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DIVISIBLE MONEY IN AN ECONOMY WITH VILLAGES

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Abstract

This paper provides a tractable search model with divisible money that encompasses the two frameworks currently used in the literature. Individuals belong to many villages. Inside a village, individuals share information so financial contracts are feasible. Money is essential to facilitate trade with individuals outside the village. The framework proposed by Lagos and Wright (2003) arises as a special case if some goods trade competitively while others trade in search markets, and preferences are quasi-linear. The framework proposed by Shi (1997) arises as a special case if individuals have complete information about their fellow villagers. The type of financial arrangements that arise in villages are not limited to insurance contracts. Credit contracts play a major role if individuals choose endogenously when to go shopping. Lotteries can substitute for insurance and credit under some conditions. The contradictory results on the welfare cost of inflation found in Shi and Lagos and Wright are reconciled.

Keywords: monetary search, divisible money, inflation cost.

JEL: E40, E52, D83.

1 Introduction

Monetary search models have provided rich insights on the foundations of money, and they have become the dominant paradigm in this field of economics. To facilitate tractability, early monetary search models made strong assumptions on the properties of money (indivisibility and limited storage capacity). These strong assumptions prevented the study of many interesting issues such as the effects of inflation. Thanks to the work of Shi (1997) and Lagos and Wright (2003), we have now two distinct frameworks that yield tractable monetary search models with divisible money. Both frameworks use a trick to obtain a tractable distribution of money balances. In the case of Shi, the trick is the assumption that individuals belong to large households. In the case of Lagos and Wright, the trick is the assumption that utility is linear on a good traded in a competitive market. Despite the many accomplishments on the rich literature that has followed the seminal contributions of Shi, and Lagos and Wright, there is a major unresolved problem. Quite often a model in one of these two frameworks obtains a conclusion that contradicts the findings of a model in the other framework. For example, Shi (1997) argues that a positive rate of inflation may well be optimal, while Lagos and Wright (2003) argues that the costs of inflation are much larger than previously thought. Since both frameworks are built around different tricks, it is difficult to trace the ultimate reasons for these disparate findings. The present paper introduces a comprehensive framework that encompasses those advanced by Shi (1997), and Lagos and Wright (2003). This framework is useful not only to compare these earlier models, but also to enrich the set of issues that can be dealt with monetary search models.

In the model of this paper, individuals belong to villages.[1] Each village contains a large number of individuals, but it is only a small part of the global economy. In a village, individuals are not altruistic as in a household, but they know their neighbors affairs. Therefore, financial contracts such as insurance and credit are feasible among individuals of the same village. Despite the existence of financial contracts inside the village, money is still essential to facilitate trade with anonymous individuals from other villages.[2] This model captures in a simple fashion that in our

[1] Jin and Temzelides (2004) advance also a random search model with villages to generate equilibria where money is used in some exchanges while credit is used in some others. However, they do not attempt to use villages to make tractable the divisibility of money. In their model, both money and goods are indivisible, and credit is a gift giving equilibrium with trigger strategies.

[2] Money is essential if it allows outcomes impossible with it (see Wallace 2001).

daily economic interactions sometimes we deal with well identified and easy to trace individuals and sometimes we deal with relative strangers that can easily disappear from our lives.

The framework advanced by Lagos and Wright (2003) is a special case of the one introduced in this paper if individuals trade competitively in the village during the day, trade in search markets outside the village during the night, and the utility for one of the commodities traded during the day is linear (quasi-linear preferences). Relaxing the quasi-linearity of preferences is important for several reasons. Quasi-linear preferences imply risk neutrality. Moreover, their usefulness in delivering a tractable distribution of money holdings rest on the absence of liquidity constraints during the day. Therefore, resting on these preferences to support tractability rules out most of the issues dealt in financial economics. Also, quasi-linear preferences rule out wealth effects on all goods except for the one that yields linear utility. This hinders the study of many interesting issues in macroeconomics where wealth effects are important.

The framework advanced by Shi (1997) is equivalent to the one introduced in this paper if there is complete information and perfect enforceability of contracts inside a village. In this environment, individuals can, among other things, insure all trading risks. The idea that the large household construct is a way to implicitly insure trading risks goes as far back as Lucas (1990). The present contribution takes seriously Lucas's idea and fleshes out the mechanisms that arise in the village to insure trading risks. There are several advantages to design these mechanisms explicitly. Typically, the mechanisms for insuring trading risks are financial contracts which are interesting on themselves. Another advantage of designing the financial contracts is that once we state explicitly the extend of risk sharing in the village, there is no ambiguity on the objectives of buyers and sellers when they bargain in a trade meeting. Finally, one can vary the information that is shared inside the villages to find out if the abstraction of trading risks is relevant or not to a particular issue.

The type of financial arrangements that arise in the village are not limited to insurance contracts. Credit contracts play a major role if individuals find optimal not to go shopping every period. Furthermore, with suitable assumptions, credit inside the village takes interesting forms that open new avenues for research. For example, in a companion paper, Faig (2004), the credit market in the village is centered around banks that offer deposits that at a cost are transferable to banks from other villages. As a result, both money and banking play essential roles in that extension.

If preferences are not quasi-linear and trading risks cannot be insured, then, in general, the distribution of money holdings is non-degenerate. This reduces the tractability of the model dramatically, so a treatment of this general case is not attempted in this paper. However, the paper studies the elimination of insurance in a simple special case where individuals choose optimally their shopping frequency. In this special case, lotteries, which involve no information about individual histories and trading opportunities, perfectly substitute for insurance and credit.

In summary, the present model provides a unifying framework for tractable search monetary models with divisible money. This framework fleshes out the financial contracts or alternative institutions required to obtain a degenerate distribution of money holdings. Also, this framework permits a clarification of the disparate claims of Shi (1997) and Lagos and Wright (2003) about the welfare cost of inflation. These claims can be reconciled as follows. As Shi pointed out, if the number of buyers shopping in the search market is endogenous, then some inflation may well be beneficial. However, this only happens if buyers have most of the bargaining power in trade meetings. Otherwise, as Lagos and Wright pointed out, the welfare cost of inflation is larger than previously thought and increasing with the bargaining power of sellers.

The rest of the paper is organized as follows. The basic model of this paper is analyzed in Section 2. This model generalizes the attractive version of Lagos and Wright (2003) found in Rocheteau and Wright (2003). Section 3 studies a variation of the model in Section 2 in which all goods are traded in monetary search markets and individuals decide how often they go shopping. The solution to this model is equivalent to the solution by Rauch (2000) of the Shi (1997) model.[3] Section 4 shows that an endogenous frequency of shopping makes the insurance of trading risks redundant under quite general conditions. Section 5 compares numerically the predictions by Lagos and Wright (2003) and Shi (1997) on the velocity of circulation of money and the welfare cost of inflation. Section 6 contains concluding remarks.

[3] Rauch (2000) adopts standard ex-post Nash bargaining, whereas Shi (1997) uses a form of ex-ante bargaining. See Berentsen, Rocheteau, and Shi (2001) for a rigorous specification of ex-ante bargaining.

2 The Model

The economy is composed of a large number of symmetric villages. Each village contains many individuals (a continuum) who can be of two types. Following Rocheteau and Wright (2003), these two types are labelled "buyers" and "sellers" because this is their role when they need money to trade. All members of a village share common information about their trading opportunities and their credit histories, but individuals are anonymous outside their village.

Time is discrete and the horizon is infinite. Each period consists of two subperiods: day and night. During the day, all the individuals can produce and consume a general nondurable good, which is traded competitively inside the village. Also during the day, individuals trade competitively with their fellow villagers a set of financial contracts to be specified below. During the night, individuals trade nondurable goods specific to each village. These goods are traded in a search market. The individuals labelled as sellers are able to produce the good specific to their own village. The individuals labelled as buyers get utility from the specific goods made in other villages. Neither buyers nor sellers get utility from the specific good made in their own village.

The measures of buyers and sellers are exogenous and normalized to one. The probabilities that a buyer and a seller meet a suitable trading partner in the search market are respectively π^b and π^s.

In this environment, there is a role for money to facilitate trade because of the following two reasons: (1) At night there is a lack of double coincidence of wants in all possible bilateral trading meetings. (2) Buyers are anonymous outside their village.[4]

Money is an intrinsically useless, perfectly divisible, and storable asset. The money supply grows at a constant factor γ such that $M_{+1} = \gamma M$, where M is the quantity of money per buyer. The subscript t is omitted in most expressions of the paper, so, for example, M stands for M_t and M_{+1} stands for M_{t+1}. New money is injected via lump-sum transfers to buyers at the beginning of each day.

The one period utility of a buyer is:

$$U^b(x, y, q) = v(x) - w(y) + u(q); \tag{1}$$

[4] The role of anonymity or lack of memory in generating a demand for money is pointed out in Levine (1991) and Kocherlakota (1998).

where x and y are respectively quantities consumed and produced of the general good, and q is the quantity consumed of specific goods from other villages. Likewise, the one period utility of a seller is:

$$U^s(x, y, q) = v(x) - w(y) - c(q). \tag{2}$$

The feasible production quantities y and q are bounded above by $\bar{y}$ and $\bar{q}$. The functions v, w, u, and c are all continuously differentiable and increasing. The functions u and v are concave, and w and c are convex. Moreover, $v(0) = w(0) = u(0) = c(0) = 0$, $w'(0) = c'(0) = 0$, $v'(0) = u'(0) = \infty$, $v'(\bar{y}) < w'(\bar{y})$, and $u'(\bar{q}) < c'(\bar{q})$. Finally, the concavity of u or the convexity of c will be strengthen below in the same fashion as in Lagos and Wright (2003) and Rocheteau and Wright (2003).

Buyers and sellers maximize their lifetime expected utilities: $E\sum_{t=0}^{\infty}\beta^t U^i\left(x_t, y_t, q_t\right)$, for $i = b$ and s, and $\beta \in (0, 1)$ is the discount factor. The discount and money growth factors obey: $\gamma > \beta$.

Individuals are not risk neutral as in Lagos and Wright (2003), so there are gains from insuring with fellow villagers the risks faced in the search market. Inside the village, it is known if buyers or sellers meet a trader or not in the search market. Therefore, individuals can insure against the risk on trading opportunities they face each night. That is, buyers can purchase a contract for the delivery of μ^b dollar next morning contingent upon meeting a seller during the night. Likewise, sellers can purchase a contract for the delivery of μ^s dollars next morning contingent upon failing to meet a buyer during the night. The fair premiums to acquire these contracts are respectively $\mu^b\pi^b$ and $\mu^s\left(1 - \pi^s\right)$. These premiums are also payable next morning.

In addition to insurance contracts, credit is viable because the credit history of all individuals is known inside their village. For concreteness, I assume that individuals can issue one period risk free nominal discount bonds that promise one unit of money next morning. These bonds trade in competitive markets inside the villages at a price $(1 + r)^{-1}$. Outside the villages of origin, these personal bonds are not valued because potential buyers do not know the issuer of these bonds. For the time being, initial bond holdings are assumed to be zero.

The concept of equilibrium combines competitive markets during the day with Nash bargaining at night. In equilibrium, buyers and sellers make optimal choices given the environment where they live. This environment includes the sequence of general good prices: $\{p_t\}_{t=0}^{\infty}$, the nominal interest rates on bonds, the rules describing the terms of trade that arise from the bargaining process, and the money balances other individuals carry to the search market. These prices, rates, rules, and balances must be consistent with optimal behavior, market clearing, and Nash bargaining. For simplicity, I focus on stationary equilibria where all individuals of the same type behave symmetrically and real allocations are constant over time.

To find an equilibrium, I adopt the following widely used strategy. First, I make a conjecture on the rate of inflation, the rate of interest on bonds, the terms of trade rules, and the money balances carried by other traders. Given this conjecture, I then solve for the optimal behavior of a representative buyer and a representative seller. Finally, I check that when individuals behave optimally the initial conjecture is consistent with market clearing, Nash bargaining, symmetry, and stationarity.

The conjecture about inflation is that the price of the general good increases at the same rate as the money supply:

$$p_{+1} = \gamma p. \tag{3}$$

The conjecture about the rate of interest on nominal bonds is that it is such that the real rate of interest is $\beta^{-1} - 1$:

$$r = \frac{\gamma - \beta}{\beta}. \tag{4}$$

The conjecture about bargaining is that the output and the money exchanged in a trading meeting (q, d) depend only on the quantity of money that the buyer carries $\left(m^b\right)$ and the price of the general good next period (p_{+1}) . The quantity q is a function of m^b/p_{+1}, and the payment d is a linearly homogeneous function of m^b and p_{+1}:

$$q = \tilde{q}\left(\frac{m^b}{p_{+1}}\right), \text{ and } d = \tilde{d}(m^b, p_{+1}), \tag{5}$$

In addition, these functions satisfy the following properties: (1) If m^b/p_{+1} is smaller or equal than a threshold z^*, then the buyer spends all the money that carries ($d = m^b$) and $\tilde{q}$ is an increasing and differentiable function of m^b/p_{+1}. (2) If m^b/p_{+1} is greater or equal than z^*, then the buyer spends a fixed amount of money $d = p_{+1}z^* \leq m^b$ to obtain a fixed amount of output $q = q^*$. (3) At $m^b = 0$, $\tilde{q}(0) = \tilde{d}(0, p_{+1}) = 0$. (4) For $m^b/p_{+1} \leq z^*$, $u\left[\tilde{q}\left(m^b/p_{+1}\right)\right]$ is a concave function of m^b/p_{+1}, and $\lim_{m^b \to 0} u'\left[\tilde{q}\left(m^b/p_{+1}\right)\right] \tilde{q}'\left(m^b/p_{+1}\right) = \infty$. As it will be seen, property 4 requires strengthening the assumptions made about u and c with an analog of Assumption 1 in Rocheteau and Wright (2003). Finally, other sellers carry no money balances to the search market, and other buyers carry an amount M^b that grows at the factor γ. This quantitative obeys: $M^b < z^* p_{+1}$.

2.1 The Behavior of Buyers

A representative buyer that starts the morning with wealth a^b faces the following budget constraint:

$$x^b + \frac{m^b + b^b (1+r)^{-1}}{p} = y^b + a^b, \quad m^b \geq 0, \tag{6}$$

where x^b and y^b are respectively the consumption and the production of general goods during the day, and m^b and b^b are respectively the dollars and bonds acquired during the day. The buyer is aware that the outcomes of bargaining in a trade meeting at night obeys the rules in (5), and can purchase fair insurance against the event of meeting a seller. The optimal choice of $\{x^b, y^b, m^b, \mu^b, b^b\}$ solves the following maximization program:

$$V^b(a^b) = \max_{\{x^b, y^b, m^b, \mu^b, b^b\}} v(x^b) - w(y^b) + \pi^b \left[u(q) + \beta V^b\left(a_{+1}^{b1}\right)\right] + \left(1-\pi^b\right)\beta V^b\left(a_{+1}^{b0}\right) \tag{7}$$

subject to (5) and (6). The terms a_{+1}^{b0} and a_{+1}^{b1} denote the wealth next morning. The superscripts $b1$ and $b0$ denote respectively the existence or not of a trading opportunity. Therefore,

$$a_{+1}^{b0} = \frac{b^b + m^b - \mu^b \pi^b + \tau}{p_{+1}}, \text{ and} \tag{8}$$

$$a_{+1}^{b1} = \frac{b^b + m^b - d^b + \mu^b\left(1-\pi^b\right) + \tau}{p_{+1}}; \tag{9}$$

where τ denotes the monetary transfers from the government, and d^b is the planned payment in a trade meeting. Another restriction to the maximization problem (7) is that next period wealth is bounded below by the condition that individuals must be able to repay their debts with probability one without reliance to unbounded borrowing (No-Ponzi game condition).

Standard recursive dynamic arguments show that V^b is a well defined value function that depends on a^b. Furthermore, V^b is concave and continuously differentiable. The Appendix provides a sketch of these arguments. These arguments are much simpler than those in Lagos and Wright (2003) because in this paper the feasible values of one period utilities are bounded. Hence, standard recursive results can be directly applied.

To solve (7), the buyer must behave as follows. The buyer must fully insure against the risk on trading opportunities because of the concavity of V^b:

$$\mu^b = d^b. \tag{10}$$

Therefore, the quantity of money held in the next morning is $m^b - d^b\pi^b + \tau$ regardless of meeting a seller or not, so $a_{+1}^{b0} = a_{+1}^{b1} \equiv a_{+1}^{b}$. Also, the buyer must equate the marginal utility of consumption of the general good to the marginal disutility of its production:

$$v'(x^b) = w'(y^b) \tag{11}$$

Finally, using the Envelope Theorem, $V^{b\prime}\left(a_{+1}^b\right) = v'(x_{+1}^b)$, the buyer must demand bonds and money to satisfy the following conditions:

$$v'(x^b) = \beta\left(1+r\right)\frac{p}{p_{+1}}v'(x_{+1}^b), \text{ and} \tag{12}$$

$$\frac{w'(y^b)}{p} = \pi^b u'\left(q\right)\tilde{q}'\left(\frac{m^b}{p_{+1}}\right)\frac{1}{p_{+1}} - \beta w'(y_{+1}^b)\left[\frac{\tilde{d}_{m^b}\left(m^b, p_{+1}\right)\pi^b - 1}{p_{+1}}\right]. \tag{13}$$

Equation (12) is a standard Euler condition equating the marginal utility of consuming the general good today with the marginal utility of acquiring bonds to consume tomorrow. Condition (13) equates the marginal cost and the marginal benefit of acquiring money. The cost is the disutility of producing the general good. The benefit is the expected increase in the consumer's surplus at night.

In the conjectured environment where (3) and (4) hold, conditions (11) and (12) imply

$$x^b = x_{+1}^b, \text{ and } y^b = y_{+1}^b. \tag{14}$$

Moreover, it is optimal for the buyer to avoid carrying money in excess of the amount ever spent in a trade meeting. Otherwise, $\tilde{q}'\left(m^b/p_{+1}\right) = \tilde{d}_{m^b}\left(m^b, p_{+1}\right) = 0$, which together with (3), (13), and (14) imply $\beta = \gamma$. Thus contradicting the assumption: $\gamma > \beta$. If the buyer exhaust money balances in a trade meeting, then $\tilde{d}_{m^b}\left(m^b, p_{+1}\right) = 1$. Hence, condition (13) simplifies to:

$$\frac{u'(q)}{\beta w'(y^b)}\tilde{q}'\left(\frac{m^b}{p_{+1}}\right) = 1 + \frac{r}{\pi^b}. \tag{15}$$

2.2 The Behavior of Sellers

A representative seller that starts the morning with wealth a^s faces the following budget constraint:

$$x^s + \frac{m^s + b^s(1+r)^{-1}}{p} = y^s + a^s, \quad m^s \geq 0, \tag{16}$$

where x^s and y^s are respectively the consumption and the production of general goods during the day, and m^s and b^s are respectively the dollars and bonds acquired during the day. The seller is aware that the outcome of bargaining in a trade meeting at night obeys (5), and she can purchase fair insurance against the event of not meeting a buyer. The optimal choice of $\{x^s, y^s, m^s, \mu^s, b^s\}$ solves the following maximization program:

$$V^s(a^s) = \max_{\{x^s,y^s,m^s,\mu^s,b^s\}} v(x^s) - w(y^s) + (1-\pi^s)\,\beta V^s\left(a^{s0}_{+1}\right) + \pi^s\left[-c(q) + \beta V^s\left(a^{s1}_{+1}\right)\right] \tag{17}$$

subject to (17), (5), and the No-Ponzi game condition. The terms a^{s0}_{+1} and a^{b1}_{+1} denote the wealth next morning. The superscripts $s1$ and $s0$ denote respectively the existence or not of a trading opportunity. Therefore,

$$a^{s0}_{+1} = \frac{b^s + m^s + \mu^s\pi^s}{p_{+1}}, \text{ and} \tag{18}$$

$$a^{s1}_{+1} = \frac{b^s + m^s + d^s - \mu^s\left(1-\pi^s\right)}{p_{+1}}; \tag{19}$$

where the expected payment received in a trade meeting is $d^s = \tilde{d}(M^b, p_{+1})$.

Recursive dynamic arguments show that V^s is a well defined value function. Furthermore, V^s is concave and continuously differentiable. (See the Appendix.)

To solve (16), the seller must fully insure the risk of failing to meet a buyer because V^s is concave:

$$\mu^s = d^s. \tag{20}$$

Therefore, the quantity of money held next morning is $m^s + d^s\pi^s$ regardless of meeting a buyer or not, so $a^{s0}_{+1} = a^{s1}_{+1} \equiv a^s_{+1}$. Also, the seller must equate the marginal utility of consumption of the general good to the marginal disutility of its production:

$$v'(x^s) = w'(y^s). \tag{21}$$

Finally, using the Envelope Theorem $V^{s\prime}\left(a^s_{+1}\right) = w'(y^s_{+1})$, the seller must demand bonds and money to satisfy the following conditions:

$$v'(x^s) = \beta\,(1+r)\,\frac{p}{p_{+1}} v'(x^s_{+1}), \text{ and} \tag{22}$$

$$\left[\frac{\beta w'\left(y^s_{+1}\right)p}{p_{+1}} - w'\left(y^s\right)\right] m^s = 0. \tag{23}$$

In an environment where (3) and (4) hold, (21), (22), and (23) imply

$$x^s = x^s_{+1},\; y^s = y^s_{+1}, \text{ and} \tag{24}$$

$$m^s = 0. \tag{25}$$

Consequently, the optimal behavior is consistent with stationarity, and sellers carry no money balances to the goods markets.

2.3 Generalized Nash Bargaining

This subsection characterizes the outcome of Nash bargaining when a buyer and a seller meet at night. Let S^b and S^s be respectively the trading surpluses of the buyer and the seller. The generalized Nash bargaining outcome $\{q, d\}$ solves the following program:

$$\max_{\{q,d\}} \left(S^b\right)^{\theta} \left(S^s\right)^{1-\theta}, \tag{26}$$

subject to the rationality constraints: $S^b \geq 0$ and $S^s \geq 0$, and the constraint that the buyer can only pay the money he carries: $d \leq m^b$. The parameter θ is a number in the interval $(0,1)$ that indicates the bargaining power of buyers relative to sellers.

If a buyer and a seller meet, their respective payoffs after they trade are: $u(q) + \beta V^b\left(a^b\right)$ and $\beta V^s\left(a^s\right) - c(q)$. In this meeting, their respective reservation utilities, which they would attain in the absence of trade, are: $V^b\left(a^b + d/p_{+1}\right)$ and $V^s\left(a^s - d/p_{+1}\right)$. Subtracting the reservation utilities from the payoffs with trade, the trading surpluses are:

$$S^b = u(q) + \beta\left[V^b\left(a^b_{+1}\right) - V^b\left(a^b_{+1} + \frac{d}{p_{+1}}\right)\right], \tag{27}$$

and

$$S^s = \beta\left[V^s\left(a^s_{+1}\right) - V^s\left(a^s_{+1} - \frac{d}{p_{+1}}\right)\right] - c(q). \tag{28}$$

To solve the model analytically, the following Taylor approximations are very convenient:

$$V^b\left(a_{+1}^b+\frac{d}{p_{+1}}\right)-V^b\left(a_{+1}^b\right)\approx V^{b\prime}(a_{+1}^b)\frac{d}{p_{+1}} \tag{29}$$

$$V^s\left(a_{+1}^s\right)-V^s\left(a_{+1}^s-\frac{d}{p_{+1}}\right)\approx V^{s\prime}(a_{+1}^s)\frac{d}{p_{+1}} \tag{30}$$

Without these approximations, the model is still well defined, but the solution is difficult to analyze without numerical methods. These approximations turn into equalities, and so their use is fully justified, in the following three cases. (1) The value function is linear in wealth. Lagos and Wright (2003) and Rocheteau and Wright (2003) provide examples where this property holds because of quasi-linear preferences. Next section provides another example where the linearity of the value function holds because of endogenous shopping. However, the value functions V^s and V^b in this section are in general not linear. (2) Individuals have insurance not only against the risk of meeting or not a trading partner, but also against the risk of failing to reach a bargaining agreement with the trading partner. Such an insurance generates the moral hazard problem that sellers are better off if bargaining negotiations fail. To avoid this problem, insurance contracts must specify in detail how individuals must act in all eventualities. Enforcing these contracts requires much more information than observing if an individual met a trading partner or not. In the framework of a representative household, such a modeling strategy has lead to the critique that individuals turn into "machines". In the framework of this paper, this modeling strategy is avoidable under the following condition. (3) The payment d is an infinitessimal fraction of the comprehensive wealth of individuals. This comprehensive wealth, which includes the present discounted value of all future production, is arbitrarily large as $\beta \longrightarrow 1$. Consequently, (29) and (30) are justified when β is in a neighborhood of one, or equivalently, when the trading period is short. This third justification requires the existence of credit inside the village and is the one implicitly adopted in this section.

Using (29) and (30) with equality, together with the Envelope Theorem and stationarity, the trading surpluses of buyers and sellers are the following:

$$S^b=u(q)-\beta w'(y^b)\frac{d}{p_{+1}} \tag{31}$$

$$S^{s} = \beta w'(y^{s})\frac{d}{p_{+1}} - c\left(q\right) \tag{32}$$

Substituting (31) and (32) into (26), the first order interior conditions that characterize generalized Nash bargaining are:

$$\frac{\theta}{1-\theta}\frac{w'(y^{b})}{w'(y^{s})} = \frac{u(q) - \beta w'(y^{b})z}{\beta w'(y^{s})z - c(q)} \tag{33}$$

$$\frac{\theta u'(q)}{\left(1-\theta\right)c'(q)} = \frac{u(q) - \beta w'(y^{b})z}{\beta w'(y^{s})z - c(q)}. \tag{34}$$

where $z \equiv d/p_{+1}$. Let (q^{*}, z^{*}) be the solution to the system (33) and (34). Using the Kuhn-Tucker theorem, if $m^{b} \geq z^{*}p_{+1}$, then the Nash bargaining solution is unconstrained by $d \leq m^{b}$ and the outcome is (q^{*}, z^{*}). In contrast, if $m^{b} \leq z^{*}p_{+1}$, then the Nash bargaining solution is constrained, so $d = m^{b}$ and q obeys (34) with $z = m^{b}/p_{+1}$.

As we saw in the previous subsections, the constrained solution is the relevant one in equilibrium. In this case, it is convenient to rewrite (34) in the following form:

$$\beta w'(y^{b})z = \frac{\left[\theta u'(q)c(q) + \left(1-\theta\right)c'(q)u(q)\right]w'(y^{b})}{\theta u'(q)w'(y^{s}) + \left(1-\theta\right)c'(q)w'(y^{b})} \equiv g(q). \tag{35}$$

In the three justifications provided for turning (29) and (30) into equalities, the values $w'(y^{s})$ and $w'(y^{b})$ are treated as constant in a particular trading match. Also, in a constrained solution $z = m^{b}/p_{+1.}$ Consequently, the function $g(q)$ in (35) is the inverse function of the terms of trade rule $\tilde{q}\left(m^{b}/p_{+1}\right)$. Using the inverse function theorem, we obtain

$$\tilde{q}'\left(\frac{m^{b}}{p_{+1}}\right) = \left(\frac{\partial z}{\partial q}\right)^{-1} = \frac{\beta w'(y^{b})}{g'(q)} \tag{36}$$

Using this result, the conjectured concavity of $u\left[\tilde{q}\left(m^{b}/p_{+1}\right)\right]$ as a function of m^{b}/p_{+1}requires that $u'\left(q\right)/g'(q)$ is a decreasing function of q. Also, the conjectured Inada condition $\lim_{m^{b}\to 0} u'\left[\tilde{q}\left(m^{b}/p_{+1}\right)\right]\tilde{q}'\left(m^{b}/p_{+1}\right) = \infty$ requires that $\lim_{q\to 0} u'\left(q\right)/g'(q) = \infty$. These two properties of the ratio $u'\left(q\right)/g'(q)$ are clearly satisfied if $\theta \approx 1$. However, in general they place stronger restrictions on u and c than those stated at the beginning of this section. This caveat is also found in Rocheteau and Wright (2003), and it is discussed at length in Lagos and Wright (2003). As in there, if one normalizes c to be linear (units of output are measured in the utils it costs to produce them), then $u'\left(q\right)/g'(q)$ is a decreasing function of q if u' is log-concave.

2.4 Equilibrium

The previous subsection shows that the conjectured properties on the terms of trade rules that we employed to solve the optimization problems of buyers and sellers are consistent with generalized Nash bargaining. Also, the conjecture that prices increase at the rate γ and the real interest rate is $\beta^{-1} - 1$ are consistent with optimal behavior and stationarity. Finally, it is optimal for a seller not to carry money to the search market and for all buyers to carry the same amount m^b, which is fully spent in a trade meeting. Therefore, the environment assumed when solving for the optimal behavior of individuals is consistent with the concept of equilibrium adopted. Everything is ready to collect the equations that characterize the equilibrium allocation $\{x^b, x^s, y^b, y^s, q\}$.

Combining the first order conditions (11), (15), and (21), with the constrained solution of Nash bargaining (36), the market clearing condition for the general good, and the budget constraint of sellers with $m^s = 0$, $b^s = 0$, and $a^s = \pi^s g(q)$, we obtain the following system of equations that characterizes a stationary equilibrium:

$$v'(x^b) = w'(y^b), \tag{37}$$

$$v'(x^s) = w'(y^s), \tag{38}$$

$$x^b + x^s = y^b + y^s, \tag{39}$$

$$y^s - x^s = \pi^s g(q), \tag{40}$$

$$\frac{u'(q)}{g'(q)} = 1 + \frac{\gamma - \beta}{\beta \pi^b}, \text{ and} \tag{41}$$

$$g(q) \equiv \frac{[\theta u'(q)c(q) + (1-\theta)\, c'(q)u(q)]\, w'(y^b)}{\theta u'(q)w'(y^s) + (1-\theta)\, c'(q)w'(y^b)}. \tag{42}$$

The two key equations in this system are (41), and (42). Except for the endogenous values of the marginal disutilities $w'(y^s)$ and $w'(y^b)$, these equations are identical to those that characterize an equilibrium in Rocheteau and Wright (2003). The extra equations, (37) to (40) are needed to determine $w'(y^s)$ and $w'(y^b)$. In general, these two values differ. However, one can simplify the system of equations that characterizes an equilibrium if instead of assuming zero initial endowments of bonds, one assumes that the initial bonds owned by buyers and sellers are such that all individuals share a common marginal utility for the general goods they consume. In this simpler case, $w'(y^s) = w'(y^b)$ and the equilibrium is described by

$$x^b = x^s = y^b = y^s, \tag{43}$$

together with (37), (41), and (42). The comparative statics of the model are almost identical to those described by Rocheteau and Wright (2003), so they are omitted here. Instead, a numerical calibration of the model that illustrates these comparative statics is presented in Section 5.

3 Endogenous Shopping

This section presents a modified version of the previous model where individuals choose which role they play at the search market. This analysis complements the previous section in two ways. In the previous section, credit contracts were not used along the equilibrium path. In this section, credit contracts are the key financial arrangement between the members of a village. In the previous section, the model extends the Rocheteau and Wright (2003) version of the Lagos and Wright (2003) model. In this section, an equilibrium is equivalent to the Rauch (2000) solution to the Shi (1997) model.

In the version of the model presented in this section, there are no general goods, and all the individuals have symmetric preferences and abilities. All individuals are able to produce a good specific to their village. Also, they all get utility from the goods produced in other villages. However, nobody desires to consume the good produced in their own village, so trade across villages is necessary to consume. This trade makes money essential because outside their own village individuals are anonymous.

Production must take place around the village of origin, so at the beginning of each period each individual has to choose to either stay around the village as a seller or visit other villages as a buyer. In each period, some individuals are buyers while the others are sellers. Over time, an individual typically alternates between these two roles because he cannot afford being always a buyer, and he never consumes by being always a seller. The emphasis of the present model is to endogenize how often an individual plays each one of these roles.

The measure of individuals is one. The endogenous fraction of sellers is denoted as n, so the fraction of buyers is $1 - n$. Each period, buyers and sellers are randomly matched. The measure of trading matches, in which a buyer meets a seller, is a function of $n : \Pi(n)$. The function Π is assumed to be non-negative, continuously differentiable, concave, and satisfies the terminal conditions $\Pi(0) = \Pi(1) = 0$. The probability that a buyer finds a seller is

$$\pi^b(n) = \frac{\Pi(n)}{1-n}. \tag{44}$$

The probability that a seller finds a buyer is

$$\pi^s(n) = \frac{\Pi(n)}{n}. \tag{45}$$

Money is injected through lump-sum transfers received by all individuals at the beginning of a period. The growth factor of the money supply is $\gamma : M_{+1} = \gamma M$. Because of the absence of the general good, p has no meaning as a price. However, to facilitate the comparison with the previous model, p is still used to denote the deflator of money. It turns out that a convenient and simple deflator is the aggregate quantity of money, so this section uses $p \equiv M$. With this deflator, the ratio m/p denotes the fraction of the money supply M that an individual holds. As discussed below, this fraction properly measures the purchasing power of the individual's money holdings.

As in the previous section, individuals can insure risks on trading opportunities and buy and sell riskless bonds inside their village. The initial bond holdings of all individuals is zero.

3.1 Optimal Behavior

The model is solved using the same strategy as the one used in the previous section. First, the optimal behavior of individuals is solved in a conjectured environment. In this environment, the terms of trade rules are summarized by the functions $\tilde{q}$ and $\tilde{d}$ in (5) that have the same properties as those conjectured in the previous section. These properties imply that p correctly deflates the value of money in terms of specific goods. The rate of change of p and the rate of interest on nominal bonds satisfy (3) and (4). Finally, other sellers carry no money balances to the market, and other buyers carry a certain amount M^b. The ratio M^b/p_{+1} is constant over time and smaller than z^*. This conjectured environment is later shown to be consistent with generalized Nash bargaining and stationarity when individuals behave optimally.

The optimal program of an individual is described as follows:

$$V(a) = \max\left\{V^b\left(a\right), V^s\left(a\right)\right\}; \tag{46}$$

where $V^b\left(a\right)$ is the utility of being a buyer and $V^s\left(a\right)$ is the utility of being a seller:

$$V^b\left(a\right) = \max_{\left\{m^b, \mu^b, b^b\right\}} \pi^b\left[u\left(q^b\right) + \beta V\left(a_{+1}^{b1}\right)\right] + \left(1 - \pi^b\right)\beta V\left(a_{+1}^{b0}\right), \text{ and} \tag{47}$$

$$V^s\left(a\right) = \max_{\{m^s, \mu^s, b^s\}} \pi^s\left[-c(q^s) + \beta V\left(a_{+1}^{s1}\right)\right] + \left(1 - \pi^s\right)\beta V\left(a_{+1}^{s0}\right). \tag{48}$$

The wealth at the beginning of next period, a_{+1}, depends on if the individual chooses to be a buyer (superscript b) or a seller (superscript s). It also depends on if the individual finds a trading partner (superscript 1) or not (superscript 0). Therefore, a_{+1} takes one of the following four values:

$$a_{+1}^{b1} = \frac{b^b + m^b - d^b + \mu^b\left(1 - \pi^b\right) + \tau}{p_{+1}}, \tag{49}$$

$$a_{+1}^{b0} = \frac{b^b + m^b - \mu^b\pi^b + \tau}{p_{+1}}, \tag{50}$$

$$a_{+1}^{s1} = \frac{b^s + m^s + d^s - \mu^s\left(1 - \pi^s\right) + \tau}{p_{+1}}, \text{ and} \tag{51}$$

$$a_{+1}^{s0} = \frac{b^s + m^s + \mu^s\pi^s + \tau}{p_{+1}}. \tag{52}$$

The program (46) faces the following constraints: $q^b = \tilde{q}(m^b/p_{+1})$, $d^b = \tilde{d}(m^b, p_{+1})$, $q^s = \tilde{q}(M^b/p_{+1})$, $d^s = \tilde{d}(M^b, p_{+1})$, $m^b \geq 0$, $m^s \geq 0$, and the budget:

$$\frac{m^i + b^i(1+r)^{-1}}{p} = a, \text{ for } i = b, s. \tag{53}$$

In addition to the flow constraints (49) to (53), next period wealth is bounded below by the condition that individuals must be able to repay their debts with probability one without reliance to unbounded borrowing (No-Ponzi game condition):

$$a_{+1} \geq \underline{a}, \tag{54}$$

where $\underline{a}$ is an endogenous lower bound on wealth. The value of $\underline{a}$ depends crucially on the government transfers and the existence of insurance. Without transfers ($\tau = 0$) and insurance, the lower bound $\underline{a}$ is zero. With these assumptions, an individual cannot guarantee having positive income in the future because sellers may have a long stream of bad outcomes in the market. Therefore, an individual cannot guarantee to be able to repay a positive debt with probability one without running a Ponzi game. Positive transfers and insurance on trading risks allow individuals to issue risk free bonds up to the point where the interest on these bonds is equal to the guaranteed income of a seller purchasing full insurance ($\mu^s = d^s$). Because of undirected search, this income cannot be increased by trying to offer more output or by giving a better deal to buyers.[5] Therefore, using (53) and (52) with $a = a^{s0}_{+1} = \underline{a}$, $\mu^s = d^s = M^b$, $m^s = 0$, $\tau = (\gamma - 1) M$, and $p_{+1} = \gamma M$, we obtain

$$\underline{a} = -\frac{\frac{\gamma-1}{\gamma} + \frac{\pi^s M^b}{\gamma M}}{1-\beta}. \tag{55}$$

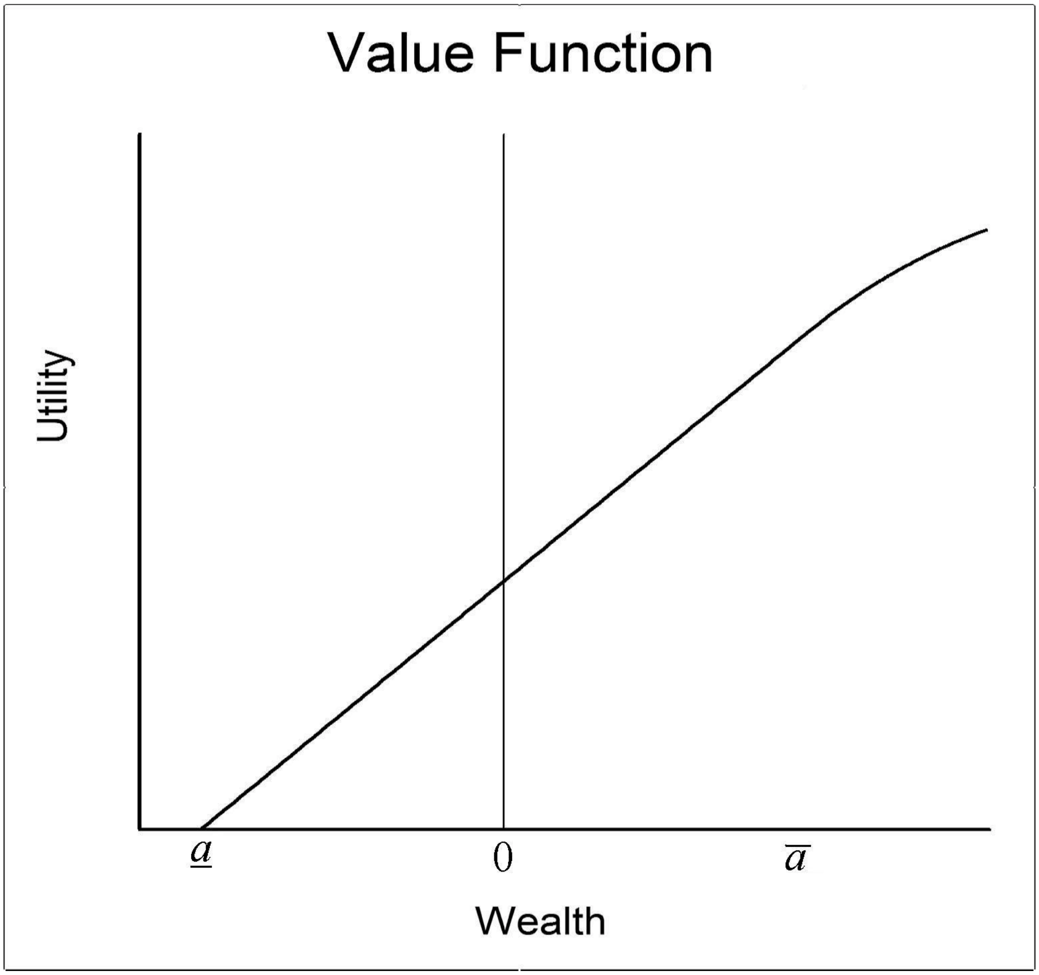
Standard recursive dynamic arguments show that V is a well defined concave value function that depends on a. Moreover, V is linear in an interval of wealth:

$$V(a) = v_0 + a, \text{ for } a \in [\underline{a}, \overline{a}]; \tag{56}$$

where $\underline{a}$ is given by the No-Ponzi condition (55) and $\overline{a}$ is the lowest wealth at which the individual can optimally afford being a buyer forever. The linearity of V can be ascertained using the method of undetermined coefficients. The Appendix provides a formal argument supporting this method in this context. Figure 1 displays this value function.

[5] With competitive search these possibilites are present which modifies that shape of the value function. See Faig (2004).

Figure 1



The function V depicted in Figure 1 is consistent with the following behavior. A poor individual whose wealth is $\underline{a}$ can only afford being a fully insured seller forever. In the other extreme, a rich individual whose wealth is above $\overline{a}$ can afford being a buyer forever. The money holdings m^b that this rich individual carries to the market are increasing with wealth. Therefore, the strict concavity of u with respect to m^b implies the strict concavity of V if wealth is above $\overline{a}$. Middle wealth individuals whose wealth is between $\underline{a}$ and $\overline{a}$ choose each period if they are going to be a buyer or a seller. In equilibrium, they are indifferent between these two roles as long as $a_{+1} \in [\underline{a}, \overline{a}]$ with probability one. Moreover, the interval $[\underline{a}, \overline{a}]$ is absorbing because the

No-Ponzi game condition prevents $a_{+1} < \underline{a}$ and optimal behavior dictates that $a_{+1} < \overline{a}$. As long as wealth is in the interval $[\underline{a}, \overline{a}]$, whenever an individual goes to the market as a buyer carries an amount of money m^b independent from wealth. Consequently, inside the middle wealth bracket $[\underline{a}, \overline{a}]$ richer individuals expect to consume more often in the rest of their lives, but they consume the same amount each time they do so. Since utility is linear on the number of times an individual consumes, the value function is linear. This argument is developed in more detail in the following paragraphs.

If the wealth of an individual belongs to the interval $[\underline{a}, \overline{a}]$, then a_{+1} also belongs to this interval with probability one because of the following two reasons. Choices that lead to $a_{+1} < \underline{a}$ with positive probability violate the No-Ponzi game condition (55), so they are ruled out. (An individual with $a \geq \underline{a}$ can avoid $a_{+1} < \underline{a}$ by going to the market as a fully insured seller.) Choices that lead to $a_{+1} > \overline{a}$ with positive probability cannot be optimal. If there is a positive probability of entering the strictly concave region of V, optimal behavior implies full insurance of trading risks. Therefore, the Euler equation of program (46) is

$$V'(a) = \frac{\beta(1+r)p}{p_{+1}} V'(a_{+1}). \tag{57}$$

In an environment where (4) holds, this equation is violated if $V'(a) > V'(a_{+1})$, so $a \leq \overline{a}$ implies $a_{+1} \leq \overline{a}$.

If a_{+1} remains in $[\underline{a}, \overline{a}]$ with probability one, the functional form (56) implies that the optimization problems for buyers (47) and sellers (48) simplify into:

$$V^b(a) = \frac{\beta(1+r)p}{p_{+1}} a + \beta\left(v_0 + \frac{\tau}{p_{+1}}\right) + \max_{m^b} \left\{ \pi^b \left[u\left(q\right) - \beta \frac{d}{p_{+1}} \right] - \beta \frac{m^b}{p_{+1}} r \right\}, \text{ and} \tag{58}$$

$$V^s(a) = \frac{\beta(1+r)p}{p_{+1}} a + \beta\left(v_0 + \frac{\tau}{p_{+1}}\right) + \max_{m^s} \left\{ \pi^s \left[\beta \frac{d}{p_{+1}} - c(q) \right] - \beta \frac{m^s}{p_{+1}} r \right\}. \tag{59}$$

The values of μ^b and μ^s drop from (58) and (59) because of the linearity of $V(a_{+1})$. However, as explained in the previous paragraph, insurance may have to be purchased to ensure $a_{+1} \in [\underline{a}, \overline{a}]$ with probability one. The optimal choice of m^b and m^s is characterized by

$$m^s = 0 \text{ since } r = \frac{\gamma - \beta}{\beta} > 0, \text{ and} \tag{60}$$

$$\frac{u'(q)\tilde{q}'\left(z\right)}{\beta} = 1 + \frac{r}{\pi^b}; \tag{61}$$

where $z \equiv m^b/p_{+1}$. The optimal values of m^s and m^b implied by (60) and (61) are independent from a. Therefore, in an environment where (4) holds, we have

$$V^b(a) = v_0^b + a \text{ and } V^s(a) = v_0^s + a, \text{ for } a \in [\underline{a}, \overline{a}]. \tag{62}$$

The values v_0^b and v_0^s are independent from a because $\tau/p_{+1} = (\gamma - 1)/\gamma$ and the properties of $\tilde{q}$ and $\tilde{d}$ imply that q and d depend only on m^b and p_{+1} which are independent from a. Therefore, the objective (46) together with (62) implies that $V(a)$ has the linear form in (56) for $a \in [\underline{a}, \overline{a}]$ with $\overline{a}$ being the wealth that permits an individual to be a buyer in perpetuity when (61) holds and the individually fully insures.

3.2 Equilibrium

The assumption that initially all individuals hold zero bonds implies that their wealth is in the interval $[\underline{a}, \overline{a}]$. In a monetary equilibrium, there must be both buyers and sellers. Therefore, for the coexistence of both roles it is necessary that $v_0^b = v_0^s$. This implies that the following condition must hold:

$$\pi^b(n)[u(q) - \beta z] - \beta z r = \pi^s(n)[\beta z - c(q)]. \tag{63}$$

That is, n must be such that buyers and sellers have equal expected trading surpluses. For existence of an equilibrium both sides of (63) must be non-negative otherwise an individual would be better off dropping out of the market. This places an implicit limit on r and so γ.

Nash bargaining proceeds along the same lines as those in the previous section with the advantage that $w'(y^b) = w'(y^s)$ and the approximations (29) and (30) can be replaced with equalities even if β is low. From this analysis we obtain:

$$\beta z = \frac{\theta u'(q)c(q) + (1-\theta)c'(q)u(q)}{\theta u'(q) + (1-\theta)c'(q)} \equiv g(q) \tag{64}$$

This validates the conjecture on the terms of trade rules that arise from bargaining. It also validates that $p \equiv M$ is a suitable deflator.

The remaining conjectures about the environment where individuals optimize are also consistent with an equilibrium. The conjecture about the inflation rate (3) is directly implied by the definition $p \equiv M$. The conjecture (4) is consistent with stationary fractions of individuals that are willing to be buyers and sellers. Finally, it is optimal for sellers not to carry money balances, and for all buyers to carry the same amount, which is fully spent in a trade meeting.

Using (4), (44), and (45) to simplify (63), we obtain the following two equations that together with (64) determine n and q in a stationary equilibrium:

$$\frac{n}{1-n} = \frac{g(q) - c(q)}{u(q) - g(q)\left(1 + \frac{\gamma-\beta}{\pi^b(n)\beta}\right)}, \text{ and} \tag{65}$$

$$\frac{u'(q)}{g'(q)} = 1 + \frac{\gamma - \beta}{\pi^b(n)\beta}. \tag{66}$$

These equations are equivalent to the solution presented by Rauch (2000) of Shi (1997).[6] The distribution of assets across individuals in a stationary equilibrium is undetermined except for the fact that the wealth of all individuals is in the interval $[\underline{a}, \overline{a}]$. However, this distribution is irrelevant to find the values of n and q. This implies that as long as the wealth of all individuals remains in $[\underline{a}, \overline{a}]$, random redistributions of money at the beginning of a period are irrelevant to characterize aggregate variables.

4 Financial Contracts versus Lotteries

One of the advantages of achieving tractability with the existence of financial contracts is that the effects of this mechanism can be evaluated by simply assuming imperfect information inside the village. In general, the absence of financial contracts leads to models where the amounts of money carried by buyers are heterogeneous. Once buyers hold divers money balances, models are typically much more difficult to solve and quite often their analyses require numerical methods.[7] Interestingly, this section shows that the simple model analyzed in the previous section is tractable even in the absence of financial contracts. In this model, if individuals are anonymous both inside and outside the village, fair lotteries are able to replace the role of financial contracts. These lotteries demand no memory about personal histories and no ability to observe of trading opportunities. Yet they lead to the same system of equations, (64) to (66), that characterizes an equilibrium in the previous section.

[6] Rauch adds barter and assumes linearity of u and symmetric bargaining.

[7] For examples of models with an endogeneous non-degenerate distribution of money see Green and Zhou (1998), Molico (1999), Camera and Corbae (1999), and Zhou (1999).

The model in this section makes two modifications to the model presented in the previous one. First, the government can give lump-sum transfers but cannot impose lump-sum taxes, so the money supply can only grow: $\gamma \geq 1$. The importance of this modification is discussed below. Second, individuals have no access to credit and insurance contracts, but individuals can play a fair lottery. A lottery ticket delivers an amount of money m with probability ψ. The cost of the ticket is ψm.

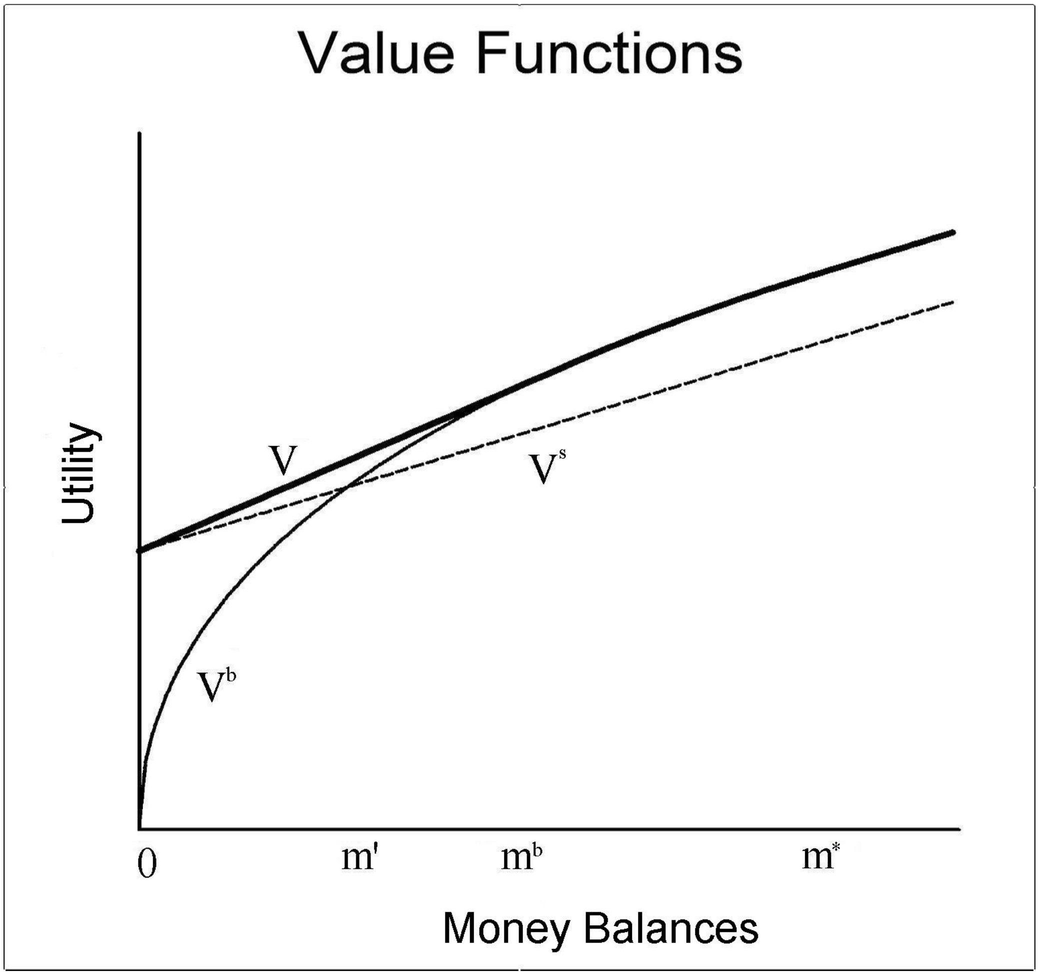
The absence of financial contracts implies that money is the only asset in the economy. Consequently, the state variable measuring an individuals wealth is the real quantity of money at the beginning of a period. As it will be confirmed below, the value function has still the linear form: $V(a) = v_0 + a$ in the relevant interval of wealth. This interval is $\left[0, m^b/p\right]$, where m^b/p is calculated below. Using this functional form as the next period value function, the utilities attained by being a buyer and a seller are respectively:

$$V^b\left(\frac{m}{p}\right) = \beta\left(v_0 + \frac{\tau}{p_{+1}}\right) + \pi^b\left[u\left(q\right) - \beta\frac{d}{p_{+1}}\right] + \beta\frac{m}{p_{+1}}, \text{ and} \tag{67}$$

$$V^s\left(\frac{m}{p}\right) = \beta\left(v_0 + \frac{\tau}{p_{+1}}\right) + \pi^s\left[\beta\frac{d}{p_{+1}} - c(q)\right] + \beta\frac{m}{p_{+1}}. \tag{68}$$

subject to the terms of trade rules (5). Figure 2 represents the shape of these functions if the individual optimizes in an equilibrium environment where both buyers and sellers are active in the market.

Figure 2



Both V^s and V^b are increasing functions of m. At the vertical axis, $V^s(0) - V^b(0)$ is equal to the expected seller's surplus, which must be positive in equilibrium, so $V^s(0) > V^b(0)$. The function V^b is at least as steep as the function V^s, because money holdings increase the buyer's surplus but have no effect on the seller's surplus. Beyond the quantity of money m^*, where Nash bargaining ceases to the constrained by the buyer's money, the two functions have the same slope: β/p_{+1}. Finally, V^s and V^b must cross otherwise there would be no buyers in the market.

In the absence of lotteries, an individual would be indifferent between a buyer or a seller at the point m' where V^s and V^b cross. Furthermore, the individual would choose to be a buyer if $m < m'$ and to be a seller if $m > m'$. However, it is clear from Figure 2, that the individual can achieve a better outcomes than those just described by randomizing between $V^s(0)$ and $V^b\left(m^b/p_{+1}\right)$ if $m \in \left(0, m^b\right)$. In this case, the individual uses all his money holdings, ap, at the beginning of each period to buy a lottery ticket with a payoff m^b and a probability of winning ap/m^b. If the individual wins the lottery, then he becomes a buyer, and if the individual loses the lottery, then he becomes a seller. Consequently, the individual solves the following program:

$$V(a) = \max_{\left\{m^b, \psi\right\}} \psi V^b\left(\frac{m^b}{p}\right) + (1-\psi)\, V^s(0), \tag{69}$$

subject to (5), $m^b \geq 0$, $\psi \in [0,1]$, and the budget constraint:

$$\psi m^b = ap. \tag{70}$$

The first order interior conditions with respect to ψ and m^b yield the same equations, (61) and (63), that characterize optimal behavior in the previous Section. These equations imply that m^b is independent from the initial wealth a if $a \in \left[0, m^b/p\right]$. As long as the initial wealth is in this interval, m^b is not affected by a while ψ is a linear function of a.That is, wealth increases the probability of going shopping but not the quantity purchased. Since V is linear in ψ, this confirms the conjecture that V is a linear function of wealth in the interval $\left[0, m^b/p\right]$. (See the Appendix for a formalization of this argument).

After an analogous characterization of Nash bargaining to the one in the previous two sections, we obtain that q and n are still determined by the system (64) to (66).

The distribution of wealth in equilibrium is the following. At the search market, sellers carry no money, and all buyers carry the same amount, m^b. At the beginning of a period, the wealth of individuals who were matched buyers or unmatched sellers in the previous period is the transfer from the government, τ/p_{+1}. Meanwhile, the wealth of the other individuals (unmatched buyers and matched sellers) is $\left(m^b + \tau\right)/p_{+1}$. Therefore, the wealth distribution at the beginning of a period is:

$$\frac{\tau}{p_{+1}} \quad \text{with frequency } (1-n)\,\pi^b + n\,(1-\pi^s)\,, \text{ and} \tag{71}$$

$$\frac{\tau + m_b}{p_{+1}} \quad \text{with frequency } (1-\ n)\left(1-\pi^b\right) + n\pi^s. \tag{72}$$

Since $p \equiv M$ and $\tau = (\gamma - 1)\, M$, (3) implies that $\tau/p_{+1} = (\gamma-1)/\gamma$. Moreover, since the measure of buyers is $(1-n)$ and they hold the whole of the money supply, the definition $p \equiv M$ implies that m^b/p is equal to $(1-n)^{-1}$. Finally, the probabilities π^b and π^s satisfy (44) and (45). Therefore, (71) and (72) simplify into:

$$\frac{\gamma-1}{\gamma} \quad \text{with frequency } n, \text{ and} \tag{73}$$

$$\frac{1}{1-n} - \frac{\gamma-1}{\gamma}\frac{n}{1-n} \text{ with frequency } (1-\ n). \tag{74}$$

As long as $\gamma \geq 1$, both mass points in this distribution are in the interval $\left[0,(1-n)^{-1}\right]$. Therefore, the interval $\left[0,m^b/p\right]$ for which the value function is linear is absorbing. That is, if the initial endowments are in this interval, the wealth of all individuals remains there forever.

In conclusion, and quite paradoxically, lotteries are perfect substitutes for financial contracts (credit and insurance) in this economy. Both lotteries and financial contracts put on the hands of buyers all available money balances. This is remarkable because lotteries require neither memory about credit histories nor the ability to observe trading opportunities.

4.1 A Difficulty with the Optimum Quantity of Money

An interesting feature of the model is that monetary authorities cannot implement reductions on the quantity of money with uniform lump-sum taxes to achieve $\gamma < 1$. Indeed, a uniform lump-sum tax means that each individual must pay every period a preordained amount of money. In the absence of financial contracts, an individual can fail to earn income for an indefinite number of periods despite going to the market as a seller every period. While this happens, the individual must continue paying the tax, so the real wealth of the individual evolves as follows:

$$a_{t+1} = \frac{\tau}{p_{+1}} + \frac{a_t p}{p_{+1}}. \tag{75}$$

Using $\tau = (\gamma-1)M$, $p = M$, and $M_{+1} = \gamma M$, the law of motion of real wealth is simplified into:

$$a_{t+1} = \frac{\gamma-1}{\gamma} + \frac{a_t}{\gamma}, \text{ for } t = 0,1,2... \tag{76}$$

If $\gamma < 1$, this is an unstable difference equation. For $a_0 < 1$, a_t drops down to minus infinity as $t \to \infty$. In a finite number of periods, and hence positive probability, a_t falls below the value of the lump-sum tax and the individual must default. Therefore, an individual must maintain $a_t \geq 1$ at all times to avoid a positive probability of defaulting on the tax obligations in the future. Since

a_t is the ratio of money held by an individual over the average quantity of money, all individuals must hold the average quantity of money at all times to be solvent. This is impossible with trading risks. Hence, there is no equilibrium with $\gamma < 1$ if money must be reduced with a uniform lump-sum tax. This is reminiscent of the difficulty in implementing the optimum quantity of money emphasized by Bewley (1983). This difficulty appears with lotteries, but it did not appear in the previous section with credit and insurance contracts.

5 Numerical Comparison

The previous sections show how Lagos and Wright (2003) and Shi (1997) can be generalized in the framework advanced in this paper. In this framework, these two earlier models appear as variations of a common theme. One variation, studied in Section 2, leads to the Rocheteau and Wright (2003) version of Lagos and Wright (2003). The other variation, presented in Section 3, leads to the Rauch (2000) solution to Shi (1997). It is time, the generalized model is used to compare the predictions of the two variations. In particular, it is time to resolve the disparity of results about the welfare cost of inflation: Lagos and Wright (2003) argues that this cost is larger than previously thought, while Shi (1997) points out that it may well be negative.

The key difference between the two variations studied in this paper is that the shopping frequency is endogenous in Section 3, while it is exogenous in Section 2. Rocheteau and Wright (2003) successfully argues that if the composition of buyers and sellers is endogeneized by allowing a costly entry of sellers, then the predictions in Lagos and Wright (2003) about the welfare cost of inflation are reinforced. However, in Rocheteau and Wright (2003) the measure of buyers active in the market remains exogenous. In contrast, in Shi (1997) the fraction of individuals that go shopping as buyers is endogenous. This section evaluates the importance of this endogenous shopping decision for the velocity of circulation of money and the welfare cost of inflation.

Another difference between the two variations studied in this paper is that some goods are traded competitively inside the village in Section 2, while these goods are absent in Section 3. This difference is not crucial. Even if the existence of these goods is essential for the logic of the original version of the Lagos and Wright (2003) model, their existence is not essential to the generalization of their model in Section 2. Moreover, the model of Section 3 can be easily extended to incorporate general goods traded inside the village in the same way that Shi (1997) could be extended to incorporate home production.

When the model of Section 3 is extended to incorporate general goods traded inside the village, the equilibrium allocation $\{x, y, r, q, n\}$ is characterized by the following system of equations:

$$v'(x) = w'(y), \tag{77}$$

$$x = y, \tag{78}$$

$$r = \frac{\gamma - \beta}{\beta}, \tag{79}$$

$$g(q) \equiv \frac{\theta u'(q)c(q) + (1-\theta)\, c'(q)u(q)}{\theta u'(q) + (1-\theta)\, c'(q)}, \tag{80}$$

$$\frac{u'(q)}{g'(q)} = 1 + \frac{r}{\pi^b(n)}, \text{ and} \tag{81}$$

$$\frac{n}{1-n} = \frac{g(q) - c(q)}{u(q) - g(q)\left(1 + \frac{r}{\pi^b(n)}\right)}. \tag{82}$$

Equations (77) to (81) are identical to those in Section 2 that characterize an equilibrium when the initial distribution of wealth is such that buyers and sellers have a common marginal utility of wealth. In Section 2, instead of the extra equation (82), n is exogenous and so is π^b.

Conveniently, the system of equations (77) to (82) is block recursive. The consumption and production of general goods, x and y, are determined by (77) and (78), and so they are independent from the rate of growth of the money supply γ. Equation (79) states the familiar one-to-one relationship between γ and the nominal interest rate r. The output of specific goods exchanged in a trade meeting, q, and the fraction of individuals that are sellers in the search market, n, are determined by equations (81) and (82) using the definition in (80). These two variables are affected by r, and so by the growth rate of the money supply γ.

The velocity of circulation of money, defined as nominal GDP divided by the quantity of money, is

$$\zeta = \frac{xp_{+1} + (1-n)\,\pi^b\,(n)\,m^b}{(1-n)\,m^b}. \tag{83}$$

The definition (83) calculates GDP aggregating the expenditures made in the search markets this afternoon with the expenditure in general goods next morning. This timing of aggregation facilitates the following intutition. Using (3), (35), (77), and $m^b = zp_{+1}$, the definition (83) simplifies to:

$$\zeta = \frac{x\beta v'(x)}{(1-n)\,g(q)} + \pi^b\,(n)\,. \tag{84}$$

Velocity can be decomposed as the sum of two terms: the fraction of money carried to the search market that changes hands there, $\pi^b\,(n)$, and the fraction of this money that would change hands next morning inside the villages if all transactions involving general goods were performed using money, $x\beta v'(x)/\left[(1-n)\,g\,(q)\right]$.

If n is exogenous, $\pi^b\,(n)$ is exogenous as well. Hence, the interest elasticity of velocity depends exclusively on the effect that r has on $g\,(q)$. An increase in r represents a higher opportunity cost for the purchase of specific goods in the search market. Consequently, as implied by (81), individuals respond by carrying less money and purchasing smaller amounts of these goods. In conclusion, if r increases, then q and $g(q)$ fall, and the ratio $x\beta v'(x)/\left[(1-n)\,g\,(q)\right]$ rises.

In contrast, if n is endogenous, both components of velocity are interest elastic. The nominal interest rate r not only has an effect on $g(q)$, but it also has an effect on n and so on $\pi^b\,(n)$. An increase in r represents an inflation tax on being a buyer because it raises the opportunity cost of the money balances that buyers must carry. Consequently, if this tax were not shifted, the increase in r would tend to induce individuals to be sellers instead of buyers, and so it would raise n unambiguously. Unfortunately, things are a bit more complicated because in general a portion of the tax is shifted to sellers through the bargaining process. As a result, n may be increasing or decreasing with r depending on the fraction of the inflation tax that is shifted to sellers. As shown numerically, this fraction depends crucially on θ (the bargaining power of buyers relative to sellers in a trade meeting).

Following Lagos and Wright (2003), the welfare cost of inflation is measured comparing the steady state at 10 percent annual rate of inflation with the steady state at the Friedman rule $(\gamma \to \beta)$. In this comparison, welfare is defined using the following utilitarian criterion:

$$W\left(x, y, q^b, q^s, n\right) = \frac{v\left(x\right) - w\left(y\right) + \Pi\left(n\right)\left[u\left(q^b\right) - c\left(q^s\right)\right]}{1-\beta}. \tag{85}$$

Each period, all individuals (measure one) consume x and produce y. Moreover, a measure $\Pi(n)$ of individuals consumes q^b and an identical fraction produces q^s. To facilitate the interpretation of welfare comparisons, the analysis reports the percentage equivalent variation of consumption in departing from the Friedman rule. That is, it reports the portion δ of consumption that individuals are willing to sacrifice to continue at the Friedman rule instead of being shifted to the allocation at 10 percent inflation:

$$W\left(x_F\left(1-\delta\right), y_F, q_F^b\left(1-\delta\right), q_F^s, n_F\right) = W\left(x_{10}, y_{10}, q_{10}^b, q_{10}^s, n_{10}\right); \tag{86}$$

where the subscript 10 denotes 10 percent inflation and the subscript F denotes the Friedman rule.[8]

For the numerical analysis, the length of one period is thought to be one month, and the functional forms are specialized as follows:

$$u(q) = \frac{q^{1-\eta}}{1-\eta}, \tag{87}$$

$$c(q) = q, \tag{88}$$

$$v(x) = A\ln\left(x\right), \tag{89}$$

$$w\left(x\right) = \ln\left(1-x\right), \text{ and} \tag{90}$$

$$\Pi\left(n\right) = Bn\left(1-n\right). \tag{91}$$

Except for (90) and (91), these functional forms are identical to those assumed by Lagos and Wright (2003). The disutility of producing x is assumed to have the logarithmic form in (90) instead of being linear as in Lagos and Wright to maintain consistency with the assumptions of the

8 This comparison is simplified by the equilibrium equalities: $x_F = y_F = x_{10} = y_{10}$, $q_F^b = q_F^s$, and $q_{10}^b = q_{10}^s$.

present model. An assumption about the matching technology in the search market is unnecessary in Lagos and Wright (2003) because n is exogenous. The matching function (91) assumes that all individuals are matched randomly without regard of their type. This is the most standard assumption in monetary random matching models. For later reference, remark that the first best values implied by these functional forms are $x^{*} = A/(1+A)$, $q^{*} = 1$, and $n^{*} = 0.5$.

There are five parameters to be calibrated: θ, β, η, A, and B. To facilitate comparison with the numerical analysis in Lagos and Wright (2003), these parameters are chosen to satisfy the following conditions:

$$\beta^{-\frac{1}{12}} - 1 = 0.03, \tag{92}$$

$$\zeta_F = 0.2, \tag{93}$$

$$\zeta_{10} = 0.7, \text{ and} \tag{94}$$

$$\pi^{b}(n_F) = 0.052. \tag{95}$$

Condition (92) is that the annual real interest rate on bonds must be 3 percent. Conditions (93) and (94) pin down the values of velocity to be those predicted by the monthly model in Lagos and Wright (2003). This model is estimated using M1, GDP, the GDP deflator, and the commercial paper rate.in the United States during the 20th century. Condition (95) states that the probability of a buyer meeting a seller at the Friedman rule is the same as the one chosen by Lagos and Wright (2003). Finally, five possible values of θ are considered. These values are chosen to illustrate the importance of the bargaining power of buyers and sellers in the results obtained.

Table 1

Numerical Simulations with n Exogenous

θ	$1/10$	$1/3$	$1/2$	$2/3$	$9/10$
β	$1.03^{-\frac{1}{12}}$	$1.03^{-\frac{1}{12}}$	$1.03^{-\frac{1}{12}}$	$1.03^{-\frac{1}{12}}$	$1.03^{-\frac{1}{12}}$
η	0.491	0.306	0.226	0.178	0.137
A	0.015	0.051	0.075	0.099	0.134
B	0.062	0.081	0.106	0.157	0.520
q_F	0.154	0.690	0.848	0.926	0.984
q_{10}	0.017	0.115	0.164	0.195	0.221
n_F	0.836	0.641	0.491	0.331	0.100
n_{10}	0.836	0.641	0.491	0.331	0.100
$\delta \times 100$	**15.92**	**6.03**	**3.58**	**2.45**	**1.67**

Table 1 reports the numerical calibration of the model with n exogenous. The results in this table are similar to those found by Lagos and Wright (2003). The quantity traded of specific goods falls dramatically with inflation ($q_F >> q_{10}$). This is due to the high elasticity of the demand for specific goods (inverse of η) which is required to match the estimated steep decline in velocity in going from the Friedman rule to 10 percent inflation. For all values of θ, the welfare costs of inflation (last row) is positive. As emphasized by Lagos and Wright, these welfare costs increase as θ declines. A lower θ implies that even without inflation there is a large wedge between the marginal cost of producing a specific good and the price that buyers pay for it. Inflation magnifies this wedge by increasing the opportunity cost of carrying the money used to purchase the good. Consequently, if θ is low, inflation distorts an already highly distorted margin of choice, so the welfare cost of inflation is very large.

Table 2

Numerical Simulation with n Endogenous

θ	$1/10$	$1/3$	$1/2$	$2/3$	$9/10$
β	$1.03^{-\frac{1}{12}}$	$1.03^{-\frac{1}{12}}$	$1.03^{-\frac{1}{12}}$	$1.03^{-\frac{1}{12}}$	$1.03^{-\frac{1}{12}}$
η	0.434	0.307	0.227	0.161	0.099
A	0.018	0.051	0.075	0.099	0.134
B	0.062	0.081	0.106	0.157	0.520
q_F	0.268	0.688	0.848	0.935	0.990
q_{10}	0.001	0.117	0.197	0.245	0.259
n_F	0.836	0.641	0.491	0.331	0.100
n_{10}	0.675	0.646	0.563	0.439	0.174
$\delta \times 100$	**13.13**	**6.04**	**3.24**	**1.48**	**-0.02**

Table 2 reports the numerical calibration of the model with n endogenous. As in Table 1, q declines steeply with inflation. The value of n is decreasing with inflation ($n_{10} < n_F$) for low values of θ (less than 1/3) and increasing with inflation ($n_{10} > n_F$) for high values of θ (higher than 1/3). Intuitively, the portion of the inflation tax that is shifted from buyers to sellers through the bargaining process depends on θ. If θ is close to 1, the portion of the inflation tax shifted is small. In this instance, sellers have little bargaining power, so their trade surplus is close to zero irrespective of inflation. Buyers appropriate most of the trade surplus and bear the largest burden of the inflation tax. As a result, inflation discourages individuals being buyers and raises n. At the other extreme, if θ is close to 1, the portion of the inflation tax shifted to sellers is relatively large. In this instance, buyers have little bargaining power, so their trade surplus is close to zero irrespective of inflation. Most of the burden of the inflation tax falls on sellers who irrespective of the rate of inflation capture most of the trade surplus. As a result, n falls with inflation.

The importance of the endogeneity of n on the welfare cost of inflation can be ascertained by comparing the last row of Tables 1 and 2. For values of θ around 1/3, the effect of inflation on n is negligible, so the endogeneity or not of n is of little consequence for the welfare cost of inflation. For values of θ below 1/3, inflation reduces n. This reduction constitutes a welfare improvement because n is above the first best n^* at these low values of θ. Hence, the endogeneity of n reduces the welfare cost of inflation. However, this effect is not sufficiently large to make a large dent on the huge welfare cost of inflation when θ is so low. For values of θ 1/2 and above, inflation increases n. This increase is again a welfare improvement because n is now below n^*. For extremely high values of θ (above 0.9), this effect is sufficiently large to reverse the sign of δ. That is, for high values of θ 10 percent inflation is a welfare improvement relative to the Friedman rule. In conclusion, as Shi (1997) points out some inflation may be beneficial if n is endogenous. However, except for large values of θ, the welfare costs of inflation are large and comparable to those found by Lagos and Wright (2003) even if n is endogenous.

6 Conclusion

This paper provides a tractable search model with divisible money that encompasses Lagos and Wright (2003) and Shi (1997). The key feature of the model is that individuals belong to many villages. Inside a village, individuals are not altruistic as in a representative household, but they share information so financial contracts are feasible. Money is essential in the model to facilitate trade with individuals outside the village. Rocheteau and Wright (2003) version of Lagos and Wright (2003) arises as a special case if competitive markets coexist with search markets, and preferences are quasi-linear. Rauch (2000) solution to Shi (1997) arises as a special case with sufficiently rich financial contracts inside the village.

Curiously, the type of financial arrangements that arise in the village are not always centered around the insurance of trading risks. In the version of the model in Section 2, insurance is the key financial institution that arises in equilibrium, but this is not the case in the versions in Sections 3 and 4. In Section 3, credit plays the key role and insurance is only necessary to ensure the solvency of debtors. In Section 4, insurance contracts are completely unnecessary.

The generalized model presented in this paper is used to reconcile the disparate claims about the welfare cost of inflation found by Shi (1997) and Lagos and Wright (2003). As Shi pointed out, inflation may be beneficial if the number of buyers in the search market is endogenous. However, this only happens if buyers have most of the bargaining power in the trade meetings. Otherwise, the welfare cost of inflation is large and increasing with the bargaining power of sellers. Moreover, for most bargaining weights, the order of magnitude of the welfare cost of inflation calculated by Lagos and Wright is robust to the endogeneity of the number of buyers in the search market.

The coexistence of money and financial contracts allows an integration of the microfoundations of money and finance, so it opens the door to new lines of research. For example, the extension of the present model in Faig (2004) assumes that the credit market in the village is centered around banks because they are the only agents that are able to implement credit contracts. At a cost, these banks can transfer deposits to banks from other villages. Therefore, simple extensions of the model presented in this paper are able to integrate money and banking along quite different lines from previous attempts in monetary search models.[9] The role of banks in these previous attempts is to provide a medium of exchange whereas the banks in Faig (2004) not only provide a medium of exchange (transferable deposits) but also act as financial intermediaries between borrowers and lenders.

9 See for example Cavalcanti, Erosa, and Temzelides (1999) and He, Huang, and Wright (2003).

Appendix

Properties of the Value Functions

Consider the problem of a buyer in Subsection 2.1. For all finite a^b, the set of feasible policies $\{x_t^b, y_t^b, m_t^b, \mu_t^b, b_t^b\}_{t=0}^{\infty}$ is non empty. Moreover, for all these policies the present discounted utility is well defined and finite because U^b is a bounded function. Consequently, we can use standard recursive methods to find the value function.

Let $C(a^b)$ be the space of bonded and continuous functions $f : a^b \to R$, with the sup norm. Use the Bellman's equation (7) to define the mapping T of $C(a^b)$ onto itself by substituting f for V^b in the left hand of (7) and denoting as $Tf(a^b)$ the right hand of (7). As in the main text, the maximization problem in (7) is subject to the constraints (6), (8), and (9). For a given a^b, the set of feasible policies $\{x^b, y^b, m^b, \mu^b, b^b\}$ is non-empty, compact-valued, and continuous. The utility function U^b is a bounded and continuous, and $0 < \beta < 1$. Therefore, Theorem 4.6 in Stokey and Lucas with Prescott (1989) implies that there is a unique fixed point to the mapping T, which is the value function V^b.

The operator T maps increasing and concave functions onto increasing and concave function, so V^b is increasing and concave. Finally, U^b is continuously differentiable and the solution is interior, so V^b is continuously differentiable.

Analogous arguments prove the same properties for V^s.

The same arguments also apply to value function V in Sections 3 and 4 . In addition, for this function the mapping T maps functions that are linear in the interval $[\underline{a}, \overline{a}]$ onto functions that are linear in this interval, so V has the linear form stated in (56).

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