Dollarization via Currency Substitution

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Abstract

We study currency substitution from first principles in order to study how government policies affect the extent of dollarization. Using a dual currency search model where agents are allowed to hold currency portfolios to buy goods, we analyze an agent's choice to spend dollars or risky domestic currency for internal trade. We focus on two equilibria, a 'high dollarization' equilibrium, in which domestic citizens spend dollars and hold domestic currency and a 'low dollarization' equilibrium in which agents spend the domestic currency first and hold onto their dollars. We show that for the low dollarization equilibrium to prevail, the purchasing power risk on the domestic currency must be low and trading frictions must be small. Otherwise, currency substitution occurs and the high dollarization equilibrium prevails. Thus, our results show that dollarization will not be a problem as long as the domestic currency is relatively safe and domestic trading environments are functioning well. Otherwise, currency substitution and dollarization will arise.

JEL: E4, E5, D7

1 Introduction

Recently, dollarization has moved to the forefront of international monetary economics. To understand dollarization, one needs to define what it means. However, the concept of dollarization has a wide variety of meanings and uses.[1] One of the most common and basic forms of dollarization

[1] For example, the World Bank uses three different categories to define the meaning of dollarization: (i) Asset Dollarization (the use of foreign currency as unit of account, means of exchange and store of value), (ii) Liability Dollarization (the domestic banking system or the government has relatively large foreign currency debt obligations), and (iii) Full Dollarization (a country abandons its own currency and adopts another's). A considerable amount of work has examined these other aspects of dollarization: the adoption of a fixed exchange rate, the establishment of

is the simultaneous use of a foreign currency alongside the home currency as a media of exchange. This phenomenon is commonly referred to as currency substitution (Calvo and Végh, 1992). The purpose of our research is to formalize from first principles this basic form of dollarization and study how government policies affect the extent of dollarization.

Why does currency substitution occur? Typically, it arises due to the existence of currency risk on the home currency. The fundamental problem is that domestic agents perceive that there is a potential loss of purchasing power from holding the home currency, be it from inflation risk, exchange rate risk, or political instability.[2] As a result, a safe foreign currency may find its way into the economy as an alternative media of exchange. Currency substitution thus involves an endogenous choice by private agents to use a foreign currency to carry out some fraction of *internal* trade.[3] More concretely, citizens holding both home and foreign currencies may choose to spend the foreign rather than the home currency in some transactions. In this situation the safe currency has greater purchasing power than the domestic currency. As a result, agents have two spending strategies: 1) spend the safe currency and hold the lower valued currency for future consumption or 2) get rid of the risky currency now and hold the safe currency for future consumption. Under the first strategy, the agent consumes more today but faces an less certain level of consumption tomorrow. Under the second strategy, the agent gets lower consumption today but a more certain level of consumption tomorrow. In order to understand the fundamentals of currency substitution, we need to know when one strategy is preferred to the other. This in turn requires that we model individual patterns of monetary exchange.

While this may seem a natural starting point for understanding the rise of dollarization, most theoretical models of currency substitution do not lend themselves to this type of analysis. The reason is they are based on a Walrasian framework which, in turn, lacks an explicit role for money. To study individual transaction patterns one needs to construct a decentralized model of exchange

a currency board, changing finance laws to allow agents to hold dollar-denominated assets or making the dollar legal tender.

[2] For example a recent Wall Street Journal article on dollarization (January 15 2001, p. A1) states "Everybody in these countries knows they can easily wake up one morning and find the value of their money gone."

[3] This raises the question of why the home currency and foreign currency coexist given their different rates of return. Kareken and Wallace (1981) showed that if markets are centralized and both currencies are acceptable in trade, then both currencies must have the same rate of return. Consequently, there must be some form of market incompleteness for the two currencies to coexist with differing rates of return.

in which money plays a fundamental role. Consequently, we conduct our analysis using a search-theoretic model of money, in the tradition of Kiyotaki and Wright (1989, 1993). We build a one country, two-currency model in which agents are allowed to hold multiple units of currency. The two currencies are fundamentally different in that the home currency is assumed to be 'risky', whereas the foreign currency is not. Hence the agents' spending patterns are driven by the relative riskiness of the currencies and not by ad-hoc transactions costs or institutional restrictions. In equilibrium the extent of currency substitution is determined endogenously and depends on *which* currency is used when a trading opportunity arises.

We first analyze equilibria in which agents prefer to spend the safe currency now and the risky currency later. We refer to this as the 'high dollarization' equilibrium since dollars are used more frequently as a medium of exchange. This equilibrium tends to exist when the foreign currency is more valued than the domestic currency. This combination of greater value and use of the foreign currency is suggestive of Hayek's (1976) notion of currency competition – good money drives out bad money. Analysis shows that this equilibrium is 'robust' in the sense that it is supported over a large portion of the parameter space and for different utility specifications.

We then study the 'low dollarization' equilibria in which agents prefer to spend the risky currency now and hold on to the safe currency. What is interesting about the low dollarization equilibrium is that because buyers prefer to spend the lower-valued domestic currency and hold onto the high-valued foreign currency, we observe something akin to Gresham's Law – bad money drives out good money. While this strategy may appear to be the most obvious one for the buyer, our analysis reveals that, in fact, this equilibrium is very 'fragile' in the sense that it is supported over a smaller segment of the parameter space. The reason is that while dumping a risky currency now makes sense for a buyer, it effectively transfers the risk onto the seller who will not accept the risk without being compensated. Compensation takes the form of having the seller produce less for the domestic currency, which lowers current consumption for the buyer. If the currency risk is large enough, the buyer gets so little that he would prefer to spend the safe currency and hold the risky currency. We show numerically that the currency risk does not have to be very large before this equilibrium breaks down. This suggests that a highly dollarized economy is not necessarily the outcome from having a risky domestic currency. But it will be the outcome if home currency risk gets out of hand.

Finally, our model generates an equilibrium distribution of real exchange rates. We can show that an increase in domestic currency risk leads to a depreciation in the real value of the domestic currency. Furthermore, there will be an increase in the dispersion of observed real exchange rates. Thus, despite the highly stylized nature of the monetary economy, we generate equilibrium behavior of real exchange rates that is quite intuitive and consistent with the data.

The structure of the paper is as follows. Section 2 contains a discussion of the existing literature on currency substitution. Section 3 describes the economic environment. Section 4 contains our definition of an equilibrium. Section 5 examines the 'high dollarization' equilibrium where agents prefer to spend the foreign currency rather than the home currency. Section 6 examines the 'low dollarization' equilibria where agents prefer to spend the home currency and hold onto the foreign currency. Section 7 contains numerical results characterizing which equilibria arise in different portions of the parameter space. Section 8 contains concluding comments.

2 Related Literature

A substantial amount of previous research has looked at currency substitution. Giovannini and Turtleboom (1994) provide a good survey of this line of research and the types of models used. They group most research on currency substitution into three classes of models: 1) cash-in-advance models, 2) transaction cost models and 3) ad-hoc models. The main problem with all of these models is that they do not have a fundamental role for money as a medium of exchange. Hence, arbitrary restrictions on the use of money and/or ad-hoc transaction costs from using a particular currency must be employed to make any progress. A more preferred approach to studying currency substitution would be to construct a model in which money plays a fundamental role as a medium of exchange. Furthermore, we want to study currency substitution in an environment in which trading frictions are not a function of the currency used. In short, we want a level trading field.

Search theoretic models of money have these properties and have been used to study the use of multiple currencies as media of exchange. Both open and closed economy models have been used to study rate of return dominance, the existence of international versus national currencies and currency exchange.[4] The fact that multiple currencies circulate for some fraction of internal

[4] Aiygari, Wallace and Wright (1996) examine rate of return dominance. Matsuyama, Kiyotaki and Matsui (1993), Trejos and Wright (1996,2000), Kocherlakota and Krueger (1999), Camera and Winkler (2000) look at international versus national currencies. Zhou (1997), Waller and Curtis (2000), Craig and Waller (2000), Head and Shi (2000)

trade means that in a sense dollarization is occurring. However, with one exception, in these papers agents can only carry one currency when conducting trade, i.e., individual buyers and sellers are not allowed to hold portfolios of currencies. Consequently, it is not possible to study equilibria in which agents have to choose between one currency or the other to buy goods.[5]. Furthermore, it is not possible to study how changes in currency risk alter these transaction patterns. We believe that transaction patterns are crucial to understanding the process of dollarization and currency substitution. Thus, the model in this paper is the first to study dollarization in this fashion.

3 Economic Environment

The environment is based on the standard monetary search model. There is a continuum of infinitely lived agents uniformly located on the unit interval who specialize in consumption and production of goods and services. There is a continuum of good types defined on the unit circle. Agents specialize in production and consumption. Specifically, an agent can produce only one type of good, but consume a subset of good types. When producing the quantity $q > 0$, the agent incurs a linear production cost measured in units of utility given by $c(q) = q$. When consuming q units of a desired consumption good the agent obtains utility $u(q)$, with $u'(q) > 0,\ u''(q) < 0$ and $u'(0) \to \infty$.

Agents meet bilaterally and at random via a Poisson process with arrival rate $\alpha > 0$. The matching process is such that, contingent on meeting, there is probability x of single coincidence of wants, xy of double coincidence. We set $y = 0$, for simplicity to rule out barter. Without barter, agents must resort to alternative means of conducting trade such as money. To concentrate on the medium of exchange function of money, we assume away the existence of alternative payments systems or financial intermediaries. We allow agents to trade with either currency and the trading frictions are the same for both currencies. In short, we assume the transaction costs are identical for each currency in terms of its use in trade.

Agents are initially randomly endowed with indivisible units of foreign and home fiat money of which there is a constant per capita supply M_f and M_h, respectively. In what follows we refer to the foreign currency as the dollar. An individual can hold at most N units of money in total.

look at currency exchange.

[5] While Craig and Waller (2000) allow agents to hold currency portfolios, the analysis is all done numerically and the transaction patterns are stunningly complex. Thus, to make any analytical progress, the model used in Craig and Waller must be reduced dramatically.

In order to study currency substitution we allow the currencies to be fundamentally different with respect to their purchasing power risk. Specifically, we proceed as in Li (1995) by assuming that agents can have their home currency randomly confiscated by the government.

An agent can meet the government with arrival rate α. Upon meeting an agent holding the home currency, the government randomly confiscates all of the agent's home currency holdings with probability $\tau \in [0, 1]$. Confiscated currency holdings are destroyed immediately. The government consumes all goods and services but does not produce them. For this reason, conditional on meeting a seller, the government buys goods from the agent with probability $\eta \in [0, 1]$, paying with a new unit of home currency.[6] While highly stylized, the randomness of confiscation captures the idea that the home currency is risky and those holding it are prone to sudden losses of purchasing power.

4 Symmetric Stationary Equilibria

We study stationary rational expectations equilibria, where symmetric Nash strategies are adopted, and identical agents use identical time-invariant pure strategies. Furthermore we study equilibria where the beliefs over strategies and traded quantities are identical across individuals, and each agent correctly evaluates the potential gains from trade in all matches.

Agents must use money to conduct trade. We examine the case in which both currencies are fully acceptable media of exchange.[7] Agents thus can hold a 'portfolio' of currencies. To simplify the analysis of the transaction patterns, we let $N = 2$. The reason for this assumption is two-fold. First, there is only one 'diversified' currency portfolio consisting of one unit of each currency. This allows us to focus our entire analysis of dollarization on the actions of these portfolio holders. Second, no pure currency trades will arise, i.e., currency does not trade for currency.[8] This allows us to focus on goods trades only and ignore nominal exchange rate determination in pure currency

[6] The government has three parameters under its control, τ, η, M_h. Two of these are free parameters while the third must adjust to maintain the balanced budget constraint. We let τ and $M_h$be set by the government with η being endogenously determined.

[7] There is always an equilibrium in which one or both currencies are not accepted. We do not focus on these equilibria because currency substitution requires that both are accepted.

[8] If the two currencies have different values, then one-for-one currency trades will not exist. With an upper bound of 2, the only remaining trade is a 2 for 1 trade. However, these trades require that the two traders 'swap' their entire portfolios, which has to make one of them worse off. So these trades do not occur either. Thus, with an upper bound of 2 units, pure currency trades will not occur.

trades. However, there is the possibility for currency to trade for the other currency plus some goods, as in Aiyagari, Wallace and Wright (1996). We rule out these trades for one important reason. Allowing them can generate equilibria in which two identical currencies trade at different values simply due to beliefs.[9] In order to focus on differences in currency values arising strictly from 'fundamentals', such as currency risk, this potential source of extrinsic valuation needs to be controlled for. Preventing these trades is one way to do it; additionally, it greatly reduces the complexity of the model.

Let m_i denote the fraction of agents in the economy holding a currency portfolio i, where $i \in \{0, f, h, 2f, 2h, fh\}$ denotes the composition of the portfolio. For example fh means that the agent has one unit of each currency, $2f$ that she has two units of the foreign currency and so on. As a result m_i must satisfy the following constraints:

$$\begin{aligned} 1 &= m_0 + m_f + m_h + m_{2f} + m_{2h} + m_{fh} \\ M_f &= m_f + 2m_{2f} + m_{fh} \\ M_h &= m_h + 2m_{2h} + m_{fh} \end{aligned} \tag{1}$$

where $M_f + M_h < 2$ since $N = 2$. In a stationary equilibrium $\dot{m}_i = 0$ for all i.[10] Furthermore, to keep the per capita stock of home currency constant the government must run a balanced budget:

$$\tau(m_h + 2m_{2h} + m_{fh}) = \eta\left[m_0 + m_f + m_h\right]. \tag{2}$$

i.e. the outflow of home currency from confiscation (left hand side) must equal the inflow of home currency due to government purchases (right hand side).[11]

The terms of trade are endogenously formed. Agents with money can be buyers or sellers in a bilateral match, depending on their trading partner. Agents with no currency, however, can only be sellers since all exchange must be quid-pro-quo, barter is not feasible, and there is no credit. Note, however, that since $N = 2$, only those agents with portfolios $0, h$, and f can be sellers Agents with two-unit portfolios can only be buyers; we denote their proportion in the economy by $\mu = m_{2f} + m_{2h} + m_{fh}$. The trading mechanism is assumed to be based on take-it-or-leave-it

[9] For example, suppose two currencies are identical except for their colors. If agents believe blue currency is more valuable than red currency, then an equilibrium consistent with this belief can be supported for some parameter values (see Aiyagari, Wallace and Wright (1996) or Cavalcanti (2000)).

[10] The laws of motion depend on the transaction pattern and are described in a later section.

[11] Due to the inventory constraint only those agents holding less than two units of currency can sell.

bargaining protocol. Specifically, when a buyer meets a seller, he offers the seller a trade of d units of currency for the quantity q of goods. The seller can accept or reject. Thus, the optimal offer pair (d, q) is such that the seller is left indifferent between accepting and rejecting it.[12] Consequently, the seller gets zero net surplus in all trades. When government buys goods it also makes a take-it-or-leave-it offers.

To define prices one must specify the equilibrium transaction pattern. We focus on the one studied by Camera and Corbae (1999), in which agents only spend one unit of money per transaction, i.e. $d = 1$. In this case the price in a transaction is given by $1/q$. While there are many transaction patterns that one can consider in this setting, we choose this particular pattern because it is the simplest way to analyze the agent's choice of spending one currency or the other.

We want to determine the conditions under which an agent holding a diversified portfolio will prefer to spend his unit of foreign currency rather than the home currency, and vice versa. The choice of which currency to trade is complicated because it is contingent on the seller's money holdings. For example, sellers holding a dollar are willing to produce a different amount for a second dollar than will a seller holding a unit of home currency. Restricting attention to a representative buyer with portfolio fh, let $p_i \in [0, 1]$ denote the probability that he chooses to spend currency f when he is matched to a seller with portfolio $i \in \{0, h, f\}$. With the complementary probability, $1 - p_i$, he spends his unit of home currency. To describe this buyer's spending strategy, when $d = 1$, we use the 3-tuple $p = (p_0, p_f, p_h)$. We denote by p^* the equilibrium strategy vector, taken as given by an agent. Note that although we will limit our analysis to only pure strategies, there are eight possible equilibrium strategy vectors p^*.

Let V_i denote the value associated with holding portfolio i for a given vector p^*. Furthermore, let q_i^j denote the equilibrium quantity produced by a seller with portfolio i in exchange for one unit of currency $j = f, h$. Under the conjecture that $d = 1$ and buyer-take-all bargaining, V_i must

[12] Because of the indivisibility of money, d must be an integer. Therefore, the optimal offer pair (d, q) may not maximize the surplus in the match. However, Berentsen, Molico and Wright (2001) show that agents may choose to engage in lotteries over d to improve the expected surplus from trade. Allowing for lotteries would substantially complicate the analysis without changing the nature of the transaction patterns.

satisfy $V_0 = 0$ and

$$\begin{aligned}
\rho V_i &= x \sum_{j\in\{0,f,h\}} m_j u(q_j^i) - x(1-\mu)(V_i - V_0) - \tau(V_i - V_0)1_{\{i=h\}} \\
\rho V_{2i} &= x \sum_{j\in\{0,f,h\}} m_j u(q_j^i) - x(1-\mu)(V_{2i} - V_i) - \tau(V_{2i} - V_0)1_{\{i=h\}} \\
\rho V_{fh} &= \max_{p_j\in\{0,1\}} x \sum_{j\in\{0,f,h\}} m_j \left[p_j u(q_j^f) + (1-p_j)\, u(q_j^h)\right] + x \sum_{j\in\{0,f,h\}} m_j p_j (V_h - V_f) \\
&\quad -x(1-\mu)(V_{fh} - V_f) - \tau(V_{fh} - V_f)
\end{aligned} \tag{3}$$

where the indicator function $1_{\{i=h\}} = 1$ (and zero otherwise), and $\rho = r/\alpha$ is the discount factor adjusted by the arrival rate. It measures the severity of the search frictions in the economy: as ρ goes to zero, search frictions vanish. The first term on the right-hand side of each of the value functions in (3), is the expected utility from current consumption matches, i.e. those with sellers who can produce one's desired good. With probability xm_j the agent meets a seller with portfolio j who can produce his desired consumption good which pays off utility $u(q_j^i)$, when currency i is used in the transaction. The second term is the expected value from changing the portfolio from spending (or acquiring) a unit of currency, which occurs with probability $x(1-\mu)$. For holders of the home currency, $i = h$, the third term, is the expected loss from having the government confiscate one's holdings of home currency. This occurs with probability τ, independent of the size of one's portfolio. For agents holding portfolio fh, the first term represents the expected utility from meeting sellers and acquire consumption goods by spending a unit of currency. The first component in the brackets is the utility from choosing to spend the dollar, with probability p_j, in a match with a seller with portfolio j. The second component is the utility derived from choosing to spend the domestic currency in that match. The second and third terms are the expected payoffs from changing portfolio states and the last term is the expected loss due to confiscation.

It is useful to manipulate the value functions in (3) in order to show that, for any equilibrium p^*, the value of multiple-unit portfolios can be expressed as linear combinations of the values associated with single-unit holdings. Specifically for homogenous portfolios of currency $i = h, f$

$$\begin{aligned}
V_i &= \tfrac{A_i}{1-\mu}\left[m_0 u(q_0^i) + m_f u(q_f^i) + m_h u(q_h^i)\right] \\
V_{2i} &= (1 + A_i)V_i
\end{aligned} \tag{4}$$

while for a diversified portfolio

$$V_{fh} = \frac{A_h}{1-\mu} \sum_{j\in\{0,f,h\}} m_j p_j^* \left[u(q_j^f) - u(q_j^h) + V_h - V_f\right] + V_h + A_{fh}V_f \tag{5}$$

where

$$A_h = \frac{x(1-\mu)}{\rho+\tau+x(1-\mu)} < A_f = \frac{x(1-\mu)}{\rho+x(1-\mu)} < A_{fh} = \frac{\tau+x(1-\mu)}{\rho+\tau+x(1-\mu)} < 1$$

such that as $\rho \to 0$, then A_f, $A_{fh} \to 1$ while $A_h < 1$ for $\tau > 0$. It is immediate that, in a monetary equilibrium, the expected lifetime utility of any portfolio is bounded below by zero. It is also concave in the size of currency holdings, whereby $V_{2i} \leq 2V_i$ and $V_{fh} \leq V_h + V_f$ for all parameters and any p^*. Furthermore, $(V_{2f} - V_f)/V_f = A_f > (V_{2h} - V_h)/V_h = A_h$, i.e. the percentage gain in expected lifetime utility from acquiring a second dollar is greater than the percentage gain in utility from acquiring a second unit of the risky home currency.

In equilibrium, the quantities exchanged in the matches are such that the cost of producing equals the expected utility from acquiring a unit of currency, that is

$$\begin{array}{cc} \text{Foreign} & \text{Domestic} \\ q_0^f = V_f & q_0^h = V_h \\ q_f^f = V_{2f} - V_f & q_f^h = V_{fh} - V_f \\ q_h^f = V_{fh} - V_h & q_h^h = V_{2h} - V_h \end{array} \tag{6}$$

Although the buyer with portfolio fh can choose among eight possible pure strategies, we concentrate on the two extreme cases in which the buyer always spends the dollar, $p^* = (1,1,1)$, or he always spend the home currency, $p^* = (0,0,0)$. We think of the $p^* = (1,1,1)$ case as corresponding to 'high dollarization' – agents prefer to spend the foreign rather than the home currency when a trading opportunity arises. What is interesting about this strategy is that the buyer gives up the safe currency and chooses to hold onto the risky currency rather than dumping the risky currency when the opportunity arises. This is also the equilibrium that best corresponds to the idea of currency competition – good money circulates more widely than bad money. We consider the $p^* = (0,0,0)$ equilibrium to be a 'low-dollarization' equilibrium because the buyer is essentially dumping the risky currency when given the opportunity and hoarding the safe currency. In short, this is the Gresham's Law equilibrium. Finally, these extreme strategies are appealing in that they are non-discriminatory, i.e. all sellers are offered the same currency. Although we concentrate on these two extreme strategies analytically, we do consider the other pure strategy equilibria but do so via numerical illustrations.

To prove existence of an equilibrium, we follow the approach of Camera and Corbae. Given the conjecture that $d = 1$ and a strategy vector p^* are optimal, we derive necessary conditions

such that the conjectured strategies are individually optimal. Then we solve for the equilibrium value functions, quantities, and distributions of portfolios, providing conditions sufficient to satisfy individual optimality, in term of the parameters of the model.

4.1 Individual optimality conditions

To determine the conditions under which the conjectured transaction pattern is individually optimal we must do the following. First, for any given p^*, $d = 1$ is optimal if agents choose to spend at least one unit of currency but no more than one unit. This implies that for buyers holding 2 units of currency, upon meeting a seller with no currency, the trade surplus from spending one unit is greater than the trade surplus from spending both units. Since sellers holding one unit of currency cannot accept two units of currency due to the inventory constraint, the only meetings that matter for this constraint are those between 2-unit buyers and sellers holding no currency. With three 2-unit buyers there are three optimality conditions that must be satisfied given by:

$$\begin{aligned} u(V_f) + V_f - V_{2f} &> u(V_{2f}) + V_0 - V_{2f} && (2f \text{ buyer}) \\ u(V_h) + V_h - V_{2h} &> u(V_{2h}) + V_0 - V_{2h} && (2h \text{ buyer}) \\ \max\{u(V_f) + V_h, u(V_h) + V_f\} - V_{fh} &> u(V_{fh}) + V_0 - V_{fh} && (fh \text{ buyer}) \end{aligned} \tag{7}$$

Second, it must be the case that the trade surplus buyers receive from spending one unit of currency is larger than the payoff from walking away. It is straightforward to show that if buyers holding one unit of currency choose not to walk away, 2-unit buyers will not walk away either. Since 'rich' sellers (those holding currency) produce less than 'poor' sellers (those with no currency), if buyers with one unit of currency buy from rich sellers, they will also buy from poor sellers. Since there are 2 poor buyer states, f and h, and 2 rich seller states, f and h, then the condition to spend at least one unit generates four optimality constraints and are given by:

$$\begin{aligned} u(V_{fh} - V_f) + V_0 - V_h &> 0 && (h \text{ buyer, } f \text{ seller}) \\ u(V_{2h} - V_h) + V_0 - V_h &> 0 && (h \text{ buyer, } h \text{ seller}) \\ u(V_{2f} - V_f) + V_0 - V_f &> 0 && (f \text{ buyer, } f \text{ seller}) \\ u(V_{fh} - V_h) + V_0 - V_f &> 0 && (f \text{ buyer, } h \text{ seller}) \end{aligned} \tag{8}$$

Finally, under the conjecture that $d = 1$, we must verify that a buyer at portfolio fh chooses to spend either the dollar or the domestic currency when he meets sellers with portfolios $0, f, h$. Consequently, the trading surplus from spending one currency must be larger than the trading

surplus from spending the other currency. Since there are three sellers, $\{0, f, h\}$, there are three conditions that need to be satisfied in order for any p to be optimal:

$$\begin{aligned} p_0 = 1 \quad &\Leftrightarrow \; u(V_f) + V_h - V_{fh} > u(V_h) + V_f - V_{fh} \\ p_f = 1 \quad &\Leftrightarrow \; u(V_{2f} - V_f) + V_h - V_{fh} > u(V_{fh} - V_f) + V_f - V_{fh} \\ p_h = 1 \quad &\Leftrightarrow \; u(V_{fh} - V_h) + V_h - V_{fh} > u(V_{2h} - V_h) + V_f - V_{fh} \end{aligned} \tag{9}$$

and $p_i = 0 \; \forall i$, if the corresponding inequality is reversed.

Despite the large number of inequalities to be satisfied, it turns out that there is one of particular interest, namely the first one in (9). It describes the fh buyer's decision to offer a dollar to a seller with no money. Rewrite it as

$$S(V_f) \equiv u(V_f) - V_f > u(V_h) - V_h \equiv S(V_h)$$

This expression has a simple and intuitive interpretation. $u\,(V_f)$ is the utility gain from spending a dollar and consuming $q_0^f = V_f$. V_f is also the value of the foregone portfolio state f. Thus, the net gain from spending a dollar is $S(V_f) \equiv u(V_f) - V_f$. Similarly, $S(V_h)$ is the net gain from using the home currency instead. Thus the agent has to compare the two strategies and chooses to spend the dollar if the net gain is larger. Note that the two sides of the inequality evaluate the same function at different points. Thus, the functional form of preferences matters in determining whether spending the dollar is optimal, and the relative value of the two currencies is the critical element. Specifically, there are two cases to consider depending on whether the net gain is monotonically increasing in V or if it decreases as V becomes large.

If $S(V)$ is monotonically increasing, then an immediate result for $p^* = (1, 1, 1)$ to individually optimal is that $V_f > V_h$. In short the dollar must be more valued than the domestic currency. This makes intuitive sense because the domestic currency is risky. By making a purchase with the domestic currency the buyer transfers the risk to the seller. However, the seller will not accept the risk unless he is compensated for it. A natural way to compensate the seller is to ask for a smaller quantity of goods. Since agents prefer current to future consumption, and since the net gain increases monotonically in the value of the transaction, the buyer will prefer to make a dollar purchase, whenever possible. On the other hand, $p^* = (0, 0, 0)$ can be an equilibrium only if $V_h > V_f$. This implies that the domestic currency, despite its fundamental risk, has greater purchasing power than the dollar. In short, not only does the seller accept a risky currency but

he chooses to compensate the buyer in the process by producing more today! It is hard to believe that this behavior can be supported as an equilibrium on a large region of the parameter space.[13]

If $S(V)$ is not monotonically increasing, then it is possible that $p^* = (0, 0, 0)$ can be supported when $V_h < V_f$. Intuitively this is because even if the dollar buys more goods, there is a high opportunity cost in spending it. Consequently, $S(V_f)$ could be very small if the net gain exhibits decreasing returns for high value transactions. In this case, despite the fact that it buys less today, it is better to spend the domestic currency and hold on to the dollar for future purchases. Doing so raises the net gain since, the lower opportunity cost more than compensates for the drop in consumption.

To illustrate how the form of preferences affects this surplus and the possible equilibrium transaction patterns, we will consider two forms of utility: quasi-linear and CRRA. Suppose utility is quasi-linear, given by $u(q) = q^\sigma + q$, $0 < \sigma < 1$. This implies that, in a match between an fh buyer and a seller with no money, the net gain, $S(V) = V^\sigma$, is monotonically increasing in V. Now consider the CRRA function $u(q) = q^\sigma$, a specification common in many search-theoretic models of money. This implies that $S(V) = V^\sigma - V$ is a hump-shaped function that is zero at $V = 0, 1$ and has a unique maximum at $\hat{V} = \sigma^{\frac{1}{1-\sigma}} < 1$.

A symmetric stationary monetary equilibrium is a set of value functions satisfying (3), a distribution of portfolios satisfying (1)-(2) and $\dot{m}_i = 0$, prices given by (6), and a set of symmetric strategies d and p satisfying (7)-(9).

5 The High-Dollarization Equilibrium: $p^* = (1, 1, 1)$

In this section we determine conditions under which the buyer with portfolio fh decides to spend the dollar in all transactions. Under the conjecture that $p^* = (1, 1, 1)$, (5) becomes

$$V_{fh} = V_f + A_h V_h \tag{10}$$

so that, using (6)

$$q_h^f = V_f - (1 - A_h)V_h \quad \text{and} \quad q_f^h = A_h V_h \tag{11}$$

where $q_h^f > 0$ only if $(1 - A_h)V_h < V_f$.

Given the conjectured pattern of transactions, the distribution of portfolio holdings must be feasible, i.e. it must satisfy (1)-(2) and $m_i \in (0, 1)$. In the steady state it must also be stationary,

[13] $V_h > V_f$ would be required for any $N > 2$ since the first inequality in (9) does not depend on N.

i.e. $\dot{m}_i = 0$ for all portfolios i. Due to the linear dependency of m_i we need to consider only three flow conditions. We focus on

$$\dot{m}_{2f} = m_f\left(m_f + m_{fh}\right) - m_{2f}\left(m_0 + m_h\right) = 0 \tag{12}$$

$$\dot{m}_{2h} = x[m_h^2 - m_{2h}\left(m_0 + m_f\right)] + \eta m_h - \tau m_{2h} = 0 \tag{13}$$

$$\dot{m}_{fh} = x[m_f m_{2h} + m_h m_{2f} + 2m_h m_f - m_{fh}\left(m_0 + m_f\right)] + \eta m_f - \tau m_{fh} = 0 \tag{14}$$

In the appendix we derive sufficient conditions for existence of a stationary distribution and show that it is unique.

Using (4), (10), and (11), we obtain:

$$V_h = \frac{A_h\left[m_0 u(V_h) + m_f u(A_h V_h) + m_h u(A_h V_h)\right]}{1-\mu} \tag{15}$$

$$V_f = \frac{A_f\left[m_0 u(V_f) + m_f u(A_f V_f) + m_h u(V_f - (1-A_h)V_h)\right]}{1-\mu} \tag{16}$$

where (16) is defined only for $V_f > (1-A_h)V_h$. Under the conjecture that high-dolalrization is a monetary equilibrium, then let (V_f^*, V_h^*) denote a positive fixed point of the map given by (15) and (16). We discuss existence of positive fixed points in the next lemma (all proofs of lemmas and propositions are in the appendix).

Lemma 1. *Suppose $d = 1$ and $p^* = (1,1,1)$ is a monetary equilibrium. Then there always exists a unique V_h^*, and $V_f^* = V_h^*$ whenever currency risk is absent. If currency risk is present, then then can be at most two distinct and mutually exclusive cases:*

(i) $V_f^* > V_h^*$, which always exists and, if ρ is sufficiently small, satisfies $\frac{V_f^*}{V_h^*} \leq \frac{1-A_h}{1-A_f}$

(ii) $V_f^* < V_h^*$ which not always exists and, in particular, does not exist either if τ is sufficiently large or if ρ is sufficiently small.

The lemma shows that if there is no fundamental difference between the two currencies ($\tau = 0$) and $p^* = (1,1,1)$ and $d = 1$ is individually optimal, then the only monetary equilibrium is such that both currencies are identically valued. Hence, the fh buyer would be indifferent between spending

the home relative to the dollar and could choose any $p_i \in [0,1]$. When $\tau > 0$, however, this is never the case. This is a general result which holds for any set of concave preferences.

One immediate implication of Lemma 1 is that, although currency exchange does not occur, the model generates an equilibrium distribution of real exchange rates (or relative prices). There is more than one relative price in the model since the same seller will be willing to produce different quantities for different currencies. Let $R_i = q_i^f / q_i^h$ denote the relative price offered by a seller with portfolio $i \in \{0, f, h\}$. This measure gives us the real value of the dollar to a unit of the domestic currency. Using (11) we obtain

$$\begin{aligned} R_0 &= \frac{V_f}{V_h} \\ R_f &= \frac{A_f}{A_h}\frac{V_f}{V_h} \\ R_h &= 1 + \frac{(V_f/V_h) - 1}{A_h} \end{aligned}$$

When $\tau = 0$ the distribution of real exchange rates is degenerate, $R_i = 1 \ \forall i$, since $V_f = V_h$ and $\frac{A_f}{A_h} = 1$. When there is some currency risk, however, then $R_0 < R_h < R_f$ for $\frac{V_f}{V_h} < \frac{1-A_h}{1-A_f}$ but $R_0 < R_f < R_h$ if $\frac{V_f}{V_h} > \frac{1-A_h}{1-A_f}$. Thus, as the risk on the domestic currency increases from zero, the observed spread of real exchange rates increases. Although this is a cross-section of real exchange rates, it loosely corresponds to the idea that greater currency risk leads to an increase in the volatility of observed real exchange rates between the domestic currency and the dollar.

5.1 Existence of the $p^* = (1,1,1)$ equilibrium

We study existence of equilibrium by considering specific preferences. Suppose utility is given by $u(q) = q^\sigma + q$ where $0 < \sigma < 1$. We are able to obtain a closed-form solution for V_h but not V_f :

$$V_h = \left\{ \frac{A_h \left[m_0 + m_f A_h^\sigma + m_h A_h^\sigma \right]}{1 - \mu - A_h \left[m_0 + (m_f + m_h) A_h \right]} \right\}^{\frac{1}{1-\sigma}} \tag{17}$$

$$V_f = \left\{ \frac{A_f \left[m_0 + m_f A_f^\sigma + m_h (1 - (1 - A_h)(V_h/V_f))^\sigma \right]}{1 - \mu - A_f \left[m_0 + m_f A_f + m_h (1 - (1 - A_h)(V_h/V_f)) \right]} \right\}^{\frac{1}{1-\sigma}} \tag{18}$$

The conditions in (8)-(9) reduce to:

$$\left[\frac{(1 + A_f)^\sigma - 1}{1 - A_f} \right]^{\frac{1}{1-\sigma}} < V_f < \left(\frac{A_f^\sigma}{1 - A_f} \right)^{\frac{1}{1-\sigma}} \tag{19}$$

$$\left[\frac{(1+A_h)^\sigma - 1}{1-A_h}\right]^{\frac{1}{1-\sigma}} < V_h < \left(\frac{A_h^\sigma}{1-A_h}\right)^{\frac{1}{1-\sigma}} \tag{20}$$

$$V_f > V_h \tag{21}$$

$$(1-A_h)\,V_h + A_f^\sigma V_f^\sigma > A_h^\sigma V_h^\sigma + (1-A_f)\,V_f \tag{22}$$

$$(1-A_h)\,V_h + V_f^\sigma > (V_f + A_h V_h)^\sigma \tag{23}$$

The two inequalities (19)-(20) are essentially the same as in Camera and Corbae (1999). In short, the value of holding a unit of currency must be high enough to prevent 2-unit buyers from spending all of their cash but not high enough to prevent expenditures by poor buyers on rich sellers. The new restrictions arising from multiple currencies are (21), (22) and (23). Inequality (21) is the condition for a buyer with a mixed portfolio to spend the dollar rather than the domestic currency on a poor seller while (22) is the condition that he spends the dollar on a rich seller holding a unit of the domestic currency. Inequality (23) ensures that a buyer fh only spends the dollar and not both. Equation (21) shows that in equilibrium a necessary condition is that the dollar is more valued than the home currency, $V_f > V_h$. This is a consequence of the net gain being monotonically increasing in V.

With these preferences we provide sufficient conditions for the existence of a unique high dollarization equilibrium.

Proposition 1. *Consider quasi-linear preferences and a stationary distribution supporting the transaction pattern $d = 1$ and $p^* = (1,1,1)$. There exist positive values σ_H and ρ_H such that if $\sigma \in (0, \sigma_H)$ and $\rho \in (0, \rho_H)$ then the high dollarization equilibrium exists and is unique.*

The intuition for these parameter values is as follows. For sufficiently low search frictions (small ρ), agents are always willing to buy now rather than wait for a better deal.[14] Low values of σ imply that the marginal utility of consumption is very high but diminishes rapidly. This ensures that agents spend at least one unit of currency but not two. If search frictions are low, the buyer holding one unit of each currency is willing to spend the safe foreign currency in all matches and hold onto the risky domestic currency. This is so because the surplus in trade is increasing in the

[14] This is because the amount of goods produced by a rich seller converges to the quantity produced by a poor seller as A_f and A_h approach 1. So there is nothing to gain by waiting to meet a poor seller.

value of the currency and agents discount future consumption less thereby increasing the value of the dollar relative to the home currency.

If preferences are CRRA, $u(q) = q^\sigma$, the equilibrium (V_f^*, V_h^*) must satisfy

$$V_h = \left\{ \frac{A_h}{1-\mu} \left[m_0 + m_f A_h^\sigma + m_h A_h^\sigma \right] \right\}^{\frac{1}{1-\sigma}}$$

$$V_f = \left\{ \frac{A_h}{1-\mu} [m_0 + m_f A_f^\sigma + m_h (1 - (1 - A_h)(V_h/V_f))^\sigma] \right\}^{\frac{1}{1-\sigma}}$$

It is straightforward to show that V_f and V_h approach 1 as $\rho, \tau \to 0$. By Lemma 1, when ρ is sufficiently small, then $V_f^* > V_h^*$, in which case the conditions in (8)-(9) reduce to:

$$\begin{aligned} [(1+A_f)^\sigma - 1]^{\frac{1}{1-\sigma}} &< V_f < A_f^{\frac{\sigma}{1-\sigma}} \\ [(1+A_h)^\sigma - 1]^{\frac{1}{1-\sigma}} &< V_h < A_h^{\frac{\sigma}{1-\sigma}} \\ V_f^\sigma - V_f &> V_h^\sigma - V_h \\ (A_h V_h + V_f - V_h)^\sigma - V_f &> A_h^\sigma V_h^\sigma - V_h \\ V_h + V_f^\sigma &> (V_f + A_h V_h)^\sigma \end{aligned} \tag{24}$$

a set of constraints that mirrors (19)-(23). Given these expressions we can state the following proposition:

Proposition 2. *Consider CRRA preferences. If ρ is sufficiently small then the high dollarization equilibrium does not exist.*

The proof is immediate: the third inequality in (24) is violated when V_f^* is close to 1. Comparing Propositions 1 and 2 we find that depending on the functional form of preferences, the high dollarization equilibrium may or may not exist for the same parameter values. This seems surprising but is the result of the properties of the net gain from spending a dollar relative to the domestic currency under these two utility specifications. The key element is whether or not $S(V)$ is monotonically increasing or not. To see this consider Figure 1 which shows $S(V)$ for the two utility specifications considered. For $u(q) = q^\sigma$, $S(V)$ is decreasing for values of V close to 1. In this case, for ρ small, $V_f^* > V_h^*$ and V_f^* is close to 1. Thus, it must be the case that $S(V_f^*) < S(V_h^*)$, so $p_0 = 0$ is optimal and the high dollarization equilibrium cannot exist. However, when $u(q) = q^\sigma + q$, $S(V)$

is monotonically increasing, hence for any $V_f^* > V_h^*$, it must be the case that $S(V_f^*) > S(V_h^*)$ so $p_0 = 1$ is always optimal, which is needed for the high dollarization equilibrium to exist.

6 The Low Dollarization Equilibrium: $p^* = (0, 0, 0)$

We now want to consider a world in which dollars circulate in the domestic economy but despite being risky, the home currency circulates more widely. This is a common occurrence in many developing and transitional economies – agents use dollars for some transactions but carry out a majority of purchases using the risky home currency. Several papers have tried to model this phenomenon [see Chang (1994), Uribe (1997), Engineer (2000), Sibert and Liu (1998)]. The main drawback of all of these models is that they all rely on an ad-hoc assumption that the foreign currency has a relatively higher 'transaction cost' (or trading friction) associated with its use as a medium of exchange. We want to consider a world in which the fundamental trading environment and all trading frictions are the same for each currency.

An appealing aspect of this equilibrium strategy is that it corresponds to the case where buyers prefer to make purchases with the risky currency when the opportunity arises rather than holding onto it as in the $p^* = (1, 1, 1)$ equilibrium. In this equilibrium, fh buyers tend to 'hoard' the dollar and spend the risky domestic currency. This has the flavor of Gresham's Law – the circulation of good money is reduced while the circulation of bad money increases.

The solution procedure in the case where $d = 1$ and $p^* = (0, 0, 0)$ follows directly from that above. The value function expressions in (4) are unchanged, (5) becomes

$$V_{fh} = V_h + A_{fh} V_f$$

while the quantities in (6) still hold with the only changes being

$$q_h^f = A_{fh} V_f \quad \text{and} \quad q_f^h = V_h - (1 - A_{fh}) V_f$$

Substituting the quantities into V_f and V_h yields

$$V_f = \frac{A_f \left[m_0 u(V_f) + m_f u(A_{fh} V_f) + m_h u(A_f V_f) \right]}{1 - \mu} \tag{25}$$

$$V_h = \frac{A_h \left[m_0 u(V_h) + m_f u(A_h V_h) + m_h u(V_h - (1 - A_{fh}) V_f) \right]}{1 - \mu} \tag{26}$$

where (26) is defined only for $V_h > (1 - A_{fh})V_f$. Once again, in a monetary equilibrium (V_f^*, V_h^*) must be a positive fixed point of the map defined by (25)-(26).

Lemma 2. *Suppose $d = 1$ and $p^* = (0, 0, 0)$ is a monetary equilibrium. Then there always exists a unique V_h^*, and $V_f^* = V_h^*$ whenever currency risk is absent. If currency risk is present, $V_f^* > V_h^*$ exists whenever search frictions are sufficiently limited. This requirement is also sufficient to guarantee that (V_f^*, V_h^*) is unique and that $\frac{V_f^*}{V_h^*} < \frac{1-A_h}{1-A_{fh}}$.*

Determining the conditions under which the conjectured buying strategies are individually optimal follows from (7)-(9). With regards to the equilibrium distribution of money holdings, the constraints (1) and (2) still hold but the steady-state flow conditions change and are listed in the appendix. As before, we can generate sufficient conditions for an equilibrium distribution to exist and can show that it is unique.

By setting $u(q) = q^\sigma + q$ we can derive a set of conditions comparable to (17)-(23), which are contained in the appendix. As noted earlier a critical requirement now is that $V_f < V_h$ for $p_0 = 0$ to be optimal. This inequality is at the core of the following proposition:

Proposition 3. *Consider quasi-linear preferences. If ρ is sufficiently small then the low dollarization equilibrium does not exist.*

This follows from the fact that limited search frictions only support $V_f^* > V_h^*$ (by Lemma 2). This, combined with the monotonicity of $S(V)$, makes $p_0 = 1$ optimal. Hence, the low dollarization equilibrium cannot exist for ρ small. In order for it to exist, either the parameters yield $V_f^* < V_h^*$, which seems unlikely when the domestic currency is risky, or $S(V)$ cannot be monotonically increasing in V.

For the utility function $u(q) = q^\sigma$, the following is proved:

Proposition 4. *Consider CRRA preferences and a stationary distribution supporting the transaction pattern $d = 1$ and $p^* = (0, 0, 0)$. there exist positive values $\hat{\sigma}_H$ and $\hat{\rho}_H$ such that if $\sigma \in (0, \hat{\sigma}_H)$ and $\rho \in (0, \hat{\rho}_H)$ then a the low dollarization equilibrium exists and is unique.*

The reason this equilibrium exists in this case is because the parameter restrictions on ρ and σ ensure that the solutions for V_f^* and V_h^* lie on the decreasing portion of $S(V)$. In this case, even though the dollar is more valuable and buys more goods, it is also a very valuable asset to give up. Hence, the net gain from spending a dollar is very low while the net gain from spending the domestic currency is higher despite its riskiness.

7 Discussion and Numerical Illustration

In this section we explore the existence of equilibria over sections of the parameter space to get an sense of the 'robustness' or 'fragility' of a particular equilibrium. The figures below were generated in the following way: 1) pick a pair of values for the variables defined on the axes of each figure, 2) use this pair of values to solve for the equilibrium distribution and value functions, 3) then check to see if the conjectured buying strategies are individually optimal. If an equilibrium exists, then that parameter pair is shaded. This was done for 1 million pairs of parameter values for all 8 pure strategies defined on $p^* = (p_0^*, p_f^*, p_h^*)$. Consequently, to generate each figure, 8 million possible equilibria were examined.

The first set of of numerical exercises that we conducted varies the degree of domestic currency risk, τ, and the ratio of the domestic money stock to the aggregate money stock in order to see how sensitive the transaction patterns are to the relative stocks of each currency in circulation. Figure 1 contains all of the equilibria for $u(q) = q^\sigma + q$. In this figure, we set $\sigma = .15$, $M = M_f + M_h = 1$, $\rho = .04$, and $x = .4$. What is surprising from this picture is that despite the fact that there are 8 possible pure strategies on $p^* = (p_0^*, p_f^*, p_h^*)$, we find equilibria for only two of them: $p^* = (1, 1, 1)$ and $p^* = (1, 0, 1)$. When we used $u(q) = q^\sigma$ for these parameter values, the only equilibrium that was observed was $p^* = (1, 1, 1)$.

The results in Figure 1 show that when the domestic currency risk is very low, the $p^* = (1, 1, 1)$ strategy is supported even when the percentage of dollars in the economy, M_f/M is less than 50%. However, as the domestic currency risk rises, the $p^* = (1, 1, 1)$ equilibrium is only supported if dollars are the predominant source of currency in the economy. This closely corresponds to the idea of the economy being 'highly dollarized' because agents always spend the dollar when they have the choice and there are a large number of dollars circulating in the economy.

On the other hand, if the domestic currency is the primary source of currency, then the $p^* = (1, 0, 1)$ equilibrium is the relevant one. In this situation, agents holding portfolio fh only spend the dollar on sellers with no money or those holding the domestic currency. The reason is that they receive a large quantity of goods for the dollar from these sellers – poor sellers want cash and rich sellers holding domestic currency want to diversify their currency portfolios. But when meeting a seller who already holds a dollar, the fh buyer opts to dump the domestic currency on him and avoid the risk of having it confiscated.

By reducing the curvature of the utility function from $\sigma = .15$ to $\sigma = .5$, we obtained the equilibria displayed in Figure 2 when $u(q) = q^\sigma$. If $u(q) = q^\sigma + q$ the only equilibrium we found for this set of parameter values was $p^* = (1, 1, 1)$. Looking at Figure 2, we see that when $u(q) = q^\sigma$ the relative amount of domestic currency in the economy is not an important factor for determining the equilibrium transaction pattern; domestic currency risk is the critical parameter. From this figure we observe an interesting pattern in spending behavior. For a given ratio of M_h/M, for low levels of domestic currency risk, the fh buyer spends the domestic currency when meeting a seller. As the domestic currency risk rises from a low level, the fh buyer begins using the dollar when buying from sellers who already hold a unit of the domestic currency. The reason is that the h sellers will produce a large quantity of goods in order to acquire a unit of safe currency and move to the diversified portfolio state fh. As risk continues to increase, the fh buyer starts spending the dollar on rich sellers in order to get the most goods. Finally, when the domestic risk is high enough, no seller is willing to produce much for the domestic currency. Hence, the fh buyer uses the dollar when meeting any seller.

The next set of numerical exercises we conducted vary the domestic currency risk against the level of search frictions in the economy. As our propositions showed, the value of ρ was a critical parameter in proving existence or non-existence of particular equilibria. Figure 3 contains all of the equilibria points for $u(q) = q^\sigma$ when we varied the level of search frictions and domestic currency risk. The parameter values are $\sigma = .5$, $M_f = .75, M_h = .25$ and $x = .4$. For these parameter values, when $u(q) = q^\sigma + q$ the only equilibrium we found was $p^* = (1, 1, 1)$ under the $d = 1$ equilibrium transaction pattern.

From Figure 3 we see that the high dollarization, $p^* = (1, 1, 1)$, equilibrium occurs when i) the domestic currency risk is high and search frictions are reasonably low, ii) search frictions are high

but τ is low or iii) both are high. The low dollarization, $p^* = (0, 0, 0)$, equilibrium occurs when i) search frictions are high and the domestic currency risk is very low, ii) search frictions are very low and the currency risk is sufficiently high, or iii) both are low.[15]

The intuition for this is as follows. When search frictions are very low, buyers know that they will meet another seller very quickly. If the domestic currency risk is also relatively low, then the quantity received for a unit of domestic currency is not much less than what is received for a dollar. By spending the domestic currency, the fh buyer accepts a slightly smaller quantity of goods today but gets rid of the risky currency. In addition, he knows that with a high probability he will meet another seller in the next period and can spend the dollar on that seller. Hence, he is more willing to accept fewer goods today in return for dumping the risky currency. When search frictions are high, the fh buyer knows that he will not consume again for awhile, hence he wants to get a sufficiently high amount of consumption today if a trading opportunity arises. This leads him to spend the dollar and hold onto the domestic currency despite the risk of losing it in the next period.

These numerical results are interesting because they confirm the following intuition: if the domestic currency risk is low and the economy is functioning well (trading is relatively easy to accomplish), then citizens will use the domestic currency first when conducting trades. However, if domestic currency risk is high and the economy is not functioning well, then currency subsitution occurs and agents will resort to spending the foreign currency. In short, risky domestic currencies and poorly functioning economies lead to dollarization. Thus our analysis is consistent with the view that a nation's citizens will continue using the domestic currency if the purchasing power risk is kept very low but once that risk gets too high, substantial dollarization will occur.

8 Conclusion

We have constructed a model of currency substitution based on first principles. We investigate how changing levels of risk on the domestic currency affect agents' transaction patterns and thus their willingness to use dollars as a preferred medium of exchange. Our results demonstrate that small changes in the degree of domestic currency risk, can result in moving from a low dollarized economy to a high dollarized economy. The positive aspect of our results that a low dollarized economy can

[15] It should be noted that as ρ goes to zero the probability of meeting someone in the next instant in time goes to 1. However, the probability of meeting a private agent versus meeting the government stays exactly the same.

avoid becoming a highly dollarized economy by implementing policies aimed at reducing currency risk and improving the trading environment so that the economy functions well. At the same time our results serve as a warning that dollarization will be unavoidable if currency risk is not kept under control.

Appendix

Existence and uniqueness of a stationary distribution of portfolios when $p^* = (1,1,1)$.

I. Sufficient conditions for existence. We use a procedure similar to that used by Zhou (1997). Specifically, consider the state space M and an equilibrium point $m^* \in M$. Define a real-valued function L on M, that satisfy the following requirements: (i) L is continuous and has continuous first-partial derivatives (ii) $L(m)$ has a unique minimum at m^* with respect to all other points in M. (iii) The function $\dot{L}(m)$ satisfies $\dot{L}(m) \leq 0$ for all $m \in M$. This function L is called a Liapunov function. We then rely on the Liapunov theorem stating that if there exists a Liapunov function the equilibrium point m^* is stable and if the function $\dot{L}(m) < 0$ at all $m \neq m^*$ then the stability is asymptotic.

Equations (1)-(2) imply that $\{m_0, m_f, m_h, \eta\}$ are single-valued functions of $\{m_{2f}, m_{2h}, mf_h\}$:

$$\begin{aligned} m_f &= M_f - m_{fh} - 2m_{2f} \\ m_h &= M_h - m_{fh} - 2m_{2h} \\ m_0 &= 1 - M_f - M_h + m_{2f} + m_{2h} + m_{fh} \end{aligned} \tag{27}$$

and the government budget constraint

$$\eta = \frac{\tau M_h}{1 - (m_{2f} + m_{2h} + m_{fh})} \tag{28}$$

Using (27) in (12)-(14) we get:

$$\begin{aligned} \dot{m}_{2f} &= (M_f - m_{fh} - 2m_{2f})(M_f - 2m_{2f}) - m_{2f}(1 - M_f + m_{2f} - m_{2h}) \\ \dot{m}_{2h} &= x\left[(M_h - m_{fh} - 2m_{2h})^2 - m_{2h}(1 - M_h - m_{2f} + m_{2h})\right] \\ &\quad + \eta(M_h - m_{fh} - 2m_{2h}) - \tau m_{2h} \\ \dot{m}_{fh} &= x[(M_f - m_{fh} - 2m_{2f})m_{2h} + (M_h - m_{fh} - 2m_{2h})m_{2f} \\ &\quad + 2(M_h - m_{fh} - 2m_{2h})(M_f - m_{fh} - 2m_{2f}) - m_{fh}(1 - M_h - m_{2f} + m_{2h})] \\ &\quad + \eta(M_f - m_{fh} - 2m_{2f}) - \tau m_{fh} \end{aligned} \tag{29}$$

Define the 3x1 vector $m = [m_1, m_2, m_3]$ where $m_1 = m_{2f}$, $m_2 = m_{2h}$, $m_3 = m_{fh}$ and $m_i \in [0,1]$ with $m_1 + m_2 + m_3 \leq 1$. Then define the system in (29) as $\dot{m} = F(m)$ where $F(m)$ is a 3×1 vector. Denote by $F(m)[i]$ the ith row of $F(m)$. Then, letting $\frac{dF(m)[i]}{dm_j} = a(i,j)$, $j, i = 1,2,3$, the Jacobian

of $F(m)$ is a 3x3 matrix

$$\frac{dF(m)}{dm} = \begin{bmatrix} a(1,1) & ... & a(1,3) \\ : & a(2,2) & : \\ a(3,1) & ... & a(3,3) \end{bmatrix}$$

where, recalling that τ is a constant,

$$a(1,1) = m_{2h} + 2m_{fh} + 6m_{2f} - 3M_f - 1 < 0$$
$$a(1,2) = m_{2f}$$
$$a(1,3) = 2m_{2f} - M_f < 0 \ \text{ (since } M_f > 2m_{2f})$$
$$a(2,1) = xm_{2h} + \tfrac{d\eta}{dm_{2f}}\left(M_h - m_{fh} - 2m_{2h}\right) > 0$$
$$a(2,2) = x\left[4m_{fh} + m_{2f} + 6m_{2h} - 3M_h - 1\right] + \tfrac{d\eta}{dm_{2h}}\left(M_h - m_{fh} - 2m_{2h}\right) - 2\eta - \tau$$
$$a(2,3) = -2x\left[M_h - m_{fh} - 2m_{2h}\right] + \tfrac{d\eta}{dm_{fh}}\left(M_h - m_{fh} - 2m_{2h}\right) - \eta$$
$$a(3,1) = x\left[4m_{fh} + 4m_{2h} - 3M_h\right] + \tfrac{d\eta}{dm_{2f}}\left(M_h - m_{fh} - 2m_{2f}\right) - 2\eta$$
$$a(3,2) = x\left[2m_{fh} + 4m_{2f} - 3M_f\right] + \tfrac{d\eta}{dm_{2h}}\left(M_f - m_{fh} - 2m_{2f}\right)$$
$$a(3,3) = x\left[4m_{fh} + 4m_{2f} + 2m_{2h} - M_h - 2M_f - 1\right] + \tfrac{d\eta}{dm_{fh}}\left(M_f - m_{fh} - 2m_{2f}\right) - \eta - \tau$$

where we note that $\frac{d\eta}{dm_i} > 0$ for all m_i. We note that $M_f > m_{fh} + 2m_{2f}$ and $M_h > m_{fh} + 2m_{2h}$ if $m_f, m_h > 0$, using (27)-(**??**). Substituting the infimum $M_f = m_{fh} + 2m_{2f}$ and $M_h = m_{fh} + 2m_{2h}$ in $a(2,2), a(3,2)$ and $a(3,3)$, it is easy to show that all of these terms are strictly negative as $\eta \to 0$ while $a(2,3) \to 0^-$. Thus there are small values of $\eta > 0$ such that $a(2,2), a(2,3), a(3,2)$ and $a(3,3)$ are all negative. Note that $\eta \to 0$ when either $\tau \to 0$ or $M_h \to 0$, and so does $\frac{d\eta}{dm_i} > 0$ for all m_i.

We want to show that $\frac{dF(m)}{dm}$ is negative definite. To do so we can consider the sign of its three principal minors:

$$D_1 = a(1,1), \quad D_2 = \begin{vmatrix} a(1,1) & a(1,2) \\ a(2,1) & a(2,2) \end{vmatrix}, \text{ and } D_3 = \begin{vmatrix} a(1,1) & a(1,2) & a(1,3) \\ a(2,1) & a(2,2) & a(2,3) \\ a(3,1) & a(3,2) & a(3,3) \end{vmatrix}$$

We note that $D_1 = a(1,1) < 0$. This is so because $M_f \geq m_{fh} + 2m_{2f}$ (with strict inequality if $m_f > 0$) using (27). Substituting $M_f = m_{fh} + 2m_{2f}$ in $a(1,1)$ provides a maximum for $a(1,1)$. This maximum is seen to be negative since $-m_{fh} + m_{2h} - 1 < 0$.

The minor $D_2 = a(1,1)a(2,2) - a(1,2)a(2,1)$. Note that $a(1,2)$ and $a(2,1)$ are both positive, and that their product tends to zero as x and η shrink to 0. Furthermore, $a(2,2) < 0$ as η tends

to zero because $-3M_h - 1 + 4m_{fh} + m_{2f} + 6m_{2h} < 0$ (since $M_h \geq m_{fh} + 2m_{2h}$). Thus $D_2 > 0$ for x and η small (i.e. either τ or M_h small).

The third minor is

$$\begin{aligned} D_3 = & a(1,1)\left[a(2,2)a(3,3) - a(2,3)a(3,2)\right] \\ & -a(1,2)\left[a(2,1)a(3,3) - a(2,3)a(3,1)\right] \\ & +a(1,3)\left[a(2,1)a(3,2) - a(2,2)a\left(3,1\right)\right] \end{aligned}$$

Note that as $\eta, x \to 0$ then the second and third line in D_3 vanish, and that the first line, is strictly negative and given by

$$\tau^2\left(-3M_f - 1 + m_{2h} + 2m_{fh} + 6m_{2f}\right)$$

We conclude that there exist an M_h and x positive but sufficiently small such that $D_1 < 0$, $D_2 > 0$ and $D_3 < 0$. Thus, for M_h and x small the matrix $\frac{dF(m)}{dm}$ is negative definite (see Chiang).

Since $F(m)$ is a 3x1 vector ('$'$' transposes it), define the function

$$L(m) = \left[F(m)\right]' F(m) = (\dot{m}_{2f})^2 + (\dot{m}_{2h})^2 + (\dot{m}_{fh})^2 \geq 0$$

We show it is a Liapunov function. It is continuous (by construction) and it has continuous first partial derivatives. Recalling that the vector $F(m) = \dot{m}$, that $d[F(m)]'/dt = \dot{m}'\frac{dF(m)}{dm}$ (a 1x3 vector) and that $dF(m)/dt = \left[\frac{dF(m)}{dm}\right]'\dot{m}$ (a 3x1 vector) then the time derivative of $L(m)$ is the quadratic form (a scalar)

$$\dot{L}(m) = \dot{m}'\frac{dF(m)}{dm}\dot{m} + \dot{m}'\left[\frac{dF(m)}{dm}\right]'\dot{m}$$

so that $\dot{L}(m) = 0$ if $\dot{m} = 0$, and < 0 if $\dot{m} \neq 0$ for x and M_h small, since $\frac{dF(m)}{dm}$ is negative definite.

To show that there exists an m^* such that $L(m^*) = 0$ we use a proof by contradiction. If $L(m) = \dot{m} \neq 0$ for all m defined above then $\dot{L}(m) \neq 0$. Since m is defined on a compact set it follows that $\dot{L}(m)$ has a maximum, say $l < 0$ (because of negative definiteness). But this cannot be since, defining $m(t)$ to be the state of the system at date t,

$$\int_0^t \dot{L}(m(s))ds = L(m(t)) - L(m(0)) \leq lt \Rightarrow L(m(t)) \leq lt + L(m(0))$$

which in turn implies $L(m(t)) \to -\infty$ as $t \to \infty$. This can't be since at every date, by construction, $L(m) \geq 0$. Thus $L(m)$ must be reaching a minimum 0 at some m^*. To show that m^* is unique, see below.

Thus $L(m)$ is a Liapunov function, and applying the Liapunov Theorem (see Azariadis, 1993, for a discrete time version) the unique equilibrium m^* is asymptotically stable if x and M_h are positive but sufficiently small. The money distribution m^* is unique and stationary.

II. Uniqueness. Using (27)-(28) and $M_f + M_h < 2$, then $m_i > 0$ and $\eta < 1$ require

$$m_{fh} + 2m_{2f} < M_f < 2 - M_h < 2 - m_{fh} - 2m_{2h}$$

$$m_{fh} + 2m_{2h} < M_h < \frac{1 - (m_{2f} + m_{2h} + m_{fh})}{\tau} \quad .$$

We now show that for a feasible pair $\{m_{2h}, m_{2f}\}$, if m^*_{fh} solves (14), then it must be unique. Using (14) and (a1)-(a4) we obtain

$$m_{fh} = \frac{(M_f - m_{fh} - 2m_{2f})\left(\frac{\tau M_h}{1-m_{2f}-m_{2h}-m_{fh}} + xm_{2h}\right)}{\tau + x(1 - M_h + m_{2h} - m_{2f})} + \frac{x(M_h - m_{fh} - 2m_{2h})[m_{2f} + 2(M_f - m_{fh} - 2m_{2f})]}{\tau + x(1 - M_h + m_{2h} - m_{2f})}.$$

The right hand side can be shown to be strictly decreasing in m_{fh} for all feasible values of m_{fh}, m_{2h}, and m_{2f}. It then follows that if there is a feasible m^*_{fh} that solves this expression, then it is unique.

We now show that for a feasible value of m_{fh}, a unique pair $\left\{m^*_{2h}, m^*_{2f}\right\}$ solves (12) and (13). Using (12) and (a1)-(a4) we obtain $m_{2h} = f(m_{fh}, m_{2f})$ where

$$f(m_{fh}, m_{2f}) = 1 - M_f + m_{2f} - \frac{(M_f - m_{fh} - 2m_{2f})(M_f - 2m_{2f})}{m_{2f}}$$

which is easily seen to be increasing in m_{2f} for feasible values $m_{2f} \leq (M_f - m_{fh})/2$. Furthermore it is concave in m_{2f}.

Using (13) and (a1)-(a4) we obtain $m_{2f} = h(m_{fh}, m_{2h})$ where

$$h(m_{fh}, m_{2h}) = \frac{\tau}{x} + 1 - M_h + m_{2h} - \frac{[\eta + x(M_h - m_{fh} - 2m_{2h})](M_h - m_{fh} - 2m_{2h})}{xm_{2h}}$$

which is easily seen to be increasing and concave in m_{2h} for feasible values $m_{2h} \leq (M_h - m_{fh})/2$,since η is increasing in m_{2h}. Note also that $f(m_{fh}, m_{2f}) \to -\infty$ as $m_{2f} \to 0$ and $h(m_{fh}, m_{2h}) \to -\infty$ as $m_{2h} \to 0$. Note that $m_{2h} \leq (M_h - m_{fh})/2 < f(m_{fh}, (M_f - m_{fh})/2)$ and $m_{2f} \leq (M_f - m_{fh})/2 < h(m_{fh}, (M_h - m_{fh})/2)$. The properties of the two functions imply there is a single crossing point for the two functions in the feasible part of the (m_{2h}, m_{2f}) plane. Thus, for any feasible value of

m_{fh} and η, there is a unique pair $\left\{m_{2h}^*, m_{2f}^*\right\}$ that solves the system

$$\begin{cases} m_{2h} = f(m_{fh}, m_{2f}) \\ m_{2f} = h(m_{fh}, m_{2h}) \end{cases}$$

Given the uniqueness of the values in (27)-(28) and m_{fh},then if a feasible distribution exists, it is unique.

Proof of Lemma 1.

Conjecture $d = 1$ and $p^* = (1, 1, 1)$ and consider $\tau > 0$. Using (10) it follows that

$$\rho V_h = x\left[m_0 u(V_h) + m_f u(A_h V_h) + m_h u(A_h V_h)\right] - x(1-\mu)V_h - \tau V_h \equiv H(V_h)$$

$H(V_h)$ is a monotonically increasing and strictly concave function of V_h, which starts at 0 and has a decreasing first derivative that vanishes as $V_h \to \infty$. Thus, it has two fixed points, one is $V_h = 0$ (the non-monetary equilibrium, which we ignore), and the other is $V_h^* > 0$.

Given V_h^* use once again (3) and (11) so that in equilibrium

$$\rho V_f = x\left[m_0 u(V_f) + m_f u(V_f - (1-A_h)V_h^*) + m_h u(A_f V_f)\right] - x(1-\mu)V_f \equiv F\left(V_f, V_h^*\right)$$

defined only for $V_f \geq (1-A_h)\, V_h^*$. $F\left(V_f, V_h^*\right)$ is strictly concave and monotonically increasing in V_f. As $V_f \to^+ (1-A_h)V_h^*$ then $F\left(V_f, V_h^*\right)$ converges to a positive value, and its slope becomes unbounded. Thus, the intermediate value theorem suggest there can be at most two positive fixed points to the map $\rho V_f = F\left(V_f, V_h^*\right)$.

1. A fixed point $V_f^* > V_h^*$ exists if

$$\rho V_f - F\left(V_f, V_h^*\right)\big|_{V_f = V_h^*} < 0 \quad \Leftrightarrow \quad H\left(V_h^*\right) < F\left(V_h^*\right) \tag{30}$$

since $F(V_f, V_h^*)$ is strictly concave and V_h^* satisfies $\rho V_h^* = H\left(V_h^*\right)$. Using the definition of $H\left(V_h^*\right)$, rearrange (30) as

$$H\left(V_h^*\right) - F\left(V_h^*\right) = -x m_f\left[u(A_f V_h^*) - u(A_h V_h^*)\right] - \tau V_h^* < 0 \tag{31}$$

always satisfied since $A_h < A_f$. Hence $V_f^* > V_h^*$ always exists when $p^* = (1, 1, 1)$ is an equilibrium and $\tau > 0$.

2. Notice that (31) holds as an equality iff $\tau = 0$, so that $V_f^* = V_h^*$ is the unique positive fixed point.

3. If $V_f^* > V_h^*$, then $\frac{V_f^*}{V_h^*} \leq \frac{1-A_h}{1-A_f}$ for all $\tau \geq 0$ if ρ is small This is so whenever

$$\left.\frac{F\left(V_f, V_h^*\right)}{H\left(V_h^*\right)}\right|_{V_f=V_h^*} \equiv \frac{m_0 u(V_h^*) + m_f u(A_h V_h^*) + m_h u(A_f V_h^*) - (1-\mu)V_h^*}{m_0 u(V_h^*) + m_f u(A_h V_h^*) + m_h u(A_h V_h^*) - (1-\mu)V_h^* - (\tau/x)\, V_h^*}$$
$$\leq \frac{1-A_h}{1-A_f}$$

satisfied with equality as $\tau \to 0$ since $F\left(V_f, V_h^*\right) \to^+ H\left(V_h^*\right)$; if $\tau > 0$ then $\lim_{\rho\to 0} \frac{1-A_h}{1-A_f} \to \infty$ but $F\left(V_h^*\right)/H\left(V_h^*\right)$ is bounded. Concavity of $F\left(V_f, V_h^*\right)$ completes the argument.

4. We now show that if another fixed point V_f^{**} of the map $\rho V_f = F\left(V_f, V_h^*\right)$exists when $\tau > 0$, it must be such that $V_f^{**} < V_h^*$. To show it note that $F\left(V, V_h^*\right) = 0$ for some positive $V = V^L < (1-A_h)\, V_h^*$. However, $H\left(V^L\right) > 0$ since it is increasing, and $H(0) = 0$. That is $\exists\ 0 < V^L < (1-A_h)\, V_h^*$ such that $F\left(V^L, V_h^*\right) = 0 < H\left(V^L\right)$. Since $V_f^* > V_h^*$ always exists, then it must be that $F\left(V, V_h^*\right)$ intersects $H\left(V\right)$ at some point $V^H \geq (1-A_h)\, V_h^*$, i.e. $F\left(V^H, V_h^*\right) = H\left(V^H\right)$, satisfied iff

$$x m_h \left[u(V^H - (1-A_h)\, V_h^*) - u(A_h V^H)\right] + x m_f \left[u(A_f V^H) - u(A_h V^H)\right] + \tau V^H = 0$$

Since $A_f > A_h$ then this last equality can be satisfied only if $u(V^H - (1-A_h)\, V_h^*) - u(A_h V^H) < 0$, i.e. if $V^H < V_h^*$. Since $F\left(V^H, V_h^*\right) = H\left(V^H\right) \geq \rho V^H$ (i.e. the functions intersect above the line traced by ρV) it must be that $V_f^{**} < V^H < V_h^*$.

5. $V_f^{**} < V_h^*$ cannot exist if

$$\rho V_f - F\left(V_f, V_h^*\right)|_{V_f=(1-A_h)V_h^*} < 0 \quad \Leftrightarrow \quad (1-A_h)\, H(V_h^*) < F\left((1-A_h)\, V_h^*, V_h^*\right) \tag{32}$$

which we can rewrite as

$$\begin{aligned} m_0 u((1-A_h)\, V_h^*) + m_f u(A_f\,(1-A_h)\, V_h^*) - (1-A_h)\left[m_0 u(V_h^*) + m_f u(A_h V_h^*)\right] \\ > (1-A_h)\left[m_h u(A_h V_h^*) - \tfrac{\tau}{x} V_h^*\right] \end{aligned} \tag{33}$$

The LHS of the inequality is always positive $\forall \tau > 0$ since $A_f > A_h$ and $u\left((1-A_h)\, k\right) > (1-A_h)\, u(k)$ for any $k > 0$, due to Jensen's inequality. The RHS of the inequality is negative

if τ is close to one since $V_h^* > xm_h u(A_h V_h^*)$; it is also decreasing in τ since A_h and V_h^* increase as τ shrinks. Hence by the intermediate value theorem (33) holds for any ρ if τ sufficiently large. However, it can be shown that (33) holds for τ if ρ is sufficiently small. To do so note that as $\rho \to 0$ then $LHS, RHS \to 0$, but $LHS > RHS$ in the limit since the LHS has a partial relative to ρ, compared to RHS, when ρ is around 0.

6. If $d = 1$ and $p^* = (1, 1, 1)$ is an equilibrium, then equilibria with $V_f^{**} < V_h^*$ and $V_f^* > V_h^*$ cannot coexist, i.e. they are mutually exclusive. To prove it consider the first contraint in (9), $S(V_f) > S(V_h)$. If $S(V)$ is monotonically increasing then $S(V_f) > S(V_h)$ only if $V_f > V_h$ (never if $V_f < V_h$). If $S(V)$ is hump-shaped then (i) if V_h is on the decreasing segment of $S(V_h)$ then $S(V_f) > S(V_h)$ only if $V_f < V_h$ (never if $V_f > V_h$) and (ii) if V_h is on the increasing segment of $S(V_h)$ then $S(V_f) > S(V_h)$ only if $V_f > V_h$ (never if $V_f < V_h$). ■

Proof of Proposition 1.

Consider an equilibrium distribution that satisfies (1)-(2) and (12)-(14). From a prior discussion we know that it exists, under certain conditions.

It is straightforward to show that (22) and (23) are satisfied as strict inequalities, whenever $1 < \frac{V_f}{V_h} \leq \frac{1-A_h}{1-A_f}$. By continuity $\frac{1-A_h}{1-A_f} < \frac{V_f}{V_h}$ also satisfies these two inequalities if $\frac{V_f}{V_h}$ is above–but close to–$\frac{1-A_h}{1-A_f}$. From Lemma 1 we know that there always exists a unique fixed point of (15)-(16) $V_f^* > V_h^*$ when $\tau > 0$, such that $V_f^*/V_h^* \leq \frac{1-A_h}{1-A_f}$ for $\rho > 0$ small. As a result, we know that there exists a $\rho_{H1} > 0$ such that (22) and (23) and Lemma 1 are satisfied for some $\rho \in (0, \rho_{H1})$.

What remains to be shown is that when $V_f^* > V_h^*$ satisfies (17) and (18), then it also satisfies (19) and (20). The intervals defined by the bounds in (19) and (20) are non-empty for all values of A_f, A_h and σ. Furthermore, A_f and A_h converge to 1 as ρ approaches zero. Comparing the expressions in (17) and (18) to the respective upper bounds in (19) and (20) it is easy to verify the existence of a value of positive $\rho_H < \rho_{H1}$ such that for $\rho \in (0, \rho_H)$, (17) is below the upper bound in (20) and (18) is below the upper bound in (19) for all $V_f^* > V_h^*$. Furthermore, as σ approaches zero, the lower bounds of (19) and (20) approach zero, while (17) and (18) converge to positive values. Consequently, there exists a $\sigma_H \in (0, 1)$ such that if $\sigma \in (0, \sigma_H)$, and $\rho \in (0, \rho_H)$, then there is always a unique positive fixed point $V_f^* > V_h^*$ that satisfies (17) and (18), and it also

satisfies (19)-(23), i.e. an equilibrium exists such that the conjectured transaction pattern $d = 1$ and $p^* = (1, 1, 1)$ is individually optimal.■

Proof of Lemma 2.

Conjecture $d = 1$ and $p^* = (0, 0, 0)$ and consider $\tau > 0$. Inspection of (25) shows that its RHS is a monotonically increasing strictly concave function of V_f, which starts at 0 and has a decreasing first derivative that vanishes as $V_f \to \infty$. Thus, it has two fixed points, one is $V_f = 0$ (the non-monetary equilibrium, which we ignore), and the other is $V_f^* > 0$ which satisfies

$$(1-\mu)\, V_f = A_f \left[m_0 u(V_f) + m_f u(A_{fh} V_f) + m_h u(A_f V_f)\right] \tag{34}$$

Given V_h^*, Consider the map defined by (26) i.e.

$$V_h = \frac{A_h \left[m_0 u(V_h) + m_f u(A_h V_h) + m_h u(V_h - (1 - A_{fh}) V_f^*)\right]}{1-\mu} \equiv H\left(V_h, V_f^*\right)$$

defined only for $V_f > (1 - A_{fh}) V_f^*$. The function $H\left(V_h, V_f^*\right)$ is strictly concave for $V_h \geq (1 - A_{fh})\, V_f^*$, and monotonically increasing in V_h. As $V_h \to^+ (1 - A_{fh}) V_f^*$ then $H\left(V_h, V_f^*\right)$ converges to a positive quantity, and its slope becomes unbounded. Thus, the intermediate value theorem suggest there can be two positive fixed points to the map $V_h = H\left(V_h, V_f^*\right)$. A fixed point such that $V_f^* > V_h^*$ exists if

$$V_h - H\left(V_h, V_f^*\right)\big|_{V_h = V_f^*} > 0 \quad \Leftrightarrow \quad V_f^* > F\left(V_f^*\right) \tag{35}$$

Furthermore, it will be unique if

$$V_h - H\left(V_h, V_f^*\right)\big|_{V_h = (1 - A_{fh}) V_f^*} < 0 \quad \Leftrightarrow \quad (1 - A_{fh})\, V_f^* < H\left((1 - A_{fh})\, V_f^*, V_f^*\right) \tag{36}$$

Using (34) we can rewrite the inequality in (35) as

$$\begin{aligned} & A_f \left[m_0 u(V_f^*) + m_f u(A_{fh} V_f^*) + m_h u(A_f V_f^*)\right] \\ > & A_h \left[m_0 u(V_f^*) + m_f u(A_h V_f^*) + m_h u(A_{fh} V_f^*)\right] \end{aligned}$$

Let V_f^* be any positive constant. Recall that $A_{fh} > A_f > A_h$. It follows that as $\rho \to \infty$ then $A_{fh}, A_h, A_h \to 0$, hence both sides of the inequality converge to zero. As $\rho \to 0$ then $A_{fh}, A_h \to 1$ but $A_h < 1$, and the RHS side of the inequality converges to a positive number smaller than the

LHS. It is easy to show that both sides of the inequality are decreasing in ρ. Since (**??**) is satisfied as $\rho \to 0$ small, by continuity there is a $\rho_{H3} > 0$ such that the inequality above holds $\forall \rho \in (0, \rho_{H3})$, in which case $V_f^* > V_h^*$ when $p^* = (1,1,1)$, $d = 1$ and $\tau > 0$.

It is a matter of algebra to show that (**??**) is likely to be violated if $\tau > 0$ small, $x \cong 0$, $m_f \cong 0$, and ρ large. That is, when the search frictions are large but the risk on the home currency is quite limited. It is also obvious that if $\tau = 0$ then (35) holds as an equality hence $V_f^* = V_h^*$ is the unique positive fixed point (the other fixed point is $V_f = V_h = 0$).

To show that the positive fixed point $\left(V_f^*, V_h^*\right)$ is unique, rewrite inequality (36) as

$$\begin{aligned}
&(1 - A_{fh})\, A_f \left[m_0 u(V_f^*) + m_f u(A_{fh} V_f^*) + m_h u(A_f V_f^*)\right] \\
&< A_h \left[m_0 u((1 - A_{fh})\, V_f^*) + m_f u(A_h\, (1 - A_{fh})\, V_f^*)\right]
\end{aligned}$$

Note that as $\rho \to 0$ then $A_{fh} \to 1$ hence both sides of the inequality converge to zero. Let V_f^* be any positive constant. Take the partial of each side of the inequality with respect to ρ, and then take the limit as $\rho \to 0$. In this way, the partial of RHS of the inequality is seen to be positive and unbounded since $u'\left((1 - A_{fh})\, V_f^*\right) \to \infty$ as $A_{fh} \to 1$. The partial of LHS of the inequality, however, is bounded. By the intermediate value theorem it follows that there is a $\rho_{H4} > 0$ such that (32) holds $\forall\ \rho \in (0, \rho_{H4})$. Hence the equilibrium $\left(V_f^*, V_h^*\right)$ is unique and such that $V_f^* > V_h^*$ given $d = 1$ and $p^* = (0,0,0)$.

For ρ small and $\tau > 0$, $\frac{1-A_h}{1-A_{fh}} > \frac{V_f}{V_h}$ always since the left-hand side converges to infinity while the right-hand side converges to a finite number. ■

Proof of Proposition 3.

If $p^* = (0,0,0)$ and $d = 1$ then the laws of motion must satisfy

$$\begin{aligned}
\dot{m}_{2f} &= m_f m_f - m_{2f}\,(m_0 + m_h) \\
\dot{m}_{2h} &= x[m_h^2 + m_h m_{fh} - m_{2h}\,(m_0 + m_f)] + \eta m_h - \tau m_{2h} \\
\dot{m}_{fh} &= x[m_f m_{2h} + m_h m_{2f} + 2 m_h m_f - m_{fh}\,(m_0 + m_h)] + \eta m_f - \tau m_{fh}
\end{aligned}$$

and we can apply the same procedure as before to show that a unique stationary distribution exists.

When $u(q) = q^\sigma + q$, we obtain the following expressions for the value functions and the optimality constraints

$$V_f = \left\{ \frac{A_f\left[m_0 + m_f A_{fh}^\sigma + m_h A_f^\sigma\right]}{1-\mu-A_f\left[m_0+m_f A_{fh}+m_h A_f\right]} \right\}^{\frac{1}{1-\sigma}}$$

$$V_h = \left\{ \frac{A_h\left[m_0 + m_f A_h^\sigma + m_h(1-(1-A_{fh})/z)^\sigma\right]}{1-\mu-A_h\left[m_0+m_f A_h+m_h(1-(1-A_{fh})/z)\right]} \right\}^{\frac{1}{1-\sigma}}$$

$$\left[\frac{(1+A_f)^\sigma - 1}{1-A_f}\right]^{\frac{1}{1-\sigma}} < V_f < \left(\frac{A_f^\sigma}{1-A_f}\right)^{\frac{1}{1-\sigma}} \tag{37}$$

$$\left[\frac{(1+A_h)^\sigma - 1}{1-A_h}\right]^{\frac{1}{1-\sigma}} < V_h < \left(\frac{A_h^\sigma}{1-A_h}\right)^{\frac{1}{1-\sigma}} \tag{38}$$

$$V_f < V_h \tag{39}$$

$$A_2^\sigma V_h^\sigma + (1-A_{fh})\, V_f > (1-A_h)\, V_h + A_3^\sigma V_f^\sigma \tag{40}$$

$$(1-A_{fh})\, V_f + V_h^\sigma > (V_h + A_{fh} V_f)^\sigma \tag{41}$$

As before,inequalities (37) and (38) on the value functions are needed to ensure that 'rich' buyers only spend one unit of currency and 'poor' buyers buy from 'rich' sellers. Inequalities (39) and (40) are the conditions needed to ensure that the $p^* = (0,0,0)$ strategy is optimal. The last inequality (41) ensures the fh buyer only spends the domestic currency and not both. The surprising feature of these constraints is that despite its riskiness, the domestic currency must be more valuable than the dollar for this equilibrium to exist. This sharp relationship regarding the magnitude of V_f relative to V_h is a result of the quasi-linear preference specification.

Note, that (37) and (38) are identical to (17) and (18). As note in the proof of Proposition 1, it follows that they can hold if ρ is sufficiently small. Contradicting this requirement, Lemma 2 has shown that (39) is violated whenever ρ is sufficiently small. It follows that $p^* = (0,0,0)$ and $d = 1$ cannot be an equilibrium if search frictions are too low.■

Proof of Proposition 4.

The value functions must solve

$$V_f = \left\{ \frac{A_h}{1-\mu}\left[m_0 + m_f A_{fh}^\sigma + m_h A_f^\sigma\right] \right\}^{\frac{1}{1-\sigma}}$$

$$V_h = \left\{ \frac{A_h}{1-\mu}[m_0 + m_f A_h^\sigma + m_h(1-(1-A_{fh})\,(V_f/V_h))^\sigma] \right\}^{\frac{1}{1-\sigma}}$$

It is straightforward to show that V_f^* and V_h^* approach 1 as $\rho, \tau \to 0$. By Lemma 2, when ρ is sufficiently small, it must be that case that $V_f^* > V_h^*$. Consequently, the conditions in (8)-(9) reduce to:

$$[(1+A_f)^\sigma - 1]^{\frac{1}{1-\sigma}} < V_f < A_f^{\frac{\sigma}{1-\sigma}}$$

$$[(1+A_h)^\sigma - 1]^{\frac{1}{1-\sigma}} < V_h < A_h^{\frac{\sigma}{1-\sigma}}$$

$$V_f^\sigma - V_f < V_h^\sigma - V_h$$

$$V_f + V_h^\sigma > (V_h + A_{fh}V_f)^\sigma$$

$$(A_f V_f)^\sigma - V_f < (A_{fh}V_f - V_f + V_h)^\sigma - V_h \quad \text{if } \frac{1-A_h}{1-A_{fh}} < \frac{V_f}{V_h} \quad \text{(ignore otherwise)}$$

$$(A_{fh}V_f)^\sigma - V_f < (A_h V_h)^\sigma - V_h$$

As before the first four conditions are satisfied when ρ and σ are sufficiently small since A_{fh} and V_f approach 1 while A_h and V_h converge to values less than one for $\tau > 0$. For ρ small and $\tau > 0$, $\frac{1-A_h}{1-A_{fh}} > \frac{V_f}{V_h}$ always (see Lemma 2). Finally, for σ close to zero, the last inequality is always satisfied when $V_f > V_h$. By the intermediate value theorem we conclude that there exist positive values $\hat{\sigma}_H$ and $\hat{\rho}_H$ such that if $\sigma \in (0, \hat{\sigma}_H)$ and $\rho \in (0, \hat{\rho}_H)$ then a the low dollarization equilibrium exists and is unique.■

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Figure 1: Risk versus domestic share of total money stock – $u(q) = q^{\sigma} + q$

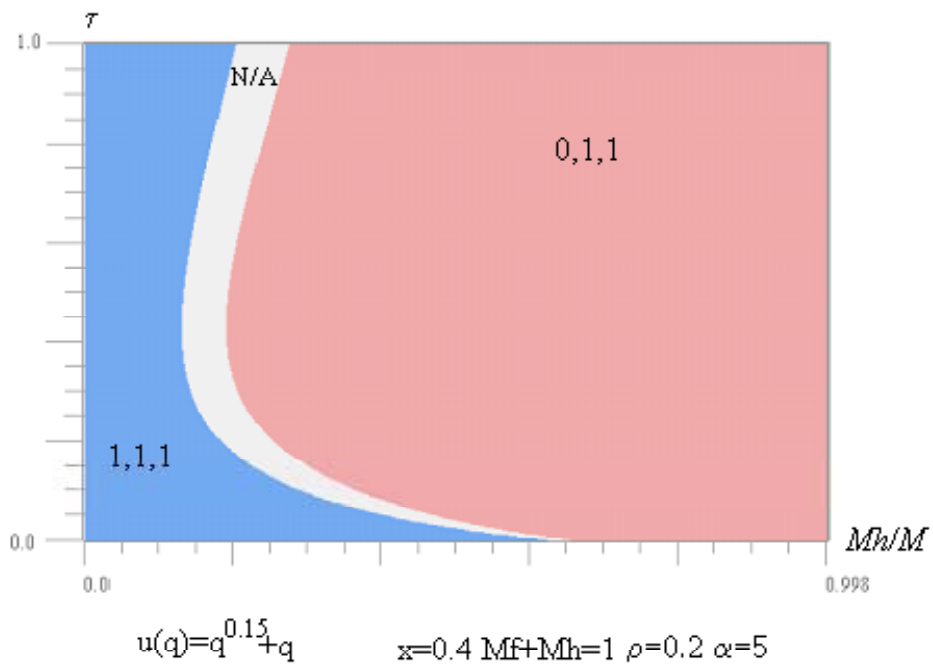

Figure 2: Risk versus domestic share of total money stock – $u(q) = q^{\sigma}$

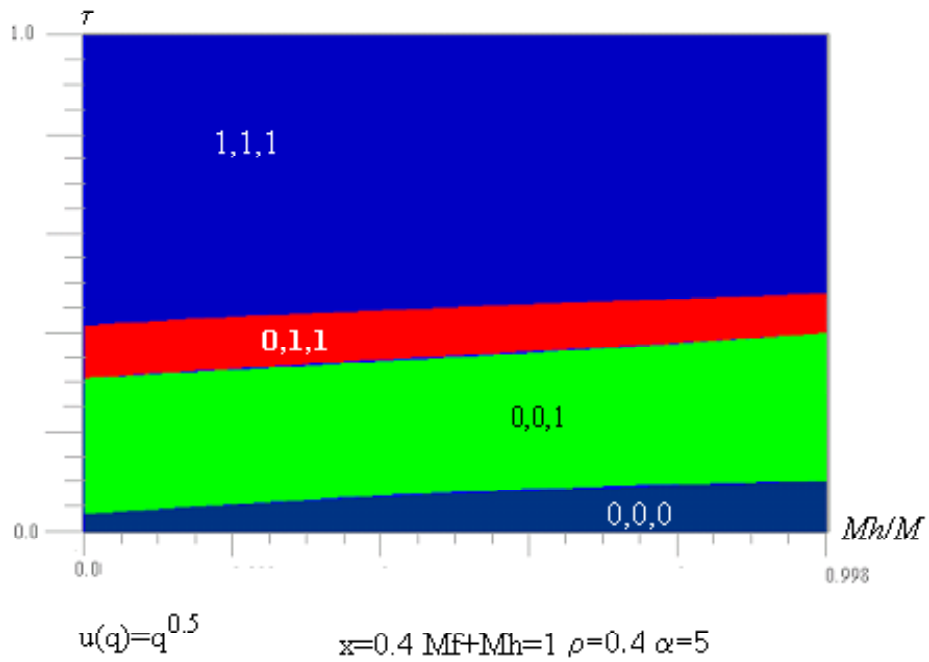
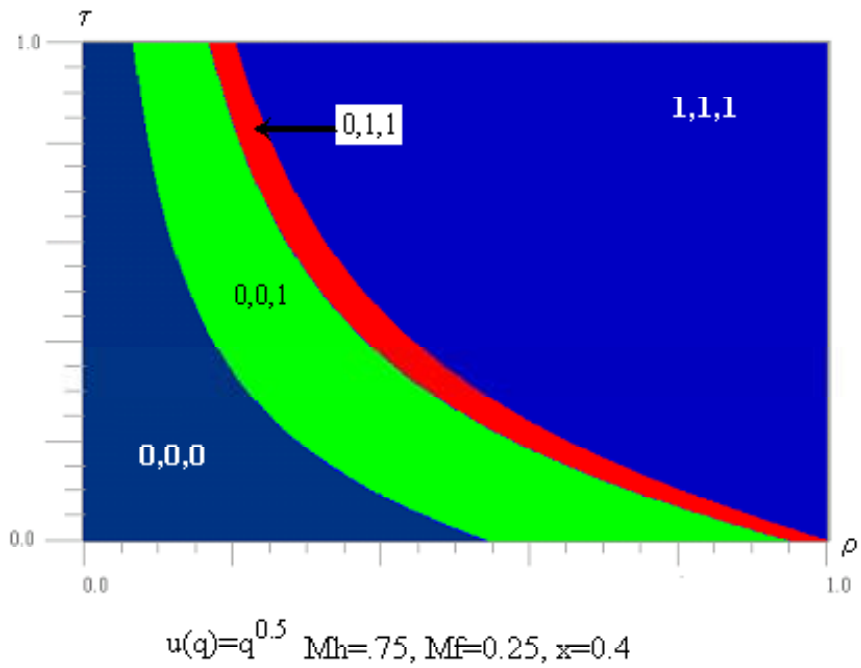
Figure 3: Risk versus trade frictions – $u(q) = q^{\sigma}$