Expectations and the Stability Problem for Optimal Monetary Policies*

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Abstract

A fundamentals based monetary policy rule, which would be the optimal monetary policy without commitment when private agents have perfectly rational expectations, is unstable if in fact these agents follow standard adaptive learning rules. This problem can be overcome if private expectations are observed and suitably incorporated into the policy maker's optimal rule. These strong results extend to the case in which there is simultaneous learning by the policy maker and the private agents. Our findings show the importance of conditioning policy appropriately, not just on fundamentals, but also directly on observed household and firm expectations.

1 Introduction

The formulation and performance of monetary policy rules has been analyzed extensively in the recent literature, for recent surveys see e.g. (Clarida, Gali,

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and Gertler 1999), (Woodford 1999) and (McCallum 1999). Much of the recent analysis has been conducted using the "New Phillips curve" model, also known as the optimizing IS-AS model, derived in a number of papers and reviewed e.g. in (Clarida, Gali, and Gertler 1999) and (Woodford 1999). This model gives a central role for private sector expectations of inflation and future output.

The corresponding monetary policy literature includes both the study of optimal policy, for specified objective functions, and analysis of the performance of simple, not necessarily optimal rules, such as Taylor's interest rate feedback rule (Taylor 1993). Examination of optimal policy in turn can be divided into the study of "time consistent" policy, under discretion, and policy in which the monetary authorities can commit to a rule that constrains future policy. As is well known, under rational expectations there can be gains, as measured by the policy maker's objective function, when a binding commitment to a rule is possible. However, it may not be considered credible for policy makers to commit to a rule from which there will be future incentives to deviate. Optimal rules without commitment have the property that the monetary authorities have no incentive to change policy even though they have the discretion to do so.

While almost all of the literature has assessed the design of monetary policy rules under rational expectations (RE), this may not be an innocuous assumption. It has been shown that some policy rules can yield indeterminacy of equilibria, i.e. multiple RE solutions, see e.g. the discussion in (Bernanke and Woodford 1997) and (Woodford 1999). Recently, (Bullard and Mitra 2000) have shown that if agents are instead assumed to follow adaptive learning rules, then the stability of the Taylor-type rules could not be taken for granted. An earlier paper by (Howitt 1992) showed the instability under learning of interest rate pegging and related rules in both flexible price and IS-LM type models. In his conclusions Howitt explicitly warned that any analysis of monetary policy under RE should be supplemented with an investigation of its stability under learning.

We take up this issue in the context of optimal policy rules without commitment, and obtain some strong results.[1] We begin by showing that, if the policy makers follow the optimal interest rate rule formulated in terms

[1] Policy with a limited form of commitment possible is discussed in Section 4.3.2. We leave for future research a general study of learning stability for optimal monetary policy rules when commitment is possible.

of observed fundamental shocks and private agents follow natural learning rules to form expectations of inflation and future output, then the economy is invariably unstable. Small expectational errors by private agents become magnified by the policy which assumes RE and the cumulative process drives the economy away from the rational expectations equilibrium (REE). This result can be viewed as an extension of Howitt's results to the New Phillips curve framework.

A central objective of the paper is to show how this problem can be overcome. In particular, we seek a monetary policy which both is stable under learning and implements optimal policy. The key to achieving a stable optimal monetary policy is to formulate a rule that explicitly takes account of private sector expectations and the economic structure. We obtain a striking result. We show that under a suitably designed rule of this type, the economy will invariably be stable under least squares learning and will converge over time to the REE corresponding to the optimal policy without commitment. Furthermore, we show that such rules continue to be stable under learning even if the policy maker also has to learn the true structural parameters required to conduct optimal monetary policy.

In practice, monetary authorities appear to base their policy on a variety of indicators that include consumer and business sentiments. From the strict RE viewpoint such practices might seem puzzling since with a unique REE these expectations are functions of the observable fundamentals. Possible ways to explain the practices of the policy maker include multiplicities of REE and informational asymmetries. This paper provides a different explanation, which does not hinge on these factors. The possibility of small deviations from perfect RE is enough to justify careful attention to the expectations of private agents.

2 The Key Results

2.1 The Basic Model

We utilize the standard log-linear framework as developed in Section 2 of (Clarida, Gali, and Gertler 1999). The structural model consists of two equations:

$$x_t = -\varphi(i_t - \hat{E}_t \pi_{t+1}) + \hat{E}_t x_{t+1} + g_t \tag{1}$$

$$\pi_t = \lambda x_t + \beta \hat{E}_t \pi_{t+1} + u_t, \tag{2}$$

where x_t is the "output gap" i.e. the difference between actual and potential output, π_t is the inflation rate, i.e. the proportional rate of change in the price level from $t-1$ to t and i_t is the nominal interest rate. $\hat{E}_t x_{t+1}$ and $\hat{E}_t \pi_{t+1}$ denote private sector expectations of inflation and output gap next period. We will use the same notation without the "^" to denote RE. All the parameters in (1) and (2) are positive. $0 < \beta < 1$ is the discount rate of the representative firm and is therefore close to one.

(1) is a dynamic "IS" curve which can be derived from the Euler equation associated with the household's savings decision. (2) is a "new Phillips" curve which can be derived from optimal pricing decisions of monopolistically competitive firms facing constraints on the frequency of future price changes. The essence of the new Phillips curve is the forward-looking character of the inflation expectations. Finally, g_t and u_t denote observable shocks following first order autoregressive processes

$$g_t = \mu g_{t-1} + \hat{g}_t \tag{3}$$

$$u_t = \rho u_{t-1} + \hat{u}_t, \tag{4}$$

where $0 < |\mu| < 1, 0 < |\rho| < 1$ and $\hat{g}_t \sim iid(0, \sigma_g^2), \hat{u}_t \sim iid(0, \sigma_u^2)$. g_t represents shocks to government purchases as well as shocks to potential GDP. u_t represents any cost push shocks to marginal costs other than those entering through x_t.

It should be emphasized that the model (1) and (2) is derived by suitably linearizing a nonlinear framework around a nonstochastic steady state. Any analysis based on linearization must be interpreted as being local in a neighborhood of the steady state. In particular, the random shocks must be sufficiently small for the linearization to make sense. We also follow the standard practice of leaving hidden the government budget constraint and the equation for the evolution of government debt.[2]

The policy maker has the following standard objective function defined in terms of the target variables x_t and π_t:

$$\min \frac{1}{2} E_t \left\{ \sum_{i=0}^{\infty} \beta^i \left[\alpha (x_{t+i} - \bar{x})^2 + (\pi_{t+i} - \bar{\pi})^2 \right] \right\}, \tag{5}$$

[2] These would need to be made explicit if we wished to examine the implications of following explosive price paths. See (Sims 1994) and (Woodford 2000).

where $\bar{x}$ allows for a possible deviation of socially optimal output from potential output and $\bar{\pi}$ is the target value for the inflation rate. (If desired, these can be assumed to be zero.) α is the relative weight for output deviations, and β is the discount rate. Thus the policy maker discounts the future at the same rate as the private sector. We note that $\alpha = 0$ would correspond to pure inflation targeting.

Optimal policy without commitment reduces to a sequence of static problems in which the nominal interest rate is chosen to deliver the values x_t and π_t which minimize $(1/2)\left[\alpha(x_t - \bar{x})^2 + (\pi_t - \bar{\pi})^2\right] + \mathcal{F}_t$ subject to $\pi_t = \lambda x_t + F_t$, where $\mathcal{F}_t, F_t$ denote remainder terms which are treated as given under discretionary policy. This leads to the first order condition

$$\lambda(\pi_t - \bar{\pi}) + \alpha(x_t - \bar{x}) = 0, \tag{6}$$

so that i_t is set to satisfy (1), (2) and (6). An explicit form will be given below.

To obtain the optimal interest rate rule under RE we show that there is a solution of the form

$$\pi_t = a_1 + d_1 u_t$$

$$x_t = a_2 + d_2 u_t.$$

Under RE we have $E_t \pi_{t+1} = a_1 + d_1 \rho u_t$, $E_t x_{t+1} = a_2 + d_2 \rho u_t$. Inserting into (1), (2) and (6) yields

$$\bar{a}_1 = \frac{\lambda(\lambda\bar{\pi} + \alpha\bar{x})}{\alpha(1-\beta) + \lambda^2}, \bar{d}_1 = \frac{\alpha}{\alpha(1-\beta\rho) + \lambda^2} \tag{7}$$

$$\bar{a}_2 = -\frac{\lambda\beta a_1 - \lambda\bar{\pi} - \alpha\bar{x}}{(\lambda^2 + \alpha)}, \bar{d}_2 = -\frac{\lambda}{\lambda^2 + \alpha(1-\beta\rho)} \tag{8}$$

$$i_t = \psi_0 + \psi_u u_t + \psi_g g_t, \text{ where } \psi_0 = \bar{a}_1, \psi_u = \frac{(1-\rho)\lambda + \alpha\rho\varphi}{\varphi[\alpha(1-\beta\rho) + \lambda^2]}, \psi_g = \varphi^{-1}. \tag{9}$$

Under this optimal policy rule the interest rate is adjusted to neutralize any shock to the IS curve. Price shocks present the policy makers with a trade-off

since they affect both inflation and output in the same direction. It can be shown that the policy makers faces a trade-off between the variabilities of output and inflation and their optimal choice depends on α as is reflected in the coefficient ψ_u. We will call (9) the *fundamentals form of the RE-optimal policy rule.*

Clearly, the optimal monetary policy without commitment can be characterized in other ways besides (9). In particular, it is immediate from the above that the optimal interest rate can be written as

$$i_t = -\frac{(1-\rho)\lambda}{\rho\alpha\varphi}\bar{a}_1 + (1 + \frac{(1-\rho)\lambda}{\rho\alpha\varphi})E_t\pi_{t+1} + \varphi^{-1}g_t, \tag{10}$$

as pointed out in (Clarida, Gali, and Gertler 1999). Indeed, under RE there are many equivalent ways of writing the optimal policy as emphasized by (Woodford 1999). However, if one departs from RE these rules are distinct and their properties can differ. We begin by examining the fundamentals form of the RE-optimal rule (9) and later consider the alternative form (10).

2.2 Instability of Fundamentals-Based Policy Rules

We now relax the assumption that private agents have rational expectations and instead endow them with a standard adaptive learning rule such as least squares. These learning rules have been widely studied and shown to converge to the usually employed REE in many standard models. This is true of the stationary solutions of, for example, the Cagan model of inflation, the Sargent-Wallace IS-LM-PC model, the Samuelson overlapping generations model and the real business cycle model. Recent overviews of the literature are provided e.g. in (Evans and Honkapohja 1999) and (Evans and Honkapohja 2001). In stochastic frameworks least squares and closely related learning rules are the most widely employed formulation, though other learning algorithms have occasionally been considered. In this paper we follow the literature and focus on least squares learning, but we do briefly demonstrate the robustness of our results to some alternative learning schemes.

A reasonable requirement for a good policy rule is that if private agents make small expectational errors the economy will converge to the posited REE as agents correct these errors over time through a learning rule. As we shall see, this issue turns out to be a major concern in the implementation of optimal monetary policy.

Throughout this section we retain the assumption that the policy maker knows the true structure (1), (2) and the parameter values of the economy. The policy maker is assumed to follow the fundamentals form of the RE-optimal rule (9), mistakenly assuming that agents have fully rational expectations. Combining (1), (2) and (9) we obtain the reduced form

$$\begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \begin{pmatrix} \beta + \lambda\varphi & \lambda \\ \varphi & 1 \end{pmatrix} \begin{pmatrix} \hat{E}_t \pi_{t+1} \\ \hat{E}_t x_{t+1} \end{pmatrix} + \begin{pmatrix} 0 \\ -\varphi \bar{a}_1 \end{pmatrix} + \begin{pmatrix} 1 - \lambda\varphi\psi_u \\ -\varphi\psi_u \end{pmatrix} u_t. \tag{11}$$

The private agents have forecast functions

$$\hat{E}_t \pi_{t+1} = a_{1,t} + k_{1,t} u_t \tag{12}$$

$$\hat{E}_t x_{t+1} = a_{2,t} + k_{2,t} u_t, \tag{13}$$

where the parameters $a_{1,t}, k_{1,t}, a_{2,t}$ and $k_{2,t}$ are updated over time according to least squares. Under rational expectations $a_{1,t} = \bar{a}_1, k_{1,t} = \rho\bar{d}_1, a_{2,t} = \bar{a}_2, k_{s,t} = \rho\bar{d}_2$. The question we investigate is whether, under a least squares learning rule, the estimated parameters converge to these values over time when the policy maker conducts monetary policy using the rule (9).[3]

The preceding formulation of learning is standard in the literature, but its motivation here requires some comment. First, since the private sector in the economy is modeled as monopolistically competitive, it is populated by a large number of "small" agents. It is therefore natural to assume, as we do, that strategic behavior in expectations formation and learning is absent and agents simply try to learn the equilibrium processes for the endogenous variables they need to forecast. We also assume that the policy maker does not make active use of the learning behavior on the part of agents. Strategic manipulation by the policy maker of agents' learning rules merits investigation in future research. Finally, since the forecasts based on learning will be directly incorporated in the model, it is implicitly assumed that the Euler equations are used as behavioral rules describing how private agents respond to their forecasts. This approach has the advantage that the system

[3] (12)-(13) omit dependence on g_t because it does not affect the REE. Agents might also include g_t in the estimated forecast functions. This would not affect any of our instability or stability results. In particular, in Propositions 3 and 5, part (ii), the coefficients on g_t would converge to 0.

can be formulated entirely in terms of the linearized reduced form model and the standard updating equations from the learning literature.[4]

It is known for learning problems of this type that under fairly general assumptions convergence to REE obtains if and only if certain stability conditions, known as E-stability conditions, are satisfied. In this section we obtain the E-stability results and defer the explicit treatment of least squares learning to the next section.

The E-stability conditions are developed as follows. For given values of the parameters of the forecast functions, called the perceived law of motion, one computes the resulting actual law of motion implied by the structure. E-stability is then determined by a differential equation in which the parameters adjust in the direction of the actual law of motion parameter values. Formally, we proceed as follows.

Inserting the perceived law of motion

$$\hat{E}_t \pi_{t+1} = a_1 + k_1 u_t \tag{14}$$

$$\hat{E}_t x_{t+1} = a_2 + k_2 u_t, \tag{15}$$

into (1) and (2) and using the rule (9) yields

$$\begin{aligned} x_t &= -\varphi(\bar{a}_1 + \psi u_t + \varphi^{-1} g_t - a_1 - k_1 u_t) + a_2 + k_2 u_t + g_t \\ &= -\varphi(\bar{a}_1 - a_1) + a_2 + (k_2 - \varphi(\psi - k_1)) u_t \end{aligned}$$

$$\begin{aligned} \pi_t &= \lambda(-\varphi(\bar{a}_1 - a_1) + a_2) + \lambda(k_2 - \varphi(\psi - k_1)) u_t + \beta(a_1 + k_1 u_t) + u_t \\ &= \beta a_1 + \lambda a_2 - \lambda\varphi(\bar{a}_1 - a_1) + (1 + \beta k_1 + \lambda(k_2 - \varphi(\psi - k_1))) u_t. \end{aligned}$$

Using (4) we arrive at the actual law of motion, i.e. the implied forecast functions $\hat{E}_t^* \pi_{t+1} = a_1^* + k_1^* u_t$ and $\hat{E}_t^* x_{t+1} = a_2^* + k_2^* u_t$, where

$$\begin{aligned} a_1^* &= \beta a_1 + \lambda a_2 - \lambda\varphi(\bar{a}_1 - a_1) \\ k_1^* &= (1 + \beta k_1 + \lambda(k_2 - \varphi(\psi - k_1)))\rho \\ a_2^* &= -\varphi(\bar{a}_1 - a_1) + a_2 \\ k_2^* &= (k_2 - \varphi(\psi - k_1))\rho. \end{aligned}$$

[4] In particular, agents just respond to expectations about the next period, and expectations further into the future are assumed not to affect current decisions. This is another issue awaiting further study.

These equations can be collected into the mapping

$$(a_1^*, k_1^*, a_2^*, k_2^*) = T_{RE}(a_1, k_1, a_2, k_2).$$

The T_{RE} mapping gives rise to the differential equation defining E-stability, namely

$$\frac{d}{d\tau}(a_1, k_1, a_2, k_2) = T_{RE}(a_1, k_1, a_2, k_2) - (a_1, k_1, a_2, k_2). \tag{16}$$

The REE parameter values are a fixed point of T_{RE} and hence an equilibrium of the differential equation. The equilibrium is said to be E-stable if it is a locally asymptotically stable point of the differential equation. Inspecting the T_{RE} mapping it is seen that the differential equation breaks up into two independent subsystems for (a_1, a_2) and (k_1, k_2), respectively. The subsystems are both linear and have slope coefficient matrices

$$\begin{aligned} DT_{RE,a} - I &= \begin{pmatrix} \beta + \lambda\varphi - 1 & \lambda \\ \varphi & 0 \end{pmatrix} \text{ and} \\ DT_{RE,k} - I &= \begin{pmatrix} (\beta + \lambda\varphi)\rho - 1 & \lambda\rho \\ \rho\varphi & \rho - 1 \end{pmatrix}. \end{aligned}$$

It can easily be verified that the subsystem for (a_1, a_2) is always unstable (the other one may or may not be stable). In the appendix we show that this implies instability of the economy under least squares learning:

Proposition 1 *The REE of the economy $\pi_t = \bar{a}_1 + \bar{d}_1 u_t$, $x_t = \bar{a}_2 + \bar{d}_2 u_t$ under the fundamentals form of the RE-optimal monetary policy rule and least squares learning by private agents is unstable for all parameter values, i.e. the economy converges to the REE with probability zero.*

We have stated this result in terms of least squares learning, but the instability also holds under alternative learning rules, as we will show below in Section 4.1. The intuition for the result is straightforward. Assuming $\lambda\varphi + \beta > 1$, a deviation of $\hat{E}_t \pi_{t+1}$ above its RE value leads, through the IS curve, to an increase in x_t and through the Phillips curve to a higher π_t. Over time this leads to upward revisions of both $\hat{E}_t \pi_{t+1}$ and $\hat{E}_t x_{t+1}$. Nothing in the interest rate rule offsets this tendency, so that the economy would move cumulatively away from the REE.[5] We remark that the estimated coefficients

[5] Using (11), a more refined intuition can be developed for the case $\lambda\varphi + \beta < 1$.

(with probability one) do not converge to *any* constant parameter vector, at least within the region for which the linearization is valid.

In fact, Proposition 1 is a special case of the next proposition which states that all interest rate rules which depend linearly only on the fundamentals are unstable:

Proposition 2 *Consider a policy rule of the form* $i_t = \eta_0 + \eta_u u_t + \eta_g g_t$ *and an associated REE of the form* $\pi_t = a_1 + d_1 u_t + n_1 g_t$, $x_t = a_2 + d_2 u_t + n_2 g_t$. *For all values of the structural parameters, the REE is unstable under least squares learning by private agents.*

The proof of this result follows from the observation that the coefficient matrix of the expectations is the same as in the reduced form (11) for optimal policy (the coefficients of the policy rule only affect the constant and disturbance terms). E-stability depends only on this matrix. A predecessor of this result is (Howitt 1992) who showed instability under learning of interest rate pegging in several monetary models.

The instability under learning is a separate difficulty from the indeterminacy of the REE, noted by (Woodford 1999), Section 5.1, under policy rules that depend solely on the fundamental shocks. The indeterminacy means that there also exist other types of REE which can, for example, depend on sunspots i.e. extraneous variables affecting the REE only via expectations.[6]

This strong instability result raises the question of whether there are alternative monetary policy rules that are robust to least squares learning and that implement the optimal policy without commitment.

2.3 An Expectations Based Optimal Rule

The instability problem of the preceding section resulted from the implicit assumption by the policy maker that private agents have perfectly rational expectations at every point in time. If there is a chance that they are not fully rational it would seem natural instead to base policy in part directly on expectations of private agents. In this section we assume that these expectations are observable and consider a policy rule that depends partly on

[6] This is easily seen from the coefficient matrix of the expectations in (11). This matrix has one root inside and the other root outside the unit circle and there are no predetermined variables in the model.

them.[7]

The relevant policy rule is obtained by solving for i_t from structural equations (1), (2) and the optimality condition (6). We obtain

$$i_t = \delta_0 + \delta_\pi \hat{E}_t \pi_{t+1} + \delta_x \hat{E}_t x_{t+1} + \delta_g g_t + \delta_u u_t, \tag{17}$$

where the coefficients are

$$\begin{aligned}
\delta_0 &= -(\lambda^2+\alpha)^{-1}\varphi^{-1}(\lambda\bar{\pi}+\alpha\bar{x}), \\
\delta_\pi &= 1+(\lambda^2+\alpha)^{-1}\varphi^{-1}\lambda\beta \\
\delta_x &= \varphi^{-1} \\
\delta_g &= \varphi^{-1} \\
\delta_u &= (\lambda^2+\alpha)^{-1}\varphi^{-1}\lambda.
\end{aligned}$$

Since this rule satisfies the static optimality condition (6), the policy is in fact optimal taking the expectations of private agents as given.[8] We will refer to (17) as the *expectations based optimal rule.*

The reduced form is

$$\begin{aligned}
\begin{pmatrix} \pi_t \\ x_t \end{pmatrix} &= \begin{pmatrix} \beta\alpha(\lambda^2+\alpha)^{-1} & 0 \\ -\beta\lambda(\lambda^2+\alpha)^{-1} & 0 \end{pmatrix} \begin{pmatrix} \hat{E}_t\pi_{t+1} \\ \hat{E}_t x_{t+1} \end{pmatrix} \\
&\quad + \begin{pmatrix} -\varphi\delta_0\lambda \\ -\varphi\delta_0 \end{pmatrix} + \begin{pmatrix} 1-\varphi\delta_u\lambda \\ -\varphi\delta_u \end{pmatrix} u_t.
\end{aligned} \tag{18}$$

We now turn to the question whether the economy will converge to the REE when private agents follow least squares learning as above. The perceived law of motion of the private agents is as before, and is thus given by equations (14) and (15). Using (1), (2) and (17) with the specified coefficients we get

$$x_t = (\lambda^2+\alpha)^{-1}(\lambda\bar{\pi}+\alpha\bar{x}-\lambda\beta a_1) - \lambda(\lambda^2+\alpha)^{-1}(\beta k_1+1)u_t \tag{19}$$

$$\begin{aligned}
\pi_t &= \beta a_1 + \lambda^2(\lambda^2+\alpha)^{-1}(\lambda\bar{\pi}+\alpha\bar{x}-\beta a_1) \\
&\quad +(\beta k_1+1)(1-\lambda^2(\lambda^2+\alpha)^{-1})u_t.
\end{aligned} \tag{20}$$

[7] Alternatively, if the policy maker knows the learning rules of private agents, it can infer the expectations from other observed data.

[8] Recall that we are ignoring the possibility that the government might attempt to exploit the learning rule of the private agents.

Thus the mapping from the perceived to the actual law of motion takes the form

$$\begin{aligned}
a_1^* &= \frac{\lambda}{\lambda^2+\alpha}(\lambda\bar{\pi}+\alpha\bar{x})+\frac{\alpha}{\lambda^2+\alpha}\beta a_1 \\
k_1^* &= \rho\frac{\alpha}{\lambda^2+\alpha}(1+\beta k_1) \\
a_2^* &= \frac{1}{\lambda^2+\alpha}(\lambda\bar{\pi}+\alpha\bar{x}-\lambda\beta a_1) \\
k_2^* &= -\rho\frac{\lambda}{\lambda^2+\alpha}(1+\beta k_1).
\end{aligned}$$

Writing

$$(a_1^*, k_1^*, a_2^*, k_2^*) = T_{EB}(a_1, k_1, a_2, k_2),$$

E-stability is defined, as before, by the differential equation

$$\frac{d}{d\tau}(a_1, k_1, a_2, k_2) = T_{EB}(a_1, k_1, a_2, k_2) - (a_1, k_1, a_2, k_2). \tag{21}$$

Again there are two independent linear subsystems for (a_1, a_2) and (k_1, k_2) with slope coefficient matrices

$$\begin{aligned}
DT_{EB,a} - I &= \begin{pmatrix} \beta(\alpha/(\lambda^2+\alpha))-1 & 0 \\ -\beta\lambda/(\lambda^2+\alpha) & -1 \end{pmatrix} \text{ and} \\
DT_{EB,k} - I &= \begin{pmatrix} \beta(\alpha/(\lambda^2+\alpha))\rho-1 & 0 \\ -\beta\lambda\rho/(\lambda^2+\alpha) & -1 \end{pmatrix}.
\end{aligned}$$

It can be readily verified that both of these subsystems are globally stable. In the appendix we show that this implies convergence under least squares learning:

Proposition 3 *The REE of the economy under the expectations based optimal rule and least squares learning by private agents is stable for all parameter values, i.e. the economy converges almost surely to the REE that corresponds to the optimal monetary policy without commitment.*[9]

[9] In fact, formally the convergence of learning is even global, i.e. it happens for all initial parameter estimates. Since the economic model is based on a local linearization we do not emphasize global stability in the proposition.

A partial intuition for this result is that, for example, an increase in $\hat{E}_t\pi_{t+1}$ leads to an increase in i_t which more than offsets the direct effect of $\hat{E}_t\pi_{t+1}$ on x_t since $\delta_\pi > 1$. In economic terms the real interest is increased which reduces the output gap. Likewise, the expectations based rule tightens policy in the face of higher $\hat{E}_t x_{t+1}$. The rule (17) with the specified parameter values succeeds in guiding the expectations of private agents and the economy to the optimal REE.

It is also be easily verified that the REE is determinate under this policy rule:

Proposition 4 *The REE of the economy under the expectations based optimal rule is determinate.*

This result follows from noting that both roots of the coefficient matrix of the expectations term in (18) are inside the unit circle. This proposition establishes that under the policy rule (17) there is a unique REE. This result contrasts sharply with the findings of (Bernanke and Woodford 1997) who also considered policies that depend on private expectations. The key to our results is that the policy rule (17) uses the economic structure and all available information.[10]

Of course, formulating the optimal expectations based rule requires knowledge of the structural parameters. There are two points to be made. First, if the policy makers follow the rule (17) with parameters deviating from the specified values by small amounts, the economy will converge over time to an REE which deviates from the optimum by small amounts. Second, the policy maker could try to learn the true values of the structural parameters. This raises the issue of whether simultaneous learning by private agents and the policy maker can still lead the economy to converge to the optimal REE. We take this up in the next section. Before doing so, we consider one other formulation of optimal monetary policy.

2.4 An Alternative Expectations Based Rule

Returning optimal policy under RE discussed in Section 2.1 it was pointed out that optimal monetary policy (without commitment) can also be written in the form (10). This form which is often emphasized by (Clarida, Gali, and

[10] We assume as a simplification that private agents do not act strategically in forecasting, even though their forecasts enter the rule (17) of the policy maker.

Gertler 1999), suggests an alternative expectations based policy rule in which rational expectations $E_t\pi_{t+1}$ is replaced by observed private expectations $\hat{E}_t\pi_{t+1}$, i.e.

$$i_t = -\frac{(1-\rho)\lambda}{\rho\alpha\varphi}\bar{a}_1 + (1 + \frac{(1-\rho)\lambda}{\rho\alpha\varphi})\hat{E}_t\pi_{t+1} + \varphi^{-1}g_t. \tag{22}$$

We note that the derivation of this rule makes use of some rational expectations assumptions which eliminate the expectations of the output gap. It is of interest to know the properties of this rule.

The reduced form is now

$$\begin{pmatrix} \pi_t \\ x_t \end{pmatrix} = \begin{pmatrix} \beta - (1-\rho)\lambda^2/\rho\alpha & \lambda \\ -(1-\rho)\lambda/\rho\alpha & 1 \end{pmatrix} \begin{pmatrix} \hat{E}_t\pi_{t+1} \\ \hat{E}_t x_{t+1} \end{pmatrix} + \begin{pmatrix} -\bar{a}_1(1-\rho)\lambda^2/\rho\alpha \\ -\bar{a}_1(1-\rho)\lambda/\rho\alpha \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_t.$$

It can be verified that the REE is E-stable, so that optimal monetary policy would also be obtained asymptotically under this expectations based rule. This stability result shows the importance of strong positive feedback from private inflation expectations to interest rates.

Despite the stability result this alternative rule is less appealing than our proposed rule (17). First, the optimal REE under the alternative rule is not necessarily determinate. Indeterminacy invariably arises for values of ρ close to zero. This reflects the very large reaction of the interest rate to inflation expectations, see (10) or (22). Second, the large coefficient for small $|\rho|$ is troubling for another reason: if ρ is not known exactly and replaced by an estimate, then small estimation errors cause large deviations from optimal REE. Finally, the policy rule (22) lacks the "static" optimality property of (17). The latter is explicitly obtained as the rule which minimizes the policy maker's period objective function for all given private expectations $\hat{E}_t\pi_{t+1}$ and $\hat{E}_t x_{t+1}$. For these reasons we focus on (17) in the rest of the paper.

3 Learning by the Policy Maker

We now consider an extension of the analysis in which the key structural parameter values are unknown. The policy maker is assumed to know the

structure of the economy but must estimate the parameter values using observed data.[11] In order to make the estimation problem more realistic we introduce unobserved shocks to the model (1), (2).

The IS and Phillips curves thus take the form

$$x_t = -\varphi(i_t - \hat{E}_t \pi_{t+1}) + \hat{E}_t x_{t+1} + g_t + e_{x,t} \tag{23}$$

$$\pi_t = \lambda x_t + \beta \hat{E}_t \pi_{t+1} + u_t + e_{\pi,t}, \tag{24}$$

where now $x_t, \pi_t, e_{x,t}$ and $e_{\pi,t}$ are not observable at time t. g_t, u_t are observable at t and x_t, π_t are observed with a lag. $e_{x,t}, e_{\pi,t}$ are independent white noise and $g_t, u_t, e_{x,t}, e_{\pi,t}$ are exogenous and mutually independent. The private expectations are assumed to be observable. We also assume that the discount factor β, a parameter shared by private agents and the policy maker, is known. $\alpha, \bar{\pi}, \bar{x}$ in the objective function are, of course, taken to be known. However, the key structural parameters φ and λ must be estimated.

To complete the description of the extended model we must specify the behavior of the policy maker. The objective function (5) remains as before, but with our current informational assumptions the first order condition becomes

$$\lambda(E_t \pi_t - \bar{\pi}) + \alpha(E_t x_t - \bar{x}) = 0. \tag{25}$$

It can be verified that with the unobserved shocks the RE solution takes the form

$$\begin{aligned} \pi_t &= \bar{a}_1 + \bar{d}_1 u_t + \lambda e_{x,t} + e_{\pi,t} \\ x_t &= \bar{a}_2 + \bar{d}_2 u_t + e_{x,t}, \end{aligned}$$

where the parameter values are given in (7) and (8). The fundamentals form of the RE-optimal rule remains unchanged, i.e. it takes the form (9) and the expectations based rule continues to be of the form (17).

We now turn to the formulation of learning. The private agents forecast functions have the same form as before (12) and (13), where the parameters are estimated by least squares. At any given time, the agents run regressions using the available data and therefore they update their parameter estimates

[11] Thus in contrast to (Sargent 1999) our policy maker estimates a correctly specified structural model.

each period as new data become available. In line with the literature on least squares learning we give the explicit equations in recursive form.[12]

Let

$$\xi_{1,t} = \begin{pmatrix} a_{1,t} \\ k_{1,t} \end{pmatrix}, \xi_{2,t} = \begin{pmatrix} a_{2,t} \\ k_{2,t} \end{pmatrix}, U_t = \begin{pmatrix} 1 \\ u_t \end{pmatrix},$$

so that the forecasts can be written as $\hat{E}_t \pi_{t+1} = \xi'_{1,t} U_t$ and $\hat{E}_t x_{t+1} = \xi'_{2,t} U_t$. Under recursive least squares (RLS) learning the parameter vectors are updated according to the formulae

$$\begin{aligned} \xi_{1,t} &= \xi_{1,t-1} + t^{-1} R_{U,t-1}^{-1} U_{t-2} (\pi_{t-1} - \xi'_{1,t-1} U_{t-2}) \\ \xi_{2,t} &= \xi_{2,t-1} + t^{-1} R_{U,t-1}^{-1} U_{t-2} (x_{t-1} - \xi'_{2,t-1} U_{t-2}) \\ R_{U,t} &= R_{U,t-1} + t^{-1} (U_{t-1} U'_{t-1} - R_{U,t-1}). \end{aligned}$$

This recursion requires starting values for the parameters at time $t = 0$ and it is well-known that for appropriate starting values RLS is equivalent to the standard least squares formula. In the formulation above the parameter estimates at time t use data through period $t - 1$.[13] The time t forecasts do, however, use the current observation of the exogenous variable u_t. Note that the procedure used by private agents estimates the parameters of the reduced form, which is all that is required for forecasting purposes.

The policy maker uses RLS to estimate the structural parameters φ, λ and ρ, which are required for setting optimal monetary policy. Let

$$\begin{aligned} y_{1,t} &= x_t - \hat{E}_t x_{t+1} - g_t \\ y_{2,t} &= \pi_t - \beta \hat{E}_t \pi_{t+1} - u_t \\ r_t &= i_t - \hat{E}_t \pi_{t+1}. \end{aligned}$$

Because r_t depends directly only on the variables g_t and u_t, and x_t depends only on g_t, u_t and $e_{x,t}$, consistent estimates of φ and λ can be obtained by

[12] The general approach used here was introduced by (Marcet and Sargent 1989). See (Evans and Honkapohja 1999) for a recent survey and (Evans and Honkapohja 2001) for a detailed treatment.

[13] This is common practice in the literature. Including current data would create a simultaneity complication which is convenient to avoid and would not alter the central results.

regressions of $y_{1,t}$ on r_t and $y_{2,t}$ on x_t, respectively.[14] Thus the policy maker uses least squares to estimate

$$\begin{aligned} y_{1,t} &= -\varphi(i_t - \hat{E}_t\pi_{t+1}) + e_{x,t} \text{ and} \\ y_{2,t} &= \lambda x_t + e_{\pi,t}. \end{aligned}$$

If required, a consistent estimate of ρ can of course be obtained by a regression of u_t on u_{t-1}.[15]

In recursive form this becomes[16]

$$\begin{aligned} \hat{\varphi}_t &= \hat{\varphi}_{t-1} + t^{-1}R_{r,t-1}^{-1}r_{t-1}(y_{1,t-1} + \hat{\varphi}_{t-1}r_{t-1}) \\ R_{r,t} &= R_{r,t-1} + t^{-1}(r_{t-1}^2 - R_{r,t-1}) \\ \hat{\lambda}_t &= \hat{\lambda}_{t-1} + t^{-1}R_{x,t-1}^{-1}x_{t-1}(y_{2,t-1} - \hat{\lambda}_{t-1}x_{t-1}) \\ R_{x,t} &= R_{x,t-1} + t^{-1}(x_{t-1}^2 - R_{x,t-1}) \\ \hat{\rho}_t &= \hat{\rho}_{t-1} + t^{-1}R_{u,t-1}^{-1}u_{t-1}(u_t - \hat{\rho}_{t-1}u_{t-1}) \\ R_{u,t} &= R_{u,t-1} + t^{-1}(u_t^2 - R_{u,t-1}). \end{aligned}$$

It remains to specify the monetary policy rules. The fundamentals form of the RE-optimal rule based on estimated parameters takes the form

$$\begin{aligned} i_t &= \hat{a}_{1,t} + \hat{\psi}_t u_t + \hat{\varphi}_t^{-1} g_t, \qquad (26) \\ \text{where } \hat{a}_{1,t} &= \frac{\hat{\lambda}_t(\hat{\lambda}_t\bar{\pi} + \alpha\bar{x})}{\alpha(1-\beta) + \hat{\lambda}_t^2} \text{ and } \hat{\psi}_t = \frac{(1-\hat{\rho}_t)\hat{\lambda}_t + \alpha\hat{\rho}_t\hat{\varphi}_t}{\hat{\varphi}_t[\alpha(1-\hat{\rho}_t\beta) + \hat{\lambda}_t^2]}. \end{aligned}$$

The expectations based optimal policy rule with estimated structural parameters is

$$i_t = \hat{\delta}_{0,t} + \hat{\delta}_{\pi,t}\hat{E}_t\pi_{t+1} + \hat{\delta}_{x,t}\hat{E}_t x_{t+1} + \hat{\delta}_{g,t}g_t + \hat{\delta}_{u,t}u_t, \qquad (27)$$

[14] If $e_{x,t}$ and $e_{\pi,t}$ were correlated then an instrumental variable estimator would be needed to obtain a consistent estimator of λ.

[15] An estimate of ρ is needed for the RE-optimal policy but not for the expectations based optimal policy.

[16] The formulations for $\hat{\varphi}_t$ and $\hat{\lambda}_t$ vary slightly from least squares since we have assumed that we have introduced an additional lag in the equations for $R_{r,t}$ and $R_{x,t}$. This is convenient for expressing the system in standard form as a stochastic recursive algorithm. Note that the regression of $y_{1,t}$ on r_t gives an estimate of $-\varphi$. This explains the plus in front of $\hat{\varphi}_{t-1}$ in the first recursion below.

where the coefficients are

$$\begin{aligned}
\hat{\delta}_{0,t} &= -(\hat{\lambda}_t^2 + \alpha)^{-1}\hat{\varphi}_t^{-1}(\hat{\lambda}_t\bar{\pi} + \alpha\bar{x}), \\
\hat{\delta}_{\pi,t} &= 1 + (\hat{\lambda}_t^2 + \alpha)^{-1}\hat{\varphi}_t^{-1}\hat{\lambda}_t\beta \\
\hat{\delta}_{x,t} &= \hat{\varphi}_t^{-1} \\
\hat{\delta}_{g,t} &= \hat{\varphi}_t^{-1} \\
\hat{\delta}_{u,t} &= (\hat{\lambda}_t^2 + \alpha)^{-1}\hat{\varphi}_t^{-1}\hat{\lambda}_t.
\end{aligned}$$

Given parameter estimates and forecasts, the temporary equilibrium is determined by (23), (24) and the policy rule (26) or (27), respectively.

In the appendix we demonstrate the following result:

Proposition 5 *Suppose that both the policy maker and the private agents are learning with parameter estimates updated according to the RLS algorithms specified above. Then*
(i) If policy makers follow the fundamentals form of the RE-optimal rule (26), the REE is unstable for all parameter values.
(ii) If policy makers follow the expectations based optimal rule (27) then, for all parameter values, the estimates $(\hat{\varphi}_t, \hat{\lambda}_t)$ *converge locally to* (φ, λ)*, the expectations of private agents converge locally to RE values, and the economy converges locally to the REE that corresponds to the optimal monetary policy without commitment.*

The notion of local convergence has several precise interpretations, as discussed in the appendix. It is not surprising that convergence is not guaranteed unless the policy maker has some *a priori* information concerning the possible values of the structural parameters. Proposition 5 shows that the fundamental contrast between the instability result in part (i) and the stability result in part (ii) continues to hold when the policy maker is learning the structure of the economy.

Our results illustrate the potential feasibility of optimal monetary policy when private expectations are observable and the rule is formulated to react to these expectations. Even when the structural parameters are unknown to policy makers and agents are not endowed with rational expectations, the economy is (locally) stable when both parties follow natural procedures to estimate the key parameters.

4 Extensions

In this section we take up several extensions. First we show that our central results extend to variations in the learning rules used by private agents. Next we examine the robustness of our results to observation errors. Finally, we consider a number of variations to the model or the formulation of monetary policy. For simplicity, in this section we assume the policy maker knows the true parameter values.

4.1 Alternative Learning Rules

Although the literature on adaptive learning in stochastic models has focused on least squares learning, a number of alternatives have also been considered. We here show that our key results are not limited to the least squares framework.[17]

We first consider learning based on SG (stochastic gradient) algorithms. These have been proposed by, among others, (Sargent 1993) and (Kuan and White 1994). This algorithm attempts to minimize the expected sum of squared forecast errors. In the current context, private agents would update parameters according to

$$\begin{aligned} \xi_{1,t} &= \xi_{1,t-1} + t^{-1}U_{t-2}(\pi_{t-1} - \xi'_{1,t-1}U_{t-2}) \\ \xi_{2,t} &= \xi_{2,t-1} + t^{-1}U_{t-2}(x_{t-1} - \xi'_{2,t-1}U_{t-2}). \end{aligned}$$

This algorithm is simpler than RLS in that these updating equations are not dependent on the matrix of second moments of the regressor U_t. For regression set-ups with fixed parameters and standard assumptions, the SG algorithm will provide consistent estimates of the parameters, though it does not possess the optimality properties of least squares. It does, however, have the advantage of being simpler and faster to compute than least squares.

The analysis in (Evans and Honkapohja 1998) can be applied here. It is shown there that the convergence conditions for SG learning for a class of models is given by the E-stability conditions that also govern convergence of least squares learning. (The framework of (Evans and Honkapohja 1998) has

[17] It can be shown that the stability results in Propositions 3 and 5 also hold for other stability criteria under learning that have been proposed in the literature. For example, the REE is both strongly and iteratively E-stable, see e.g. (Evans and Honkapohja 2001), Chapters 9 and 15 for these concepts.

expectations of current rather than future variables. However, because here we consider RE solutions of an analogous form, the same argument can be applied.) It follows that both our instability and stability results carry over to SG learning.

As a second example, we consider a simple misspecified learning rule. Suppose that private agents ignore the dependence of π_t and x_t on u_t and forecast each variable using simple averages. Formulating this recursively we have $\hat{E}_t\pi_{t+1} = a_{1,t}$ and $\hat{E}_t x_{t+1} = a_{2,t}$ where

$$\begin{aligned} a_{1,t} &= a_{1,t-1} + t^{-1}(\pi_{t-1} - a_{1,t-1}), \\ a_{2,t} &= a_{2,t-1} + t^{-1}(x_{t-1} - a_{2,t-1}). \end{aligned}$$

As in Section 3 this can be set up as a stochastic recursive algorithm. It is straightforward to show that the system remains unstable under the fundamentals based monetary policy rule and that the economy is stable under the expectations based policy rule. In the latter case private expectations $(a_{1,t}, a_{2,t})$ will converge to $(\bar{a}_1, \bar{a}_2)$ and the economy converges to a "restricted perceptions equilibrium."[18] In this equilibrium private sector expectations are equal to the unconditional mathematical expectations of π_t and x_t. However, they are not fully rational (when $\rho \neq 0$) since they ignore the dependence on u_t and are therefore not equal to the conditional mathematical expectations. Nevertheless, the expectations based policy rule continues to provide the optimal monetary policy without commitment, given the way agents form their expectations.

4.2 Observation Errors

We next consider the issue of observation errors, starting with the case in which the exogenous variables g_t, u_t are subject to measurement error. We use the set-up (23)-(24), but assume that policy makers know the values of the structural parameter.. For simplicity we restrict attention to the case where g_t and u_t are serially uncorrelated. Suppose that the policy maker observes $\tilde{g}_t$ and $\tilde{u}_t$, where

$$\begin{aligned} \tilde{g}_t &= g_t + \varepsilon_t, \text{ where } \varepsilon_t \sim iid(0, \sigma_\varepsilon^2) \\ \tilde{u}_t &= u_t + v_t, \text{ where } v_t \sim iid(0, \sigma_v^2). \end{aligned}$$

[18] This is the terminology we use in (Evans and Honkapohja 2001). The phrase "self-confirming equilibium" is also used, e.g. see (Sargent 1999).

Provided the variances are known, the policy maker faces a signal extraction problem with a straightforward solution. The linear projection of g_t on $\tilde{g}_t$ is given by

$$P[g_t \mid \tilde{g}_t] = \zeta_g \tilde{g}_t, \text{ where } \zeta_g = \frac{\sigma_g^2}{\sigma_g^2 + \sigma_\varepsilon^2}.$$

Analogously, the linear projection of u_t is

$$P[u_t \mid \tilde{u}_t] = \zeta_u \tilde{u}_t, \text{ where } \zeta_u = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2}.$$

The first order condition for optimal policy is again given by (25). Under optimal policy the REE takes the form

$$\begin{aligned} \pi_t &= a_1 + b_1 g_t + c_1 \varepsilon_t + d_1 u_t + f_1 v_t + \lambda e_{x,t} + e_{\pi,t} \\ x_t &= a_2 + b_2 g_t + c_2 \varepsilon_t + d_2 u_t + f_2 v_t + e_{x,t}, \end{aligned}$$

where

$$\begin{aligned} a_1 &= \bar{a}_1, a_2 = \bar{a}_2 \\ b_1 &= \lambda(1 - \zeta_g), b_2 = 1 - \zeta_g \\ c_1 &= -\lambda \zeta_g, c_2 = -\zeta_g \\ d_1 &= 1 - (\alpha + \lambda^2)^{-1} \lambda^2 \zeta_u, d_2 = -(\alpha + \lambda^2)^{-1} \lambda \zeta_u \\ f_1 &= d_1 - 1, f_2 = d_2. \end{aligned}$$

We note that, under optimal policy, x_t and π_t now respond to g_t as well as u_t shocks because of the measurement error. The analysis of learning is, however, virtually unchanged and the key results carry through as before: the economy is unstable under the RE optimal policy rule and stable under the expectations based optimal rule.

Another potentially important complication arises in connection with the expectations based rule if private sector expectations are measured with error. Clearly the optimality properties of our rule would be undermined if these errors are sizeable, but we can show that our rule is robust to small measurement errors in private expectations. Thus suppose that the expectations based policy rule (17) is modified to be

$$i_t = \delta_0 + \delta_\pi(\hat{E}_t \pi_{t+1} + w_{\pi,t}) + \delta_x(\hat{E}_t x_{t+1} + w_{x,t}) + \delta_g g_t + \delta_u u_t,$$

where $w_{\pi,t}$ and $w_{x,t}$ denote white noise measurement errors with finite variances. The parameter values of the rule are unchanged. For private agents we continue to assume perceived laws of motion of the form (14)-(15). For the basic model (1)-(2) of Section 2.3, the stochastic processes for x_t and π_t generated by these forecast rules and this policy rule are given by (19)-(20) augmented by additive linear terms in $w_{x,t}$ and $w_{\pi,t}$. The mapping from the perceived to the actual law of motion is therefore identical to that given in Section 2.3. Therefore this expectations based policy rule remains stable, with private agents converging to rational expectations. This result holds even if the observation errors $w_{x,t}$ and $w_{\pi,t}$ are large, though with large observation errors the resulting equilibrium would be far from optimal. However, it is easily seen that if $w_{x,t}$ and $w_{\pi,t}$ have small variances then the economy under the expectations based rule is close to the overall optimum.

The preceding paragraph assumed that the policy makers know the true structural parameters. If instead they estimate structural parameters using RLS as in Section 3 there is the additional complication that, while $\hat{\lambda}_t$ remains consistent, the estimator $\hat{\varphi}_t$ becomes inconsistent due to measurement error in the regressor r_t. However, if $w_{\pi,t}$ has a small variance then the inconsistency will be small and the equilibrium will be approximately optimal asymptotically. Alternatively, policy makers could obtain a consistent estimate of φ using recursive instrumental variables with lagged measured r_t as instrument.

In summary, although observation errors of the fundamental shocks or private expectations introduce complications for our proposed policy rule, we have seen that these difficulties can be largely overcome. The case of substantial measurement errors in private expectations is sufficiently important to warrant a separate treatment and is the subject of current research.

4.3 Other Directions

4.3.1 Output Inertia

A number of variations of the new Phillips curve model have been taken up in the policy literature, including endogenous inflation and output inertia. We consider briefly the more straightforward case of output persistence due to costs of adjustment. As shown in (Clarida, Gali, and Gertler 1999), Section

6, with output persistence the IS curve becomes

$$x_t = -\varphi(i_t - \hat{E}_t\pi_{t+1}) + \theta x_{t-1} + (1-\theta)\hat{E}_t x_{t+1} + g_t, \tag{28}$$

where $0 < \theta < 1$. The corresponding expectations based policy rule is then

$$i_t = \delta_0 + \delta_\pi \hat{E}_t\pi_{t+1} + \delta_x(1-\theta)\hat{E}_t x_{t+1} + \varphi^{-1}\theta x_{t-1} + \delta_g g_t + \delta_u u_t, \tag{29}$$

where the parameters δ_i for $i = 0, \pi, x, g, u$ are the same as in (17). We restrict attention to the case $\rho = 0$, as do (Clarida, Gali, and Gertler 1999). In this case the private sector forecasts under rational expectations are appropriate constants.

Under our rule (29) the reduced form is

$$\begin{aligned} x_t &= (\lambda^2+\alpha)^{-1}(\lambda\bar{\pi} + \alpha\bar{x}) - \frac{\lambda\beta}{\alpha+\lambda^2}\hat{E}_t\pi_{t+1} \\ \pi_t &= (\lambda^2+\alpha)^{-1}\lambda(\lambda\bar{\pi} + \alpha\bar{x}) + \frac{\beta\alpha}{\alpha+\lambda^2}\hat{E}_t\pi_{t+1} + u_t. \end{aligned}$$

Assuming now that agents form their forecast $\hat{E}_t\pi_{t+1}$ as the average of past inflation rates, it is easily verified that forecasts converge to rational expectations and the economy converges to the optimal equilibrium. Thus output persistence presents no difficulties. The case of inflation persistence is more complicated and is therefore omitted.

4.3.2 Policy Under Limited Commitment

Our discussion of optimal policies has been under the assumption that the policy maker does not have commitment power. The ability to commit to a given rule raises several additional issues, as discussed Section 4 of (Clarida, Gali, and Gertler 1999). For brevity we assume here that $\bar{x} = 0 = \bar{\pi}$, so that the classic inflationary bias issue does not arise. Nevertheless, commitment can still result in gains over the discretionary equilibrium.

(Clarida, Gali, and Gertler 1999), Section 4.2.1 argue that a simple family of rules yielding an REE of the form

$$x_t = \omega u_t$$

can dominate the optimum without commitment. (Note that with $\bar{x} = 0 = \bar{\pi}$ the optimal REE without commitment also has this form.) Suppose that

policy makers have limited commitment power in that they are able only to commit to policies that take this form. The optimal policy with commitment in this class is formally identical to the model without commitment when the relative weight on output α is replaced by a smaller value $\alpha^c = \alpha(1-\beta\rho)$. It is thus apparent that our instability and stability results can be applied for this family of rules with limited commitment.

Optimal policy under full commitment power does not generally have this form, as emphasized by (Woodford 1999). The general analysis of issues raised in this paper for monetary policy with full commitment will require a separate study.

4.3.3 Rules Based on Inflation and Output Data

A natural question to ask is whether it is possible to devise monetary policy rules that are based on inflation and output data, rather than on observed private expectations, and which are stable under learning and lead to the optimal policy without commitment. That is, can one side-step the apparent need to observe private expectations in order to implement the optimal policy rule? There are two separate issues, which we take up in turn.

First, is it possible using observed current data to infer the values of private expectations and insert them into (17) to obtain an optimal policy which is stable under learning? There are several problems with this. The most telling difficulty arises because of the simultaneity implied by using observations on $x_t, \pi_t, i_t, u_t, g_t$ to deduce private expectations while setting i_t as a function of these private expectations. Consider the basic framework (1)-(2) with known structural parameters. Under the interest rate rule (17) we are led to a *reductio ad absurdum:* it is impossible to infer the value of $\hat{E}_t x_{t+1}$ because under the rule (17) the effects of $\hat{E}_t x_{t+1}$ are completely offset and have no effects on observables.[19]

We do not wish to overstate this particular argument. With additional auxiliary assumptions it may be possible to deduce the values of private expectations from the history of data. For example, if it were known that all agents form expectations by following the least squares learning rules that we have analyzed, then it would be straightforward to use this information to infer $\hat{E}_t \pi_{t+1}$ and $\hat{E}_t x_{t+1}$ and implement (17). Such a policy would be entirely in the spirit of this paper. But whether such an inference is possible

[19] Other practical difficulties are that the values of structural parameters may be unknown and that there may be unobserved shocks as in (23)-(24).

is a delicate matter that depends on details of the model, the information structure and auxiliary assumptions. It would appear preferable to obtain direct observations on expectations.

Another question is whether judicious use of the rational expectations assumption can be used to substitute observations of actual inflation and output gap for expected inflation and expected output gap in the optimal interest rate rule. We consider two possibilities within the context of the basic model of Section 2.1. Under rational expectations and optimal policy we have $\pi_t = \bar{a}_1 + \bar{d}_1 u_t$ and $x_t = \bar{a}_2 + \bar{d}_2 u_t$. From $E_t \pi_{t+1} = \bar{a}_1 + \bar{d}_1 \rho u_t$ and $E_t x_{t+1} = \bar{a}_2 + \bar{d}_2 \rho u_t$. it follows that $E_t \pi_{t+1} = \bar{a}_1 + \rho(\pi_t - \bar{a}_1)$ and $E_t x_{t+1} = \bar{a}_2 + \rho(x_t - \bar{a}_2)$. Using these expressions to replace $\hat{E}_t \pi_{t+1}$ and $\hat{E}_t x_{t+1}$ in the expectations based rule (17) suggests the policy

$$i_t = \delta_0' + \delta_\pi \rho \pi_t + \delta_x \rho x_t + \delta_g g_t + \delta_u u_t, \tag{30}$$

where the values of $\delta_i, i = \pi, x, g, u$ are as in equation (17) and δ_0' is an appropriate constant. Alternatively one could start with the fundamentals based rule (9) and use the relationship $u_t = \bar{d}_1^{-1}(\pi_t - \bar{a}_1)$ to obtain the policy rule

$$i_t = \psi_0' + \left(\varphi^{-1}\alpha^{-1}\lambda(1-\rho) + \rho\right)\pi_t + \varphi^{-1} g_t, \tag{31}$$

where ψ_0' is an appropriate constant.

Neither of the policy rules (30) or (31) require observations of private expectations, but each was obtained making some use of the rational expectations assumption. Will these rules be stable under least-squares learning by private agents? It is easily established that stability is not guaranteed. For the case $\rho = 0$ the rule (30) reduces to $i_t = \delta_0' + \delta_g g_t + \delta_u u_t$. But we have already established in Section 2.2 that any policy that depends linearly only on the fundamental shocks cannot be E-stable and hence is unstable under least squares learning. Using continuity of eigenvalues, it follows that the instability result holds for all nonzero $|\rho|$ sufficiently small. A similar argument applies to the rule (31) for λ/α and $|\rho|$ sufficiently small. In contrast, the expectations based rule (17) is stable under learning for all possible parameter values.

The rules (30) and (31) are particular cases of rules based on actual inflation and output gap data. There are a number of simple interest rate rules along such lines that do not explicitly aim for optimal policy. (Bullard and Mitra 2000) consider various general classes of such rules and derive

conditions on policy rule parameter values that yield stability under learning. Another class of rules is based on controlling the money stock, and one could also consider the stability under learning of the REE for monetary feedback rules. See Part III of (Evans and Honkapohja 2001) for examples of the latter. These examples do not exhaust the list of policy rules that have been proposed in the recent literature. We emphasize that it is important to analyze, for any proposed policy rule, whether the RE equilibrium of interest is robust to expectational errors, i.e. is stable under learning.

5 Conclusions

The central message of the paper is both simple and fundamental. First, optimal monetary policy should not assume perfectly rational expectations on the part of private agents: Even if the initial deviations from perfect rationality are small, the economy will diverge when the fundamentals form of the RE-optimal policy rule is followed.

Second, the instability problem can be overcome if the set of conditioning variables is augmented to include observed private expectations. We have shown how to use the economic structure to devise an expectations based optimal policy rule. Under this policy rule the economy converges to the REE corresponding to the optimal policy rule without commitment.

These results hold even when the policy maker is learning the required values of the structural parameters at the same time that private agents are following least squares learning. The propositions have been demonstrated under the standard assumption that private expectations are revised in accordance with least squares learning as well as under some other natural learning algorithms.

Our analysis has been conducted under the assumption that accurate data on private expectations are available. This allows us to present sharp and powerful results. Appropriately conditioning policy on private expectations gives due weight to the role of these expectations in determining the dynamics of the economy, which the assumption of perfect RE suppresses. Obtaining accurate data on private expectations should therefore be a high priority for monetary policy makers. Finding the appropriate form of dependence of policy on private expectations is also crucial, since not all rules that depend on private forecasts lead to stability under learning.

Appendix

In this appendix we give the proof of Proposition 5 and comment on the modifications for the proofs for Propositions 1 and 3. Consider the case of the expectations based policy rule (27). Combining this equation with the IS curve (23) and the private agent forecast rules (12)-(13) we obtain the law of motion for x_t under learning:

$$\begin{aligned} x_t &= -\varphi\hat{\varphi}_t^{-1}(a_{2,t} + k_{2,t}u_t) - \varphi\hat{\varphi}_t^{-1}g_t \\ &\quad -\varphi\hat{\varphi}_t^{-1}(\hat{\lambda}_t^2 + \alpha)^{-1}\left(\hat{\lambda}_t\beta(a_{1,t} + k_{1,t}u_t) - \hat{\lambda}_t(u_t + \bar{\pi}) - \alpha\bar{x}\right) \\ &\quad +a_{2,t} + k_{2,t}u_t + g_t + e_{x,t}, \end{aligned}$$

which is of the form

$$x_t = x(\hat{\varphi}_t, \hat{\lambda}_t, \xi_{1,t}, \xi_{2,t}; u_t, g_t, e_{x,t}, e_{\pi,t}).$$

Similarly, using also the PC curve (24) and the definitions of $r_t, y_{1,t}$ and $y_{2,t}$ we can obtain

$$\begin{aligned} \pi_t &= \pi(\hat{\varphi}_t, \hat{\lambda}_t, \xi_{1,t}, \xi_{2,t}; u_t, g_t, e_{x,t}, e_{\pi,t}) \\ r_t &= r(\hat{\varphi}_t, \hat{\lambda}_t, \xi_{1,t}, \xi_{2,t}; u_t, g_t, e_{x,t}, e_{\pi,t}) \\ y_{1,t} &= y_1(\hat{\varphi}_t, \hat{\lambda}_t, \xi_{1,t}, \xi_{2,t}; u_t, g_t, e_{x,t}, e_{\pi,t}) \\ y_{2,t} &= y_2(\hat{\varphi}_t, \hat{\lambda}_t, \xi_{1,t}, \xi_{2,t}; u_t, g_t, e_{x,t}, e_{\pi,t}). \end{aligned}$$

Define the parameter vector

$$\theta_t' = (\hat{\varphi}_t, \hat{\lambda}_t, \xi_{1,t}', \xi_{2,t}', R_{x,t}, R_{\pi,t}, \mathrm{vec}(R_{U,t}))$$

and the state vector

$$X_t = (u_t, u_{t-1}, u_{t-2}, g_{t-1}, e_{x,t-1}, e_{\pi,t-1}).$$

The recursive equations for the parameter estimates can then be written in the form

$$\theta_t = \theta_{t-1} + t^{-1}\mathcal{H}(\theta_{t-1}, X_t),$$

so that theorems on the convergence of stochastic recursive algorithms can be applied, e.g. part II of (Evans and Honkapohja 2001). These results

state that, under appropriate regularity conditions, convergence of these algorithms is governed by the stability of the associated ordinary differential equation (ODE)

$$\frac{d\theta}{d\tau} = h(\theta) \text{ where } h(\theta) = \lim_{t\to\infty} E\mathcal{H}(\theta, X_t).$$

For the case of the expectations based optimal policy one can compute for the policy maker's parameter estimates

$$\begin{aligned}
\frac{d\hat{\varphi}}{d\tau} &= R_r^{-1} E r_{t-1}(\theta)^2 (\varphi - \hat{\varphi}), \\
\frac{dR_r}{d\tau} &= E r_{t-1}(\theta)^2 - R_r \\
\frac{d\hat{\lambda}}{d\tau} &= R_x^{-1} E x_{t-1}(\theta)^2 (\lambda - \hat{\lambda}) \\
\frac{dR_x}{d\tau} &= E x_{t-1}(\theta)^2 - R_x,
\end{aligned}$$

where

$$r_t(\theta) = r(\hat{\varphi}, \hat{\lambda}, \xi_1, \xi_2, \hat{\rho}; u_t, g_t, e_{x,t}, e_{\pi,t})$$

and

$$x_t(\theta) = x(\hat{\varphi}, \hat{\lambda}, \xi_1, \xi_2, \hat{\rho}; u_t, g_t, e_{x,t}, e_{\pi,t}).$$

For the private agents' parameter estimates we get

$$\begin{aligned}
\frac{d\xi_1}{d\tau} &= R_U^{-1}(EU_{t-2}U'_{t-2})\lambda \\
&\left[-\varphi\hat{\varphi}^{-1}\begin{pmatrix} a_2 \\ k_2\rho \end{pmatrix} - \frac{\varphi\hat{\varphi}^{-1}\hat{\lambda}}{\hat{\lambda}^2 + \alpha}\begin{pmatrix} \beta a_1 - \bar{\pi} - \hat{\lambda}^{-1}\alpha\bar{x} \\ (\beta k_1 + 1)\rho \end{pmatrix} + \begin{pmatrix} a_2 \\ k_2\rho \end{pmatrix}\right] \\
&+ R_U^{-1}(EU_{t-2}U'_{t-2})\left[\begin{pmatrix} \beta a_1 \\ (\beta k_1 + 1)\rho \end{pmatrix} - \begin{pmatrix} a_1 \\ k_1 \end{pmatrix}\right]
\end{aligned}$$

$$\begin{aligned}
\frac{d\xi_2}{d\tau} &= R_U^{-1}(EU_{t-2}U'_{t-2}) \\
&\left[-\varphi\hat{\varphi}^{-1}\begin{pmatrix} a_2 \\ k_2\rho \end{pmatrix} - \frac{\varphi\hat{\varphi}^{-1}\hat{\lambda}}{\hat{\lambda}^2 + \alpha}\begin{pmatrix} \beta a_1 - \bar{\pi} - \hat{\lambda}^{-1}\alpha\bar{x} \\ (\beta k_1 + 1)\rho \end{pmatrix} + \begin{pmatrix} 0 \\ k_2(\rho - 1) \end{pmatrix}\right]
\end{aligned}$$

$$\frac{dR_U}{d\tau} = EU_{t-2}U'_{t-2} - R_U.$$

It can be verified that this system has a unique equilibrium point θ^* at which

$$\begin{aligned}
\hat{\varphi} &= \varphi, \hat{\lambda} = \lambda, \xi_1 = \bar{\xi}_1 \equiv (\bar{a}_1, \bar{k}_1)', \xi_2 = \bar{\xi}_2 \equiv (\bar{a}_2, \bar{k}_2)', \\
R_r &= Er_{t-1}(\theta^*)^2, R_x = Ex_{t-1}(\theta^*)^2, R_U = EU_{t-2}U'_{t-2}.
\end{aligned}$$

It can be shown that local stability is governed by the following "small" ODE

$$\begin{aligned}
\frac{d\hat{\varphi}}{d\tau} &= \varphi - \hat{\varphi} \\
\frac{d\hat{\lambda}}{d\tau} &= \lambda - \hat{\lambda}
\end{aligned}$$

$$\begin{aligned}
\frac{d\xi_1}{d\tau} &= \lambda \left[-\varphi\hat{\varphi}^{-1} \begin{pmatrix} a_2 \\ k_2\rho \end{pmatrix} - \frac{\varphi\hat{\varphi}^{-1}\hat{\lambda}}{\hat{\lambda}^2 + \alpha} \begin{pmatrix} \beta a_1 - \bar{\pi} - \hat{\lambda}^{-1}\alpha\bar{x} \\ (\beta k_1 + 1)\rho \end{pmatrix} + \begin{pmatrix} a_2 \\ k_2\rho \end{pmatrix} \right] \\
&+ \left[\begin{pmatrix} \beta a_1 \\ (\beta k_1 + 1)\rho \end{pmatrix} - \begin{pmatrix} a_1 \\ k_1 \end{pmatrix} \right]
\end{aligned}$$

$$\frac{d\xi_2}{d\tau} = -\varphi\hat{\varphi}^{-1} \begin{pmatrix} a_2 \\ k_2\rho \end{pmatrix} - \frac{\varphi\hat{\varphi}^{-1}\hat{\lambda}}{\hat{\lambda}^2 + \alpha} \begin{pmatrix} \beta a_1 - \bar{\pi} - \hat{\lambda}^{-1}\alpha\bar{x} \\ (\beta k_1 + 1)\rho \end{pmatrix} + \begin{pmatrix} 0 \\ k_2(\rho - 1) \end{pmatrix}.$$

Clearly the subsystem in $(\hat{\varphi}, \hat{\lambda})$ is stable with $\hat{\varphi} \to \varphi, \hat{\lambda} \to \lambda$. Hence local stability of the REE is determined by the ODE

$$\frac{d\xi_1}{d\tau} = -\frac{\lambda}{\lambda^2 + \alpha} \begin{pmatrix} \beta a_1 - \bar{\pi} - \lambda^{-1}\alpha\bar{x} \\ (\beta k_1 + 1)\rho \end{pmatrix} + \begin{pmatrix} \beta a_1 \\ (\beta k_1 + 1)\rho \end{pmatrix} - \begin{pmatrix} a_1 \\ k_1 \end{pmatrix}$$

$$\frac{d\xi_2}{d\tau} = -\frac{\lambda}{\lambda^2 + \alpha} \begin{pmatrix} \beta a_1 - \bar{\pi} - \lambda^{-1}\alpha\bar{x} \\ (\beta k_1 + 1)\rho \end{pmatrix} - \begin{pmatrix} a_2 \\ k_2 \end{pmatrix}$$

which is identical to the E-stability equation (21) which was shown in section 2.3 to be asymptotically stable.

The standard results from the learning literature can be now applied. The regularity conditions on the function $\mathcal{H}(\theta, X_t)$, and bounded moment conditions for the exogenous innovations, required for the local convergence results in Chapter 6 of (Evans and Honkapohja 2001), are easily seen to be satisfied. Above we saw that the equilibrium of the large ODE θ^* is locally asymptotically stable. Hence the basic local convergence theorems apply. There are various senses of local probabilistic convergence. For example, almost sure convergence obtains if the algorithm is augmented to have a projection facility constraining estimates to a neighborhood of the REE. For details see Chapter 6 of (Evans and Honkapohja 2001) or (Evans and Honkapohja 1999). This proves part (ii) of Proposition 5.

The proof of part (i) of Proposition 5 follows a similar sequence of steps, but with the alternative policy rule to obtain law of motion for $x_t, \pi_t, r_t, y_{1,t}$ and $y_{2,t}$. For this rule the parameter vector θ_t must be augmented to include $\hat{\rho}_t$ and $R_{u,t}$. The stability properties of the associated ODE can be shown to be depend on the E-stability property of the small ODE. The latter is locally unstable and the nonconvergence results found in Chapter 6 of (Evans and Honkapohja 2001) or (Evans and Honkapohja 1999) can be applied. These results state that θ_t converges to θ^* (or to any other point) with probability zero.

Turning to Propositions 1 and 3 the parameter vector θ_t is now

$$\theta_t' = (\xi_{1,t}', \xi_{2,t}', \mathrm{vec}(R_{U,t})).$$

For the expectation based rule the associated ODE is

$$\begin{aligned}\frac{d\xi_1}{d\tau} &= R_U^{-1}(EU_{t-2}U_{t-2}') - \frac{\lambda}{\lambda^2+\alpha}\begin{pmatrix} \beta a_1 - \bar{\pi} - \lambda^{-1}\alpha\bar{x} \\ (\beta k_1 + 1)\rho \end{pmatrix} + \begin{pmatrix} \beta a_1 \\ (\beta k_1 + 1)\rho \end{pmatrix} \\ &\quad - \begin{pmatrix} a_1 \\ k_1 \end{pmatrix}\end{aligned}$$

$$\frac{d\xi_2}{d\tau} = R_U^{-1}(EU_{t-2}U_{t-2}') - \frac{\lambda}{\lambda^2+\alpha}\begin{pmatrix} \beta a_1 - \bar{\pi} - \lambda^{-1}\alpha\bar{x} \\ (\beta k_1 + 1)\rho \end{pmatrix} - \begin{pmatrix} a_2 \\ k_2 \end{pmatrix}$$

$$\frac{dR_U}{d\tau} = EU_{t-2}U_{t-2}' - R_U.$$

It can be seen that for this system the equilibrium point

$$(\theta^*)' = (\bar{\xi}_1', \bar{\xi}_2', \mathrm{vec}(EU_{t-2}U_{t-2}'))$$

is globally asymptotically stable. It can be verified that the assumptions are satisfied for the global stability result found in Chapter 6 of (Evans and Honkapohja 2001) or (Evans and Honkapohja 1999), which should be consulted for further technical details. It follows that $\theta_t \to \theta^*$ with probability one, proving Proposition 3. For the fundamentals form of the RE-optimal rule we obtain a differential equation system in $(\frac{d\xi_1}{d\tau}, \frac{d\xi_2}{d\tau}, \frac{dR_U}{d\tau})$, which is locally unstable at θ^*. Thus $\theta_t \to \theta^*$ with probability zero, establishing Proposition 1.

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