline{s}}^{\check{s}} \frac{\alpha\varepsilon\,[u''c' - (n/s)\, c''u']}{(c')^2} \frac{\partial q(s,z)}{\partial z} f(s)\, ds < 0.$$

Since the right-hand side is decreasing in z, we have a unique z that solves (29). Consequently, we have

$$q < q^*(s) \ \text{ if } s < \breve{s} \text{ and } q = q^*(s) \ \text{ otherwise.}$$

Finally, it is straightforward to show that the hours worked in market 2 are bounded away from zero. ■

Proof of Proposition 6. Here $\omega = \varepsilon$. From (19) we have

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \left[\frac{\alpha \varepsilon u'[q(\varepsilon, z)]}{c'[(n/s)\, q(\varepsilon, z)]} - 1 \right] f(\varepsilon)\, d\varepsilon. \tag{30}$$

From Lemma 1 $\frac{\partial g(\varepsilon)}{\partial \varepsilon} \geq 0$. If $g(\underline{\varepsilon}) > \upsilon z$, then agents are constrained in all states. If $g(\overline{\varepsilon}) < \upsilon z$, then agents are never constrained. If $g(\overline{\varepsilon}) \geq \upsilon z \geq g(\underline{\varepsilon})$, for a given value of z there is a unique critical value $\tilde{\varepsilon}$ such that

$$g(\tilde{\varepsilon}) = \upsilon z \tag{31}$$

This implies that $q = q^*(\varepsilon)$ for $\varepsilon \leq \tilde{\varepsilon}$ and $q < q^*(\varepsilon)$ for $\varepsilon > \tilde{\varepsilon}$. Note that $\frac{\partial \tilde{\varepsilon}}{\partial z} \geq 0$.

Existence: Using (31) we can write (30) as follows

$$\frac{\gamma - \beta}{\beta} = \int_{\breve{\varepsilon}}^{\overline{\varepsilon}} \left\{ \frac{\alpha \varepsilon u'[q(\varepsilon, z)]}{c'[(n/s)\, q(\varepsilon, z)]} - 1 \right\} f(\varepsilon)\, d\varepsilon \equiv RHS \tag{32}$$

where $\breve{\varepsilon} = \max\{\tilde{\varepsilon}, \underline{\varepsilon}\}$. Only the right-hand side is a function of z. Note that $\lim_{z \to 0} RHS = \infty$. For $\upsilon \overline{z} = g(\overline{\varepsilon})$ we have $\tilde{\varepsilon} = \overline{\varepsilon}$ and therefore $RHS\,|_{z=\overline{z}} = 0 \leq \frac{\gamma - \beta}{\beta}$. Since RHS is continuous in z an equilibrium exists.

Uniqueness: The right-hand side of (32) is monotonically decreasing in z. To see this use Leibnitz's rule and not that by construction $q(\breve{\varepsilon}, z) = q^*(\breve{\varepsilon})$ to get

$$\frac{\partial RHS}{\partial z} = \int_{\breve{\varepsilon}}^{\overline{\varepsilon}} \frac{\alpha \varepsilon \left[u'' c' - (n/s)\, c'' u'\right]}{(c')^2} \frac{\partial q(\varepsilon, z)}{\partial z} f(\varepsilon)\, d\varepsilon < 0.$$

Since the right-hand side is strictly decreasing in z, we have a unique z that solves (32). Consequently, we have

$$q = q^*(\varepsilon) \ \text{ if } \varepsilon \leq \breve{\varepsilon} \text{ and } q < q^*(\varepsilon) \ \text{ otherwise.}$$

Finally, it is straightforward to show that the hours worked in market 2 are bounded away from zero. ■

Proof of Proposition 7. Here $\omega = \alpha$. From (19) we have

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{\alpha}}^{\overline{\alpha}} \left[\frac{\alpha \varepsilon u' \left[q \left(\alpha, z \right) \right]}{c' \left[\left(n/s \right) q \left(\alpha, z \right) \right]} - 1 \right] f \left(\alpha \right) d\alpha. \tag{33}$$

From Lemma 1 $\frac{\partial g(\alpha)}{\partial \alpha} < 0$. If $g(\overline{\alpha}) > \upsilon z$, then agents are constrained in all states. If $g(\underline{\alpha}) < \upsilon z$, then agents are never constrained. If $g(\underline{\alpha}) \geq \upsilon z \geq g(\overline{\alpha})$, for a given value of z there is a unique critical value $\tilde{\alpha}$ such that

$$g(\tilde{\alpha}) = \upsilon z \tag{34}$$

This implies that $q = q^*(\alpha)$ for $\alpha \geq \tilde{\alpha}$ and $q < q^*(\alpha)$ for $\alpha < \tilde{\alpha}$. Note that $\frac{\partial \tilde{\alpha}}{\partial z} < 0$.

Existence: Using (34) we can write (33) as follows

$$\frac{\gamma - \beta}{\beta} = \int_{\underline{\alpha}}^{\breve{\alpha}} \left\{ \frac{\alpha \varepsilon u' \left[q \left(\alpha, z \right) \right]}{c' \left[\left(n/s \right) q \left(\alpha, z \right) \right]} - 1 \right\} f \left(\alpha \right) d\alpha \equiv RHS \tag{35}$$

where $\breve{\alpha} = \min \{\tilde{\alpha}, \overline{\alpha}\}$. Only the right-hand side is a function of z. Note that $\lim_{z \to 0} RHS = \infty$. For $\upsilon \overline{z} = g(\overline{\alpha})$ we have $\tilde{\alpha} = \overline{\alpha}$ and therefore $RHS\,|_{z = \overline{z}} = 0 \leq \frac{\gamma - \beta}{\beta}$. Since RHS is continuous in z an equilibrium exists.

Uniqueness: The right-hand side of (35) is monotonically decreasing in z. To see this use Leibnitz's rule and note that $q(\breve{\alpha}, z) = q^*(\breve{\alpha})$ to get

$$\frac{\partial RHS}{\partial z} = \int_{\underline{\alpha}}^{\breve{\alpha}} \frac{\alpha \varepsilon \left[u'' c' - (n/s) c'' u' \right]}{(c')^2} \frac{\partial q(\alpha, z)}{\partial z} f(\alpha) \, d\alpha < 0.$$

Since the right-hand side is decreasing in z, we have a unique z that solves (35). Consequently, we have

$$q < q^*(\alpha) \text{ if } \alpha < \breve{\alpha} \text{ and } q = q^*(\alpha) \text{ otherwise.}$$

Finally, it is straightforward to show that the hours worked in market 2 are bounded away from zero. ■