Some Benefits of Cyclical Monetary Policy

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Abstract

In this paper, we present a simple random-matching model of seasons, where different seasons translate into different propensities to consume and produce. We find that the cyclical creation and destruction of money is beneficial for welfare under a wide variety of circumstances. Our model of seasons can be interpreted as providing support for the creation of the Federal Reserve System, with its mandate of supplying an elastic currency for the nation.

1 Introduction

Should monetary policy be cyclical? Although this is an old question, there is no general consensus as to the correct answer. Recent research on the "pure theory of money" has contributed very little, if anything, to the debate that surrounds this question. By "pure theory of money," we refer to that line of research where money, instead of being treated as a primitive such as preferences and technology, arises endogenously as a solution to a trading problem. Perhaps it is not so surprising that modern theories of money have remained silent on the desirability of cyclical monetary policy; although the environments that are suitable for modeling a role for fiat money—environments with infinite horizons and diverse trading opportunities—are quite tractable when they are stationary, they become quite intractable when the stationarity assumption is relaxed. In this paper, we present a simple model with seasons where money is essential and where cyclical money creation and destruction is, under some circumstances, welfare enhancing.

In our model economy, individual agents experience seasonal preference shocks and trade between pairs of agents is characterized by a lack of double coincidence of wants. Agents in the economy belong to one of two equally sized groups. When one group has a production opportunity the other group has a consumption opportunity and, on a period by period basis, each group alternates between having production and consumption opportunities. In pairwise meetings, the consumer faces an idiosyncratic preference shock which affects his desire and/or intensity to consume. The notion of seasons is introduced by having the (economy wide) distribution of consumer preference shocks differ over even and odd dates. For example, the even period will be a high demand season and the odd period a low demand season if the total number of consumer agents who actually want to consume in even periods is greater than the total number of consumer agents who want to consume in odds periods. There are frictions in the environment, specified below, which imply that (fiat) money is essential for trade. Monetary policy is restricted to take the form of a reoccurring pattern of taxing money holdings in one period and injecting the proceeds in a lump-sum fashion in the subsequent period. If taxes and subsidies are non-zero, then monetary policy will be cyclical; if taxes and subsidies are equal to zero, then the money supply will be constant. We first show that under a constant monetary policy rule, the seasonal frequency of trade is constant. When we compare a cycle monetary policy with a constant money supply policy we find two basic effects. First, cyclical policies may reduce the return on money and, hence, tighten the producers' participation constraints. This effect, the so-called *intensive margin* effect, may reduce the social surplus associated with each trade. Second, if there is a sufficient asymmetry in the distribution of aggregate preference shocks, so that one season has a relatively high desire or demand for consumption compared to the other season, then a cyclical monetary policy will increase the average frequency of trades, or the *extensive margin*, compared to a constant money supply policy. We find that under a wide variety of circumstances, the optimal monetary policy will be cyclical. So, although a cyclical monetary policy may result in a lower and inefficient level of production at the match level, the fact that the economy wide frequency of trades increases implies that a cyclical monetary policy can increase the welfare of society, when compared to a constant money supply policy.

The results from our model can be loosely be interpreted as providing some support for the creation of the Federal Reserve System in 1913. The preamble to the Federal Reserve Act states that the Federal reserve banks were established to, among other things, "furnish an elastic currency." According to Meltzer (2003) there are two meanings for the word "elasticity." One meaning is in regard to the ability of a central bank to pool reserves and lend them out in the event of a banking or financial crisis. The second meaning is in reference to seasonal fluctuations, which is the topic of this paper. In practice, the two meanings of elasticity are not unrelated because the data indicates that seasonal fluctuations in money demand can exacerbate a (potential) banking or financial panic. For example, farmers needed cash in the autumn in order to harvest their crops but, given the structure of the banking system before the founding of the Fed, there was essentially only a fixed amount of reserves to go around. As a result, the increase in demand for cash in the autumn could potentially turn a quite independent and manageable liquidity problem in financial markets into a financial panic or banking crisis. Miron (1986) concludes that the founding of the Fed had positive welfare consequences for the economy since its policy of furnishing an elastic currency greatly reduced the possibility of financial panics. In particular, financial panics were commonplace and sometimes quite severe before the founding of the Fed and this was no longer so after the founding. Note that the two meanings of elasticity are at play here. By consolidating reserves at a central place, the Fed was able to provide reserves to banks who needed them in a time of financial stringency. Furthermore, by discounting real bills, the Fed was able to provide (additional) liquidity to farmers, implying that their increase in demand for money need not exacerbate a potential liquidity problem in financial markets. Miron (1986) points out, however, if an economy has deposit insurance, then an elastic currency policy would not be welfare improving since existence of deposit insurance would greatly reduce, if not eliminate, the possibility of financial panics, which is the source of the welfare gain in his analysis. In this paper, we completely abstract any notion of financial panics and find that there are other sources for welfare gains associated with an elastic currency (cyclical monetary) policy and that is the provision of an elastic currency can increase the average frequency of trade in the economy.

Since the "fine tuning" of monetary policy is a broad topic with voluminous contributions, it is important to relate our model to some well known papers up front, so as to highlight the particular debt our work owes to this broad literature. Lucas (1972) was the first to present a pure theory of short-run effects of monetary policy, but an important ingredient in his analysis is an exogenous and random supply of money. In a competitive environment, the optimal monetary policy invariably leads to the Friedman (1959) rule in the form of a deflation that eliminates the opportunity cost of holding money. Bewley (19..), Levine (1991), and Sheikman and Weiss (1986), among others, departed from a representative consumer structure and found that there exist welfare gains associated with an ongoing inflation. In these models, traders face uninsurable shocks and can benefit from some redistribution of wealth generated by inflation. The literature that has followed the seminal work in the pure theory of money, Kiyotaki and Wright (1989), has more or less been limited to reproducing these inflation gains.[1]

The rest of the paper is divided as follows. In section 2, we describe the environment with two seasons. In section 3, we define symmetric and stationary allocations, as well as the welfare criteria that guides the discussion of optimal monetary policies. In section 4, we define participation constraints and implementable allocations. In section 5 we characterize the stationary distributions of money, according to consumer/producer status and seasons. That characterization is independent of preferences, but a complicating feature of the analysis arises from the fact that the distribution of shocks is seasonal. We pay particular attention to the effects of small rates of money creation, and destruction, on the average extensive margin. In section 5, we describe implementable allocations in more detail. In section 6 we discuss the implications of the previous results regarding optima. Section 7 concludes with brief remarks about time consistency of optimal policy.

[1] See Molico (1999), and Deviatov and Wallace (2001). There has been also work on the effects of inflation on search intensity, such as Li (1995) and Shi (1999), among others. More recently, Cavalcanti and Dazelmone (2003) have computed optima in the Cavalcanti and Wallace (1999) model, featuring nonstationary rates of return of inside money in constant supply.

2 The environment

Time is discrete, the horizon is infinite, and there is a $[0,1]$ continuum of each of 2 types of people. There is one type of divisible and perishable consumption good per date, but people rank goods in odd dates differently from goods in even dates. Each type is specialized in consumption and production: A type e person consumes even-date goods and produces odd-date goods, and a type d person consumes odd-date goods and produces even-date goods. Each type maximizes expected discounted utility, with a common discount factor $\beta \in (0,1)$. We find it useful to have a notation for the two-period discount factor, $\delta \equiv \beta^2$. We also find it convenient to refer to a type e individual in an even (odd) date, or a type d individual in an odd (even) date, as a *consumer* (*producer*). We let $s \in \{e,d\}$ indicate the seasons and/or types of people.

People meet randomly in pairs and face idiosyncratic preference shocks. For $s \in \{e,d\}$, the utility function for consumers at season s, with domain on the set of nonnegative real numbers, is $\varepsilon_s u_s(y_s)$, where ε_s is the idiosyncratic shock affecting this consumer, and y_s is the amount consumed. The shock ε_s is Bernoulli and the probability of $\varepsilon_s = 1$, $\pi_s \in (0,1)$, is indexed by seasons. The producers in a meeting in season s can produce any choice of $y_s \geq 0$ units of the corresponding date good, at a utility cost normalized to be y_s itself. Utility in a period is thus $\varepsilon_s u_s(y_s)$ when consuming, and $-y_s$ when producing. The utility function u_s, conditional on $\varepsilon_s = 1$, is standard, assumed to be increasing and twice differentiable, and satisfies $u_s(0) = 0$, $u_s'' < 0$, $u_s'(0) = \infty$ and $u_s'(\infty) < 1$. We assume that $\pi_e \geq \pi_d$, and $u_e' \geq u_d'$ in a relevant range, to be specified later, so that even dates feature a high desire for consumption, relative to odd dates. It should be emphasized that a strict inequality for either these gives rise to a cyclical demand for liquidity.

In each period, each type e person is matched randomly with a type d person. During meetings, the realization of preference shocks occurs and production may take place. All individuals are anonymous, in the sense that they all have private histories. We also assume that people cannot commit to future actions, so that those who produce have to get a future reward for doing so. The reward, in this paper, takes the form of outside money. To keep the model simple, we assume that each person can carry from one meeting to the next either 0 or 1 unit of fiat money. A consequence of this assumption, which makes the distribution of people tractable, is that trade can only take place when the consumer has money (and realizes $\varepsilon_s = 1$) and the producer has none.

Monetary policy takes the simple form of a choice of a pair (σ, τ), where σ is the probability that a person without money finds one unit of money before meetings, and τ is the probability that a person with money looses the money before meetings. Let M_e denote the measure of individuals holding money in even periods and M_d the measure of individuals holding money in odd periods. We restrict attention to cases in which either $\tau = \sigma = 0$ all the time, or $\sigma > 0$ in even dates and $\tau > 0$ in odd dates. This simple formulation is designed to limit our analysis to the specific question of whether periods of high desire for consumption should experience an increase in the supply of money, offset by taxation in the following period.

3 Stationarity and welfare criteria

We let the measure of consumers with money during meetings in season s be denoted q_s, so that those consumers without money add to $1 - q_s$. We let the measure of producers without money during meetings be denoted p_s, so that those producers with money add to $1 - p_s$. In order to save on notation, we let $y = (y_e, y_d)$ denote the list of output levels, $x \equiv (p_e, q_e, p_d, q_d)$ denote an arbitrary *distribution*, and use the superscript $+$, as in $x^+ \equiv (p_e^+, q_e^+, p_d^+, q_d^+)$, when the qualification that $\sigma > 0$ for that distribution becomes essential. A distribution $x \in [0, 1]^4$ is considered *invariant* if and only if there exists $(\sigma, \tau) \in [0, 1]^2$ such that

$$p_e = (1 - \sigma)(1 - q_d + \pi_d p_d q_d), \tag{1}$$

$$p_d = (1 - \tau)(1 - q_e + \pi_e p_e q_e) + \tau, \tag{2}$$

$$q_e = (1 - \sigma)(1 - p_d + \pi_d p_d q_d) + \sigma \tag{3}$$

and

$$q_d = (1 - \tau)(1 - p_e + \pi_e p_e q_e), \tag{4}$$

where the distribution x is described after money is created or destroy has occurred.

The stationarity requirement (1) can be explained as follows. During odd-date meetings, trade takes place after the destruction of money. The measure of consumers with money is q_d and the measure of producers without money is p_d. Consumers without money, who number $1-q_d$, cannot buy goods; each of them faces a probability σ of finding money at the beginning of the *next* date. Hence, $1-\sigma$ times $1-q_d$ is the total flow of those consumers who become producers without money in the next (even) date. Similarly, the measure of consumers with money in the odd date is q_d. Only a fraction, π_d, of these consumers will want to consume in the odd date and only a fraction q_d of these consumers will meet a producer without money. Therefore, $\pi_d p_d q_d$ represents the measure of consumers with money that will trade in date d and $(1-\sigma)\pi_d p_d q_d$ represents the number of these consumers that become producers without money in the next (even) period, after money creation takes place.

Likewise, regarding the requirement (2), we first notice that a measure $1-q_e+\pi_e p_e q_e$ producers arrive at the beginning of date d without money. Adding now to that the mass of money destroyed from those date e consumers with money who did not trade, $\tau q_e(1-\pi_e p_e)$, yields the right-hand of (2). The same principle explains the requirement (3). The measure of consumers with money at date e consists of the measure of producers who leave date d with money, $1-p_d+\pi_d p_d q_d$, and the measure of producers who leave date d without money but obtain some when additional money is created at the beginning of date e, $\sigma p_d(1-\pi_d q_d)$. Finally, the requirement (4) follows from imposing stationarity on the measure of consumers with money arriving at date d, $1-p_e+\pi_e p_e q_e$, after the destruction of money takes place with probability τ.

It is now clear that our notion of stationarity amounts to restricting that output, y_s, as well as the measures p_s and q_s, to be constant functions of the season s only. These functions are used symmetrically in a measure of welfare as follows. We adopt an *ex ante* welfare criteria, with an average expected discounted utility computed according to an invariant distribution and output function. Whenever trade takes place in a season, it is because money is changing hands from a fraction p_s of the mass of consumers $\pi_s q_s$ in position to trade. Since, for each consumer there is a producer, the flow of average utility in season s is $\pi_s p_s q_s[u_s(y_s) - y_s]\frac{1}{2}$. We call the term $\pi_s p_s q_s$ the *extensive margin* at s, and $u_s(y_s) - y_s$ the *intensive margin* at s. The extensive margin is a property of x, while the intensive margin is a property of y. An *allocation* is a pair (x, y), where x and y are invariant and y has nonnegative coordinates. The *welfare* U attained by an allocation is defined as the present discounted value

$$U(x, y) = \frac{1}{2(1-\beta)} \sum_s \pi_s p_s q_s[u_s(y_s) - y_s].$$

The intensive margin at s is maximized by y_s^* such that $u_s'(y_s^*) = 1$, which is uniquely defined by assumption. We refer to $y^* = (y_e^*, y_d^*)$ as the *first-best* output list.

4 Implementable Allocations

The definition of the values of y consistent with incentives is set according to the notion of sequential individual rationality employed by Cavalcanti and Wallace (1999), and also applied by Cavalcanti (2003). Underlying their definition of participation constraints is the idea that a social planner proposes an allocation, but that anonymous individuals may defect from that proposal by not trading in a given meeting, without losing any money holdings that was brought to the meeting. We adopt the same concept here, with the exception to the taxation of money holdings, which we assume that cannot be avoided by individuals with money. The participation constraints are then defined by a set of allocations, according to the expected discounted utilities implied by the allocations. In order to be able to state these constraints, we need first to describe the Bellman equations of the economy.

The value functions will be computed *before* the realization of the effects of creation and destruction for each individual in a given date. (Recall that money is created at the beginning of even dates and is destroyed at the beginning of odd dates.) The value function for consumers with money at s is v_s, and that for producers without money is w_s. For ease of exposition, we define $\bar{v}_s$ as the value for consumers without money at s, and $\bar{w}_s$ as that of producers with money. The Bellman equations for $(v, w) = (v_e, v_d, w_e, w_d)$ read

$$\begin{aligned} v_e &= \pi_e p_e(u_e + \beta w_d) + (1 - \pi_e p_e)\beta \bar{w}_d \\ w_e &= \sigma\beta v_d + (1-\sigma)[\pi_e q_e(-y_e + \beta v_d) + (1 - \pi_e q_e)\beta \bar{v}_d] \\ v_d &= \tau\beta w_e + (1-\tau)[\pi_d p_d(u_d + \beta w_e) + (1 - \pi_d p_d)\beta \bar{w}_e] \\ w_d &= \pi_d q_d(-y_d + \beta v_e) + (1 - \pi_d q_d)\beta \bar{v}_e \end{aligned} \tag{5}$$

where u_e and u_d, by an abuse of notation, stand for $u_e(y_e)$ and $u_d(y_d)$, respectively. The definition is completed by substituting the values of $(\bar{v}, \bar{w})$, given by,

$$\begin{aligned} \bar{v}_e &= \sigma v_e + (1-\sigma)\beta w_d \\ \bar{w}_e &= \beta v_d \\ \bar{v}_d &= \beta w_e \\ \bar{w}_d &= \tau w_d + (1-\tau)\beta v_e \end{aligned} \tag{6}$$

into the previous system.

The participation constraint for producers at even dates is simply

$$-y_e + \beta v_d \geq \beta \bar{v}_d = \delta w_e, \tag{7}$$

since this producer is bringing no money to the meetings, and only has the option of leaving the meeting for becoming a producer two periods later. Producers at odd dates have to take into account that if they disagree with producing the planned output y_d and walk away from trades, then they have a chance of receiving money next period from the money-creation policy. Thus, the participation constraint for producers at odd dates can be stated as

$$-y_d + \beta v_e \geq \beta \bar{v}_e = \beta\sigma v_e + \delta(1-\sigma) w_d. \tag{8}$$

For completeness, we state the participation constraint for consumers, which can be shown to be implied by those of producers. They are

$$u_e + \beta w_d \geq \beta \bar{w}_d \tag{9}$$

and

$$u_d + \beta w_e \geq \beta \bar{w}_e. \tag{10}$$

An allocation (x, y) is *implementable* if $x \equiv (p_e, q_e, p_d, q_d)$ is invariant for some policy (σ, τ), and such that there exists (v, w) and $(\bar{v}, \bar{w})$ for which (5-10) hold. An allocation is *optimal* if it maximizes $U(x, y)$ among the set of implementable allocations.

5 Extensive-margin effects

Monetary policy in this set up is a choice of an invariant distribution x. Changes in x due to changes in (σ, τ) have direct effects on extensive margins, $\pi_s p_s q_s$, and indirect effects on intensive margins, $u_s(y_s) - y_s$, because participation constraints for y depend on x. The latter effects can be ignored if, for $s = e$ and $s = d$, the maximizer of $u_s(y_s) - y_s$, the first-best level of output y_s^*, satisfies participation constraints. In this section, we investigate whether the maximizer of the sum $\sum_s \pi_s p_s q_s$, among all invariant distributions x, is a *cyclical* policy x^+, that is, one with positive σ. We shall see that a cyclical monetary policy tends to increase the extensive margin at e, and to decrease that at d. Since $u_e' \geq u_d'$, it will follow that if y^* satisfies participation constraints and the maximizer of the sum $\sum_s \pi_s p_s q_s$ is cyclical, then the optimal allocation is indeed a cyclical monetary policy.

We start by pointing out an important property of the set of invariant distributions. If x is invariant with $\sigma = 0$, i.e., when the money supply is constant, we say that x is *acyclical*, a label motivated by the following lemma.

Lemma 1 *Assume that x is acyclical. Then $\pi_s p_s q_s$ is constant in s.*

Proof. Set $\sigma = \tau = 0$ in equations (1) and (4). It follows that $\pi_e p_e q_e = \pi_d p_d q_d$. ■

The property of constant extensive margins holds regardless of the relative values of π_s. We can offer an intuitive explanation for this property as follows. Let us consider the inflow and outflow of money for a set of individuals of the same type, say type e. Then, on one hand, the stationary measure of consumers of this type spending money is $(\pi_e p_e) q_e$, an event taking place at even dates. On the other hand, the stationary measure of producers of this type acquiring money is $(\pi_d q_d) p_d$, an event taking place at odd dates. Since the quantity of money in the hands of this group must be stationary, and all seasons have the same frequency, then these two margins must be equalized, as stated in the lemma.

Some useful observations about acyclical distributions can be made with regard to the relative values of $p_s$and q_s.

Lemma 2 *Assume that x is acyclical. Then (i) $p_e - q_e = p_d - q_d$, and (ii) $p_e \leq p_d$ if and only if $\pi_d \leq \pi_e$.*

Proof. (i) Set $\sigma = \tau = 0$ in equations (1) and (2). Since, by Lemma 1, $\pi_e p_e q_e = \pi_d p_d q_d$, equations (1) and (2) imply that $p_e - q_e = p_d - q_d$. (ii) Since, by Lemma 1, $\pi_e p_e q_e = \pi_d p_d q_d$, $\pi_e \geq \pi_d$ if and only if $p_e q_e \leq p_d q_d$. Part (i) of this lemma implies that if $p_e q_e \leq p_d q_d$, then $p_e \leq p_d$ and $q_e \leq p_d$. ■

There is an alternative way to think about part (i) of lemma 1. The measure of individuals that hold money in period s, M_s, is the sum of consumers with money, q_s, and producers with money, $1 - p_s$. When $\sigma = \tau = 0$, the measures of individuals that hold money in odd and even periods are the same, i.e., $M_e = M_d$, which implies that $1 - p_e + q_e = 1 - p_d + q_d$, or that $p_e - q_e = p_d - q_d$.

A simple application of lemma 2 allow as to describe in simple terms the set of acyclical distributions when $\pi_e = \pi_d$.

Lemma 3 *Assume $\pi_e = \pi_d = \pi$. Then the set of acyclical distributions is fully described by $p_e = p_d = p$, $q_e = q_d = q$, and $p = 1 - q + \pi pq$ for $q \in [0, 1]$.*

Proof. Since $\pi_e = \pi_d$ then, by lemma 2, $p_e = p_d$, and consequently, by lemma 1, $q_e = q_d$. Equation (1) with $\sigma = 0$ thus proves the lemma. ■

The one-dimensional set described by lemma 3 is the symmetric set of distributions that appears in Cavalcanti (2003). The equality $p = 1 - q + \pi pq$ defines a strictly concave function for $q \in [0, 1]$, and the extensive margin $pq\pi$ is maximized at (p, q) such that $p = q = [1 - (1 - \pi)^{\frac{1}{2}}]/\pi$. Similar properties are also present in the generalization with $\pi_e \geq \pi_d$, so that every acyclical x can be indexed by a one-dimensional choice of q_d, and the margin $\pi_s p_s q_s$, for any s, is maximized when $p_d = q_d$. The algebra required to prove these properties in the general case is a lot more involved. We can however still describe analytically the set of acyclical distributions with one equation, which will prove useful for the numerical examples that are explored later.

Lemma 4 *There exists, for each q_s, an unique acyclical x. Moreover, x can be solved for analytically. The statement holds for any s in $\{e, d\}$.*

Proof. We shall make repeated used of the system (1-4) with $\sigma = \tau = 0$. According to lemma 2, $p_s = q_s + a$ for some a that does not depend on s. We shall first solve for a analytically. For this purpose, let $A \equiv 1 + \pi_s p_s q_s$, which, by force of lemma 1, does not depend on s as well. Equation (1) now reads (i) $p_e = A - q_d$. Using (ii) $p_d = q_d + a$, we can write (2) as (iii) $q_e = A - (a + q_d)$. The equality $p_e q_e = \theta p_d q_d$ for $\theta = \pi_d / \pi_e$, can be written, using (i), (ii) and (iii), as $(A - q_d)^2 - a(A - q_d) - \theta q_d (a + q_d) = 0$. The only relevant solution of this quadratic equation is given by (iv) $2(A - q_d) = a + \sqrt{a^2 + 4\theta b}$, where $b = q_d(a + q_d)$. Since $A = 1 + \pi_d q_d p_d = 1 + \pi_d b$, we can rewrite (iv) as (v) $a^2 + 4\theta b = [2\pi_d b + 2(1 - q_d) - a]^2$. Expanding now (v) as a quadratic equation in b, we find that the only relevant solution is given by (vi) $2\pi_d^2 b = \theta + a\pi_d - 2\pi_d(1 - q_d) + \sqrt{\theta^2 - 4\pi_d(1 - q_d)\theta + \pi_d^2 a^2 + 2\theta \pi_d a}$. Substituting in (vi) the expression $b = q_d(a + q_d)$, produces a quadratic equation in a as a function of q_d. The only relevant solution of the latter is (vii) $a = [-k_2 - \sqrt{k_2^2 - 4k_1 k_3}]/(2k_1)$, where $k_1 = \pi_d^2[(2\pi_d q_d - 1)^2 - 1]$, $k_2 = 2\pi_d\{(2\pi_d q_d - 1)[2(\pi_d q_d)^2 + 2\pi_d(1 - q_d) - \theta] - \theta\}$ and $k_3 = [2(\pi_d q_d)^2 + 2\pi_d(1 - q_d) - \theta]^2 - \theta^2 + 4\pi_d(1 - q_d)\theta$. If q_d is fixed, then $p_d = q_d + a$ determines p_d. Using now (1) and $p_e = q_e + a$, the values of p_e and q_e are also determined. Since the system (1-4) is symmetric in e and d, when $\sigma = \tau = 0$, similar conclusions follows when q_e is given, instead of q_d. ■

The system (1-4) defines the boundary of a convex set, which is symmetric when $\sigma = \tau = 0$. Since the level curves of $\pi_s p_s q_s$ are convex, the following holds in the general case in which $\pi_e > \pi_d$ is allowed.

Proposition 5 *The maximizer of $\pi_s p_s q_s$, among the set of acyclical distributions, is the unique x such that $p_{s'} = q_{s'}$. The statement holds for any s and s' in $\{e, d\}$.*

Proof. The set of acyclical distributions is closed, and $\pi_s p_s q_s$ is continuous in x for each s, so that a maximizer exists. Let us fix $x = x^1$, with $p_s^1 \neq q_s^1$ for some s, and show that x^1 cannot be the maximizer. Note that, by lemma 2, $p_s^1 \neq q_s^1$ if and only if $p_{s'}^1 \neq q_{s'}^1$. We start by constructing x^2, the "mirror image" of x^1, with the equalities $p_s^2 = q_s^1$ and $q_s^2 = p_s^1$ for $s \in \{e, d\}$. Also, for $\alpha \in (0,1)$, let $x^\alpha \equiv \alpha x^1 + (1-\alpha)x^2$. It is clear that, for all s, $\alpha p_s^1 q_s^1 + (1-\alpha)p_s^2 q_s^2 < p_s^\alpha q_s^\alpha$. Thus the distribution x^α attains a higher extensive margin than that of x^1, although x^α is not invariant if it does not satisfies (1-4) with equality. However, using now lemma 1, one can rewrite each equation in the system (1-4), when $\sigma = \tau = 0$, as $p_s + q_{s'} = 1 + \pi_s p_s q_s$ or $p_{s'} + q_s = 1 + \pi_s p_s q_s$ where $s' \neq s$, so that each right-hand side is increasing in the extensive margin. Since $p_s^\alpha + q_{s'}^\alpha < 1 + \pi_s p_s^\alpha q_s^\alpha$ and $p_{s'}^\alpha + q_s^\alpha < 1 + \pi_s p_s^\alpha q_s^\alpha$, then there exists an acyclical $\bar{x}$, with $\bar{x} \geq x^\alpha$, that attains a higher extensive margin than that of x. The proof is now complete. ■

A standard result in many search models of money is it is optimal for half of the population to hold money; this aggregate level of money holdings maximizes the number of productive matches. Our model also has this feature. When $\sigma = \tau = 0$ and when $\pi_s p_s q_s$ is maximized, i.e., $p_s = q_s$ for $s \in \{e, d\}$, then measure of individuals holding money at date s is $1 - p_s + q_s = 1$. Since the total measure of individuals in the economy is 2, having half the population holding money maximizes the extensive margin when $\sigma = \tau = 0$. The value of x when $p_d = q_d$ is easily computed.

Lemma 6 *If x is acyclical and $p_s = q_s$, then*

$$p_d = \frac{1 + \sqrt{\theta} - \sqrt{(1+\sqrt{\theta})^2 - 4\pi_d}}{2\pi_d},$$

and

$$p_e = 1 - p_d + \pi_d p_d^2$$

where $\theta = \pi_d / \pi_e$.

Proof. Since by lemma 1, $\pi_e p_e q_e = \pi_d p_d q_d$, equation (2) with $\tau = 0$ yields $p_d = 1 - q_e + \pi_e p_e q_e$. Because $q_s = p_s$, then $p_e = \sqrt{\theta} p_d$ and $p_d = 1 - p_e + \pi_d p_d^2$. The last two expressions yields a quadratic equation in p_d whose only relevant solution is as stated. The value for p_e can be computed from the last expression once p_d is determined. ■

We now consider small pertubations in the quantity of money. We consider cyclical distributions x^+ in a neighborhood of a given acyclical x. Our goal is to describe the derivative of the sum $\sum_s \pi_s p_s q_s$ with respect to σ, evaluated at $\sigma = 0$ and $p_s = q_s$. It follows, by force of proposition 1, that if this derivative is positive, then the maximizer of the sum must be cyclical. The system (1-4) defining x^+ depends on σ and τ. Existence of x^+ follows from a simple fixed-point argument.

Lemma 7 *Let $(\tau, \sigma) \in (0,1)^2$ be fixed. Then there exists an invariant distribution x^+.*

Proof. The right-hand side of (1-4) defines a continuous function of x^+, with domain on the compact and convex set $[0,1]^4$. The result then follows from Brower's fixed point theorem. ■

If x^+ is invariant, then the quantity of money destroyed in season d must equal the quantity created in e, that is,

$$\tau(1 - p_e^+ + q_e^+) = \sigma(1 - q_d^+ + p_d^+). \tag{11}$$

It can be shown that the equality (11) is implied by the system (1-4). The quantity of money during season e meetings, just before trade, is given by the mass $1 - p_e^+$ with producers, plus the mass q_e^+ with consumers. Since trade itself does change this quantity of money, and each money holder at the beginning of next season faces a probability τ of losing his money, then the total destroyed is given by the left-hand side of (11). Likewise, the measure of individuals without money at the end of season d is $1 - q_d^+ + p_d^+$, and since each of those finds money at the beginning of season e with probability σ, then the quantity of money created is that expressed in the right-hand side of (11).

While there is a continuum of acyclical distributions, one x for each q_d, the same does not hold with the cyclical ones. When σ and τ are positive, there is an inflow of money that has to be matched by an outflow of the same quantity. Our numerical experiments indicate that only one level of q_d^+ produces quantities of money that is capable of equalizing inflows and outflows for a given pair (σ, τ). However, we can pin down a neighborhood in which q_d^+ lies as follows. Because we want to associate x^+ to a given x, we find it useful to define the constant ϕ with the property that, for $\tau = \phi\sigma$, x^+ converges to x as σ approaches zero. Since the pair (σ, τ) must be consistent with the stationary quantities of money in the economy, expressed above by equation (11), the desired ratio of τ to σ, for a given $x = (p_e, q_e, p_d, q_d)$, is

$$\phi = \frac{1 - q_d + p_d}{1 - p_e + q_e}. \tag{12}$$

By lemma 1 and proposition 1, the maximizer of the sum $\sum_s \pi_s p_s q_s$ among the set of acyclical distributions is the unique x for which $\phi = 1$. We assess the effects of pertubations by differentiating the system (1-4) with respect to σ for ϕ fixed. In order to facilitate the algebra with the equations in x, we find it useful to define the function f_s as follows.

Lemma 8 *If x is invariant and $\tau = \phi\sigma$, then $p_s - q_s = f_s(\sigma)$, where $f_e(\alpha) = \frac{\phi - \phi\alpha - 1}{1 + \phi - \phi\alpha}$ and $f_d(\alpha) = \frac{\phi + \phi\alpha - 1}{1 + \phi - \phi\alpha}$.*

Proof. The system (1-4) can be rewritten as

$$\hat{p}_e = 1 - (1 - \tau)\hat{q}_d + \pi_d p_d q_d, \tag{13}$$

$$\hat{p}_d = 1 - (1 - \sigma)\hat{q}_e + \pi_d p_e q_e + \frac{\tau}{1 - \tau}, \tag{14}$$

$$\hat{q}_e = 1 - (1 - \tau)\hat{p}_d + \pi_d p_d q_d + \frac{\sigma}{1 - \sigma} \tag{15}$$

and

$$\hat{q}_d = 1 - (1 - \sigma)\hat{p}_e + \pi_e p_e q_e, \tag{16}$$

where $\hat{p}_e = p_e/(1-\sigma)$, $\hat{p}_d = p_d/(1-\tau)$, $\hat{q}_e = q_e/(1-\sigma)$ and $\hat{q}_d = q_d/(1-\tau)$. Eliminating $\pi_d p_d q_d$ between equations (13) and (15), and $\pi_e p_e q_e$ between (14) and (16), yields

$$\hat{p}_e - \hat{q}_e = (1 - \tau)(\hat{p}_d - \hat{q}_d) - \frac{\sigma}{1 - \sigma}$$

and

$$\hat{p}_d - \hat{q}_d = (1 - \sigma)(\hat{p}_e - \hat{q}_e) + \frac{\tau}{1 - \tau},$$

which can now be solved as

$$\hat{p}_e - \hat{q}_e = \frac{(1-\sigma)\tau - \sigma}{(1-\sigma)[1-(1-\tau)(1-\sigma)]} \tag{17}$$

and

$$\hat{p}_d - \hat{q}_d = \frac{\tau - (1-\tau)\sigma}{(1-\tau)[1-(1-\tau)(1-\sigma)]}. \tag{18}$$

One can now multiply both sides of (17) by $1-\sigma$ to obtain the expression $p_e - q_e = f_e(\sigma)$, and multiply both sides of (18) by $1-\tau$ to obtain the expression $p_d - q_d = f_d(\sigma)$. ■

Using now the expression $q_s = p_s - f_s$ to reduce (1-4) to a system in (p_e, p_d), allows us to write the derivatives of p_s with respect to σ as follows.

Lemma 9 *If x is invariant and $\tau = \phi\sigma$, then the derivatives of p_s with respect to σ, evaluated at $\sigma = 0$, satisfy*

$$\begin{bmatrix} 1 & 1-\pi_d(2p_d - f_d) \\ 1-\pi_e(2p_e - f_e) & 1 \end{bmatrix} \begin{bmatrix} p'_e \\ p'_d \end{bmatrix} = \begin{bmatrix} (1-\pi_d p_d) f'_d - p_e \\ (1-\pi_e p_e) f'_e - \phi p_d + \phi \end{bmatrix}.$$

Proof. Equations (1) and (2) can be written as

$$\frac{p_e^+}{1-\sigma} = 1 - p_d^+ + f_d + E_d \tag{19}$$

and

$$\frac{p_d^+}{1-\phi\sigma} - \frac{\phi\alpha}{1-\phi\sigma} = 1 - p_e^+ + f_e + E_e, \tag{20}$$

where $E_d = \pi_d p_d^+ (p_d^+ - f_d)$ and $E_e = \pi_e p_e^+ (p_e^+ - f_e)$. Taking derivatives on both sides of (19) and (20), with respect to σ, yields, for $\sigma = 0$,

$$p_e + p'_e = -p'_d + f'_d + E'_d \tag{21}$$

and

$$\phi p_d + p'_d - \phi = -p'_e + f'_e + E'_e, \tag{22}$$

where $E'_d = \pi_d p'_d (2p_d - f_d) - \pi_d p_d f'_d$ and $E'_e = \pi_e p'_e (2p_e - f_e) - \pi_e p_e f'_e$. Substituting the expressions for E'_d and E'_e into equations (21) and (22), yields the result. ■

The total effect of changes in σ on extensive margins can also be expressed in a compact form.

Lemma 10 *If x is invariant and $\tau = \phi\sigma$, then the derivative of the sum $\sum_s \pi_s p_s q_s$, with respect to σ, evaluated at $\sigma = 0$, is equal to $p_e + \phi p_d - \phi - f'_e - f'_d + 2(p'_e + p'_d)$.*

Proof. Using equations (21) and (22), derived in the proof of the previous lemma, yields the results, since the derivative of the sum $\sum_s \pi_s p_s q_s$ is precisely $E'_d + E'_e$. ■

Using now the last three lemmas, we can characterize the sign of the derivative of the sum $\sum_s \pi_s p_s q_s$, for $p_s = q_s$, as follows.

Proposition 11 *The maximizer of sum $\sum_s \pi_s p_s q_s$ is cyclical if and only if $\pi_d \in [0, \bar{\pi}]$, where $\bar{\pi} \in (0, \pi_e)$ can be solved for analytically as a function of π_e.*

Proof. The last three lemmas allow the substitution of expressions for $p'_e + p'_d$ and $f'_e + f'_d$ into the expression of the derivative of $\sum_s \pi_s p_s q_s$, evaluated at $\sigma = 0$, $p_s = q_s$ and $\phi = 1$. Substituting also the analytical solution for p_e and p_d, when $p_s = q_s$ and $\sigma = 0$ from lemma 5, yields an expression for the derivative involving only parameters. After some tedious but straightforward algebra, the condition according to which this derivative is positive can be written as

$$2\pi_d \leq (1-\theta)\sqrt{2} - (1-\sqrt{\theta})^2,$$

where $\theta = \pi_d/\pi_e$. The inequality is not satisfied for $\theta = 1$ and $\pi_d > 0$. Hence the cutoff value of π_d for which the derivative is positive must be below π_e. Imposing now equality in this expression and substituting for the value of θ yields, after solving for the unique relevant solution of the implied quadratic equation in π_d^2,

$$\bar{\pi} = \frac{1}{4}\left[\frac{2/\sqrt{\pi_e} + \sqrt{4/\pi_e - 4(2 + (1+\sqrt{2})/\pi_e)(1-\sqrt{2})}}{2 + (1+\sqrt{2})/\pi_e}\right]^2,$$

which has the properties stated in the proposition. ■

The maximizer is therefore acyclical if $\pi_d = \pi_e$. Intuition behind which policy—acyclical or cyclical—maximizes the average extensive margin is straightforward. Suppose that $\pi_d = \pi_e = \pi$. Then, the optimal policy, i.e., one that maximizes the average extensive margin, is given by $p_e = p_d = q_e = q_d \equiv t$. Now if σ is slightly increase from zero, there will be a stationary cyclical distribution x^+ in the neighborhood of

x. When $\sigma > 0$, then $M_d = 1 - p_d^+ + q_d^+ < 1 < M_e = 1 - p_e^+ + q_e^+$. Hence, it must be the case that q_e increases by more than p_e decreases and q_d decreases by more than p_d increases when σ (and τ) is increased from zero. Therefore, $p_e^+ q_e^+ > t^2$ and $p_d^+ q_d^+ < t^2$: For a cyclical monetary policy, the extensive margin will increase in season e and will decrease in season d, compared to the acyclical policy. Since, in a world with "no seasons," i.e.,$\pi_d = \pi_e = \pi$, a constant stock of money is optimal, it must be the case that the negative extensive margin effect associated with season d out weighs the positive extensive margin effect associated with season e. Another way of thinking about this result is that when $p_s q_s$ is "equally weighted", i.e., $\pi_e = \pi_d$, the (negative) odd season effect dominates the (positive) even season effect. Suppose now that $\pi_d < \pi_e$. It will still be the case that $p_e^+ q_e^+ > p_e q_e$ and $p_d^+ q_d^+ < p_d q_d$, where (p_e, q_e, p_d, q_d) is the distribution associated with the acyclical monetary policy. However, since the differences between "p" and "q" do not depend upon π_e and π_d, see equations (17) and (18), it may now be the case that the (positive) even season effect dominates the (negative) odd season effect. And this is because the even season matching probability, $p_e q_e$, is weighted more heavily than the odd season matching probability, $p_d q_d$, i.e., $\pi_e > \pi_d$. Hence, if the fraction of potential consumers in odd periods is sufficiently smaller than the fraction of potential consumers in even periods—or if demand in the "high" season is sufficiently greater than demand in the "low" season, then a cyclical monetary policy will deliver a higher average extensive margin than an acyclical policy.

6 Intensive-margin effects

The only participation constraints that are relevant, given our notion of stationarity, are those of producers. In this section, we derive representations of producer constraints as functions of preference parameters, policy parameters, and allocations, without references to value functions. While the first order effect of cyclical interventions is a tightening of participation constraints, these negative effects can be negligible or even absent if the discount factor is sufficiently high.

Substituting the values of $(\bar{v}, \bar{w})$ from equation (6) into equation (5), allow us to work with two independent systems of Bellman equations in (v, w), represented in matrix format as

$$\left[\begin{array}{c} v_s \\ w_{s'} \end{array}\right] = \frac{1}{\det(M_{ss'})} M_{ss'} \left[\begin{array}{c} \mu_{us}\pi_s p_s u_s \\ -\mu_{ys'}\pi_{s'} q_{s'} y_{s'} \end{array}\right] \tag{23}$$

where $s, s' \in \{e, d\}$, $s' \neq s$, $\mu_{ue} = \mu_{yd} = 1$, $\mu_{ud} = 1 - \tau$, $\mu_{ye} = 1 - \sigma$, and

$$M_{ss'} = \left[\begin{array}{cc} 1 - (1 - \pi_{s'} q_{s'})\delta(1 - \sigma) & \tau\beta + (1 - \tau)\pi_s p_s \beta \\ \sigma\beta + (1 - \sigma)\pi_{s'} q_{s'} \beta & 1 - (1 - \pi_s p_s)\delta(1 - \tau) \end{array}\right].$$

We start with the following lemma that allow us to ignore $\det(M_{ss'})$ in the algebra that follows.

Lemma 12 *The determinant of $M_{ss'}$ is positive.*

Proof. For $a_d \equiv 1 - \pi_d q_d$ and $a_e \equiv 1 - \pi_e p_e$, the determinant of M_{ed} equals

$$(1 - \delta a_d + \sigma\delta a_d)(1 - \delta a_e + \tau\delta a_e) - \delta(\pi_d q_d + \sigma a_d)(\pi_e p_e + \tau a_e),$$

which can be written as the sum of two terms, k_0 and k_1, where k_0 contains all the terms without σ or τ, and k_1 contains the other terms. The expression for k_0 is

$$k_0 = [1 - \delta(1 - \pi_d q_d)][1 - \delta(1 - \pi_e q_e)] - \delta\pi_d q_d \pi_e q_e.$$

After some simple algebra, that expression becomes

$$k_0 = (1 - \delta)(1 - \delta + \delta\pi_d q_d + \delta\pi_e q_e - \delta\pi_d q_d \pi_e q_e),$$

which is positive if x is invariant. Likewise, since for $a_d \equiv 1 - \pi_d q_d$ and $a_e \equiv 1 - \pi_e p_e$, one can write k_1 as

$$\begin{aligned} &\tau\delta a_e(1 - \delta a_d - \pi_d q_d) + \sigma\delta a_d(1 - \delta a_e - \pi_e p_e) + \delta\sigma a_d \tau a_e(\delta - 1), \text{ or} \\ &\tau\delta a_e(1 - \delta)(1 - \pi_d q_d) + \sigma\delta a_d(1 - \delta)(1 - \pi_e p_e) - \sigma\delta a_d(1 - \delta)\tau a_e, \text{ or} \\ &\tau\delta a_e(1 - \delta)(1 - \pi_d q_d) + \sigma\delta a_d(1 - \delta)(1 - \pi_e p_e)(1 - \tau), \end{aligned}$$

which is nonnegative. A similar argument shows that $\det(M_{de})$ is also positive. ■

Next, we use the Bellman equation for $w_e$to write (7) in an equivalent format that does not depend on y_e explicitly.

Lemma 13 *The participation constraint for date-s producers is equivalent to* $[1-(1-\sigma)\delta]w_s \geq \sigma\beta v_{s'}$.

Proof. Let $s = e$. The Bellman equation for w_e can be written as

$$[1-(1-\sigma)\delta]w_e - \sigma\beta v_d = (1-\sigma)\pi_e q_e(-y_e + \beta v_d - \beta\bar{v}_d)$$

then the result follows directly from (7). The argument for $s = d$ also follows from the same steps. ■

We now use the previous two lemmas to write the slack in the producer constraint in matrix algebra as

$$\begin{bmatrix} -\sigma\beta & 1-(1-\sigma)\delta \end{bmatrix} \begin{bmatrix} v_s \\ w_{s'} \end{bmatrix} = \frac{1}{\det(M_{ss'})} \begin{bmatrix} m_{us} & m_{ys'} \end{bmatrix} \begin{bmatrix} \mu_{us}\pi_s p_s u_s \\ -\mu_{ys'}\pi_{s'} q_{s'} y_{s'} \end{bmatrix} \quad (24)$$

where the scalars m_{us} and $m_{ys'}$ are to be computed, so that the sign of the participation constraint does not depend on the magnitude of $\det(M_{ss'})$. After some straightforward algebra is used to produce a simple expression for m_{us} and $m_{ys'}$, the desired inequalities are derived as follows.

Proposition 14 *The participation constraints are satisfied if and only if*

$$u_d(y_d) \geq \frac{y_e}{\beta}\left[\frac{1}{(1-\tau)\pi_d p_d} - (1-\sigma)\delta\frac{1-\pi_d p_d}{\pi_d p_d}\right] \quad (25)$$

and

$$u_e(y_e) \geq \frac{y_d}{\beta}\left[\frac{1}{(1-\sigma)\pi_e p_e} - (1-\tau)\delta\frac{1-\pi_e p_e}{\pi_e p_e}\right] \quad (26)$$

Proof. The steps for deriving the inequality (25) are simple; we omit the proof for inequality (26) it is identical to the proof of inequality (25). Regarding participation constraint for date-e producers, we find it useful to set $\rho = \pi_d p_d$ and $\xi = \pi_e q_e$ so that the expression for m_{ud} can be written as

$$\begin{aligned} -m_{ud} &= \sigma\beta - \sigma\beta(1-\sigma)(1-\xi)\delta - \sigma\beta - (1-\sigma)\xi\beta + \sigma\beta(1-\sigma)\delta + \\ &\quad (1-\sigma)\xi\beta(1-\sigma)\delta \\ &= -(1-\sigma)\xi\beta + (1-\sigma)\xi\beta\delta \\ &= -(1-\delta)(1-\sigma)\xi\beta. \end{aligned}$$

That for m_{ye} is

$$\begin{aligned}
-m_{ye} &= \sigma\beta\tau\beta + \sigma\beta(1-\tau)\rho\beta - 1 + (1-\tau)(1-\rho)\delta + (1-\sigma)\delta + \\
&\quad -(1-\sigma)\delta(1-\tau)(1-\rho)\delta \\
&= \sigma\beta\tau\beta + \sigma(1-\tau)\rho\delta - 1 + (1-\sigma)\delta + \\
&\quad -(1-\tau)(1-\rho)\delta[(1-\sigma)\delta - 1] \\
&= -1 + \delta - \sigma\delta[1-\tau-(1-\tau)\rho] + \sigma\delta(1-\tau)(1-\rho)\delta + \\
&\quad (1-\delta)(1-\tau)(1-\rho)\delta \\
&= -1 + \delta - \sigma\delta(1-\tau)(1-\rho) + \sigma\delta(1-\tau)(1-\rho)\delta + \\
&\quad (1-\delta)(1-\tau)(1-\rho)\delta \\
&= -1 + \delta - (1-\delta)\sigma\delta(1-\tau)(1-\rho) + (1-\delta)(1-\tau)(1-\rho)\delta \\
&= -(1-\delta)[1-(1-\sigma)\delta(1-\tau)(1-\rho)].
\end{aligned}$$

Thus, the right-hand side of (24) equals

$$\frac{(1-\delta)(1-\sigma)\pi_e q_e}{\det(M_{de})}\begin{bmatrix} \beta & 1-\delta(1-\sigma)(1-\tau)(1-\pi_d p_d) \end{bmatrix}\begin{bmatrix} (1-\tau)\pi_d p_d u_d \\ -y_e \end{bmatrix},$$

so that (25) follows. ∎

Inequalities (25) and (26) indicate that cyclical policies have a potentially negative effect on intensive margins, since the right-hand side of both inequalities is increasing in σ and τ. The intuition behind these potential negative effects is straightforward: In either case—whether money is injected or withdrawn from the economy—the value of money in a trade will fall compared to the situation where $\sigma = \tau = 0$. In the case where the money supply is contracted after production and trade, the value of currency falls because there is a chance that the producer will be unable to use his unit of currency in a future trade; in the case where the money supply is expanded after production and trade, the fact that a producer may receive a unit of currency if he does not produce reduces the value of a unit of currency for a producer who does. A fall in the value of money implies that the amount of output received per unit of currency is reduced. If, however, β is sufficiently high, then the inequalities do not bind for $y = y^*$, the output levels that maximize the intensive margins, $y = y^*$, and the potential negative affects on the intensive margins do not materialize for small monetary interventions.

Suppose that neither participation constraint binds and that $\sigma = 0$. It turns out that if β is then reduced to lower levels, then the first participation constraint to be violated is the participation constraint for date-e producer, (25). Hence,

Lemma 15 *If the participation constraint for date-e producers is satisfied for x acyclical and $y = y^*$, then (x, y) is implementable.*

Proof. Since $u_e' \geq u_d'$ and $u_e(0) = u_d(0)$ then $u_e'(y_d^*) \geq 1$, so that $y_e^* \geq y_d^*$ and $u_e^*(y_e^*) \geq u_d^*(y_d^*)$. Now, it has been established in the previous section that, if x is acyclical, then $\pi_d \leq \pi_e$ implies $q_d \leq q_e$. As a result, the equality $\pi_e p_e q_e = \pi_d p_d q_d$, which holds for x, and $\pi_d \leq \pi_e$ implies $\pi_d p_d \leq \pi_e p_e$. Since the right-hand side of (25) or (26) is increasing in $\pi_s p_s$, and since $u_e^*(y_e^*)/y_d^* \geq u_d^*(y_d^*)/y_e^*$, then the result follows. ■

The result of the previous lemma indicates that it suffices to look at the participation constraint for date-e producers in order to find a value of β such that small interventions have no negative effects on intensive margins.

Proposition 16 *Let x take the value of the acyclical distribution with $p_s = q_s$, and let $\beta > \bar{\beta}$, where*

$$\bar{\beta} = \frac{-\frac{u_d(y_d^*)}{y_e^*} + \sqrt{\left(\frac{u_d(y_d^*)}{y_e^*}\right)^2 + 4\frac{1-\pi_d p_d}{(\pi_d p_d)^2}}}{2\frac{1-\pi_d p_d}{\pi_d p_d}}.$$

Then if σ is sufficiently small, the cyclical allocation (x^+, y^*), for x^+ in a neighborhood of x, is implementable.

Proof. The cutoff value $\bar{\beta}$ was constructed so that (x, y^*) is implementable for $\beta = \bar{\beta}$. Since the participation-constraint sets vary continuously with (σ, τ), the result follows. ■

7 Optimal policies

On one hand, our results regarding extensive-margin effects show that there exists a cutoff value for π_d, called $\bar{\pi}$, such that the maximizer of the average extensive margin is cyclical if and only if $\pi_d < \bar{\pi}$. On the other hand, our results on intensive margins show that there exists a cutoff value of β, called $\bar{\beta}$, such that for $\beta > \bar{\beta}$, small interventions around the allocation (x, y^*), where $p_s = q_s$, are implementable. If follows, therefore, that the optimum is cyclical for a large set of parameters, including those π_s and β such that $\pi_d < \bar{\pi}$ and $\beta > \bar{\beta}$.

Proposition 17 *If $\pi_d < \bar{\pi}$ and $\beta \geq \bar{\beta}$, then the optimum monetary policy is cyclical.*

Proof. Welfare is proportional to $\sum_s E_s I_s$, where E_s is the extensive margin at s, $\pi_s p_s q_s$, and I_s is the intensive margin at s, $u_s(y_s) - y_s$. By lemma 1, $E_e = E_d$ for all acyclical policies, so that for fixed (I_e, I_d), the acyclical x that maximizes welfare features $p_s = q_s$. Since $\beta \geq \bar{\beta}$, then y^* satisfies participation constraints evaluated at this maximizer, so that the allocation that attains the highest welfare among acyclical policies is (x, y^*). Since a small intervention increases E_e and $E_e + E_d$ when $\pi_d < \bar{\pi}$, and $I_e \geq I_d$ for $y = y^*$, and such intervention is implementable according to our last proposition, then the optimal cannot be acyclical. ■

The proof of the previous proposition follows even when $u_e = u_d$. For a fixed allocation (x, y), increases in the value of the intensive margin at date e relative to that at date d resulting from an increase in the valuation of date e consumption relative to date d consumption, i.e., $u_e(c) > u_d(c)$ and $u'_e(c) > u'_d(c)$, also increases the attractiveness of cyclical policies when they increase the extensive margin at e. We document this property with numerical simulations as follows. We set $u_d(y_d) = 1 - \exp(-Ay_d)$, for $A = \exp(3)$, which the property that $y_d^* = \log(A)/A$. We then set $u_e = zu_d$, where $z \geq 1$ is a constant that we let vary in the experiments. Figure 1 illustrates the "base case" that sets $\beta = \bar{\beta}$, $\pi_d = \bar{\pi}$, and $z = 1$. For the acyclical policy, welfare is maximized at $q_d = p_d = .725$. Similarly, for a cyclical policy with $\sigma = .005$ welfare is maximized in the neighborhood $q_d = .725$ and, as would be expected, welfare is (approximately equal) to that of the acyclical policy. Note that if the amount of money is set "too low" under the acyclical policy, i.e., $p_s > q_s$ for $s \in \{e, d\}$, then the there exists a cyclical policy—that is in the neighborhood of the acyclical policy—that delivers higher welfare than the acyclical policy. (In figure 1, the amount of money is the economy is too low under the acyclical policy when

$q_d < .725$.) Since the amount of money in the economy under the acyclical policy is "too low" there is a benefit associated with the cyclical policy since in even dates the money supply is increased from is suboptimally low level. It is true that in odd periods, the money supply will now be even lower, but since the weight given to the odd period, π_d, is sufficiently small, i.e., $\pi_d = \bar{\pi}$, the beneficial effect of a higher money supply in the even periods dominates the negative effect of a lower money supply in odd periods. When the money supply is "too high" under the cyclical policy, i.e., when $q_d > .725$, then the acyclical policy always dominates a cyclical policy and this is because the money supply is already too high and increasing the money supply in even periods only makes matters worse. Since the relative weight given to the even periods, π_e/π_d, is sufficiently large, the negative effect associated with higher levels of money supply in the even periods dominates the positive effect of a lower money supply in odd periods.

Figures 2 and 3 provide a variation on Proposition 5. In particular, these figures demonstrate that if $u_e = u_d$, and if either $\beta < \bar{\beta}$ when $\pi_d = \bar{\pi}$ or $\pi_d > \bar{\pi}$ when $\beta = \bar{\beta}$ holds, then an acyclical policy will dominate a cyclical policy. Figure 2 considers the case where $\beta < \bar{\beta}$ and $\pi_d = \bar{\pi}$. Here, the optimal acyclical policy is characterized by $q_d = p_p =?$. There does not exist a cyclical policy that can attain the level of welfare that is associated with the optimal acyclical policy. Note that similar to figure 1, if the money supply gets sufficiently small, then a cyclical policy can deliver higher welfare than the acyclical policy; but if the money supply is "too large" the acyclical policy always dominates the cyclical policy. Figure 3 considers the case where $\pi_d > \bar{\pi}$ and $\beta = \bar{\beta}$. Here, even when the money supply under the acyclical policy is too low, the welfare associated with an acyclical policy exceeds that associated with the cyclical policy. The reason is that now the weight placed on the odd periods, π_d, is "sufficiently high," i.e., $\pi_d > \bar{\pi}$, so the negative effect of decreasing the money supply in odd periods when the money supply is already "too low" dominates the positive effect of increasing the money supply in even periods.

Figures 4 and 5 consider the case where $u_e(c) > u_d(c)$ and $u'_e(c) > u'_d(c)$. In particular we set $u_e = 5u_d$. Figure 4 illustrates the difference between policies when only one parameter is changed from the base case in figure 1. In figure 5 has $\beta = \bar{\beta}$

and $\pi_d = \bar{\pi}$, but $z = 5$. Given that both total and marginal utility of consumers are higher in even periods than odd periods, there is now a benefit associated with cyclical monetary policy. The idea here is that although a cyclical policy will increase the number of matches in even periods and decrease the number of matches in odds periods, because consumption has higher value in even periods, the first effect will dominate the second effect. Figure 5 can be interpreted as demonstrating that when $u_e(c) > u_d(c)$ and $u'_e(c) > u'_d(c)$, the conditions described in proposition 5 are sufficient, but not necessary, for the optimal monetary policy to be cyclical. The intuition behind the result contained in figure 5 is exactly the same as the intuition that underlies figure 4.

8 Conclusion

We have constructed a random matching model of seasons, where different seasons are characterized by both differing desires and intensities of the buyer to consume. Even when buyer's intensity to consume is constant over seasons—and only the desire to consume varies over seasons—we show that a monetary policy that injects money into the economy when the desire to consume is high and withdraws it when the desire is low may be beneficial. A cyclical policy increases the chances of a single coincidence meetings in the high season and decreases their chances in the low season, compared to a constant monetary policy. A cyclical policy will be beneficial if the proportion of consumers who want to consume is small in the low season relative to the high season. In this situation the *average* number of successful matches over both seasons will increase—which in turn increases welfare—because the measure of single coincidence matches in the high season is weighted by a larger factor than that in the low season. When the seasons are characterized by both differing desires *and* intensities to consume by the buyer, then a cycle monetary policy can be optimal even when the difference between the proportions of the buyers that want to consume in the high and low seasons is not very large. Our theory provides some additional support for the founding of the Fed. Previous explanations relied on the reduction in financial panics that came about after the founding of the Fed; our explain relies on the improved distribution of production and consumption that results when the Fed follows a cyclical policy.

9 References