# Liquidity, Interest Rates and Output

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**Abstract**

This paper integrates limited participation into monetary search theory to analyze the liquidity effects of open market operations. The centralized bonds market features limited participation and shocks to government bond sales, while the decentralized goods market features bilateral matches. Unmatured bonds can be used together with money to purchase goods in a fraction of matches, but in other matches a legal restriction forbids the use of bonds as the means of payments. In this economy, a shock to bond sales has two distinct liquidity effects. One is the immediate liquidity effect on the bond price and the nominal interest rate. The other is a liquidity effect in the goods market starting one period later, i.e., the effect on the amount of unmatured bonds circulating in the goods market. Thus, even independent shocks can affect the household's money allocation between the two markets, affect real output and the term structure of interest rates, and cause nominal interest rates to be serially correlated. I establish the existence of the equilibrium and, with numerical examples, examine equilibrium properties.

Keywords: Liquidity; Interest rates; Money; Search.

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## 1. Introduction

This paper integrates goods market search into a model with limited participation to analyze the liquidity effect of open market operations. Limited participation refers to the assumption that the participants in the bonds market cannot adjust their money holdings immediately to shocks to government bond sales. In an influential paper, Lucas (1990) has shown that limited participation enables open market operations to generate a liquidity effect, i.e., positive shocks to bond sales depress the bond price and drive up the nominal interest rate while negative shocks reduce the nominal interest rate. A central assumption in Lucas's model, and in the literature inspired by Lucas's work,[1] is that all transactions in the goods market are constrained by cash. In this paper, I will replace the Walrasian goods market in his model by a decentralized search market in order to support a role for money and to relax the cash-in-advance constraint.

The main motivation of this paper is to develop monetary search theory to a tractable form for policy analysis. Search theory, originated in Kiyotaki and Wright (1989), builds a microfoundation of monetary theory by generating an endogenous role of fiat money that is absent in traditional monetary models including Lucas's (1990) model. As Wallace (2001) forcefully argued, a strong microfoundation is important for monetary theory, because it can make policy analysis consistent with the underlying environment that supports the role of money. Despite this advantage and recent developments,[2] monetary search theory so far has not been able to incorporate nominal bonds in a tractable way, not mentioning the liquidity effect arising from stochastic bond sales. I will eliminate this deficiency of monetary search theory and then analyze the effects of stochastic shocks to open market operations. I hope that the tractable model will make the microfoundation of monetary theory more accessible to macroeconomists.

Let me briefly describe the model. The economy in this paper has a bonds market and a goods market, which operate separately in each period. The bonds market is centralized and functions in exactly the same way as in Lucas's model. That is, the government sells nominal bonds at the market price and accepts only money as payments. The amount of new bonds is stochastic, which is the only aggregate uncertainty in the economy. This shock is realized after the households have already allocated the assets between the two markets, and hence the familiar liquidity effect arises in the bonds market.

The goods market is decentralized, where agents meet randomly and determine how much to trade bilaterally. There are two types of matches. One is unrestricted matches, where the buyer

[1] For the large literature on limited participation models, see the references in Christiano et al. (1999).

[2] Examples of search models are Trejos and Wright (1995), Shi (1995, 1997), Green and Zhou (1998), Lagos and Wright (2002), and Faig (2002).

can use both money and bonds to buy goods. The other is restricted matches, where a legal restriction forbids the use of bonds as the means of payments for goods. Restricted matches are a fraction $g \in (0, 1)$ of all matches. With this legal restriction, bonds are redeemed immediately at maturity, but unmatured bonds can circulate as an imperfect substitute for money. I set the bonds' maturity to be two periods – the shortest length that allows unmatured bonds to circulate in the goods market.

Allowing agents to use unmatured bonds to purchase goods is a crude way to capture the idea that an actual economy has more than cash as the means of payments. However, it may not be useful to insist on the literal interpretation that unmatured bonds act directly as a medium of exchange. In reality, bonds are a large part of money market checking accounts that the households have in financial institutions, on which checks can be written to pay for goods.

By replacing the cash-in-advance constraint in Lucas (1990) with explicit modelling of the decentralized exchange, this model does generate some new results. The most important one is that open market operations generate a delayed liquidity effect in the goods market, in addition to the immediate liquidity effect in the bonds market. In particular, a high shock to bond sales in the previous period increases the quantity of unmatured bonds circulating in the current goods market. The additional bonds provide liquidity to the buyers who are in unrestricted trades. Thus, in addition to changing the price level, shocks to bond sales in the previous period change the current dispersion of real quantities of goods produced and traded in unrestricted matches versus restricted matches. As a result, aggregate output depends on the shock in the previous period, even though shocks are independent. This liquidity effect in the goods market induces a number of new features regarding output, interest rates and the term structure, that I will summarize in the concluding section.

It is useful to clarify the role of the legal restriction in the model. The legal restriction is imposed here to prevent matured bonds from circulating as money. To achieve this effect, the legal restriction does not have to be severe in order to drive matured bonds out of the circulation. On the contrary, even an arbitrarily small but positive measure ($g$) of restricted trades will do. This strong result arises from the decentralized exchange, as I showed in a precursor (Shi, 2002) to the current paper. In contrast, a traditional monetary model with a Walrasian goods market would not be able to prevent matured bonds from circulating as money, unless the measure of restricted trades is sufficiently large. I do not endogenize the legal restriction because this paper is a positive, rather than a normative, analysis of monetary policy. The welfare implications of the legal restriction are examined in Shi (2003) in a deterministic environment.

## 2. A Search Economy with Legal Restrictions

In this section I describe an economy with a legal restriction in the goods market, analyze individuals' decisions, and define the equilibrium.

### 2.1. Households, Matches, and Markets

The economy has discrete time and many types of households. The number of households in each type is large and normalized to one. The households in each type are specialized in producing a specific good, which they do not consume, and exchange for consumption goods in the market. Goods are perishable between periods. The utility of consumption is $u(.)$ for consumption goods and 0 for other goods, with the properties $u'(0) = \infty$ and $u'(\infty) < \infty$. The disutility of production is $\psi(.)$. To simplify the algebra, I will use the form $\psi(q) = \psi_0 q^{\Psi}$, where $\Psi > 1$ and $\psi_0 > 0$.

Each household consists of a large number of members, whose measure is normalized to one. A fraction $\sigma$ of these members are sellers and the remaining are buyers, where $\sigma \in (0, 1)$. A seller produces and sells goods, and a buyer purchases consumption goods. The members share consumption and regard the household's utility function as the common objective. As a result, individual matching risks are smoothed out within each household, and the distribution of asset holdings across households is degenerate. This degeneracy maintains tractability as it enables me to focus on the equilibrium that is symmetric across households.[3]

There are two assets in the economy – money and nominal bonds issued by the government. These assets can be stored without cost. Both are intrinsically worthless; i.e., they yield no direct utility or productive capacity. Nominal bonds are default-free and their maturity is two periods. A bond before the maturity is called an *unmatured bond*. Each bond can be redeemed for one unit of money at maturity. I assume that matured bonds cannot be redeemed once they pass the maturity. This assumption is innocuous in the described environment because households will optimally choose to redeem all matured bonds immediately (see the introduction for a discussion).

Let me describe the goods market first. In this market agents meet their trading partners bilaterally and randomly. Of interest are *trade matches*, in which the buyer likes the seller's goods. These matches are the only ones in which a trade can take place. A buyer encounters a trade match at rate $\alpha\sigma$, and a seller at rate $\alpha(1 - \sigma)$, where $\alpha < 1$. The total number of trades matches that all buyers (or sellers) of a household have in a period is $\alpha\sigma(1 - \sigma)$. There is no chance for a double coincidence of wants to support barter, nor public record-keeping of

[3] The assumption of large households, used by Shi (1997, 1999), is a modelling device extended from Lucas (1990). Lagos and Wright (2002) provide a different set of assumptions to achieve essentially the same purpose of making the distribution of money holdings degenerate in a search model.

transactions to support credit trades. As a result, every trade entails a medium of exchange. This simplification allows me to focus on the competition between money or unmatured bonds as the media of exchange.

There are two types of matches. One is an *unrestricted trade*, where the buyer can use both money and bonds to buy goods. The other is a *restricted trade*, where a legal restriction forbids the use of bonds as a means of payments for goods. The legal restriction is imposed in a fraction $g \in (0,1)$ of matches. One interpretation of the legal restriction is that a fraction $g$ of all agents are government agents who face the same matching rates as private agents do but who accept only money as payments. Although I will use this interpretation later in the numerical exercises, I do not conduct such explicit modelling here (see Shi, 2002, for such modelling in a deterministic environment). Notice that both restricted trades and unrestricted trades are decentralized exchange.[4]

I model the legal restriction as a matching shock. More precisely, in each period, *all* members of a household will be located in restricted matches with probability $g$ and in unrestricted matches with probability $1-g$. These shocks are independent across households and over time. Thus, in each period, a fraction $g$ of all households are in restricted matches and a fraction $(1-g)$ in unrestricted matches. An individual household in a period experiences either restricted trades or unrestricted trades, but not both, although the household experiences both types of trades over time. This way of modelling the matching shock simplifies the analysis.[5]

In contrast to the goods market, the bonds market has no transaction cost and trades take zero measure of agents. In this market, the government conducts open market operations by selling new bonds at the competitive price. As in Lucas (1990), the government only accepts money as payments for the bonds. However, agents can bring unmatured bonds into the bonds market, sell them to other households for money, and then use the receipt to purchase new bonds, although the net amount of such transactions is zero in any symmetric equilibrium.

The amount of newly issued bonds is stochastic, which is the only aggregate uncertainty in the economy. To specify this stochastic process, let $M_{+1}$ be the average amount of money holdings per household in the *next period after* monetary transfers in that period are made but before the markets open (see Figure 1 later for a depiction of the timing). The amount of bonds newly

[4]As an alternative to this setup, one may suggest that restricted trades be modelled as trades in a Walrasian market, while unrestricted trades as trades in a decentralized market. I do not adopt this alternative setup because it cannot prevent matured bonds from circulating as money, unless the government sector is sufficiently large (see Shi, 2002).

[5]This formulation simplifies the algebra, but it is not critical for the results. In a deterministic version of the current model (Shi, 2003), I explored a different formulation where a household experiences both restricted and unrestricted trades in each period.

issued in the current period is $zM_{+1}$, where $z$ is a random variable following a Markov process.[6] The realizations of $z$ lie in a compact set $Z$, with a lower bound $z_L > 0$ and an upper bound $z_H < \infty$. The transition function of $z$ is $\Phi(dz, z_{-1})$, where the subscript $-1$ indicates the previous period. Assume that $\Phi$ has the Feller property (i.e., that $f : Z \times Z \rightarrow R$ is continuous implies $\int f(z, z_{-1})\Phi(dz, z_{-1})$ is continuous).

Open market operations can affect the money growth rate. To focus on the "pure" liquidity effect, Lucas (1990) eliminates the effect of the shocks on money growth by assuming that the government uses lump-sum transfers to maintain a constant money growth rate. In order to compare my results with Lucas's, I adopt this assumption in most of my analysis, with the constant (gross) rate of money growth being $\gamma$. In section 6.2 I will examine the alternative assumption that monetary transfers are a constant fraction of the money stock, in which case the money growth rate depends on the shocks.

### 2.2. Timing of Events

Let me clarify four pieces of notation. First, like Lucas (1990), I normalize all nominal quantities by the aggregate money holding per household, $M$. Second, I pick an arbitrary household as the representative household and use lower-case letters to denote the decisions of this household. The corresponding capital-case letters denote other households' decisions or aggregate variables. Third, I suppress the generic time subscript $t$, denoting $t \pm j$ as $\pm j$ for $j \geq 1$. Fourth, all integrals in this paper are over $Z$, with $Z$ being suppressed.

Figure 1 depicts the timing of events in each period. At the beginning of the period the household redeems bonds that were issued two periods ago and receives a lump-sum monetary transfer, $L$. After these events, the household's holding of money (divided by $M$) is measured as $m$, and of unmatured bonds as $b$.

Then, the household chooses a fraction of money, $a$, and a fraction of unmatured bonds, $l$, that will be taken to the goods market. This part of the assets the household divides evenly among the buyers; so, each buyer carries $am/(1-\sigma)$ units of money and $lb/(1-\sigma)$ units of unmatured bonds. The household takes the remaining assets to the bonds market. At the time of choosing the portfolio divisions $(a, l)$, the household also chooses the quantities of goods and

[6] The purpose of specifying the amount of new bonds as $zM_{+1}$, rather than $zM$ as in Lucas (1990), is for the convenience of unifying the formulas in the case where open market operations affect the money growth rate and the case where the money growth rate is fixed by monetary transfers (see a discussion in the next paragraph). The two specifications are equivalent, up to rescaling, in the case where the monetary growth rate is constant. Notice that the specification of $zM_{+1}$ does not create any problem of measurability. Because there is no new shock between the realization of the current $z$ and the measurement of future money stock $M_{+1}$, the stock $M_{+1}$ and hence the amount of newly issued bonds is measurable with respect to the stochastic process of $z$.

money in a trade. These quantities are contingent on whether the household members will be located in restricted or unrestricted trades. To indicate this contingency, I denote the quantities in a restricted trade as $(q^g, x^g)$ and the quantities in an unrestricted trade as $(q^n, x^n)$.

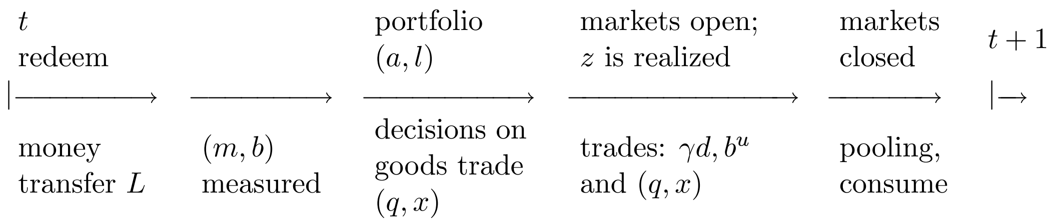


Figure 1 Timing of events in a period

Next, the two markets open simultaneously and separately. It is not possible to communicate between the two markets. In the goods market, the matching shock implied by the legal restriction is realized, and agents trade according to the quantities $(q^g, x^g)$ and $(q^n, x^n)$ prescribed by the household. In the bonds market, the shock $z$ is realized as the government issues an amount $\gamma z$ of new two-period bonds. Let $\gamma d$ be the amount of such bonds demanded by the household so that $d$ is normalized in the same way as $z$ is. The household can also trade unmatured bonds in the bonds market. Let $b^u$ be the amount of unmatured bonds that the household carries out of the bonds market when the market closes. Let the price of two-period bonds be $S$ and the price of unmatured bonds be $S^u$.

Then, the markets close and agents go home. The household pools the receipts from the trades and allocates consumption evenly among all members. After consumption, time proceeds to the next period.

With the above timing, one-period bonds (if they are introduced) do not have a chance to circulate in the goods market before maturity. Once matured, they will be redeemed immediately by the households, rather than being kept to buy goods in a fraction of future trades. Thus, only unmatured long-term bonds can circulate in the goods market.

The temporary separation between the bonds market and the goods market implies that there is an opportunity cost for bringing assets into the bonds market. Also, because the household must choose the portfolio divisions $(a, l)$ before the current state $z$ is realized, these decisions can depend on the past shock $z_{-1}$ but not on the current shock $z$. In contrast, the household chooses the amounts of bonds to purchase after observing the current state $z$. Thus, $d$ and $b^u$ can depend on $z$, as well as on $z_{-1}$. Similarly, prices of bonds, $(S, S^u)$, depend on both $z$ and $z_{-1}$.

### 2.3. Quantities of Trade in the Goods Market

The household chooses the quantities of money and goods in each trade match. To describe these choices, let $v(m, b, z_{-1})$ be the household's value function at the time where $m$ and $b$ are measured (see Figure 1).[7] The discount factor is $\beta \in (0, 1)$. Let $\omega^m(z_{-1})$ be the expected shadow value of next period's money discounted to the current period, where the expectation is calculated before observing the current shock $z$. Similarly, let $\omega^b(z_{-1})$ be the expected shadow value of unmatured bonds. Then,

$$\omega^i(z_{-1}) = \frac{\beta}{\gamma} \int v_{i_{+1}}(m_{+1}, b_{+1}, z) \Phi(dz, z_{-1}), \; i = m, b, \tag{2.1}$$

where $v_{i_{+1}} = \partial v(m_{+1}, b_{+1}, z)/\partial i_{+1}$. Notice that the discount on the value of future assets involves the money growth rate $\gamma$, because the variables $m$ and $b$ are normalized by the aggregate money stock which grows at rate $\gamma$. The expected values, $\omega^m$ and $\omega^b$, are computed before the current shock $z$ is realized, in order to make them relevant for the money allocation in the current period. Other households' expected value of future money is $\Omega^m$ and of future unmatured bonds is $\Omega^b$.

In a trade match, the buyer makes a take-it-or-leave-it offer. The offer specifies the quantity of goods that the buyer asks the seller to supply, $q$, and the quantity of assets that the buyer gives, $x$. These quantities are $(q^g, x^g)$ in a restricted trade and $(q^n, x^n)$ in an unrestricted trade. In a restricted trade, the assets used must be money. In an unrestricted trade, the assets used can be both money and unmatured bonds. However, it is not necessary to specify the division of the amount $x^n$ into money and unmatured bonds, because the two assets are equivalent to anyone who exits the trade with them. When the trade is closed, no one can use these assets to purchase goods in the current period and, at the beginning of the next period, the bonds mature and can be redeemed for money at par.[8]

Because of the assumption that the buyer makes a take-it-or-leave-it offer, the quantities $(q^i, x^i)$ yield zero surplus for the seller. The seller's surplus is $[\Omega^m x^i - \psi(q^i)]$, where $\Omega^m x^i$ is the value of the assets that the seller receives from the trade. Thus,

$$x^i(z_{-1}) = \frac{\psi(q^i(z_{-1}))}{\Omega^m}, \; i = n, g. \tag{2.2}$$

Also, the buyer is constrained by the sum of money and unmatured bonds in an unrestricted trade, and by the amount of money in a restricted trade. These *asset constraints* are:

$$x^n(z_{-1}) \leq \frac{a(z_{-1})m + l(z_{-1})b}{1 - \sigma}, \tag{2.3}$$

[7] I suppress the dependence of the value function on aggregate variables and other households' decisions.

[8] For the same reason, a trade in the goods market between a money holder and a bond holder is inconsequential, and so it is omitted here. Of course, this simplicity will be lost if bonds have maturity longer than two periods.

$$x^g(z_{-1}) \leq \frac{a(z_{-1})m}{1-\sigma}. \tag{2.4}$$

When an asset constraint binds, I say that the asset yields *liquidity service* in the goods market. Similarly, money may generate liquidity in the bonds market.

### 2.4. A Household's Decision Problem

In a typical period, the household's choices are the portfolio division, $(a, l)$, the quantities of trade, $(q^n, x^n, q^g, x^g)$, the amount of new bonds to purchase, $d$, the amount of unmatured bonds exiting the bonds market with, $b^u$, consumption, $(c^n, c^g)$, future money holdings, $m_{+1}$, and future holdings of unmatured bonds, $b_{+1}$. The decisions $(a, l, q, x, c)$ are functions of only the previous period's state $z_{-1}$, but $(d, b^u)$ can depend on the current state $z$ as well. Future money holdings are denoted $m^g_{+1}$ if the household has restricted trades in the current period and $m^n_{+1}$ if the household has unrestricted trades. The household takes as given other households' decisions, aggregate variables and bond prices $(S, S^u)$.

The representative household solves the following problem:

$$\begin{aligned}
&(PH) \quad v(m, b, z_{-1}) = \\
&\max_{(a,l,q,x,c)(z_{-1})} \left\{ g \left[ u(c^g(z_{-1})) - \alpha\sigma(1-\sigma)\psi(Q^g) + \beta \int \max_{(d,b^u)(z,z_{-1})} v(m^g_{+1}, b_{+1}, z)\Phi(dz, z_{-1}) \right] \right. \\
&\quad \left. +(1-g) \left[ u(c^n(z_{-1})) - \alpha\sigma(1-\sigma)\psi(Q^n) + \beta \int \max_{(d,b^u)(z,z_{-1})} v(m^n_{+1}, b_{+1}, z)\Phi(dz, z_{-1}) \right] \right\}.
\end{aligned}$$

The constraints are as follows:

(i) the constraints in the goods market, (2.2) – (2.4), and

$$c^i(z_{-1}) = \alpha\sigma(1-\sigma)q^i(z_{-1}), \quad i = n, g; \tag{2.5}$$

(ii) the constraints in the bonds market: $b^u(z) \geq 0$ and

$$S(z, z_{-1})\gamma d(z, z_{-1}) \leq [1 - a(z_{-1})]\, m + S^u(z, z_{-1}) \{[1 - l(z_{-1})]\, b - b^u(z, z_{-1})\}; \tag{2.6}$$

(iii) the laws of motion of asset holdings:

$$b_{+1} = d(z, z_{-1}), \tag{2.7}$$

$$\begin{aligned}
m^i_{+1} = \; & \tfrac{1}{\gamma} \{m - S(z, z_{-1})\gamma d(z, z_{-1}) + S^u(z, z_{-1}) [(1 - l(z_{-1}))\, b - b^u(z, z_{-1})] \\
& +\alpha\sigma(1-\sigma) [X^i - x^i(z_{-1})] + [l(z_{-1})b + b^u(z, z_{-1})] + L_{+1}\}, \; i = g, n.
\end{aligned} \tag{2.8}$$

(iv) and other constraints: $0 \leq a(z_{-1}) \leq 1$ and $0 \leq l(z_{-1}) \leq 1$.

The objective function in the above problem contains two groups of terms, one for the case where the household's members are located in restricted matches and the other for the case in unrestricted matches.[9] The outer maximization determines the choices $(a, l, q, x, c)$, which are made before the realization of the shock $z$. The inner maximization determines the choices $(d, b^u)$, which maximize the future value function for each realization of $z$.

The constraints in (i) and (iv), and the law of motion of unmatured bonds, (2.7), are self-explanatory. In (ii), there are two constraints in the bonds market. First, the household cannot hold a negative amount of unmatured bonds. Second, the household must finance the purchase of new bonds by the assets it brings into the bonds market, as (2.6) requires. The last term in (2.6) is the receipt of money that the household obtains by selling some of the unmatured bonds it brings to the bonds market.

To explain the law of motion of money, (2.8), recall that the household's money holding is measured at the time immediately after receiving monetary transfers and redeeming matured bonds (see Figure 1). Between two adjacent points of time of this measurement, money holdings can change as a result of the following transactions: purchasing newly issued bonds, selling unmatured bonds in the bonds market, selling and buying goods, redeeming matured bonds and receiving the monetary transfer next period. The terms following $m$ on the right-hand side of (2.8) list the net changes in money holdings from these transactions. Here, the factor $1/\gamma$ appears on the right-hand side because $m_{+1}$ is normalized by $M_{+1}$ while the money receipts in the current period are normalized by $M$.

In all symmetric equilibria, $m^g_{+1} = m^n_{+1}$. To see this, note first that the way in which the matching shock is modelled implies that, if two agents are matched, then their households both received the same matching shock and hence are symmetric. This implies $x^i = X^i$ in equilibrium for $i = g, n$. Also, because it is not possible to communicate between the two separate markets, the trading decisions in the bonds market, $(d, b^u)$, cannot depend on the realization of the current matching shock. Because matching shocks are independent over time, the trading decisions in the bonds market cannot depend on past matching shocks, either. With these features, one can easily show from (2.8) that $m^g_{+1} = m^n_{+1}$. Thus, I will suppress the superscripts $(g, n)$ on $m$.

To characterize optimal decisions, let $\rho(z, z_{-1})$ be the (state-contingent) Lagrangian multiplier of the constraint in the bonds market, (2.6). Let $\lambda^n(z_{-1})$ be the multiplier of the asset constraint in an unrestricted trade, (2.3), and $\lambda^g(z_{-1})$ the multiplier of the asset constraint in a restricted trade, (2.4). To simplify the equations, multiply $\lambda^n$ by $\alpha\sigma(1-\sigma)(1-g)$ and $\lambda^g$ by $\alpha\sigma(1-\sigma)g$.

[9] The implicit assumption here is that the goods in a restricted trade yield the same marginal utility as the goods in an unrestricted trade. For a relaxation of this assumption, see Shi (2003).

For the moment, I suppress the dependence of the dependence of $(d, b^u, S, S^u, \rho)$ on $(z, z_{-1})$ and $(a, l, q, x, c, \lambda)$ on $z_{-1}$. The following conditions characterize the household's optimal choices.
(i) Quantities $q^n$ and $q^g$:

$$u'(c^i) = \left(\omega^m + \lambda^i\right) \frac{\psi'(q^i)}{\Omega^m}, \quad i = n, g. \tag{2.9}$$

(ii) Portfolio divisions $(a, l)$ and bonds market decisions $(b^u, d)$:

$$\text{for } a\text{:} \quad \alpha\sigma \left[(1-g)\lambda^n + g\lambda^g\right] = \int \rho\Phi\left(dz, z_{-1}\right); \tag{2.10}$$

$$\text{for } l\text{:} \quad \alpha\sigma(1-g)\lambda^n + \omega^m = \int \left(\rho + \frac{\beta}{\gamma} v_{m+1}\right) S^u \Phi(dz, z_{-1}); \tag{2.11}$$

$$\text{for } b^u\text{:} \quad \frac{\beta}{\gamma} v_{m+1} = \left(\rho + \frac{\beta}{\gamma} v_{m+1}\right) S^u; \tag{2.12}$$

$$\text{for } d\text{:} \quad \frac{\beta}{\gamma} v_{b+1} = \left(\rho + \frac{\beta}{\gamma} v_{m+1}\right) S. \tag{2.13}$$

In each of these conditions, the variable attains the lowest value in its domain if the equality is replaced by "<", and the highest value if ">", where $a, l \in [0, 1]$ and $d, b^u \in [0, \infty)$.
(iii) The envelope conditions for $m$ and $b$:

$$v_m = \omega^m + a\alpha\sigma\left[(1-g)\lambda^n + g\lambda^g\right] + (1-a) \int \rho\Phi\left(dz, z_{-1}\right); \tag{2.14}$$

$$v_b = l\left[\omega^m + \alpha\sigma(1-g)\lambda^n\right] + (1-l) \int \left(\rho + \frac{\beta}{\gamma} v_{m+1}\right) S^u \Phi(dz, z_{-1}). \tag{2.15}$$

The condition (2.9) requires that the net gain to a buyer from asking for an additional amount of goods be zero. By getting an additional unit of good, the household's utility increases by $u'(c)$. The cost is to pay an additional amount $\psi'(q)/\Omega^m$ of assets in order to induce the seller to trade (see (2.2)). By giving an additional unit of asset, the buyer foregoes the discounted future value of the asset, $\omega^m$, and causes the asset constraint in the trade to be more binding. Thus, $(\omega^m + \lambda)$ is the shadow cost of each additional unit of asset to the buyer's household and the right-hand side of (2.9) is the cost of getting an additional unit of good from the seller.

In (ii), (2.10) says that for the household to allocate money to both the goods market and the bonds market, money must generate the same expected liquidity service in the two markets. The liquidity service derives from the role of the asset in relaxing the trading constraints, as reflected by the shadow costs of the corresponding constraints.

The condition (2.11) is a similar requirement on the allocation of unmatured bonds between the two markets. If the household takes a unit of unmatured bond to the goods market, the bond can generate liquidity services $\alpha\sigma(1-g)\lambda^n$ by relieving the asset constraints and will have a future value $\frac{\beta}{\gamma}v_{m+1}$ upon redemption. If the household instead takes the unit of unmatured bond to the bonds market, the bond can be sold for $S^u$ units of money, which will generate liquidity service $\rho$ in the bonds market and will have a future value $\frac{\beta}{\gamma}v_{m+1}$. Because the household must choose $l$ before seeing the realization of $z$, it compares the expected values of allocating a marginal unit of unmatured bonds to the two markets. This comparison leads to (2.11).

The condition (2.12) specifies the optimal demand for unmatured bonds in the bonds market. The value of keeping a unit of unmatured bond for future redemption is the discounted future value of one unit of money, $\frac{\beta}{\gamma}v_{m+1}$. The value of selling a unit of unmatured bond for money is $\left(\rho + \frac{\beta}{\gamma}v_{m+1}\right) S^u$, as explained above. For the choice $b^u$ to be interior, these two values must be equal to each other. The condition (2.13) is a similar requirement for the quantity of new bonds purchased, except that the price and future value of a new bond are different from those of an unmatured bond. Notice that (2.12) and (2.13) must hold for every realization of $z$.

Finally, the envelope conditions require the current value of each asset to be equal to the sum of the expected future value of the asset and the expected liquidity service generated by the asset in the current markets. Take the condition for money for example. The current value of money is $v_m$. The right-hand side of (2.14) consists of the expected future value of money, $\omega^m$, the liquidity service generated by money in the current bonds market, $\rho$, and the liquidity service generated by money in the current goods market, $\lambda$. The liquidity services in the two markets are weighted by the division of money into the two markets.

### 2.5. Equilibrium Definition and Interest Rates

A (symmetric) monetary equilibrium consists of a value function $v$: $R_+ \times R_+ \times Z \to R$, portfolio division functions $a, l$: $Z \to [0, 1]$, functions of trade quantities in matches $q^n, x^n, q^g, x^g$: $Z \to R_+$, consumption function $c$: $Z \to R_+$, bonds purchase functions $d, b^u$: $Z \times Z \to R_+$, bonds price functions $S, S^u$: $Z \times Z \to R_{++}$ such that the following requirements are met:

(i) Given other households' choices and $(m, b)$, the household's choices solve $(PH)$;

(ii) The choices are the same across households and, in particular, $m = 1$;

(iii) The bonds market clears, i.e., $d(z, z_{-1}) = z$ and $b^u(z, z_{-1}) = [1 - l(z_{-1})]\, b$ for all $(z, z_{-1}) \in Z \times Z$;

(iv) $0 < \omega^m(z), \omega^b(z) < \infty$ for all $z \in Z$.

The requirements (i), (ii) and (iii) are self-explanatory. In part (iv), the restriction that the value of each asset be positive is necessary for a meaningful examination of the coexistence of money and bonds. The restriction that these values be bounded away from infinity is necessary for the first-order conditions to characterize optimal decisions.

Moreover, I restrict attention to equilibria in which money generates liquidity in the goods market in all states of the economy. That is, for all $z_{-1}$, at least one of $\lambda^n(z_{-1})$ and $\lambda^g(z_{-1})$ must be positive. Note that this restriction also implies $a(z_{-1}) > 0$ for all $z_{-1}$.

By invoking equilibrium conditions, I can simplify some of the optimality conditions. First, because $d(z) = z \in (0, \infty)$ in equilibrium, the optimal condition for $d$ must hold as equality, as in (2.13). Second, because $m = 1$ and $b = d_{-1} = z_{-1}$ in equilibrium, I can shorten the notation for the shadow values of the assets as follows:

$$\mu^i(z_{-1}) \equiv v_i\left(1, z_{-1}, z_{-1}\right), \; i = m, b. \tag{2.16}$$

The expected value of $\mu$, defined in (2.1), can be expressed as $\omega^i = O(\mu^i)$, where

$$O(\mu^i)(z_{-1}) = \frac{\beta}{\gamma} \int \mu^i(z) \Phi\left(dz, z_{-1}\right), \; i = m, b. \tag{2.17}$$

Third, for all $S > 0$, the bonds market clearing conditions imply $a < 1$. Under the restriction $a > 0$, then $0 < a < 1$, and the equality in (2.10) holds. The condition (2.14) can be simplified as

$$\mu^m(z_{-1}) = \omega^m(z_{-1}) + \alpha\sigma\left[(1-g)\lambda^n(z_{-1}) + g\lambda^g(z_{-1})\right]. \tag{2.18}$$

Define the two-period (net) nominal interest rate as $r = \frac{1}{S} - 1$. If money yields liquidity in the bonds market (i.e., if $\rho > 0$), then (2.6) binds and $S = (1-a)/(\gamma z)$. In this case, (2.13) implies $S < \mu^b(z)/\mu^m(z)$. If $\rho = 0$, then (2.6) does not bind. In this case, $S \leq (1-a)/(\gamma z)$, and (2.13) implies $S = \mu^b(z)/\mu^m(z)$. Combining the two cases, I express the two-period bond price as

$$S\left(z, z_{-1}\right) = \min\left\{\frac{1-a(z_{-1})}{\gamma z}, \frac{\mu^b(z)}{\mu^m(z)}\right\}. \tag{2.19}$$

The two-period nominal interest rate is

$$r(z, z_{-1}) = \max\left\{\frac{\gamma z}{1-a(z_{-1})} - 1, \frac{\mu^m(z)}{\mu^b(z)} - 1\right\}. \tag{2.20}$$

From (2.13), I can compute the shadow price of the asset constraint in the bonds market as

$$\rho\left(z, z_{-1}\right) = \frac{\beta}{\gamma} \max\left\{\frac{\gamma z}{1-a(z_{-1})}\mu^b(z) - \mu^m(z), 0\right\}. \tag{2.21}$$

This shadow price may be zero for particular realizations of $z$, but the expected value of $\rho(z, z_{-1})$ over $z$ must be positive for all $z_{-1} \in Z$, in order to satisfy the earlier restriction that at least one of $\lambda^g(z_{-1})$ and $\lambda^n(z_{-1})$ be positive (see (2.10)).

The price of unmatured bonds in the bonds market, $S^u$, depends on whether the household takes all unmatured bonds to the goods market. If $l = 1$, the supply of and the demand for unmatured bonds in the bonds market are both zero, in which case $S^u$ is indeterminate. If $l < 1$, the supply of unmatured bonds in the bonds market is positive. In this case, the equality in (2.12) holds and so

$$S^u(z, z_{-1}) = \frac{\mu^m(z)}{\rho(z, z_{-1})\frac{\gamma}{\beta} + \mu^m(z)}. \tag{2.22}$$

Unmatured bonds are discounted if $\rho(z, z_{-1}) > 0$.

Although the price of unmatured bonds may be indeterminate, the price of newly issued one-period bonds is determinate. If one-period bonds were issued, the price would be given by the right-hand side of (2.22) (regardless of whether $l < 1$). Denote this price as $S^I(z, z_{-1})$. Then,

$$\frac{S(z, z_{-1})}{S^I(z, z_{-1})} = \frac{\mu^b(z)}{\mu^m(z)}. \tag{2.23}$$

The ratio $\mu^m/\mu^b$ is the expected future discount on unmatured bonds. As shown later, $\mu^b(z) < \mu^m(z)$ in the equilibrium, because unmatured bonds are not perfect substitutes for money in the goods market. Thus, there is a deeper discount on two-period bonds than on one-period bonds.

## 3. Characterization and Existence of the Equilibrium

### 3.1. Characterization

The equilibrium can be one of two cases, $0 \leq l < 1$ and $l = 1$. When $l = 1$, the household takes all unmatured bonds to the goods market and such bonds generate liquidity in a fraction $(1 - g)$ of trades; i.e., $\lambda^n > 0$. In the case $0 \leq l < 1$, unmatured bonds do not generate liquidity (at the margin) in the goods market, i.e., $\lambda^n = 0$, although some unmatured bonds may still be used to buy goods.[10] Likewise, money generates liquidity service in unrestricted trades if and only if $l = 1$. In contrast to bonds, money also generates liquidity service in restricted trades if $\lambda^g > 0$.

Before characterizing the equilibrium, let me express more explicitly the conditions under which the asset constraints bind. The condition $\lambda^n > 0$ is equivalent to $u'(c^n) > \psi'(q^n)$ (see (2.9)

[10] The proof for $\lambda^n = 0$ in this case is as follows. When $0 \leq l < 1$, the optimality condition for $l$ holds as "$\leq$"; That is, (2.11) holds as "$\leq$". Because $b^u = (1 - l)b \in (0, \infty)$ when $0 \leq l < 1$, the equality in (2.12) holds, which leads to (2.22). Substituting (2.22) into the inequality form of (2.11) yields $\lambda^n \leq 0$. Thus, $\lambda^n = 0$.

for $i = n$). Define $Q_0$ as the solution to the following equation:

$$u'(\alpha\sigma(1-\sigma)Q_0) = \psi'(Q_0). \tag{3.1}$$

Then, $\lambda^n > 0$ (and hence $l = 1$) iff $q^n < Q_0$. It is more convenient to express this condition in terms of the shadow value of money. Since $l = 1$ when $\lambda^n > 0$, then (2.3) and (2.2) imply that $\lambda^n > 0$ iff $0 < \omega^m < w_1$ where

$$w_1(a, z_{-1}) = \frac{1-\sigma}{a + z_{-1}}\psi(Q_0). \tag{3.2}$$

Similarly, the asset constraint binds in a restricted trade (i.e., $\lambda^g > 0$) iff $q^g < Q_0$, which can be rewritten as $0 < \omega^m < w_2$ where

$$w_2(a) = \frac{1-\sigma}{a}\psi(Q_0) > w_1. \tag{3.3}$$

If $\omega^m \geq w_2$, then no asset generates any liquidity service (at the margin). For $\omega^m < w_2$, the equilibrium falls into the following two cases.

**Case 1:** $0 < \omega^m < w_1$. In this case, $\lambda^n > 0$ and $\lambda^g > 0$. Also, $l = 1$. Since the asset constraints (2.3) and (2.4) bind, the quantity of goods traded is $q^n = Q_1$ in an unrestricted trade and $q^g = Q_2$ ($< Q_1$) in a restricted trade, where

$$Q_1(\omega^m; a, z_{-1}) = \psi^{-1}\left(\frac{a + z_{-1}}{1-\sigma}\omega^m\right), \tag{3.4}$$

$$Q_2(\omega^m; a) = \psi^{-1}\left(\frac{a\omega^m}{1-\sigma}\right). \tag{3.5}$$

The total amount of liquidity service that unmatured bonds generate is $\alpha\sigma(1-g)\lambda^n$. After substituting $\lambda^n$ from (2.9), I can express this amount as $\omega^m(z_{-1})F^n$, where

$$F^n(\omega^m; a, z_{-1}) = \alpha\sigma(1-g)\left[\frac{u'(\alpha\sigma(1-\sigma)Q_1(\omega^m; a, z_{-1}))}{\psi'(Q_1(\omega^m; a, z_{-1}))} - 1\right]. \tag{3.6}$$

Similarly, the total amount of liquidity service that money generates is $\alpha\sigma(1-g)\lambda^n + \alpha\sigma g\lambda^g$, which can expressed as $\omega^m(z_{-1})(F^n + F^g)$ where $F^g$ is defined as follows:

$$F^g(\omega^m; a) = \alpha\sigma g\left[\frac{u'(\alpha\sigma(1-\sigma)Q_2(\omega^m; a))}{\psi'(Q_2(\omega^m; a))} - 1\right]. \tag{3.7}$$

**Case 2:** $w_1 \leq \omega^m < w_2$. In this case, $\lambda^n = 0 < \lambda^g$. Since $\lambda^n = 0$, the quantity of goods traded in an unrestricted trade is $q^n = Q_0$. Since $\lambda^g > 0$, the quantity of goods traded in a restricted trade is $q^g = Q_2$. Bonds do not generate liquidity service. In contrast, money generates liquidity service, the total amount of which is $\omega^m(z_{-1})F^g$.

Now I express the equilibrium as a fixed point for the functions of the assets' values, $(\mu^m, \mu^b, \omega^m)$, and the money allocation function $a$. First, I combine the above cases to express the total amount of liquidity service generated by money as $\omega^m(z_{-1})F$, where

$$F(\omega^m; a, z_{-1}) = \begin{cases} F^g(\omega^m; a) + F^n(\omega^m; a, z_{-1}), & \text{if } 0 < \omega^m \leq w_1 \\ F^g(\omega^m; a), & \text{if } w_1 \leq \omega^m \leq w_2 \\ 0, & \text{if } \omega^m \geq w_2. \end{cases} \tag{3.8}$$

Using this amount to substitute for the term $[\alpha\sigma(1-g)\lambda^n + \alpha\sigma g \lambda^g]$, I can write (2.18) as:

$$\mu^m(z_{-1}) = \omega^m(z_{-1}) \left[1 + F(\omega^m(z_{-1}); a(z_{-1}), z_{-1})\right]. \tag{3.9}$$

This is a functional equation for $\mu^m$, since $\omega^m = O(\mu^m)$. Similarly, the functional equation for $\mu^b$ comes from rewriting (2.15) as follows:[11]

$$\mu^b(z_{-1}) = \begin{cases} \omega^m(z_{-1}) \left[1 + F^n(\omega^m(z_{-1}); a(z_{-1}), z_{-1})\right], & \text{if } 0 < \omega^m \leq w_1 \\ \omega^m(z_{-1}), & \text{if } \omega^m \geq w_1. \end{cases} \tag{3.10}$$

Next, I use (2.10) to eliminate $(\lambda^g, \lambda^n)$ in (2.18), substitute the definition of $\omega^m$, and use (2.21) to eliminate $\rho$. This produces the following functional equation for $a$:

$$a(z_{-1}) = 1 - \frac{\beta/\gamma}{\mu^m(z_{-1})} \int \max\left\{\gamma z \mu^b(z),\ [1 - a(z_{-1})]\, \mu^m(z)\right\} \Phi(dz, z_{-1}). \tag{3.11}$$

The equilibrium requires $\omega^m < w_2$; otherwise, $\mu^m = \omega^m = O(\mu^m)$, which would not have a stationary solution for $\mu^m$ when $\gamma > \beta$. For all $\omega^m < w_2$, (3.9) and (3.10) imply that $\mu^m(z) > \mu^b(z)$ for all $z$. Thus, unmatured bonds are not perfect substitutes for money in the goods market and, as (2.23) shows, this imperfect substitutability induces a deeper discount on two-period bonds than on one-period bonds. Of course, the imperfect substitutability relies on the existence of the legal restriction.

The strategy for determining the equilibrium is as follows. Start with an arbitrary continuous function $a(.)$ bounded in the interior of $[0,1]$ and solve the fixed point for $\mu^m$ from (3.9). Substitute the solution into $\omega^m = O(\mu^m)$ to get $\omega^m$ and into (3.10) to get $\mu^b$. Then, substitute $(\mu^m, \mu^b)$ into the right-hand side of (3.11) to obtain a new function, denoted as $\Gamma_a(z_{-1})$. The equilibrium solution for $a(.)$ solves $a(z_{-1}) = \Gamma_a(z_{-1})$. Once the functions $(\mu^m, \mu^b, \omega^m, a)$ are determined, I can recover the traded quantities of goods and consumption (output) through (3.4) and (3.5), the bond price $S$ through (2.19) and the nominal interest rate through (2.20). The fraction of unmatured bonds taken to the goods market is $l = 1$ if $\omega^m < w_1$ and $l \in [0, 1)$ if $\omega^m > w_1$.[12]

[11] The derivation is straightforward when $\omega^m < w_1$ (i.e., when $l = 1$). When $\omega^m > w_1$, $0 \leq l < 1$ and $\lambda^n = 0$, as discussed above. Then $b^u = (1-l)b \in (0, \infty)$, and so the equality in (2.12) holds. This equality and the fact $\lambda^n = 0$ reduce (2.15) to $\mu^b = \omega^m$.

[12] When $\omega^m > w_1$, the equilibrium is consistent with a range of values of $l$ in $[0, 1)$. This indeterminacy of $l$

### 3.2. Existence of the Equilibrium

All proofs for this subsection are collected in Appendix A. Although the existence proof follows the general route used by Lucas (1990), the details necessarily differ for the following reasons. First, the goods market here is non-Walrasian, and so prices are determined bilaterally. Second, output is endogenous, rather than being given by endowments. Third, there are two types of trades in the goods market – the restricted trades and unrestricted trades – and so prices are different in these trades.

Let me begin by defining the bounds on various functions. Let $a$ be bounded in $[a_L, a_H]$ and $\mu^m$ bounded in $[\frac{\gamma}{\beta}\omega_L, \frac{\gamma}{\beta}\omega_H]$, where

$$0 < a_L \leq a_H < 1,\ 0 < \omega_L \leq \omega_H < \infty. \tag{3.12}$$

Then, by (2.17), $\omega^m$ is bounded in $[\omega_L, \omega_H]$. There will be further restrictions imposed on these bounds later in Lemma 3.1 and Theorem 3.2. Let $\mathcal{V}$ denote the set of continuous functions whose values lie in $[\frac{\gamma}{\beta}\omega_L, \frac{\gamma}{\beta}\omega_H]$ and $\mathcal{A}$ the set of continuous functions whose values lie in $[a_L, a_H]$. Endow $\mathcal{V}$ and $\mathcal{A}$ with the supnorm. I restrict attention to $\mu^m(.) \in \mathcal{V}$ and $a(.) \in \mathcal{A}$.

To determine the equilibrium, I first solve for $\mu^m$ from (3.9) under a fixed function $a(.)$. Denote the right-hand side of (3.9) as $T(\omega^m; a, z_{-1})$ and define

$$TO(\mu^m; a, z_{-1}) = T(O(\mu^m); a, z_{-1}). \tag{3.13}$$

Then, (3.9) requires $\mu^m$ to be the fixed point of $TO$. I will find conditions under which $TO$ is a monotone, contraction mapping from $\mathcal{V}$ to $\mathcal{V}$. I impose the following assumption.

**Assumption 1.** *Denote the relative risk aversion as* $\delta(c) = -cu''(c)/u'(c)$. *Assume that (i)* $\delta(c) \leq 1$, *and (ii) the function* $[1 - \delta(c)]\, u'(c) \Big/ \psi'\left(\frac{c}{\alpha\sigma(1-\sigma)}\right)$ *is decreasing in* $c$.

Part (i) of the assumption is sufficient for $T(\omega^m; a, z_{-1})$ to be increasing in $\omega^m$. The upper bound of unity on the relative risk aversion may seem quite restrictive, but the numerical examples in section 6.1 will show that the bound strengthens the results and that the features of the model remain the same for higher values of risk aversion. However, Part (ii) is necessary and sufficient for $T$ to be concave in $\omega^m$ in each of the three segments $(0, w_1)$, $(w_1, w_2)$, and $(w_2, \infty)$. It is satisfied if, for example, the utility function exhibits constant relative risk aversion. Figure 2 depicts $T$ as a function of $\omega^m$.

has no effect on real variables, because unmatured bonds do not generate liquidity service at the margin. The indeterminacy does not affect the equilibrium value of $a$, either, and hence the bond price $S$ and the corresponding interest rate $r$ do not depend on such indeterminacy.

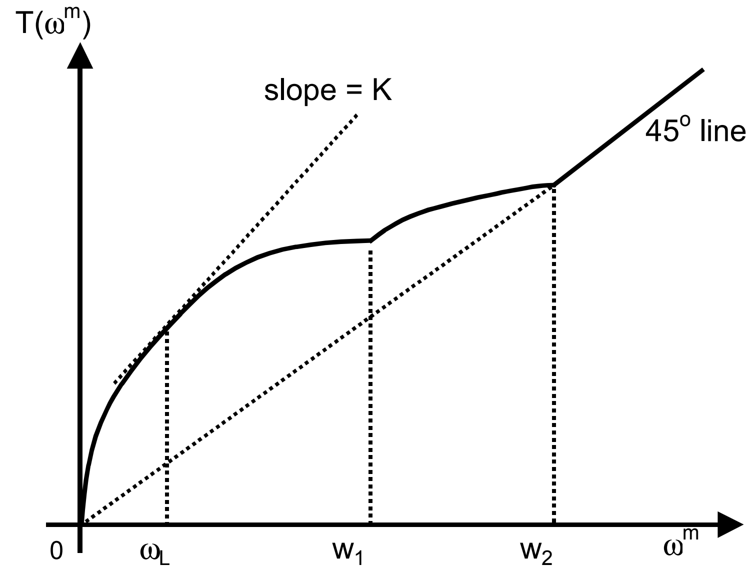


Figure 2

I can select a constant $K$ that is sufficiently close to but greater than 1, and use $K$ to construct the lower bound $\omega_L$ (see Figure 2 for an illustration and Appendix A for detailed construction). Then, the following lemma holds.

**Lemma 3.1.** *Let $\varepsilon > 0$ be a small number. Let $K$ be sufficiently close to but greater than 1. Given any function $a(.) \in \mathcal{A}$, the mapping $TO$ defined in (3.13) is a monotone contraction mapping from $\mathcal{V}$ to $\mathcal{V}$ if the following condition holds:*

$$\max \{K - 1 + \varepsilon, F(\omega_H, a_L, z_L)\} \leq \frac{\gamma}{\beta} - 1 \leq F(\omega_L, a_H, z_H). \tag{3.14}$$

*There is a non-empty set of parameter values that satisfy (3.14). Under this condition, $TO$ has a unique fixed point $\mu_a^m(.) \in \mathcal{V}$.*

The notation $\mu_a^m$ emphasizes the dependence of the fixed point on the arbitrarily chosen function $a$. Similarly, the expected future shadow value of money is $\omega_a^m(z_{-1}) = O(\mu_a^m)(z_{-1})$ and the shadow value of unmatured bonds is $\mu_a^b(.)$, obtained by substituting $\mu_a^m$ into (3.10). Clearly, $\omega_a^m(.)$ and $\mu_a^b(.)$ are continuous. Also, $\omega_a^m(z) \in [\omega_L, \omega_H]$ for all $z$.

Now, substituting $\mu_a^m$ and $\mu_a^b$ into (3.11), I have $a(z_{-1}) = \Gamma_a(z_{-1})$, where

$$\Gamma_a(z_{-1}) \equiv 1 - \frac{\beta/\gamma}{\mu_a^m(z_{-1})} \int \max \left\{ \gamma z \mu_a^b(z),\ [1 - a(z_{-1})]\, \mu_a^m(z) \right\} \Phi(dz, z_{-1}). \tag{3.15}$$

I can treat $\Gamma$ as a mapping for $a$. That is, the equilibrium function $a$ is a fixed point of $\Gamma$. Once this fixed point is shown to exist, then the functions $\mu_a^m(.)$ and $\mu_a^b(.)$ will recover the shadow values of money and unmatured bonds.

The following theorem summarizes the existence of the equilibrium.

**Theorem 3.2.** *Maintain Assumption 1. Choose $(\gamma, K, \omega_H)$ to satisfy (3.14). Let $a_H$ be close to 1. There is a nonempty set of values of $(z_H, a_L)$ that satisfy the following condition:*

$$F(\omega_H, a_H, z_H) \geq \max\left\{\frac{\gamma z_H}{1-a_L} - 1, 0\right\}. \tag{3.16}$$

*Under these conditions, $\Gamma$ is a continuous mapping from $\mathcal{A}$ to $\mathcal{A}$. Thus, an equilibrium exists, which satisfies $\mu^m(.) \in \mathcal{V}$, $a(.) \in \mathcal{A}$, and $\omega_L \leq \omega^m(z) \leq \omega_H$ for all $z \in Z$.*

The restriction (3.16) requires $z_H$ to be sufficiently small. This restriction is necessary to ensure that the households allocate a positive fraction of money to the goods market. If the size of the open market operation were very large, instead, new bonds would be heavily discounted; given that the money growth rate is fixed, the households would allocate all money to the bonds market to obtain the discount.

## 4. A Special Case: Independent Shocks

It is instructive to examine the special case where the shocks to bond sales are independent over time. This special case helps illustrating some key differences between the current model and Lucas's (1990) model.

The equilibrium behaves differently depending on whether unmatured bonds generate liquidity service in the goods market. Consider first the case where unmatured bonds do not generate liquidity service, i.e., where $\omega^m > w_1$. Since only the money constraint binds in this case, the equilibrium behaves like that in Lucas's model. With independent shocks, the shadow values of assets $(\mu^m, \mu^b)$ and the fraction $a$ are numbers, rather than functions. To solve for these constants, note that $F = F^g$ in this case. Also, $\omega^m = O(\mu^m) = \frac{\beta}{\gamma}\mu^m$. Then, (3.9) becomes $F^g(\omega^m; a) = \frac{\gamma}{\beta} - 1$. Substituting $F^g$ from (3.7), this equation solves for the quantity of goods in a restricted trade, which is a constant. Because the quantity of goods in an unrestricted trade is equal to the constant $Q_0$ when $\omega^m > w_1$, consumption and output are constant. Moreover, $\mu^b = \omega^m$ by (3.10), and so $\mu^b = \frac{\beta}{\gamma}\mu^m$. With $(\mu^b, \mu^m)$, (3.11) solves for the constant $a$.

In this case, the interest rate depends only on the current shock, because (2.20) becomes:

$$r(z) = \max\left\{\frac{\gamma z}{1-a} - 1, \frac{\gamma}{\beta} - 1\right\}.$$

A tightening open market operation, modelled as an increase in $z$, raises the interest rate when $z < (1-a)/\beta$. This liquidity effect in the bonds market does not translate into any effect on real activities. Nor does it affect the additional discount on two-period bonds relative to one-period bonds, which is $\gamma/\beta - 1$ (see (2.23)).

Continue the examination of the economy with independent shocks but consider the case where unmatured bonds generate liquidity service, i.e., where $0 < \omega^m < w_1$. This case of the equilibrium behaves quite differently from Lucas's model. In particular, $\mu^m$ and $a$ are no longer constants. Because the asset constraint binds in an unrestricted trade, the quantity of goods in such a trade depends on the amount of unmatured bonds, as well as the money stock. Since the amount of unmatured bonds in a period is equal to the quantity of new bonds issued in the previous period, the quantity of goods in an unrestricted match depends on the realization of the previous period's shock, $z_{-1}$ (see (3.4)). That is, the previous period's shock affects the amount of liquidity in the current goods market. As a result, the current shadow values of the two assets are functions of the previous shock (see (3.9) and (3.10)). Since these asset values affect the allocation of money between the two markets, $a$ is a function of $z_{-1}$, even though the shocks are independent over time.

The nominal interest rate now depends on both the current shock and the previous period's shock. Thus, nominal interest rates are serially correlated even though the shocks are independent over time. Moreover, open market operations affect the relative value of unmatured bonds to money, and hence affect the term structure of interest rates.

## 5. Numerical Examples

Let me return to the general case where the shocks can be dependent. Because the equilibrium function $a$ is a fixed point of an implicit mapping $\Gamma$, it is difficult to check whether the solution is monotone. Likewise, it is difficult to check whether consumption is a monotonic function of the past shock. To study equilibrium properties, I turn to numerical examples.

Assume the following forms of utility and cost:

$$u(c) = u_0 \frac{c^{1-\delta} - 1}{1-\delta}, \quad \psi(q) = \psi_0 q^{\Psi}.$$

Let the shock $z$ have two realizations, $z_1$ and $z_2$, with $z_2 > z_1$. Refer to $z_2$ as the high shock and $z_1$ as the low shock. The transition probability from $z_i$ to $z_i$ is $\theta$ and to $z_{i'}$ $(i' \neq i)$ is $1 - \theta$, where $i, i' = 1, 2$. Consider the following parameter values as the baseline:

| | |
|---|---|
| preference: | $\delta = 0.5$, $u_0 = 4$, $\psi_0 = 1$, $\Psi = 2$, $\beta = 0.995$; |
| goods market: | $\alpha = 1$, $\sigma = 0.5$, $g = 0.2$; |
| monetary policy: | $z_1 = 0.02$, $z_2 = 0.08$, $\gamma = 1.005$. |

The value of $g$ matches the size of the government relative to the economy, using the interpretation that the legal restriction in the goods market is imposed in trades between private households and the government.[13] The values of $(\beta, z_1, z_2)$ are the ones chosen by Lucas (1990). With the particular value of $\beta$, I can interpret the length of a period as one month and the interest rate $r$ as the bi-monthly interest rate. Also following Lucas, I explore a large range of values of $\theta$: $0.01, 0.1, 0.3, 0.5, 0.7, 0.9$ and $0.99$. I will also analyze the sensitivity of the results to other parameters in section 6.1.

The bounds on the variables are set at $a_L = 0.90$, $a_H = 0.98$, $\kappa = 1$, $\omega_L = 0.543$ and $\omega_H = 3.375$. These bounds satisfy all the conditions in Theorem 3.2. Moreover, the equilibrium lies in the region $\omega^m \in (0, w_1)$. That is, unmatured bonds generate liquidity in the goods market and the household takes all unmatured bonds to the goods market.[14]

To display the results, let me add a subscript $i$ to variables that depend only on the previous period's shock $z_i$, where $i = 1, 2$. Add subscripts $ji$ to variables that depend on both the current shock $z_j$ and the previous period's shock $z_i$, where $i, j = 1, 2$. To aggregate consumption of the goods over the two types of trades, denote the price of goods in a restricted trade, normalized by the money stock, as $p^g$ and the normalized price in an unrestricted trade as $p^n$. Aggregate real consumption (output) is defined as follows:

$$c_i = \alpha\sigma(1-\sigma)\left[\frac{gp^g(z_i)q^g(z_i) + (1-g)p^n(z_i)q^n(z_i)}{gp^g(z_i) + (1-g)p^n(z_i)}\right].$$

Notice that current output depends only on the shock in the previous period, but not on the current shock. This is because consumption in the current period is purchased with the assets that are allocated to the goods market before the current shock is realized. Then, following the convention in asset pricing models, I can define the (ex ante) real interest rate between the current and the next period period as

$$\text{Ereal}_i = \left[\beta E \frac{u'(c_j)}{u'(c_i)}\right]^{-1} - 1, \quad i = 1, 2, \tag{5.1}$$

where the expectations are taken over the current shock $z_j$, conditional on the past shock $z_i$.

The term structure of interest rates is represented by the percentage difference between the yield to newly issued two-period bonds, $S^{-1/2}$, and the yield to one-period bonds, $1/S^I$. Letting $r^I$ be the one-period interest rate corresponding to $S^I$ and using (2.23), I can write this percentage

[13] It is tempting to interpret $g$ alternatively as the fraction of goods purchased with money. However, this alternative interpretation is incorrect, because money is used in the current model to buy both restricted goods and unrestricted goods. The fraction of goods purchased with money is much larger than $g$.

[14] This is the case for a large range of parameter values. In fact, the range $(w_1, w_2) \ni \omega^m$, in which unmatured bonds do not generate liquidity service in the goods market, is very narrow.

difference as:

$$term_{ji} = \left( \frac{\mu^m(z_j)/\mu^b(z_j)}{1 + r^I(z_j, z_i)} \right)^{1/2} - 1. \tag{5.2}$$

Table 1 describes equilibrium properties of the fraction of money taken to the bonds market $(1-a)$, nominal and real interest rates, consumption, and the term structure of nominal interest rates. The mean, the standard deviation, and serial autocorrelations are calculated using the unique invariant measure $prob(z_i) = 1/2$ for $i = 1, 2$.

There are three important similarities between the results in Table 1 and those reported by Lucas (1990). First, interest rates change significantly with the persistence of the shock when the current shock is high. Also, interest rates have a large (unconditional) standard deviation. However, the mean of interest rates does not vary significantly with the persistence of the shock, even if the degree of persistence varies between 0.1 and 0.9. Thus, if one is interested only in the mean of interest rates, one can ignore the persistence and simply examine the case of independent shocks (i.e., $\theta = 0.5$).

Second, the fraction of money allocated to the bonds market is insensitive to the previous period's shock. As in Lucas's model, this insensitivity is surprising especially when the shocks are negatively dependent. With negatively dependent shocks, a high shock in the previous period implies that the amount of bond sales is likely to be low in the current period and the bond price likely to be high. Since the discount on bonds will be small, there is not much need to allocate more money to the bonds market to take advantage of the discount on new bonds. Thus, when the shocks are negatively correlated, one would expect that the household would reduce $(1-a)$ significantly upon observing a high shock in the previous period. This does not happen in the numerical examples.

Notice that the goods market provides an additional reason for the household to adjust the money allocation, which makes the insensitivity of such allocation to shocks more puzzling here than in Lucas's endowment model. In particular, a high past shock increases the amount of assets used in an unrestricted trade relative to the assets in a restricted trade. This widens the gap between the quantities of goods obtained in the two types of trades, and hence increases the variation in a household's consumption. To smooth consumption between the two types of trades, the household should increase the fraction of money allocated to the goods market, so as to maintain a stable ratio of assets used in an unrestricted trade relative to a restricted trade. Despite this additional motivation for changing $a$, the negative response of $(1-a)$ to the past shock is not significant. Even when $\theta = 0.1$, an increase of $z_{-1}$ from $z_1$ to $z_2$ reduces $(1-a)$ from 7.87% to 7.70%. This reduction is small in comparison with the variation in the shock.

Table 1. Simulation results under a constant money growth rate

| | $\theta$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.01 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.99 |
| $1-a_1$ (%) | 7.879 | 7.867 | 0.828 | 7.761 | 7.615 | 7.044 | 3.952 |
| $1-a_2$ (%) | 6.224 | 7.698 | 7.892 | 7.930 | 7.941 | 7.926 | 7.903 |
| $r_{11}$ (%) | 0.474 | 0.451 | 0.445 | 0.439 | 0.428 | 0.389 | 0.327 |
| $r_{21}$ (%) | 2.038 | 2.201 | 2.705 | 3.593 | 5.580 | 14.133 | 103.46 |
| $r_{12}$ (%) | 0.474 | 0.451 | 0.445 | 0.439 | 0.428 | 0.389 | 0.327 |
| $r_{22}$ (%) | 29.178 | 4.443 | 1.877 | 1.384 | 1.248 | 1.441 | 1.732 |
| $E(r)$ (%) | 1.392 | 1.438 | 1.451 | 1.464 | 1.488 | 1.549 | 1.538 |
| $StD(r)$ (%) | 2.119 | 1.096 | 1.041 | 1.289 | 1.759 | 2.932 | 7.259 |
| $corr(r, r_{-1})$ | -0.129 | -0.483 | -0.535 | -0.341 | -0.166 | -0.029 | 0.004 |
| $corr(r, r_{-2})$ | 0.126 | 0.387 | 0.214 | 0 | -0.067 | -0.023 | 0.004 |
| $corr(r, r_{-3})$ | -0.124 | -0.309 | -0.086 | 0 | -0.027 | -0.018 | 0.004 |
| $c_1$ | 0.616 | 0.617 | 0.617 | 0.617 | 0.618 | 0.619 | 0.622 |
| $c_2$ | 0.628 | 0.627 | 0.626 | 0.626 | 0.626 | 0.625 | 0.622 |
| $E(c)$ | 0.622 | 0.622 | 0.622 | 0.622 | 0.622 | 0.622 | 0.622 |
| $StD(c)$ | 0.006 | 0.005 | 0.005 | 0.005 | 0.004 | 0.003 | 0.000 |
| $\text{Ereal}_1$ (%) | 1.450 | 1.231 | 1.042 | 0.869 | 0.702 | 0.545 | 0.503 |
| $\text{Ereal}_2$ (%) | -0.436 | -0.221 | -0.035 | 0.136 | 0.303 | 0.460 | 0.502 |
| $corr(c, r)$ | -0.298 | -0.537 | -0.554 | -0.429 | -0.276 | -0.073 | 0.025 |
| $corr(c_{+1}, r)$ | 0.433 | 0.901 | 0.966 | 0.795 | 0.603 | 0.396 | 0.167 |
| $corr(c_{+2}, r)$ | -0.424 | -0.721 | -0.386 | 0 | 0.241 | 0.317 | 0.163 |
| $corr(c_{+3}, r)$ | 0.416 | 0.577 | 0.155 | 0 | 0.096 | 0.253 | 0.160 |
| $\text{term}_{11}$ (%) | 0.237 | 0.225 | 0.222 | 0.219 | 0.214 | 0.194 | 0.163 |
| $\text{term}_{21}$ (%) | -0.720 | -0.687 | -0.902 | -1.299 | -2.181 | -5.782 | -29.391 |
| $\text{term}_{12}$ (%) | 0.237 | 0.225 | 0.222 | 0.219 | 0.214 | 0.194 | 0.163 |
| $\text{term}_{22}$ (%) | -11.764 | -1.758 | -0.500 | -0.230 | -0.111 | -0.061 | -0.144 |

Third, the insensitivity of the money allocation leads to a strong liquidity effect in the bonds market, as reflected by changes in nominal interest rates. Interest rates are significantly higher when the current shock is high than when the current shock is low; that is, $r_{2i}$ is much higher than $r_{1i}$ for $i = 1, 2$. This is because when the money allocation is insensitive, a higher supply of new bonds must be absorbed by a fall in the bond price, resulting in a higher interest rate. Notice that, when the current shock is low, the one-period interest rate is zero (see (2.22)) and the two-period interest rate does not depend on past shocks. This is because there is more money than what is needed in the bonds market when the current shock is low.

The model generates several results that are absent in Lucas (1990). I will describe these results below for the case of independent shocks, since the contrasts with Lucas's model are the sharpest in this case.

First, open market operations have a real effect – A high shock in the previous period increases current real output. The standard deviation of output is about 0.7% of the mean. This real effect arises because (unmatured) bonds generate liquidity in the goods market. A high shock in the previous period increases the stock of unmatured bonds in the current goods market. This allows a buyer to purchase a larger quantity of goods in an unrestricted trade than if the previous period's shock was low, i.e., $q_2^n > q_1^n$. The presence of a larger quantity of nominal assets in the goods market also pushes up the price level and reduces the quantity of goods purchased in a restricted trade, i.e., $q_2^g < q_1^g$. In the numerical examples, the increase in $q^n$ dominates the decrease in $q^g$, and so aggregate output rises.[15]

As a result of the effect on consumption, the past shock also affects the real interest rate. When the shock was high in the previous period, current consumption is high and so the real interest rate is low in the current period. On the other hand, the real interest rate is high in the current period when the shock was low in the previous period. The difference in the real interest rate between the two states of the past shock is about 70 basis points, which is sizable. However, because of the liquidity effect, this difference is smaller than the one in the nominal interest rate between the two states.

Second, a high past shock reduces the current interest rate when the current shock is high. In contrast, past (independent) shocks in Lucas's model do not affect the current interest rate. To explain this new effect, recall that a high past shock increases the amount of unmatured bonds circulating in the current goods market and increases the price level. The higher price level reduces real values of both money and unmatured bonds. However, because the increased amount of unmatured bonds increases liquidity in unrestricted trades, the real value of unmatured bonds ($\mu^b$) falls by less than does the real value of money ($\mu^m$). (In fact, $\mu^b$ in the examples barely changes at all with past shocks.) Thus, the relative value of unmatured bonds to money increases, which induces the households to allocate more money to purchase new bonds. When the current shock is high, the additional money in the bonds market pushes up the bond price and depresses the current interest rate. When the current shock is low, the additional money in the bonds market does not affect the current interest rate, as discussed above.

Third, the above effects of past shocks on current activities induce the following correlations: (i) Contemporaneous output and interest rates are negatively correlated; (ii) Interest rates in two adjacent periods are negatively correlated; (iii) Future output is positively correlated with the

[15] Also, real output (consumption) is serially correlated (not reported in Table 1). The coefficient of correlation between current consumption and $k$-period past consumption is equal to $(2\theta - 1)^k$. Thus, positively correlated shocks induce positive autocorrelations in consumption.

current interest rate.[16] These correlations arise because a shock in the previous period increases current output, increases the interest rate in the previous period, and reduces the current interest rate. The negative contemporaneous correlation between output and interest rates is realistic in the US data (see Christiano et al., 1999). However, the positive correlation between future output and current interest rates is unrealistic. In section 6.2 I will examine a natural variation of the model that will eliminate this unrealistic feature.

Fourth, the term structure of interest rates responds to open market operations. The yield curve is negatively sloped when the current shock is high and positively sloped when the current shock is low. In light of (5.2), this negative response of the yield curve to the current shock is not surprising. For example, a high current shock increases the one-period interest rate; at the same time, it reduces the expected future discount on unmatured bonds ($\mu^m/\mu^b$) by generating liquidity in next period's goods market. Both effects reduce the slope of the yield curve.

Moreover, the slope of the yield curve can depend on the previous period's shock when the current shock is high. In this case, a high past shock makes the yield curve less negatively sloped. To explain this result, recall that a high past shock increases the money allocation to the current bonds market. This higher money allocation reduces current interest rates when the current shock is high. However, the expected future discount on unmatured bonds depends only on the current shock, not on past shocks. Thus, by (5.2), the yield curve becomes less negatively sloped.[17] Clearly, the role of unmatured bonds in the goods market is important for this dependence of the yield curve on past shocks, because it is the reason why the households condition their money allocation on past shocks. In contrast, in Lucas's model, bonds play no role in the goods market regardless of the maturity, and so the yield curve is independent of past shocks when the shocks are independent.

Most of the above features with independent shocks continue to exist when shocks are dependent. However, there are a few changes. First, when shocks are highly negatively dependent (i.e., $\theta \leq 0.1$), a high past shock increases (rather than decreases) the current interest rate when the current shock is high. That is, $r_{22} > r_{21}$. This is because, given the high past shock and the negative serial dependence, the households anticipate bond sales to be low in the current period and so they allocate less money to the bonds market. When the current bond sales turn out to be

[16] The formulas for the correlations between $r$ and $c$ are as follows:

$$corr(r,y) = \frac{y_2-y_1}{2}\left[\theta(r_{22}-r_{11})+(1-\theta)(r_{12}-r_{21})\right],$$
$$corr(r,y_{+1}) = \frac{y_2-y_1}{2}\left[\theta(r_{22}-r_{11})+(1-\theta)(r_{21}-r_{12})\right].$$

Moreover, $corr(r,y_{-j}) = (2\theta-1)^j corr(r,y)$ and $corr(r,y_{+j}) = (2\theta-1)^{j-1}corr(r,y_{+1})$, for $j = 1,2,\ldots$.

[17] Of course, when the current shock is low, the one-period interest rate is zero and unaffected by the money allocation, in which case the slope of the yield curve is independent of past shocks.

high, the interest rate will be high. Second, when the shocks are highly persistent (i.e., $\theta \geq 0.99$), the correlation between the current and one-period past interest rates becomes positive. This is not surprising because a permanent shock will generate a positive correlation between interest rates in all periods. Similarly, the contemporaneous correlation between output and interest rates becomes positive when the shocks are highly persistent.

## 6. Sensitivity Analysis

In this section, I first examine the sensitivity of the results to the parameters, while maintaining the assumption that the money growth rate is constant. Then I examine an alternative assumption under which the government sets monetary transfers as a fixed fraction of the money stock and lets the money growth rate vary with the shocks.

### 6.1. Sensitivity Analysis under a Constant Money Growth Rate

Table 2 reports the sensitivity of the results to the parameters. In this analysis, shocks are independent and I perturb the constant money growth rate ($\gamma$), the scope of the legal restriction ($g$), the relative risk aversion ($\delta$) and the variation in the shock. Overall, these perturbations have small effects on real variables and they do not change the main features of the baseline model. In particular, there is still a strong liquidity effect in the bonds market, as the allocation of money between the markets is insensitive to the shock. The quantitative effects of these perturbations are summarized below.

First, an increase in the money growth rate increases the mean of interest rates and reduces the mean of real consumption (output); it also increases the standard deviations of interest rates and real output. By eliminating net money growth from the baseline case, the mean and standard deviation of interest rates fall by about a half, and the standard deviation in output falls by more than a half. Real output and the nominal interest rate are still negatively correlated with each other but the magnitude seems to first increase, and then decrease, with money growth.

Second, an increase in the scope of the legal restriction increases the mean of interest rates but affects the standard deviation of interest rates in a hump-shaped pattern. The mean of real consumption barely changes with the increase in the scope of the legal restriction, the standard deviation of consumption decreases, and the negative correlation between consumption and interest rates weakens. Real consumption responds in this way because the wider coverage of the legal restriction reduces the liquidity effect of unmatured bonds in the current goods market and reduces the variation in the quantity of goods between a restricted trade and an unrestricted trade. Because consumption varies less and interest rates vary more between different states, the two variables become less correlated with each other.

Table 2. Sensitivity results under a constant money growth rate

| | | $\gamma$ | | $g$ | | $\delta$ | | $z_1 = 0.001$ |
|---|---|---|---|---|---|---|---|---|
| | baseline | 1 | 1.05 | 0.01 | 0.5 | 0.05 | 2 | $z_2 = 0.099$ |
| $1 - a_1$ (%) | 7.761 | 7.864 | 7.323 | 7.729 | 7.789 | 7.758 | 7.770 | 9.595 |
| $1 - a_2$ (%) | 7.930 | 7.933 | 7.583 | 8.033 | 7.847 | 7.939 | 7.902 | 9.843 |
| $E(r)$ (%) | 1.464 | 0.783 | 7.056 | 1.044 | 1.823 | 1.429 | 1.578 | 1.361 |
| $StD(r)$ (%) | 1.289 | 0.598 | 5.850 | 1.725 | 1.058 | 1.319 | 1.197 | 1.382 |
| $corr(r, r_{-1})$ | -0.341 | -0.316 | -0.163 | -0.337 | -0.175 | -0.346 | -0.311 | -0.352 |
| $E(c)$ | 0.622 | 0.626 | 0.588 | 0.622 | 0.622 | 0.507 | 0.789 | 0.622 |
| $StD(c)$ | 0.005 | 0.002 | 0.007 | 0.008 | 0.002 | 0.006 | 0.002 | 0.005 |
| $corr(c, r)$ | -0.429 | -0.371 | -0.168 | -0.571 | -0.181 | -0.447 | -0.361 | -0.473 |
| $corr(c_{+1}, r)$ | 0.795 | 0.851 | 0.971 | 0.589 | 0.967 | 0.775 | 0.860 | 0.744 |
| $\text{term}_{11}$ (%) | 0.219 | 0.137 | 0.685 | 0.014 | 0.399 | 0.203 | 0.274 | 0.166 |
| $\text{term}_{21}$ (%) | -1.299 | -0.577 | -5.301 | -1.927 | -0.774 | -1.353 | -1.125 | -1.446 |
| $\text{term}_{12}$ (%) | 0.219 | 0.137 | 0.685 | 0.014 | 0.399 | 0.203 | 0.274 | 0.166 |
| $\text{term}_{22}$ (%) | -0.230 | -0.141 | -3.605 | -0.015 | -0.404 | -0.212 | -0.288 | -0.180 |

Baseline: $\theta = 0.5$, $\gamma = 1.005$, $g = 0.2$, $\delta = 0.5$, $z_1 = 0.02$, $z_2 = 0.08$.

Third, an increase in the relative risk aversion increases the mean and reduces the standard deviation of interest rates and output. It also reduces the magnitude of the (negative) correlation between output and the nominal interest rate. The important features of the model do not change much with the changes in the relative risk aversion. Even when the relative risk aversion is very small, e.g., when $\delta = 0.05$, the money allocation remains insensitive to the previous period's shock. Also, notice that, when $\delta = 2$, the money allocation between the two markets is even more insensitive to the previous period's shock than in the baseline model. In this sense, by restricting the relative risk aversion in the baseline model to be not greater than 1, I have strengthened the model's predictions.[18]

Fourth, an increase in the mean-preserving spread in the shock reduces the mean and increases the variation in interest rates. It also increases the variation in output, without affecting the mean of output much, and strengthens the negative correlation between output and the interest rate.

Finally, the above changes in the parameter values change the magnitude of the slope of the yield curve but have very little effect on the sign of the slope. Not surprisingly, when the scope of the legal restriction becomes very narrow ($g = 0.01$), the yield curve becomes very flat in most cases. When $g \to 1$, the equilibrium approaches the one analyzed by Lucas (1990).

[18] For $\delta = 2$, the function $T(\omega^m; a, z_{-1})$ is first decreasing and then increasing in $\omega^m$ as $\omega^m$ increases. To ensure that $T$ is increasing in $\omega^m$, the lower bound on $\omega^m$ is chosen to be sufficiently large.

### 6.2. Allowing the Money Growth Rate to Vary

Now I change the assumption about monetary transfers to the specification $L_{+1} = \tau M$, where $\tau$ is constant. In this case, the money growth rate will vary across the states of the shock. Denote $\gamma = M_{+1}/M$. By (2.7) and the requirements (ii) and (iii) in the equilibrium definition (see section 2.5), the following relationship arises in the equilibrium:

$$\gamma = \gamma(z_{-1}) \equiv a(z_{-1}) + z_{-1} + \tau. \tag{6.1}$$

That is, the growth rate of the aggregate money stock between the current period and the next period is a function of the past shock, $z_{-1}$, but of the current shock. This is because the current shock affects neither the amount of money that the household spends in the current bonds market nor the amount of bonds that the household will redeem at the beginning of the next period.

Once $\gamma$ is replaced with $\gamma(z_{-1})$, all equilibrium conditions in section 2 continue to hold and an equilibrium can still be characterized as in section 3.1. For the existence of an equilibrium, the conditions in Theorem 3.2 need be modified to incorporate the fact that now $\gamma$ is not a constant. These modifications are straightforward and hence are omitted here.

To check the numerical results, let the parameter values (except $\gamma$) be the same as specified at the beginning of section 5. Let the fraction of monetary transfers be equal to the average of the two realizations of $z$, i.e., $\tau = 0.05$. The results are reported in Table 3.

There are some important similarities between the results here and those under a fixed money growth rate. The most important one is that open market operations continue to generate a strong liquidity effect on interest rates. As before, an important reason for the strong liquidity effect is that the allocation of money between the bonds market and the goods market is insensitive to the shock. Also as before, the liquidity effect induces real effects. Moreover, the term structure of interest rates exhibits the same pattern of dependence on the shocks as in the baseline model.

An obvious difference between this variation and the baseline model is that the past shock now has a positive effect on the money growth rate between the current and the next period. This difference arises from (6.1). Because the money allocation is insensitive to the past shock and because $\tau$ is fixed, a high shock in the previous period implies a large amount of redemption at the beginning of the next period, and hence a high money growth rate.

This effect of the past shock on the money growth rate changes the autocorrelation in interest rates. By increasing the money growth rate, a high past shock tends to increase the current interest rate. This induces interest rates to be positively autocorrelated in the case of independent shocks or positively dependent shocks, rather than negatively correlated as in the baseline model.

Table 3. Simulation results under varying money growth rates

| | | $\theta$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | baseline | 0.01 | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 0.99 |
| $1-a_1$ (%) | 7.761 | 7.356 | 7.537 | 7.544 | 7.477 | 7.339 | 6.748 | 2.026 |
| $1-a_2$ (%) | 7.930 | 2.152 | 6.462 | 7.629 | 7.841 | 7.900 | 7.873 | 7.826 |
| $r_{11}$ (%) | 0.439 | 1.436 | 0.906 | 0.723 | 0.640 | 0.545 | 0.444 | 3.627 |
| $r_{21}$ (%) | 3.593 | 8.364 | 5.577 | 5.472 | 6.482 | 8.640 | 18.845 | 314.51 |
| $r_{12}$ (%) | 0.439 | 3.029 | 0.906 | 0.723 | 0.640 | 0.545 | 0.444 | 1.183 |
| $r_{22}$ (%) | 1.384 | 312.12 | 31.891 | 10.489 | 7.298 | 6.434 | 6.825 | 7.505 |
| $E(r)$ (%) | 1.464 | 7.208 | 4.557 | 3.850 | 3.765 | 3.821 | 4.236 | 7.089 |
| $StD(r)$ (%) | 1.289 | 21.779 | 6.670 | 3.524 | 3.138 | 3.352 | 4.569 | 21.880 |
| $corr(r, r_{-1})$ | -0.341 | -0.010 | -0.045 | -0.050 | 0.065 | 0.247 | 0.354 | 0.003 |
| $c_1$ | 0.617 | 0.578 | 0.599 | 0.606 | 0.609 | 0.613 | 0.617 | 0.588 |
| $c_2$ | 0.626 | 0.596 | 0.607 | 0.609 | 0.606 | 0.602 | 0.594 | 0.587 |
| $E(c)$ | 0.622 | 0.587 | 0.603 | 0.607 | 0.608 | 0.608 | 0.605 | 0.587 |
| $StD(c)$ | 0.005 | 0.009 | 0.004 | 0.001 | 0.001 | 0.005 | 0.001 | 0.001 |
| $corr(c, r)$ | -0.429 | -0.050 | -0.083 | -0.056 | -0.065 | -0.253 | -0.427 | -0.016 |
| $corr(c_{+1}, r)$ | 0.795 | 0.193 | 0.547 | 0.887 | -0.996 | -0.977 | -0.830 | -0.159 |
| $corr(c_{+2}, r)$ | 0 | -0.189 | -0.438 | -0.355 | 0 | -0.391 | -0.664 | -0.156 |
| $corr(c_{+3}, r)$ | 0 | 0.185 | 0.350 | 0.142 | 0 | -0.156 | -0.531 | -0.153 |
| $\text{term}_{11}$ (%) | 0.219 | 0.715 | 0.452 | 0.351 | 0.320 | 0.272 | 0.222 | -0.603 |
| $\text{term}_{21}$ (%) | -1.299 | -2.624 | -1.610 | -1.596 | -2.012 | -2.885 | -6.936 | -50.076 |
| $\text{term}_{12}$ (%) | 0.219 | -0.067 | 0.452 | 0.351 | 0.320 | 0.272 | 0.222 | 0.590 |
| $\text{term}_{22}$ (%) | -0.230 | -50.067 | -11.970 | -3.956 | -2.385 | -1.884 | -1.840 | -1.969 |
| $\gamma_1 - 1$ (%) | 0.500 | -0.356 | -0.537 | -0.544 | -0.477 | -0.339 | -0.252 | 4.974 |
| $\gamma_2 - 1$ (%) | 0.500 | 10.848 | 6.538 | 5.371 | 5.159 | 5.100 | 5.127 | 5.174 |

Baseline: $\theta = 0.5$, $\gamma = 1.005$ (fixed), $g = 0.2$, $\delta = 0.5$, $z_1 = 0.02$, $z_2 = 0.08$.

The behavior of the money growth rate also generates two other different results. I list these differences in the case where the shocks are either independent or positively dependent. First, a high shock in the previous shock is likely to induce low current output and a high real interest rate. Second, future output (as well as current output) is negatively correlated with the current interest rate. To attribute these differences to the behavior of the money growth rate, notice that an increase in money growth reduces output by reducing the real value of money. Since a high shock in the previous period increases the money growth rate, then it leads to low current output. Similarly, a high current shock increases the future money growth rate, and hence low future output. Because a high current shock also increases the current interest rate, future output is negatively correlated with the current interest rate.

The negative correlation between future output and the current interest rate is realistic (see Christiano et al., 1999), as opposed to the counterfactual and positive correlation in the baseline model. Notice that, in the case $\theta = 0.7$, output three periods into the future is still negatively correlated with the current interest rate. Thus, a positive shock can have persistent and negative effects on real output.

## 7. Conclusion

In this paper I combine a decentralized goods market and a centralized bonds market to analyze the liquidity effects of open market operations. The bonds market features limited participation, while the goods market features bilateral matches. In a fraction of trades, a legal restriction forbids the use of bonds as the means of payments for goods. In such a restricted trade, the buyer faces a money constraint. In an unrestricted trade, the buyer can use both money and unmatured bonds to buy goods, and so unmatured bonds can provide liquidity. A shock to bond sales in this economy has two distinct liquidity effects. One is the immediate liquidity effect in the bonds market and the other is a liquidity effect in the goods market starting one period later.

The liquidity effect in the bonds market arises for the same reason as in Lucas (1990). That is, there is limited participation in the bonds market, and the households' money allocation between the markets is insensitive to past shocks even when shocks are highly persistent. As a result of this insensitive allocation, the bond price and hence the nominal interest rate absorbs most of the shock to current bond sales. This liquidity effect is short-lived, as in Lucas's model.

The liquidity effect in the goods market is new and it occurs with a delay.[19] In particular, a high shock to bond sales in the previous period increases the amount of unmatured bonds circulating in the current goods market, relaxes the asset constraints in unrestricted trades, and hence increases the quantity of goods traded in an unrestricted trade. Although inflation also rises in this case, which tends to reduce the quantity of goods traded in a restricted trade, aggregate output can rise with a high past shock. Important for this liquidity effect is the temporary separation between trades in the goods market, implied by random matches. If all exchanges in the goods market were centralized in the Walrasian style, then a high past shock would simply push up prices to such a level that would eliminate any response from real output. In contrast with the liquidity effect in the bonds market, the liquidity effect in the goods market can be long-lived and, in principle, the duration of this effect increases with the length of maturity of the bonds that are used in open market operations.[20]

[19] A popular variation of Lucas's model assumes that there is a separate cash-in-advance constraint on firms' wage payment and that open market operations affect firms' available funds before affecting the price level (e.g., Fuerst, 1992). This variation allows open market operations to affect real activities, but such real effects are quite different from the liquidity effect in the goods market that I emphasize here.

[20] If the government attaches repurchase agreements to bond sales, then the duration of the liquidity effect of

The liquidity effect in the goods market generates a number of other new features. In the case of independent shocks, these feature are as follows: (i) Current real output and (nominal) interest rates are negatively correlated; (ii) The shock in the previous period affects the allocation of money to the current bonds market, which affects the current interest rate and hence generates autocorrelation in interest rates; (iii) The real interest rate varies with past shocks; (iii) There is a non-trivial term structure of interest rates and the slope of the yield curve depends on both past and current shocks. These features continue to exist when the shocks are dependent. In addition, if monetary transfers do not insulate money growth from positively dependent shocks, then future output is negatively correlated with the current interest rate. Thus, a positive shock to bond sales can generate persistent reductions in output.

Perhaps the most important message that this paper tries to convey is that it is tractable to use a monetary model with a strong microfoundation to analyze monetary policy and to generate interesting predictions. Often, search monetary models have been described (e.g., Kiyotaki and Moore, 2001) as internationally consistent but difficult to be integrated with the rest of macroeconomic theory. The model described in this paper is no more difficult than many of the models in the literature of limited participation (see Christiano et al., 1999, for references). Thus, I hope that this paper has eliminated a major road block to the integration of the microfoundation of monetary theory into mainstream macroeconomics.

There are many extensions one can explore. For example, one can relax the following assumptions that I have retained from Lucas's model. First, the shock to bond sales is the only shock in the economy and, in particular, there are no shock to money demand or to the production technology. Second, there is no element (other than the one-period separation between markets) to delay the transmission of shocks from the bonds market to the goods market. Third, there is no capital accumulation that can prolong the effects of monetary shocks. A virtue of the model is that, despite the absence of these elements, the model is still able to generate strong liquidity effects in both the bonds market and the goods market. Nevertheless, one may want to introduce these realistic elements to examine the monetary propagation mechanism. Capital accumulation can be introduced as in Shi (1999) and money demand shocks can be modelled as stochastic changes in the scope of the legal restriction, $g$. Finally, there may be a need to model explicitly how financial institutions create inside money.

bonds in the goods market will be reduced.

**Appendix**

## A. Proofs for Section 3.2

### A.1. Proof of Lemma 3.1

To begin, I construct an upper bound on $T_\omega \equiv \partial T(\omega^m; a, z_{-1})/\partial\omega^m$. This upper bound is necessary for $TO$ to satisfy the contraction mapping requirement. It is easy to verify that $T_\omega > 0$ under Assumption 1. Also, $T$ is concave in $\omega^m$ in each of the three segments, $(0, w_1)$, $(w_1, w_2)$ and $(w_2, \infty)$ (see Figure 2). Thus, $T_\omega \leq \max\{T_\omega(w_1+; a, z_{-1}), 1\}$ for all $\omega^m \geq w_1$. Also, because $T_\omega(w_1\text{-}; a, z_{-1}) < T_\omega(w_1+; a, z_{-1})$, there exists $w_3 < w_1$ such that for all $\omega^m \geq w_3$, $T_\omega \leq \max\{T_\omega(w_1+; a, z_{-1}), 1\}$. Under (ii) of Assumption 1, $T_\omega(w_1+; a, z_{-1})$ decreases in $a$ and increases in $z_{-1}$, after the dependence of $w_1$ on $(a, z_{-1})$ is taken into account. Setting $a = a_L$, $z_{-1} = z_H$ and $w_1 = w_1(a_L, z_H)$, I have $T_\omega(w_1+; a, z_{-1}) \leq \bar{T}_\omega$ for all $(a, z_{-1})$, where

$$\bar{T}_\omega \equiv 1 - \alpha\sigma g + \frac{\alpha\sigma g}{\Psi} \frac{[1 - \delta(\widehat{c})]\, u'(\widehat{c})}{\psi'\left(\frac{\widehat{c}}{\alpha\sigma(1-\sigma)}\right)},$$

$$\widehat{c} = \alpha\sigma(1-\sigma)\psi^{-1}\left(\frac{a_L \psi(Q_0)}{a_L + z_H}\right).$$

I choose the upper bound on $T_\omega$ as

$$K = \kappa \max\{\bar{T}_\omega, 1\}, \text{ where } 1 \leq \kappa < \infty.$$

The upper bound $K$ leads to a lower bound on $\omega^m$. Let $\omega_0(a, z_{-1})$ ($< w_1$) solve $T_\omega(\omega_0; a, z_{-1}) = K$. Because $T_\omega(\omega; a, z_{-1})$ is decreasing in $(\omega, a, z_{-1})$, $\omega_0$ is decreasing in $(a, z_{-1})$. The lower bound of $\omega^m$ is then defined as $\omega_L = \omega_0(a_L, z_L)$. Clearly, $\omega_L$ is smaller if a larger $\kappa$ is chosen (see Figure 2), and $\omega_L > 0$ for all finite $\kappa$. Also, for all $\omega^m \geq \omega_L$, $0 < T_\omega \leq K$.

Next, I show that $TO$ maps from $\mathcal{V}$ to $\mathcal{V}$. For any $\mu^m \in \mathcal{V}$, $O(\mu^m)$ is continuous because $\Phi$ has the Feller property. Since $a(.) \in \mathcal{A}$ is continuous, $TO(\mu^m)$ is continuous. Because $\mu^m \geq \frac{\gamma}{\beta}\omega_L$, then $O(\mu^m) \geq \omega_L$. Hence,

$$TO(\mu^m) \geq \omega_L \left[1 + F(\omega_L; a, z_{-1})\right] \geq \omega_L \left[1 + F(\omega_L; a_H, z_H)\right].$$

The first inequality comes from the fact that $T$ is an increasing function of $\omega^m$ and the second inequality from the fact that, for given $\omega^m$, $F(\omega^m; a, z_{-1})$ is a decreasing function of $(a, z_{-1})$. Similarly, $O(\mu^m) \leq \omega_H$ and

$$TO(\mu^m) \leq \omega_H \left[1 + F(\omega_H; a, z_{-1})\right] \leq \omega_H \left[1 + F(\omega_H; a_L, z_L)\right].$$

Thus, $TO(\mu^m) \in \left[\frac{\gamma}{\beta}\omega_L, \frac{\gamma}{\beta}\omega_H\right]$ if

$$F(\omega_H, a_L, z_L) \geq \frac{\gamma}{\beta} - 1 \geq F(\omega_H; a_L, z_L).$$

This is part of the condition (3.14) in the lemma.

Moreover, $TO$ is a contraction mapping under the supnorm. To see this, take any $\mu', \mu'' \in \mathcal{V}$. Let $\omega' = O(\mu')$ and $\omega'' = O(\mu'')$. Then, $\omega', \omega'' \geq \omega_L$ and

$$|\omega' - \omega''| = |O(\mu') - O(\mu'')| \leq \frac{\beta}{\gamma} \|\mu' - \mu''\|.$$

Because $T$ is concave in each of its segments and because $T_\omega$ is bounded above by $K$ for all $\omega^m \geq \omega_L$, I have:

$$|T(\omega') - T(\omega'')| \leq K |O(\mu') - O(\mu'')| \leq \frac{\beta}{\gamma} K \|\mu' - \mu''\|.$$

Thus, $\|TO(\mu') - TO(\mu'')\| \leq \frac{\beta}{\gamma} K \|\mu' - \mu''\|$. The mapping $TO$ is a contraction if $\gamma/\beta \geq K + \varepsilon$, where $\varepsilon > 0$. This completes the condition (3.14) in the lemma.

Because $TO$: $\mathcal{V} \to \mathcal{V}$ is a contraction mapping under (3.14), and $\mathcal{V}$ (with the supnorm) is a complete metric space, $TO$ has a unique fixed point $\mu_a^m \in \mathcal{V}$.

Finally, there is a nonempty set of parameter values that satisfy (3.14). To show this, note that $F(\omega_L, a_H, z_H) > 0$ by construction. By choosing $\kappa$ sufficiently close to 1, I can ensure that $K$ is sufficiently close to one, and so $K + \varepsilon < F(\omega_L, a_H, z_H) + 1$. Then, there are values of $\gamma$ $(> \beta)$ that satisfy $K + \varepsilon \leq \gamma/\beta \leq F(\omega_L, a_H, z_H) + 1$. Also, because $F(\omega; a, z) = 0$ when $\omega$ is large, I can choose a large value for $\omega_H$ to ensure $F(\omega_H, a_L, z_L) + 1 \leq \gamma/\beta$. Clearly, these conditions require $\gamma > \beta$ and $\omega_H > \omega_L$. **QED**

### A.2. Proof of Theorem 3.2

To prove Theorem 3.2, I first show that $\Gamma$ defined by (3.15) maps $\mathcal{A}$ into $\mathcal{A}$. The following lemma gives the sufficient conditions for this result.

**Lemma A.1.** *Given any $a \in \mathcal{A}$, $\Gamma_a \in \mathcal{A}$ if $a_H$ is close to one and if (3.16) is satisfied.*

**Proof**. Since $\mu_a^m \geq \frac{\gamma}{\beta}\omega_L > 0$, $(\mu_a^m, \mu_a^b)$ are continuous, and $\Phi$ has the Feller property, then $\Gamma_a(.)$ defined by (3.15) is continuous. To show $\Gamma_a \in \mathcal{A}$, it suffices to show $\Gamma_a(z) \in [a_L, a_H]$ for all $z \in Z$. Notice that the right-hand side of (3.15) is increasing in $a(z_{-1})$ for given $(\mu_a^m, \mu_a^b)$. Since $a(z_{-1}) \in [a_L, a_H]$, the sufficient conditions for $\Gamma_a(z_{-1}) \in [a_L, a_H]$ are:

$$\mu_a^m(z_{-1}) \leq \frac{\beta}{\gamma} \int \max\left\{\frac{\gamma z}{1 - a_H}\mu_a^b(z_{-1}),\ \mu_a^m(z)\right\} \Phi(dz, z_{-1}), \tag{A.1}$$

$$\mu_a^m(z_{-1}) \geq \frac{\beta}{\gamma} \int \max\left\{\frac{\gamma z}{1 - a_L}\mu_a^b(z),\ \mu_a^m(z)\right\} \Phi(dz, z_{-1}). \tag{A.2}$$

The first condition is satisfied when $a_H$ is close to 1. For the second condition, note that $\mu_a^b(z) \leq \mu_a^m(z)$ for all $z$ (see (3.9) and (3.10)), and so

$$\begin{aligned} \text{RHS(A.2)} &\leq max\left\{\tfrac{\gamma z_H}{1-a_L},\ 1\right\} \tfrac{\beta}{\gamma} \textstyle\int \mu_a^m(z)\Phi(dz, z_{-1}) \\ &= \omega_a^m(z_{-1}) max\left\{\tfrac{\gamma z_H}{1-a_L},\ 1\right\}. \end{aligned}$$

Also, because $F(\omega^m; a, z_{-1})$ is decreasing in $(\omega^m; a, z_{-1})$, then

$$\mu_a^m(z_{-1}) = \omega_a^m(z_{-1})\left[1 + F(\omega_a^m(z_{-1}); a(z_{-1}), z_{-1})\right] \geq \omega_a^m(z_{-1})\left[1 + F(\omega_H; a_H, z_H\right].$$

Therefore, (3.16) is a sufficient condition for (A.2). This completes the proof of Lemma A.1.

It is possible to satisfy (3.16) by choosing $z_H$ and $a_L$ sufficiently close to 0. Thus, there is a nonempty set of parameter values in which $\Gamma$ maps $\mathcal{A}$ into $\mathcal{A}$.

Next, I show that $\Gamma : \mathcal{A} \to \mathcal{A}$ is continuous. Treat $\mu_a^m$, $\mu_a^b$ and $\omega_a^m$ as functions of $a$. I show that $(\mu_a^m, \mu_a^b, \omega_a^m)$ are continuous in $a$ in the supnorm. Once this is done, it is clear from (3.15) that $\Gamma$ is continuous in $a$ in the supnorm. Because the proofs for $(\mu_a^m, \mu_a^b, \omega_a^m)$ to be continuous in $a$ are similar, I describe only the proof for $\mu_a^m$. For the latter, I need to show that for any $\varepsilon > 0$, there exists $\Delta > 0$ such that $\|\mu_{a_2}^m - \mu_{a_1}^m\| < \varepsilon$ whenever $\|a_2 - a_1\| < \Delta$, where the norm is the supnorm. Let $\varepsilon > 0$ be an arbitrary number. Define

$$B(\omega^m, z_{-1}) = \max_{a, \widehat{a} \in \mathcal{A}} \left| \frac{F(\omega^m, a(z_{-1}), z_{-1}) - F(\omega^m, \widehat{a}(z_{-1}), z_{-1})}{\widehat{a}(z_{-1}) - a(z_{-1})} \right|,$$

where $F$ is defined in (3.8). Since $F$ is decreasing in $a$, then $B > 0$. Also, because the intervals $[a_L, a_H]$, $[\omega_L, \omega_H]$, and $[z_L, z_H]$ are bounded away from zero and bounded above, it can be verified that $B(\omega^m, z_{-1}) < \infty$. For any $a_1, a_2 \in \mathcal{A}$, if $\|a_2 - a_1\| < \Delta$, then

$$\begin{aligned} &|F(\omega^m, a_2(z_{-1}), z_{-1}) - F(\omega^m, a_1(z_{-1}), z_{-1})| \\ &\leq B(\omega^m, z_{-1})\,|a_2(z_{-1}) - a_1(z_{-1})| \leq B(\omega^m, z_{-1})\,\|a_2 - a_1\| < B(\omega^m, z_{-1})\Delta. \end{aligned}$$

Because $T(\omega^m, a, z_{-1}) = \omega^m(1 + F)$ and $\omega^m \leq \omega_H$, then

$$\begin{aligned} &|T(\omega^m, a_2(z_{-1}), z_{-1}) - T(\omega^m, a_1(z_{-1}), z_{-1})| \\ &= \omega^m\,|F(\omega^m, a_2(z_{-1}), z_{-1}) - F(\omega^m, a_1(z_{-1}), z_{-1})| < \omega_H B(\omega^m, z_{-1})\Delta. \end{aligned}$$

Since $\mu_a^m = T(\omega_a^m, a(z_{-1}), z_{-1})$ and $\|TO(\mu') - TO(\mu'')\| \leq \frac{\beta}{\gamma} K\,\|\mu' - \mu''\|$, I get:

$$\begin{aligned} & \left|\mu_{a_2}^m(z_{-1}) - \mu_{a_1}^m(z_{-1})\right| \\ = & \left|T\left(\omega_{a_2}^m, a_2(z_{-1}), z_{-1}\right) - T(\omega_{a_1}^m, a_1(z_{-1}), z_{-1})\right| \\ \leq & \left|TO\left(\mu_{a_2}^m, a_2(z_{-1}), z_{-1}\right) - TO(\mu_{a_1}^m, a_2(z_{-1}), z_{-1})\right| \\ & \quad + \left|T\left(\omega_{a_1}^m, a_2(z_{-1}), z_{-1}\right) - T(\omega_{a_1}^m, a_1(z_{-1}), z_{-1})\right| \\ < & \frac{\beta}{\gamma} K \left\|\mu_{a_2}^m - \mu_{a_1}^m\right\| + \omega_H B(\omega_{a_1}^m(z_{-1}), z_{-1})\Delta. \end{aligned}$$

Taking the maximum over $z_{-1}$ on both sides of the inequality yields

$$\left\|\mu_{a_2}^m - \mu_{a_1}^m\right\| < \frac{\Delta}{1 - \frac{\beta}{\gamma} K} \omega_H \|B\|.$$

Let $\Delta = \varepsilon \left(1 - \frac{\beta}{\gamma} K\right) \Big/ [\omega_H \|B\|]$. Because $\gamma/\beta > K$, $\|B\| < \infty$ and $0 < \omega_H < \infty$, then $\Delta > 0$. For all $a_1, a_2 \in \mathcal{A}$ such that $\|a_2 - a_1\| < \Delta$, $\left\|\mu_{a_2}^m - \mu_{a_1}^m\right\| < \varepsilon$. Therefore, $\Gamma : \mathcal{A} \to \mathcal{A}$ is continuous.

Finally, since $\Gamma$ is continuous and since $\mathcal{A}$ is compact and convex, Brouwer's fixed point theorem implies that $\Gamma$ has a fixed point in $\mathcal{A}$. This fixed point is the equilibrium function $a(.)$. This completes the proof of Theorem 3.2. **QED**

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