Bargaining in Monetary Economies.*

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Abstract

Search models of monetary exchange have typically relied on Nash (1950) bargaining, or equivalent strategic solutions, to determine the terms of trade. By considering alternative bargaining solutions in a simple search model with divisible money, we show how this choice matters for important results such as the ability of the optimal monetary policy to generate an efficient allocation. We show that the quantities traded in bilateral matches are always inefficiently low under the Nash (1950) and Kalai-Smorodinsky (1975) solutions whereas under egalitarian (Kalai, 1977) and gradual Nash (O'Neill et al., 2004) solutions, the Friedman Rule achieves the first best allocation. We identify the inefficiency under the Nash and Kalai-Smorodinsky solutions with a non-monotonicity property of these solutions. We evaluate quantitatively the welfare cost of inflation under the different bargaining solutions, and we extend the model to allow for endogenous market composition in order to study how optimal monetary policy varies with the bargaining solution.

Keywords : Bargaining, Search, Money, Inflation

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1 Introduction

Since Shi (1995) and Trejos and Wright (1995), monetary search models with bilateral trading have typically relied on bargaining to determine the terms of trade. The usual approach is to impose some version of the generalized Nash solution, or to use a strategic bargaining game that yields a similar outcome.[1] By investigating alternative axiomatic bargaining solutions in a search model of money we will show that the choice of the Nash solution is far from innocuous and that important results regarding the welfare effects of inflation hinge on it.

For example, a key result in Lagos and Wright (2003) – hereafter denoted LW – is that the Friedman rule cannot replicate the first best allocation and the quantity of goods traded is inefficiently low unless buyers have all the bargaining power. This inefficiency on the intensive margin has been attributed to a holdup problem in the bargaining. Since holdup problems are not specific to a particular bargaining solution, this would lead one to believe that this result is a robust feature of models with bargaining. However, we argue that this inefficiency is not the outcome of a holdup problem. The reason is that at the Friedman rule there is no sunk cost of acquiring money – agents can sell a unit of money in the next period for more goods than they gave up to acquire it and this exactly compensates them for discounting. This suggests that this inefficiency may have something to do with the particular bargaining solution used in LW. Therefore, it is important to explore other bargaining solutions.

Given the concerns on how the bargaining solution may affect the efficiency of the intensive margin (the quantity traded in a match), one may also wonder how the bargaining solution affects the efficiency of the extensive margin (the number of matches). For example, some models have shown that deviations from the Friedman rule could be optimal when the composition of trades is endogenous (Shi, 1997). Although intuitive, this result is difficult to establish analytically and it seems to be present only for a small set of parameter values (Rauch 2000). Again, exploring how alternative bargaining solutions affect the extensive margin appears to be needed.

The objective of this paper is precisely to examine how alternative axiomatic bargaining solutions affect the intensive and extensive margins in a monetary search model. We demonstrate that the source of the LW result on the inefficiency of the intensive margin is due the fact that

[1] There are exceptions. In his early paper on money in search equilibrium, Diamond (1984) uses the egalitarian solution. Also, other papers papers have relied on alternative pricing mechanisms such as price posting, e.g.,Curtis and Wright (2004) and Rocheteau and Wright (2003). Finally, Coles and Wright (1998) have shown that the equivalence between alternating offers bargaining games and the Nash solution breaks down when one looks at dynamic equilibria.

the Nash solution does not require agents' payoffs to be monotonic as the bargaining set expands. Since the monotonicity of the buyers' surplus matters for his choice of real balances, we carry out our analysis by comparing the Nash solution to two other standard axiomatic bargaining solutions — the Kalai-Smorodinsky (1975) solution, the Kalai (1977) proportional solution — and a recent solution proposed by O'Neill et al. (2004), called the gradual Nash (or ordinal) solution, that are based on notions of monotonicity.[2]

We show that, at the Friedman rule, the quantity traded is always inefficiently low under the Kalai-Smorodinsky solution whereas under proportional and gradual Nash the Friedman Rule achieves the first best. With regards to the extensive margin, with the proportional and gradual Nash solutions, we are able to establish a simple condition on the magnitude of the bargaining weight such that the Friedman rule is not optimal. Finally, we also investigate the quantitative implications of alternative symmetric bargaining solutions. We show that the bargaining solutions matter greatly for the welfare costs of small inflation (say 3%) but for substantial inflation rates (10%), the welfare costs of inflation are of similar magnitude across bargaining solutions.

The paper is organized as follows. Section 2 contains the monetary model. In Section 3 we define the bargaining problem in a match and the four bargaining solutions we consider. Section 4 contains the characterization of steady-state monetary equilibria for the four bargaining solutions. Finally, in Section 5 we explore some implications of these alternative bargaining solutions for the welfare effects of inflation. All proofs are in the appendix.

2 The model

The basic model we consider is essentially identical to the one in LW. Time is discrete and continues forever. Each period is divided into two subperiods, called *day* and *night* where different activities take place. There is a measure one of agents who are specialized in terms of the goods they produce and consume during the day, but all agents produce and consume the same good at night. During the day, trading is decentralized and agents are matched bilaterally. Each agent meets someone who produces a good he wishes to consume with probability $\sigma \leq 1/2$ and meets someone who likes the good he produces with the same probability σ. For simplicity, we rule-out double coincidence of wants meetings. At night there is a centralized Walrasian market where agents can trade goods and money. All goods are nonstorable and perishable.

[2] The term gradual Nash was used originally by Wiener and Winter (1998).

Agents' preferences are represented by the following utility function:

$$\mathcal{U}(q^b, q^s, x, y) = u(q^b) - c(q^s) + U(x) - y, \tag{1}$$

where q^b and q^s are the quantities consumed and produced during the day, and x and y are the quantities produced and consumed at night. We assume $U'(x) > 0$, $U''(x) < 0$, $u'(q) > 0$, $u''(q) < 0$, $u(0) = c(0) = c'(0) = 0$, $c'(q) > 0$, $c''(q) > 0$, and $c(\bar{q}) = u(\bar{q})$ for some $\bar{q} > 0$. Let q^* denote the solution to $u'(q^*) = c'(q^*)$ and x^* the solution to $U'(x^*) = 1$; $q^* \in (0, \bar{q})$ exists by the previous assumptions, and we assume such an $x^* > 0$ also exists. All agents have the same discount factor $\beta \equiv (1+r)^{-1} \in (0,1)$.

Agents trade *anonymously*; hence, they cannot get credit in the decentralized market because they could default without fear of punishment. Let the quantity of fiat money per capita at the beginning of period t be $M_t > 0$ and assume $M_{t+1} = \gamma M_t$, where $\gamma \equiv 1 + \pi$ is constant and new money is injected by lump-sum transfers. The price of goods in terms of money in the centralized market is p_t. In the following, we will omit time indices and will replace $t+1$ by $+1$, $t+2$ by $+2$ and so on.

All through the paper, we will restrict our attention to steady state equilibria where the real value of aggregate money balances M/p is constant. This implies $p_{+1} = \gamma p$. Bellman's equation for an agent in the decentralized market holding $z = m/p$ units of real balances is

$$V(z) = \sigma \int \{u[q(z,\tilde{z})] + W[z - d(z,\tilde{z})]\} dF(\tilde{z})$$

$$+\sigma \int \{-c[q(\tilde{z}, z)] + W[z + d(\tilde{z}, z)]\} dF(\tilde{z}) + (1 - 2\sigma) W(z), \tag{2}$$

where $F(\tilde{z})$ is the distribution of real balances across agents, and $W(z)$ is the value function of the agent in the centralized market. Equation (2) has the following interpretation. An agent meets someone who produces a good he likes with probability σ. He consumes q units of goods and delivers d units of real balances to his trading partner where q and d depend on his real balances z and the real balances $\tilde{z}$ of his partner in the match. With probability σ, the agent meets someone who likes his good. He is then the seller in the match. With probability $1 - 2\sigma$, no trade takes place. In the centralized market the problem of the agent is

$$W(z) = \max_{\hat{z},x,y} \{U(x) - y + \beta V(\hat{z})\} \tag{3}$$

$$\text{s.t.} \quad x + \gamma \hat{z} = y + z + T, \tag{4}$$

where T the lump-sum transfer (expressed in general goods), and $\hat{z}$ the real balances taken into the next day.[3] In the budget identity (4), we have used the fact that the relative price of real balances next period in terms of the general good is $p_{+1}/p = \gamma$. Substituting y from (4) into (3) we obtain[4]

$$W(z) = \max_{\hat{z},x} \left\{ U(x) - x - (\gamma\hat{z} - T - z) + \beta V(\hat{z}) \right\}. \tag{5}$$

From (5), $x = x^*$ and the maximizing choice of $\hat{z}$ is independent of z; and W is linear in z with $W_z = 1$. Substituting $V(\hat{z})$ by its expression given by (2), we can reformulate the buyer's problem as

$$\max_{\hat{z}} \left\{ -i\hat{z} + \sigma \int \left\{ u\left[q(\hat{z},\tilde{z})\right] - d\left(\hat{z},\tilde{z}\right) \right\} dF(\tilde{z}) + \sigma \int \left\{ d(\tilde{z},\hat{z}) - c[q(\tilde{z},\hat{z})] \right\} dF(\tilde{z}) \right\} \tag{6}$$

where we have used that $1+i = (1+\pi)(1+r)$ where i is the nominal interest rate.[5] According to (6), agents choose their real balances in order to maximize their expected surplus in the search market minus the opportunity cost of carrying real balances.

3 Bargaining

In this section, we define the bargaining problem for any match between a buyer and a seller and present the alternative bargaining solutions.

3.1 The bargaining problem

We start this section by describing the bargaining problem in a match between a buyer holding z units of real balances and a seller holding $\tilde{z}$ units of real balances. An agreement is a pair (q, d) where q is the amount of goods produced by the seller and d is the amount of real money transferred by the buyer to the seller. The utility of the buyer if an agreement is reached is $u^b = u(q) + W(z-d)$ whereas the utility of the seller is $u^s = -c(q) + W(\tilde{z}+d)$. If no agreement is reached, the utility of the buyer is $u_0^b = W(z)$ and the utility of the seller is $u_0^s = W(\tilde{z})$. It is important to note that u_0^b and u_0^s are taken as given within the bargaining problem but are endogenously determined in the equilibrium of the economy.

[3] Note that $\hat{m}/p = (p_{+1}/p)\,(\hat{m}/p_{+1}) = \gamma\hat{z}$.

[4] We do not impose nonnegativity on y, but it is easy to choose fundamentals in order to guarantee that $y \geq 0$ in equilibrium.

[5] One could introduce bonds in the model and let agents trade these bonds in the centralized market but not take them into the decentralized market. The nominal interest rate would then be given by the Fisher equation in the paper.

From the linearity of $W(z)$, $u^b = u_0^b + u(q) - d$ and $u^s = u_0^s + d - c(q)$. The set $\mathcal{S}$ of feasible utility levels is defined as

$$\mathcal{S} = \left\{ (u(q) - d + u_0^b, d - c(q) + u_0^s) : d \in [-\tilde{z}, z] \text{ and } q \geq 0 \right\}.$$

The Pareto frontier $\bar{\mathcal{S}}$ of $\mathcal{S}$ is the set of pairs of utility levels in $\mathcal{S}$ so that the buyer and the seller cannot be made better-off simultaneously. The equation for $\bar{\mathcal{S}}$ is derived from the program $u^b = \max_{q,d} [u(q) - d] + u_0^b$ s.t. $-c(q) + d \geq u^s - u_0^s$ and $d \leq z$ for some u^s. Similarly

$$u^s - u_0^s = \begin{cases} u(q^*) - c(q^*) - (u^b - u_0^b) & \text{if } u^s - u_0^s \leq z - c(q^*) \\ z - c\left[u^{-1}(u^b - u_0^b + z)\right] & \text{otherwise} \end{cases} \tag{7}$$

Therefore, $d^2u^s/(du^b)^2 = 0$ if $u^s \leq z - c(q^*)$ and $d^2u^s/(du^b)^2 < 0$ otherwise.

Formally, a bargaining game is described by a pair $(\mathcal{S}, u_0)$ where $\mathcal{S}$ is the set of feasible utility levels and $u_0 = (u_0^b, u_0^s)$ is the disagreement outcome. The definition of a bargaining game has recently been extended to situations where there are several issues to negotiate. In our context, the several issues correspond to the different units of real balances of the buyer. To formalize this idea, consider a bargaining problem where the buyer commits to spend no more than $\tau \leq z$ real balances. The bargaining set $\mathcal{S}(\tau)$ associated with this problem is

$$\mathcal{S}(\tau) = \left\{ (u(q) - d + u_0^b, d - c(q) + u_0^s) : d \in [-\tilde{z}, \tau] \text{ and } q \geq 0 \right\}$$

In Figure 1, we represent the bargaining game $\mathcal{S}(\tau)$ for three values of τ, i.e. $\tau_3 > \tau_2 > \tau_1$. Note that $\mathcal{S}(\tau) \subset \mathcal{S}(\tau')$ for all $\tau < \tau'$.[6] It then follows that a gradual bargaining problem is a pair $(\mathcal{S}(\tau), u_0)$ where $\mathcal{S}(\tau)$ is a family of sets of feasible utility levels and u_0 is the initial status quo point at which the bargaining starts.

A solution to the bargaining problem is a function that assigns a pair of utility levels to each bargaining game. In the case of gradual bargaining problems, a solution corresponds to a path of agreements $u(\tau)$, one for each τ, called gradual agreement.

3.2 The Nash solution

The Nash solution, μ^N, satisfies the following four axioms: Pareto optimality, independence of affine transformations, symmetry and independence of irrelevant alternatives.[7] It is given by

$$\mu^N(\mathcal{S}, u_0) = \arg \max_{(u^b, u^s) \in \mathcal{S}} (u^b - u_0^b)(u^s - u_0^s). \tag{8}$$

[6] Such a family of bargaining sets $\mathcal{S}(\tau)$ that varies continuously with τ is related to the notion of an *agenda* (see O'Neill, et al, 2004).

[7] For a discussion of these axioms, see Roth (1979) and Osborne and Rubinstein (1990).

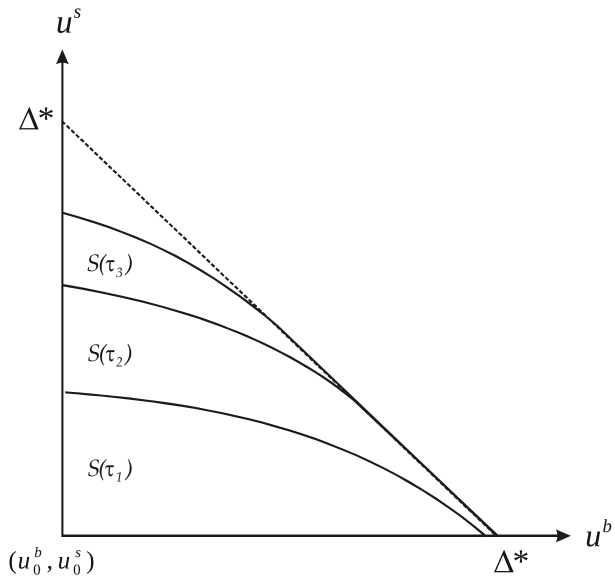

Figure 1: The bargaining set

Since $u^b - u_0^b = u(q) - d$ and $u^s - u_0^s = -c(q) + d$ the pair (q, d) satisfies

$$(q,d) = \arg\max_{q,d} \left[u(q) - d\right]\left[-c(q) + d\right]$$

subject to $d \leq z$. The solution is $q = q^*$ and $d = \left[u(q^*) + c(q^*)\right]/2$ if $z \geq z^* \equiv \left[u(q^*) + c(q^*)\right]/2$, and $d = z$ and

$$z = z(q) \equiv \frac{u'(q)c(q) + c'(q)u(q)}{u'(q) + c'(q)}, \tag{9}$$

otherwise. Note that $u(q) - z(q) = \Theta(q)[u(q) - c(q)]$ where $\Theta(q) = u'(q)/[u'(q) + c'(q)]$. It is easy to show that $u(q) - z(q)$ is non-monotonic in q and negatively sloped in the vicinity of $q = q^*$. This is illustrated in the following Figure where the buyer's utility falls as the bargaining set expands.

3.3 The Kalai-Smorodinsky solution

The Kalai-Smorodinsky solution preserves all the axioms of the Nash solution except the independence of irrelevant alternatives that is replaced by the axiom of *individual monotonicity*. Consider two bargaining problems $(\mathcal{S}_1, u_0)$ and $(\mathcal{S}_2, u_0)$ such that $\mathcal{S}_1 \subset \mathcal{S}_2$ where the maximum utility level of player j is the same in the two bargaining problems. Individual monotonicity implies that the utility of player $i \neq j$ is higher in the second bargaining problem. The Kalai and Smorodinsky solution, μ^{KS}, satisfies

$$\mu^{KS}(\mathcal{S}, u_0) = u_0 + \lambda\left(\hat{u} - u_0\right) \tag{10}$$

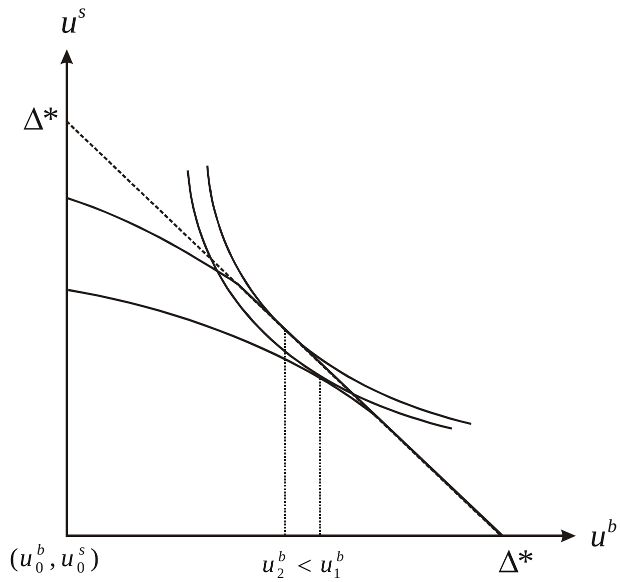

Figure 2: Nash solution.

where $\lambda \in [0,1]$ is chosen so that $\mu^{KS}(\mathcal{S}, u_0) \in \bar{\mathcal{S}}$, and where $\hat{u} = \left(\hat{u}^b, \hat{u}^s\right) \geq u_0$ specifies the best alternative in $\mathcal{S}$ for each player. From (10), (u^b, u^s) satisfies

$$\frac{u^s - u_0^s}{u^b - u_0^b} = \frac{\hat{u}^s - u_0^s}{\hat{u}^b - u_0^b}. \tag{11}$$

The component $\hat{u}^b$ is the utility level the buyer would reach if he were able to extract all the surplus of a match, i.e., $\hat{u}^b = u_0^b + \max_{q,d}[u(q) - d]$ subject to $-c(q) + d = 0$ and $d \leq z$. Therefore,

$$\hat{u}^b = \begin{cases} u_0^b + u(q^*) - c(q^*) & \text{if} \quad z \geq c(q^*), \\ u_0^b + u\left[c^{-1}(z)\right] - z & \text{otherwise.} \end{cases}$$

Similarly, $\hat{u}^s$ satisfies

$$\hat{u}^s = \begin{cases} u_0^s + u(q^*) - c(q^*) & \text{if} \quad z \geq u(q^*), \\ u_0^s + z - c\left[u^{-1}(z)\right] & \text{otherwise.} \end{cases} .$$

From (11) (q, d) satisfies $d = z$ and

$$\frac{u(q) - z}{u\left[c^{-1}(z)\right] - z} = \frac{-c(q) + z}{z - c\left[u^{-1}(z)\right]}, \quad \text{if} \quad z \leq c(q^*), \tag{12}$$

$$\frac{u(q) - z}{u(q^*) - c(q^*)} = \frac{-c(q) + z}{z - c\left[u^{-1}(z)\right]}, \quad \text{if} \quad z \in \left[c(q^*), z^*\right], \tag{13}$$

where $z^* \in [c(q^*), u(q^*)]$ satisfies

$$\frac{u(q^*) - z}{u(q^*) - c(q^*)} = \frac{-c(q^*) + z}{z - c\left[u^{-1}(z)\right]}.$$

If $z \in [z^*, u(q^*)]$, the pair (q, d) satisfies $q = q^*$ and

$$\frac{u(q^*) - d}{u(q^*) - c(q^*)} = \frac{-c(q^*) + d}{z - c\left[u^{-1}(z)\right]}. \tag{14}$$

Equations (12)-(14) define an implicit relationship between q and z, i.e. $z = z(q)$.

Despite this axiom of individual monotonicity, for our bargaining problem, it does not guarantee that the buyer's payoff is monotonically increasing q (See Figure 3).

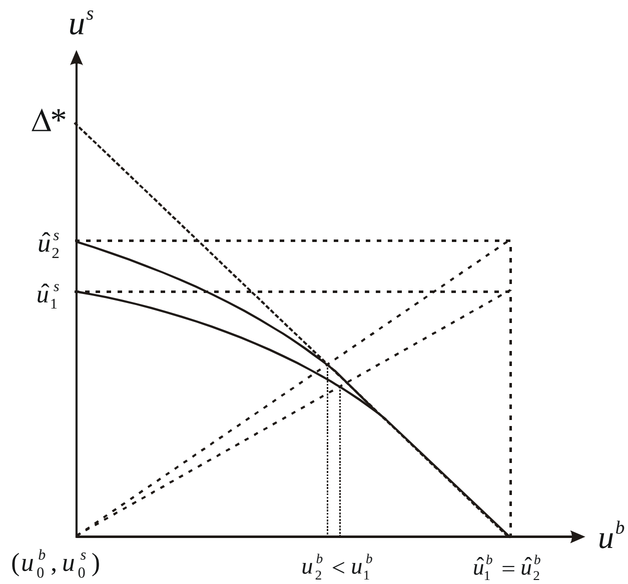

Figure 3: Kalai-Smorodinsky solution.

3.4 The proportional solution

A stronger notion of monotonicity imposes that no players are made worse-off if additional alternatives are made available to the players. Formally this means the following. Consider two bargaining problems $(\mathcal{S}_1, u_0)$ and $(\mathcal{S}_2, u_0)$ such that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Then a solution μ is monotonic iff $\mu(\mathcal{S}_1) \leq \mu(\mathcal{S}_2)$. Kalai (1977) shows that a solution μ is monotonic iff it is proportional, i.e. the solution can be written as $\mu^E(\mathcal{S}, u_0) = u_0 + \lambda(\bar{u}^b, \bar{u}^s)$ where $\bar{u}^b$ and $\bar{u}^s$ are positive constants and λ is chosen so that $\mu^E(\mathcal{S}, u_0) \in \bar{\mathcal{S}}$.[8] In our context, both players have the same utility function, and therefore symmetry implies $\bar{u}^b = \bar{u}^s$. Therefore the proportional solution satisfies

$$u^b - u_0^b = u^s - u_0^s. \tag{15}$$

[8] In contrast to the Nash and Kalai-Smorodinsky solutions, the proportional solutions are invariant only under simultaneous rescaling of the utility functions of the two players with the same rescaling factor. Therefore, the proportional solution is not unique and it depends on the choice of $(\bar{u}^b, \bar{u}^s)$. Implicitly, proportional solutions allow players to make interpersonal comparison of utility. See the discussions in Kalai (1977) and Kalai and Samet (1985).

From (15), (q, d) satisfies $-c(q) + d = u(q) + d$ and $d = z$ if $q < q^*$. Therefore, $q = q^*$ and $d = [u(q^*) + c(q^*)]/2$ if $z \geq z^* \equiv [u(q^*) + c(q^*)]/2$, and $d = z$ with

$$z = z(q) \equiv \frac{c(q) + u(q)}{2}, \tag{16}$$

otherwise.

Interestingly, this bargaining solution is invariant under decomposition of the bargaining process into stages.[9] As an illustration, suppose the buyer has z units of money. Agents could first bargain over the quantity to produce in exchange for τ_1 units of money, i.e., $\mathcal{S}_1 = \mathcal{S}(\tau_1)$, and in a second step they would bargain over the quantity to produce in exchange for the $z - \tau_1$ units that are left, i.e., $\mathcal{S}_2 = \mathcal{S}(z)$. The status quo point in the second step would be the utility that players would reach if they would agree in the first step. Under the proportional solution, this procedure is equivalent to a one-time bargaining (See Figure 4).

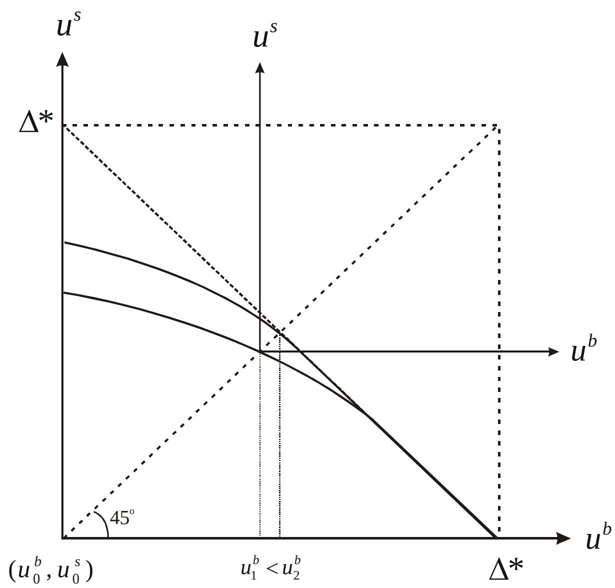

Figure 4: Proportional solution.

3.5 The gradual Nash solution

The *gradual Nash* (or *ordinal*) *solution* is a solution for gradual bargaining problems based that satisfies the axioms of Pareto efficiency, symmetry, scale invariance and an axiom of path consistency. Path consistency requires the path of agreements to be unchanged if one takes any point on

[9] This is the axiom of negotiation-by-steps. See Kalai (1977) for the formal statement of the axiom.

the agreement path as the starting point of the bargaining. In other words, the agreement reached at each stage is seen as final for the issues (real balances) on the table at that stage.[10] The gradual Nash solution satisfies the following differential equation:

$$\frac{du^s}{du^b} = \frac{\partial H(u^b, u^s, \tau)/\partial u^b}{\partial H(u^b, u^s, \tau)/\partial u^s}, \tag{17}$$

where $H(u^b, u^s, \tau) = 0$ is the equation of the Pareto frontier of $\mathcal{S}(\tau)$, and the initial condition $(u^b, u^s)(0) = u_0$. According to (17), at each point of the agreement path, the ratio of the buyer and the seller marginal utility gains is the rate of substitution of their utility on the current efficient frontier.[11]

To apply this solution to our problem, from (7), the equation for the Pareto frontier is given by

$$H(u^b, u^s, \tau) = \begin{cases} (u^s - u_0^s) + (u^b - u_0^b) - [u(q^*) - c(q^*)] & \text{if } \tau \geq u^s - u_0^s + c(q^*) \\ u^s - u_0^s + c\left[u^{-1}(u^b - u_0^b + \tau)\right] - \tau & \text{otherwise.} \end{cases} \tag{18}$$

If $\tau \geq u^s - u_0^s + c(q^*)$ then $q = q^*$ so that an increase in τ does not allow for mutual gains. Therefore, $du^s/d\tau = du^b/d\tau = 0$. If $\tau < u^s - u_0^s + c(q^*)$ then $q < q^*$ and $d = \tau$ and (17) and (18) imply

$$\frac{du^s}{du^b} = \frac{c'(q)}{u'(q)} \tag{19}$$

Differentiating $u^b = u(q) - \tau + u_0^b$ and $u^s = \tau - c(q) + u_0^s$, one can rewrite (19) as

$$\frac{d\tau}{dq} = \frac{2c'(q)u'(q)}{u'(q) + c'(q)} \quad \text{for all } q < q^*. \tag{20}$$

Integrating (20), and using the fact that $q = 0$ at $z = 0$ we have

$$z = z(q) = \int_0^q \frac{2c'(x)u'(x)}{u'(x) + c'(x)} dx. \tag{21}$$

Note from (20) that the change in the buyer's surplus along the bargaining path is $d(u^b - u_0^b)/dq = \Theta(q)\left[u'(q) - c'(q)\right]$ where $\Theta(q) = \frac{u'(q)}{u'(q)+c'(q)}$. In contrast the buyer's surplus under the

[10] This is related to the axiom of negotiation-by-steps of the proportional solution. See Kalai (1977). At each stage of the negotiation, the status quo point of the bargaining is the agreement reached at the previous stage. However, in contrast to the proportional solution there is no equivalence between one-time bargaining and bargaining in stages.

[11] This extends the Nash solution in a very natural way. The Nash solution requires the tangency between the Nash product curve $\left(u^b - u_0^b\right)\left(u^s - u_0^s\right)$ and $\bar{\mathcal{S}}(z)$ which implies

$$\frac{u_s - u_0^s}{u^b - u_0^b} = \frac{\partial H(u^b, u^s, z)/\partial u^b}{\partial H(u^b, u^s, z)/\partial u^s}.$$

This solution has several interesting properties. First, if one interprets the bargaining as a bargaining in different stages, the solution used in each stage that would be consistent with gradual Nash could be the Nash solution, the Kalai-Smorodinsky solution or any solution that is invariant to affine transformation of the utility functions. Second, the solution is ordinal in the sense of being covariant with monotonic transformations of each player's utility.

Nash solution is given by $u^b - u^b_0 = \Theta(q)\left[u(q) - c(q)\right].$ So gradual Nash is essentially applying the Nash solution at the margin. This also means the gradual Nash solution is monotonic. The determination of the bargaining path is illustrated in Figure 5. At a given point on the bargaining path, the slope of the bargaining direction is equal to the absolute value of the tangent to the bargaining set at that point.

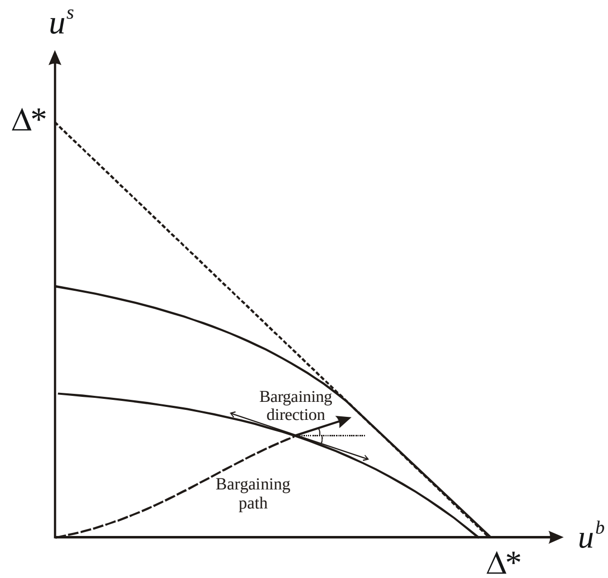

Figure 5: Gradual Nash solution

4 Equilibrium and efficiency

In the bargaining, the status quo point (u^b_0, u^s_0) was taken as given by the buyer and the seller. However, in equilibrium these quantities are endogenous and depend of the agent's choice of real balances.

Lemma 1 *For all the bargaining solutions considered in the previous section, the agent's problem (6) can be reformulated as*

$$\max_{q \in [0, q^*]} \left\{ -iz(q) + \sigma \left[u(q) - z(q)\right] \right\} \tag{22}$$

The maximization problem in (22) has a simple interpretation. The agent chooses the quantity q to trade in the decentralized market in order to maximize his expected surplus as a buyer minus the cost of holding real balances. The relationship between z and q comes from the bargaining solution. If (22) has more than one solution, we restrict our attention to symmetric equilibria where all agents choose the same real balances.[12]

Definition 1 *A steady-state monetary equilibrium is a* $q > 0$ *solution to (22).*

Proposition 1 *There exists an* $\bar{\imath} > 0$ *such that an equilibrium exists for all* $i < \bar{\imath}$*.*

We now investigate some implications of the choice of the bargaining solution for the efficiency of monetary equilibrium.

Lemma 2 *(i) If* $u(q) - z(q)$ *is (strictly) increasing then* $q = q^*$ *is an (the) equilibrium at* $i = 0$*. (ii) If* $u'(q^*) < z'(q^*)$ *then* $q = q^*$ *is not an equilibrium for any* $i \geq 0$*.*

From Lemma 2, if a bargaining solution is such that the buyer's payoff is monotonic, in the sense that it is non-decreasing when the bargaining set expands, then the Friedman rule achieves the first-best allocation. In contrast, if the solution is non-monotonic, and if the buyer's surplus decreases when q gets close to q^*, then the Friedman rule fails to achieve the efficient allocation.

Proposition 2 *For all* $i \geq 0$*, equilibria under the Nash and Kalai-Smorodinsky solutions are inefficient,* $q < q^*$*. Equilibria under proportional and gradual Nash solutions are efficient iff* $i = 0$*.*

As noticed in LW, the quantity traded under Nash bargaining is inefficiently low even at the Friedman Rule ($i = 0$). As mentioned in the introduction, this inefficiency cannot be explained by an holdup problem since at the Friedman Rule there is no sunk cost associated with the decision to hold real balances – the investment in real balances is perfectly reversible. The reason for the inefficiently low q comes from the fact that the Nash solution is non-monotonic and the buyer's surplus $u(q) - z(q)$ reaches a maximum for a $q < q^*$ (See Lemma 2). Maybe surprisingly, a similar inefficiency occurs under Kalai-Smorodinsky bargaining. The reason is as follows. When q is close to q^* if the buyer brings additional money balances when q is close to q^* then the maximum utility level he can attain remains unchanged whereas the maximum utility level of the seller increases. This increases the share of the match surplus that the seller can obtain.

Under the proportional and gradual Nash bargaining solutions, the Friedman rule achieves the first best allocation. This is a consequence of the monotonicity axiom of the proportional solution

[12] There would be no conceptual difficulty in considering asymmetric equilibria. An asymmetric equilibrium would be defined as a list (q_i), a q_i for each agent, where each q_i is solution to (22).

and the fact that under gradual Nash the buyer can always get a fraction of the increment in the surplus of the match as he increases his real balances. Since the buyer's surplus is monotonic with his money holdings, and strictly increasing if $z < z^*$, the buyer will invest up to z^* when $i = 0$.

Aside from efficiency arguments, monotonicity is important for another reason – it eliminates incentives for the buyer to hide his cash balances when meeting a seller. If the bargaining solution yields a non-monotonic payoff for the buyer, then he has an incentive to hide some of his cash balances whenever it would cause his surplus to decline. Under Nash and Kalai-Smorodinsky it is implicitly assumed the buyer cannot hide his cash holdings from the seller, otherwise he would do so. For instance, if terms of trade are determined by Nash bargaining, and if the utility of the agent is equal to $\varepsilon u(q)$ where ε is random, then the agent would want to conceal part of his money balances for low realizations of ε. If the buyer's surplus is monotonic, then he always gains from revealing a marginal unit of his cash and thus has no incentive to hide any of his money balances.

In summary, a key point of this analysis is to recognize that at $i = 0$ there is no holdup problem because there is no sunk cost associated with acquiring money. So any inefficiencies at $i = 0$ are not the result of a holdup problem. However, for $i > 0$ there is a sunk cost – even though an agent can dispose of his money in the following centralized market, the value of money does not increase at the time rate of discount. Hence, there is a real loss from holding money from one period to the next. It is this loss of value that sellers are able to extract via bargaining. Thus, we want to stress that the holdup problem is not eliminated by any of our bargaining solutions – rather it is the Friedman rule that eliminates it.

Asymmetric bargaining solutions The results in Proposition 2 can easily be extended to the case of asymmetric bargaining solutions. For instance, consider the generalized Nash solution

$$(q, d) = \arg\max \left[u(q) - d\right]^{\theta} \left[-c(q) + d\right]^{1-\theta}$$

where $\theta \in [0, 1]$ is the buyer's bargaining weight. Then,

$$z(q) = \frac{\theta u'(q)c(q) + (1-\theta)c'(q)u(q)}{\theta u'(q) + (1-\theta)c'(q)} \tag{23}$$

Unless $\theta = 1$, the quantity traded under is always too low at $i = 0$. If $\theta = 1$, the buyer can extract the entire surplus of the match. In this case, the buyer's surplus is monotonic and the Friedman rule is optimal. We can show that the same logic applies to the asymmetric Kalai-Smorodinsky solution.

The proportional solution is generalized by assuming $(1-\theta)[u(q)-d]=\theta[-c(q)+d]$. Then,

$$z(q)=(1-\theta)u(q)+\theta c(q). \tag{24}$$

Thus, $u(q)-z(q)=\theta[u(q)-c(q)]$ and, from Lemma 2, $q=q^*$ iff $i=0$ for any $\theta>0$. Finally, the asymmetric gradual Nash solution implies

$$\frac{du^s}{du^b}=\left(\frac{1-\theta}{\theta}\right)\frac{\partial H(u^b,u^s,z)/\partial u^b}{\partial H(u^b,u^s,z)/\partial u^s}$$

This gives $dz/dq=z'(q)=u'(q)c'(q)/[\theta u'(q)+(1-\theta)c'(q)]$ so that

$$z(q)=\int_0^q \frac{u'(x)c'(x)}{\theta u'(x)+(1-\theta)c'(x)}dx \tag{25}$$

It is easy to check that $u'(q)-z'(q)=\Theta(q)[u'(q)-c'(q)]$ where $\Theta(q)=\theta u'(q)/[\theta u'(q)+(1-\theta)c'(q)]$. Consequently, $q=q^*$ at the Friedman rule for any $\theta>0$.

5 Extensive margin effects

The previous section has shown the importance of the bargaining solution for the efficiency of the intensive margin, i.e., the quantity traded in bilateral matches. As shown by Shi (1997), search models of money can be extended to endogenize the frequency of trades, i.e., the extensive margin. We want to investigate in this section how the effect of inflation on the extensive margin depends of the choice of the bargaining solution. We examine the endogenous composition of buyers and sellers under the asymmetric proportional and gradual Nash bargaining solutions. The reason we focus on these two bargaining solutions is twofold: (i) As discussed before, the monotonicity property of these solutions is appealing and it avoids problems related to agents' incentives to hide money; (ii) Unlike the Nash or Kalai-Smorodinsky solutions, they yield simple analytical conditions as to when the composition of buyers and sellers, and therefore the number of trades, is efficient.

Assume that each agent can choose to be a buyer or a seller in the decentralized market and let n be the fraction of sellers.[13] Following the literature since Kiyotaki and Wright (1993), the matching probabilities of buyers and sellers are $\sigma^b=n$ and $\sigma^s=1-n$, respectively. Substituting $V(z)$ by its expression given by (2) into (5), and using the fact that buyers only consume in the

[13]Instead of assuming that agents can choose their types, one can follow Rocheteau and Wright (2003) and assume that agents need to specialize in a production technology. Either they produce the good traded in the decentralized market (in which case they are sellers) or they produce a final good that uses the good in the decentralized market as an input (in which case they are buyers).

decentralized market, the value of a buyer in the centralized market with z units of real balances satisfies

$$W^b(z) = U(x^*) - x^* + z + \max_q \{-(\gamma - \beta)z(q) + \beta n[u(q) - z(q)]\} + \beta W^b(0). \tag{26}$$

Similarly, the value of being a seller with z units of real balances is given by

$$W^s(z) = U(x^*) - x^* + z + \beta \{(1-n)[z(q) - c(q)] + W^s(0)\}. \tag{27}$$

In equilibrium, agents must be indifferent between being a seller or a buyer. Consequently, from (26) and (27), n satisfies

$$(1-n)[z(q) - c(q)] = n[u(q) - z(q)] - iz(q). \tag{28}$$

Solving for n we get

$$n = \frac{(1+i)z(q) - c(q)}{u(q) - c(q)}. \tag{29}$$

From (26), q solves

$$\max_q \{-iz(q) + n[u(q) - z(q)]\}. \tag{30}$$

Terms of trade in bilateral matches, and therefore $z(q)$, are determined according to a proportional bargaining solution that assigns a fraction $\theta \in (0,1)$ of the surplus to buyers – See Eq. (24) – or according to the gradual Nash solution where θ is the buyer's bargaining power – See Eq. (25).

Definition 2 *A steady-state monetary equilibrium is a pair (q, n) such that q is solution to (30) and n satisfies (29).*

At the Friedman rule, $i = 0$ and $q = q^*$. The effects of a change in i on q and n in the neighborhood of $i = 0$ are given by

$$\begin{aligned} \frac{dq}{di} &= \frac{u'(q^*)}{n^*[u''(q^*) - z''(q^*)]} < 0, \\ \frac{dn}{di} &= \frac{z(q^*)}{u(q^*) - c(q^*)} > 0. \end{aligned}$$

Inflation has a direct effect by raising the cost of holding real balances. Therefore, buyers reduce their real balances and q falls. Also, since sellers do not incur the cost of holding real balances, n tends to increase.

Welfare is measured by the sum of the instantaneous utilities of buyers and sellers, i.e., $\mathcal{W} = n(1-n)[u(q) - c(q)]$. Welfare is maximized for $q = q^*$ and $n = 0.5$.

Proposition 3 *(i) Under proportional bargaining, a deviation from the Friedman rule is optimal whenever $\theta > 0.5$. (ii) Under gradual Nash bargaining, a deviation from the Friedman rule is optimal whenever $\theta > \bar{\theta}$ where $\bar{\theta} < 0.5$.*

If $\theta > 0.5$ then inflation is welfare-improving under both proportional and gradual Nash solutions. In this case, a social planner would be willing to trade off efficiency on the intensive margin to improve the extensive margin. How does this compare to the Nash solution? To make the comparison, we need to specify functional forms and parameterize them. If we let $u(q) = q^{0.7}/0.7$ and $c(q) = q$, we find that the threshold $\bar{\theta}$ for the gradual Nash solution above which inflation is welfare-improving is approximately 1/3 while under the Nash solution, a deviation is optimal for $\theta > 2/3$. Therefore, the Friedman rule is more likely to be optimal under the Nash solution than under the gradual Nash solution. This is so because under Nash bargaining the Friedman rule fails to achieve the efficient q which makes a deviation from the Friedman rule more costly.

6 Quantitative implications

The result derived in Proposition 2, namely, under monotonic bargaining solutions the Friedman rule generates the efficient q, has important implications for the welfare effects of inflation. In this section we quantify the welfare costs of inflation under all of the bargaining solutions we study.

Our quantitative experiments follow the methodology in LW, which is based on Lucas (2000). We calibrate the model in order to match money demand in the data. We set a period to a year and $\beta^{-1} = 1.03$. The functional forms are $U(x) = A \ln x$ so that $x^* = A$, $u(q) = q^{1-a}/(1-a)$ and $c(q) = q$. Finally, we set $\sigma = 0.5$. We choose (a, A) to match the money demand data as defined by $L = M/PY = L(i)$ where P is the nominal price level and Y real output. We measure i by the short term commercial paper rate, Y by GDP, P by the GDP deflator, and M by $M1$, as in Lucas. We consider the period 1900-2000.

In the model L is constructed as follows. Nominal output in the decentralized market is $0.5M$. Nominal output in the centralized market is px^*. Hence, $PY = 0.5M + px^*$. Using the fact that $x^* = A$, we have

$$L = \frac{M/p}{A + 0.5M/p}. \tag{31}$$

We measure the welfare cost of a π percent inflation by asking how much buyers' total consumption should be reduced to in order to have the same welfare at $i = (1+\pi)/\beta - 1$ and $i = 0$.

Expected utility for an agent given i is measured by $\mathcal{W}_i$. Suppose we reduce i to 0 but also reduce buyer's consumption of all goods by a factor Δ. Expected utility becomes

$$\mathcal{W}_0(\Delta) = \sigma[u(q_0\Delta) - c(q_0)] + U(x^*\Delta) - x^*,$$

where q_i is the equilibrium values for q given i. The welfare cost of inflation is the value of Δ that solves $\mathcal{W}_0(\Delta) = \mathcal{W}_i(1)$. In the following, we let $\bar{\Delta} = 100(1 - \Delta)$; i.e. $\bar{\Delta}$ is the percentage they would give up to have the Friedman Rule instead of i.

We first calibrate the model assuming Nash bargaining. We find $(a, A) = (0.297, 1.91)$. Keeping (a, A) unchanged we vary the bargaining solutions in order to evaluate the welfare cost associated with different values for i. The labels N, KS, K and GN indicate the Nash, Kalai-Smorodinsky, Kalai and gradual Nash solutions, respectively.

	i	0	0.03	0.05	0.08	0.13
q	N	0.78	0.50	0.39	0.27	0.16
	KS	0.78	0.50	0.39	0.27	0.16
	K	1	0.67	0.51	0.34	0.17
	GN	1	0.68	0.54	0.39	0.25
$\bar{\Delta}$	N	0	0.85	1.52	2.49	3.83
	KS	0	0.86	1.53	2.50	3.83
	K	0	0.4	0.96	2	3.79
	GN	0	0.36	0.82	1.60	2.84

Table 1. Welfare cost of inflation under alternative bargaining solutions.

The key insight of this quantitative exercise is that the welfare cost of small deviations from the Friedman rule is much bigger for the non-monotonic bargaining solutions than the monotonic ones. For instance, for $i = 0.03$ the welfare costs under Nash and Kalai-Smorodinsky are more than double those for the proportional and gradual Nash solutions. Interestingly, for large deviations, the welfare costs are very similar. For $i = 0.13$, the first three solutions are nearly identical while the gradual Nash is about one percentage point lower. Nevertheless, the magnitudes are very large compared to the estimates of Lucas (2000). This reflects the fact that a holdup problem exists for all symmetric bargaining solutions and the inefficiency associated with it is exacerbated as inflation increases.

In the following table, we recalibrate (a, A) for each bargaining solution and we compute the welfare cost associated with $i = 0.13$.

	a	A	$\Delta_{0.03}$	$\Delta_{0.05}$	$\Delta_{0.08}$	$\Delta_{0.13}$
N	0.297	1.91	0.85	1.52	2.49	3.83
KS	0.298	1.90	0.86	1.54	2.51	3.85
K	0.292	2.41	0.34	0.81	1.67	3.14
GN	0.271	2.21	0.34	0.77	1.48	2.56

Table 2. Welfare cost of inflation under alternative bargaining solutions.

The welfare cost of inflation is almost identical under Nash and Kalai-Smorodinsky bargaining solutions. The smallest welfare cost of inflation is obtained under gradual Nash bargaining. Nevertheless, the estimates are not that much different than in Table 1.

The lesson we take away from this quantitative exercise is the following. For small deviations from the Friedman rule, the bargaining solution that is used matters a lot for measuring the welfare cost of inflation. This is due to the monotonicity properties of the bargaining solution near the Friedman rule. However, for large deviations, all of our symmetric bargaining solutions give similar numbers which show that the welfare costs of inflation are quite large.

7 Conclusion

Bargaining has become an integral part of monetary search models. Yet very little work has been done to understand how various bargaining solutions affect the qualitative and quantitative predictions of the models. In this paper we examined a series of bargaining solutions to do just that. Our qualitative analysis provides insight as to how non-monotonic payoffs in some bargaining solutions affected the equilibrium of the model in important ways. By studying bargaining solutions other than the Nash solution, we were able to separate effects due to holdup problems from those occurring because of the non-monotonicity of payoffs. This had not been done before and, as a result, erroneous conclusions were being drawn as to why the Friedman rule could not replicate the first best allocation.

On the quantitative side, we showed that monotonic bargaining solutions are associated with lower welfare costs of inflation near the Friedman rule than non-monotonic bargaining solutions. However, the costs are very similar for inflation rates sufficiently far away from it. We also showed how alternative bargaining solutions affect the extensive margin of trading and that deviations from the Friedman rule may be optimal over a wide range of parameter values.

Appendix

A1. Proof of Lemma 1

For all the bargaining solutions we have considered (q,d) only depends on the real balances of the buyer in the match. So we can omit the dependence on sellers' real balances and (6) yields

$$\max_{\hat{z}} \left\{-i\hat{z} + \sigma \left\{u\left[q(\hat{z})\right] - d\left(\hat{z}\right)\right\}\right\}.$$

Furthermore, there exists a threshold value for z denoted z^* such that for all $z < z^*$, $q(z) < q^*$ and $d(z) = z$, and for all $z \geq z^*$, $q(z) = q^*$. Therefore, it is easy to check that $z \leq z^*$ for all $i > 0$. Finally, $q(z)$ and $d(z)$ are increasing in z and strictly increasing for all $z < z^*$. So, it is equivalent to express the agent's problem as a choice of q.

A2. Proof of Proposition 1

The function $-iz(q)+\sigma\left[u(q) - z(q)\right]$ is continuous and maximized over the compact set $[0, q^*]$, so a solution exists. At $i = 0$, the solution to (22) is strictly positive since $\max_{q\in[0,q^*]}\left\{\sigma\left[u(q) - z(q)\right]\right\}$ $\geq u(q^*) - z(q^*) > 0$. From the Theorem of the Maximum, $\max_{q\in[0,q^*]}\left\{-iz(q) + \sigma\left[u(q) - z(q)\right]\right\}$ varies continuously with i. Denote $\bar{\imath} = \sup_i \max_{q\in[0,q^*]}\left\{-iz(q) + \sigma\left[u(q) - z(q)\right]\right\} > 0$. Then $\bar{\imath} > 0$ and for all $i < \bar{\imath}$ there exists a $q > 0$ solution to (22).

A3. Proof of Lemma 2

Direct from (22).

A4. Proof of Proposition 2

From Lemma 2, it is sufficient to show that $u'(q^*) < z'(q^*)$ for the Nash and Kalai-Smorodinsky solutions. For the Nash solution, $z'(q^*) = \Theta'(q^*)[u(q^*) - c(q^*)] + u'(q^*) < u'(q^*)$. For the Kalai-Smorodinsky solution,

$$\frac{dq}{dz} = \frac{\hat{u}^s - u^b\left(1 - c'[u^{-1}(z)]/u'[u^{-1}(z)]\right) + \hat{u}^b}{u'(q)\hat{u}^s + c'(q)\hat{u}^b}, \qquad \text{if} \quad z \in \left[c(q^*), z^*\right].$$

Therefore, $z'(q^*) = u'(q^*)\left(\hat{u}^s + \hat{u}^b\right) / \left[\hat{u}^s - u^b\left(1 - c'[u^{-1}(z)]/u'[u^{-1}(z)]\right) + \hat{u}^b\right] < u'(q^*)$. Consider next the proportional solution. The first-order condition for q gives

$$\frac{i}{\sigma} = \frac{u'(q) - c'(q)}{c'(q) + u'(q)}. \tag{32}$$

From (32), $q = q^*$ iff $i = 0$. Under the gradual Nash solution, Equations (22) and (21) imply

$$\frac{i}{\sigma} = \frac{u'(q) - c'(q)}{2c'(q)}. \tag{33}$$

From (33), $q = q^*$ iff $i = 0$.

A5. Proof of Proposition 3

Since $q = q^*$ at $i = 0$, the welfare effect of a deviation from the Friedman rule is given by

$$\begin{aligned} \frac{d\mathcal{W}}{di} &= \frac{dn}{di}\left[u(q^*) - c(q^*)\right](1 - 2n^*) \\ &= z(q^*)(1 - 2n^*) \end{aligned}$$

Under proportional bargaining, $n^* = 1 - \theta$ and $d\mathcal{W}/di = z(q^*)(2\theta - 1)$. Under gradual Nash bargaining, $n^* = 1 - [u(q^*) - z(q^*)]/[u(q^*) - c(q^*)]$. Furthermore,

$$u(q^*) - z(q^*) = \int_0^{q^*} \Theta(q)[u'(q) - c'(q)]dq,$$

where $\Theta(q) = \theta u'(q)/[\theta u'(q) + (1 - \theta)c'(q)] > \theta$ for all $q < q^*$. Therefore, $u(q^*) - z(q^*) > \theta[u(q^*) - c(q^*)]$ which implies $n^* < 1 - \theta$. Furthermore n^* is decreasing in θ. Let $\bar{\theta}$ be the value of θ such that $n^* = 1/2$. Then, $\bar{\theta} < 0.5$.

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