Monetary Stability and Liquidity Crises: The Role of the Lender of Last Resort[¤]

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Abstract

We study an economy where consumers are subject to liquidity demand shocks and banks arise endogenously to provide insurance against these shocks. We evaluate the desirability of having an elastic currency generated by a lender of last resort that prints money and makes liquidity loans to banks in distress. In the absence of such lending, the economy has a unique equilibrium, but it is not Pareto optimal. The introduction of unlimited lending at a zero nominal interest rate generates a steady state equilibrium allocation that is Pareto optimal. However, this policy is destabilizing in the sense that it also introduces a continuum of non-optimal in‡ationary equilibria. We then explore two restrictions on the provision of liquidity loans that can rule out such monetary instability while preserving some of the steady-state bene...ts of lending. If the lender of last resort faces an upper bound on loanable funds that is low enough, no in‡ationary equilibria can arise. For some (but not all) economies, the unique equilibrium under this policy is Pareto optimal. Finally, if the lender charges a high enough interest rate on liquidity loans, no in‡ationary equilibria can arise.

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1. Introduction

Recent developments in a number of countries have served to renew interest in the role of a lender of last resort. According to Fischer [7, p. 86] , "there is considerable agreement on the need for a domestic lender of last resort," even though there is some disagreement about exactly what this lender should do. However, several recent papers have identified the lender of last resort as a cause of excess volatility in emerging economies' financial markets and of the currency crises that have plagued many of these economies in the 1990s.[1] In response to these crises, proposals have been made in a number of countries to either establish a currency board or abolish the national currency altogether and adopt some other country's currency as legal tender (this second arrangement is often called *dollarization*). While adopting such policies may be successful in eradicating excess volatility stemming from speculation against a domestic currency, they clearly do not come without cost. In particular, both of these arrangements severely limit the ability of the central bank to act as a lender of last resort. In light of these proposals, we think it is important to understand the implications (both benefits and costs) of having a lender of last resort that is able to freely print money and lend to the banking system.

One of the important roles of a lender of last resort is the provision of an elastic currency, that is, the adjusting of the money supply in response to transitory changes in liquidity demand. This role was important enough to merit high billing in the act establishing the Federal Reserve System in the United States, "An act to provide for the establishment of Federal Reserve Banks, to furnish an elastic currency, ⋮ and for other purposes." Beginning with Sargent and Wallace [15], several papers have examined the effects of having an elastic currency supply.[2] These papers focus on stationary equilibria and show how an elastic currency promotes a more efficient allocation of resources in these equilibria. In the present paper, we show that when nonstationary equilibria are considered, the picture can change dramatically. We build on the model of Champ, Smith, and Williamson [4], where aggregate liquidity shocks create a role for a lender of last resort. We show how having a lender of last resort that prints money and lends freely to the banking system at a zero nominal interest rate allows the economy to completely overcome the liquidity shocks, so that the stationary equilibrium is Pareto optimal. This is in line with the previous results in the literature.

1 See, for example, Chang and Velasco [5] , Mishkin [12] , and Fischer [7] .
2 Among them are Champ, Smith, and Williamson [4] , Williamson [21] , and Freeman [8] .

However, we also show that there is a continuum of non-optimal inflationary equilibria under this regime. Hence, while having an unrestricted lender of last resort allows the economy to posses an efficient equilibrium allocation, it also opens the door to currency instability.

Having identified the lender of last resort as a potential source of instability, we ask the following question: What measures could be implemented to eliminate the bad equilibria associated with unlimited and costless lending, while retaining some of the benefits of such lending? We show that in some cases this may be achieved by placing a sufficiently low ceiling on the amount of real lending that can be undertaken, and that it always can be achieved by charging a sufficiently high real interest rate on liquidity loans.

Our model is a pure exchange, two-period-lived overlapping generations economy, where some agents are lenders and others are borrowers. There is a store-of-value role for money. Agents are assigned to either of two locations at birth, and in each period a fraction of lenders is forced to move to the other location. Limited communication prevents claims on specific agents from being traded across locations and only money has value in exchange after relocation. As in Townsend [19] , Mitsui and Watanabe [13] , and Hornstein and Krusell [11] , this generates a transactions role for currency and allows equilibria where money is dominated in rate of return by other assets. In this set-up, stochastic relocations act like the portfolio preference shocks in Diamond and Dybvig [6] and banks will arise endogenously to insure consumers against such random liquidity shocks. Thus, banks write deposit contracts that insure lenders against the possibility of relocation, hold reserves, and provide intermediation between borrowers and lenders.

In this framework, we obtain the following results. In the absence of a lender of last resort, the economy has a unique equilibrium. This equilibrium is stationary, with a constant price level and with banks holding the same fraction of their portfolio in the form of reserves at all times. There is a critical value of the relocation shock below which these precautionary reserves suffice to fully cover the demand for liquidity and to equalize the return on deposits for all agents. However, for realizations of the relocation shock above this critical value, banks face a “liquidity crisis”. This corresponds to a situation of complete exhaustion of banks’ cash reserves and, since other bank assets are illiquid, precludes depositors from being fully insured. A wedge is driven between the returns earned by depositors who are subject to the relocation shock and those who are not. Since the aggregate resources of the economy are nonstochastic, this allocation is not Pareto efficient.

Inflation is inconsistent with equilibrium in this setting because stochastic relocation generates a strong demand for currency, even when the rate of return to holding money is low. When the money supply is constant, a sustained inflation causes the real stock of money to go to zero. Real money demand cannot go to zero in this model, and hence inflation is not consistent with equilibrium.

When we allow for the unrestricted provision of lender of last resort services at a zero nominal interest rate, the set of equilibria is substantially different. In this case the steady state equilibrium is Pareto optimal. Compared to the equilibrium for the benchmark case, banks hold a lower fraction of their portfolio as real balances. They contract a liquidity loan from the lender of last resort whenever they face a crisis where reserves are insufficient to cover the demand for liquidity. By doing so they are able to fully insure agents against random liquidity shocks in all states of the world: relocated and non-relocated agents earn the same return under all circumstances. However, in addition to the Pareto optimal stationary equilibrium, the economy also has a continuum of inflationary equilibria, none of which are Pareto optimal. With a lender of last resort, the effective money supply (base money plus liquidity loans) is no longer fixed. The entire point of having a lender in this setting is to make the money supply elastic so that it responds to the stochastic movements in currency demand. In the stationary equilibrium, this leads to an efficient allocation of resources. However, it also opens the door to nonoptimal, inflationary equilibria. If there is a sustained inflation, the real value of the stock of base money goes to zero, as before. This leads to an excess demand for money, which in turn leads the lender of last resort to provide liquidity loans that increase the total money supply. In other words, *a lender of last resort implicitly promises to respond to an inflation by printing money*. In the long run, the inflation rate and the growth rate of the total money supply approach the same constant value.

We explore whether certain restrictions on the provision of liquidity loans would allow the economy to preserve some of the desirable features of having lender of last resort services without permitting inflationary equilibria. A seemingly natural constraint would be to place an upper bound on the amount of lending that can be done. Ideally, this cap would be high enough that it never binds in the steady state (preserving the Pareto optimality of this equilibrium), but low enough that it eliminates all nonstationary equilibria. We show that whether or not this is possible depends on the distribution of the aggregate liquidity shock. We then study an economy with a lender of last resort that charges a real interest rate on liquidity loans to banks facing a crisis. When this interest rate

is high enough, inflationary equilibria are ruled out regardless of the distribution of the liquidity shock.

Before turning to the model, we would like to point out some differences between this paper and much of the existing literature on the lender of last resort. Our model has elements in common with Diamond and Dybvig [6] , but the banking panics studied there and the liquidity crises arising here are of a very different nature. In Diamond and Dybvig [6] , the presence of real assets that can be liquidated prematurely and the imposition of a sequential service constraint combine to create the potential for self-fulfilling panics. These panics can be ruled out by providing deposit insurance. In contrast, the liquidity crises in our model are the result of fundamentals, and it is the presence of the lender of last resort that permits self-fulfilling, inflationary equilibria to arise. Moreover, Diamond and Dybvig [6] consider a purely real economy, while Champ, Smith and Williamson [4] have argued that models of banking crises should be expanded to consider monetary factors. We follow the latter paper in studying a world where money has a role, both as a store of value and in the completion of transactions. This allows us to examine the relationship between the terms of last resort lending and currency stability, the central question of our paper.

We also want to underscore that we have abstracted from the problems of moral hazard and "excessively risky" behavior associated with the presence of a lender of last resort. Many authors, ranging from Bagehot [1] to Solow [18] , Fischer [7] , and Mishkin [12] , have argued that these problems are crucial in understanding the potential for instability related to the provision of lender of last resort arrangements. In our model, currency instability may arise even when there is no scope for moral hazard and "overly risky" portfolio allocation.

The remainder of the paper proceeds as follows. The next section lays out the basic model. Section 3 describes equilibrium without a lender of last resort, while Section 4 presents the case of unrestricted lending at a zero nominal interest rate. Section 5 describes the behavior of an economy with a lender of last resort that faces an upper bound on the amount it can lend to banks in distress, while Section 6 looks at a policy of charging interest on liquidity loans. Some concluding comments are offered in Section 7.

2. The Basic Model

In this section we describe those elements of the model that are independent of the type of lender-of-last-resort services that are available to banks. The sections that follow then tailor the model to the specific policy regimes we consider.

2.1 The Environment

We begin with the pure-exchange monetary economy developed by Champ, Smith, and Williamson [4] . The economy consists of an infinite sequence of two-period lived, overlapping generations, plus an initial old generation. There is a single, perishable consumption good. At each date $t = 0; 1; \ldots$, a continuum of agents with unit mass is born at each of two identical locations. Half of these agents are "lenders" and the remaining half are "borrowers". The former have endowments $(!_1; !_2) = (x; 0)$, while the latter's endowment vector is $(!_1; !_2) = (0; y)$: All consumers have R^2_{++} as their consumption set and have preferences given by $u(c_1; c_2) = \ln(c_1) + \bar{} \ln(c_2)$: We assume that $\bar{} x > y$ holds, which implies that this is a "Samuelson case" economy (see Gale [9]) and hence there is a role for money as a store of value in this world. At $t = 0$ there is a continuum of old agents with unit mass in each location. Each of these agents is endowed with $M > 0$ units of fiat money, which we will refer to as "base money". The stock of base money is constant over time.

In addition to the store of value role for money, spatial separation and limited communication generate a transactions role for money in a way reminiscent of Townsend [19] , Mitsui and Watanabe [13] , and Hornstein and Krusell [11] . This allows money to be dominated in rate of return by other assets. At the beginning of each period, agents cannot move between or communicate across locations. Goods can never be transported between locations. Hence, goods and asset transactions occur autarkically within each location at the beginning of each period. After this trade is concluded at time t, a fraction $¼_t$ of young lenders in each location is forced to move to the other location. Limited communication prevents the cross-location exchange of privately issued liabilities. Currency, on the other hand, is universally recognizable and non-counterfeitable, and is therefore acceptable in inter-location exchange. Hence the relocation process acts like a stochastic cash-in-advance constraint for young lenders. The old-age consumption of a mover will be equal to

the real value of the money that she takes with her to the new location.[3] The relocation probability π_t is a random variable in each period with support $[0, 1]$, and is drawn from the twice continuously differentiable, strictly increasing distribution function F with associated density function f. The relocation probability is independently and identically distributed over time.

Stochastic relocations also act like shocks to portfolio preferences which have the same consequences as the "liquidity preference shocks" in Diamond and Dybvig [6] . Hence they motivate a role for banks to insure lenders against random liquidity needs. Banks take deposits, hold cash reserves, and intermediate lending. We now describe the behavior of borrowers, lenders, and banks.

2.2 Consumers

Borrowers, who never move, face a gross market interest rate of R_t. They choose their quantity of borrowing ℓ_t to solve the problem

$$\max_{\ell_t} \ln(\ell_t) + \beta \ln(y - R_t \ell_t).$$

The solution to this problem is given by

$$\ell_t = \frac{y}{(1+\beta) R_t}. \tag{1}$$

Lenders face a more complicated problem. Given that they are confronted with random relocation, they deposit all of their savings in a bank and the return they receive depends on both whether or not they move and what fraction of all young lenders move.[4] Specifically, they are promised a real return $r_t(\pi)$ if they do not move and $r_t^m(\pi)$ if they do move. Notice that, given our assumptions on the distribution of π, these return schedules depend only on the relocation probability at time t, and not on the history of realizations of relocation probabilities. Lenders then choose the amount they save and deposit d_t to maximize expected utility, that is, to solve

$$\max_{d_t} \ln(x - d_t) + \beta \int_0^1 \pi \ln[r_t^m(\pi) d_t] f(\pi) d\pi + \beta \int_0^1 (1-\pi) \ln[r_t(\pi) d_t] f(\pi) d\pi.$$

[3] Note that since the consumption set is $\mathbb{R}^2_{++}$, this implies that money must have positive value in equilibrium.

[4] We assume that an individual's relocation status is public information. Since in equilibrium no agent ever has an incentive to misreport her status, this seems innocuous.

The solution to this problem sets

$$d_t = d = \frac{\bar{\ }}{1 + \ } = x; \tag{2}$$

that is, saving is independent of the distribution of the rates of return. This result clearly depends on the assumptions of log utility and no old-age income for lenders, which imply that the income and substitution effects of a change in the rate of return exactly offset each other.

2.3 Banks

Banks take deposits, make loans, hold reserves, and announce return schedules. We assume that any borrower can establish a bank and that banks behave competitively in the sense that they take the real return on assets as given. On the deposit side, we assume that banks behave as Nash competitors, which leads them to choose deposit returns to maximize the expected utility of young lenders. The constraints that banks face in this maximization problem depend on what lender-of-last-resort services are available to them.

Below we consider four different scenarios. First, as a benchmark case, we consider a world without a lender of last resort. We then turn our attention to the economy with a lender that provides unlimited lender of last resort services at a zero nominal interest rate. Next, we examine the case of a lender of last resort constrained by an upper bound on the amount that it can lend to a bank in distress. Finally, we analyze an economy with a lender of last resort that charges a positive interest rate when providing liquidity to banks.

3. No Lender of Last Resort

In this section we discuss equilibrium for an economy in which banks are unable to borrow from anyone other than lenders. We begin by describing the bank's problem for this benchmark case.

3.1 The Bank's Problem

A young lender deposits her entire savings d with a bank. Per young depositor, the bank acquires an amount z_t of real balances, and makes loans with a real value $d - z_t$. The bank faces two constraints with respect to the return it promises to movers r_t^m and the return it promises to non-movers r_t. First,

relocated agents, of which there are φ_t, must be given currency, since that is the only asset which will allow these agents to consume at time $t+1$ in their new location. This is accomplished using a fraction $\alpha_t(\varphi)$ of the bank's holdings of cash reserves. Hence, letting p_t denote the general price level at time t,[5] the return to holding money between time t and $t+1$ is given by $\frac{p_t}{p_{t+1}}$ and

$$\varphi d r_t^m(\varphi) \leq \alpha_t(\varphi) z_t \frac{p_t}{p_{t+1}}$$

must hold. If we denote by $\theta_t \equiv \frac{z_t}{d}$ the ratio of reserves to deposits, then we can rewrite this constraint as

$$\varphi r_t^m(\varphi) \leq \alpha_t(\varphi) \theta_t \frac{p_t}{p_{t+1}}. \tag{3}$$

Second, real payments to nonmovers, which occur at time $t+1$, cannot exceed the value of the bank's remaining portfolio – remaining reserves plus loan repayments. Since loans earn the gross real rate of return R_t, this can be written as

$$(1-\varphi) d r_t(\varphi) \leq [1-\alpha_t(\varphi)] z_t \frac{p_t}{p_{t+1}} + (d - z_t) R_t$$

or

$$(1-\varphi) r_t(\varphi) \leq [1-\alpha_t(\varphi)] \theta_t \frac{p_t}{p_{t+1}} + (1-\theta_t) R_t. \tag{4}$$

Of course, $0 \leq \theta_t \leq 1$ and $0 \leq \alpha_t(\varphi) \leq 1$ must hold.

Since φ is i.i.d., the bank's problem is the same in every period and hence the optimal functions r_t, r_t^m, and α_t are independent of time and of the history of realizations of φ. Moreover, since there is free entry in banking, and since banks behave as Nash competitors, they will maximize young lenders' utility, taking deposit demand d as given. Banks will earn no profits, and constraints (3) and (4) will hold with equality. Given (2), the bank's problem is then to choose $r(\varphi)$ and $r^m(\varphi)$ to solve

$$\max \ln\left(\frac{x}{1+\bar{x}}\right) + \int_0^1 \left[\varphi \ln\left(r_t^m(\varphi)\frac{\bar{x}}{1+\bar{x}}\right) + (1-\varphi)\ln\left(r_t(\varphi)\frac{\bar{x}}{1+\bar{x}}\right)\right] f(\varphi)\, d\varphi \tag{5}$$

subject to

[5] That is, p_t is the price of consumption in units of currency. Some authors (such as Wallace [20] and Balasko and Shell [3]) work instead with the inverse of p_t; the price of money in units of consumption, because it better handles situations where money has no value. In our model, the physical environment combined with the assumed consumption sets precludes equilibria with an infinite price level, and hence the two ways of defining the price system are equivalent.

$$
\begin{aligned}
r_t^m(\varphi) &= \frac{1}{\varphi}\alpha_t(\varphi)\,\gamma_t\frac{p_t}{p_{t+1}} \\
r_t(\varphi) &= \frac{1}{1-\varphi}\left[(1-\alpha_t(\varphi))\,\gamma_t\frac{p_t}{p_{t+1}} + (1-\gamma_t)R_t\right] \\
0 &\le \gamma_t \le 1 \\
0 &\le \alpha_t(\varphi) \le 1.
\end{aligned}
$$

Substituting in the first two constraints and dropping the constant terms yields the problem

$$
\max_{\alpha_t(\varphi);\,\gamma_t} \int_0^1 \varphi \ln\left(\frac{\alpha(\varphi)\,\gamma_t}{\varphi}\frac{p_t}{p_{t+1}}\right) + (1-\varphi)\ln\left(\frac{[1-\alpha(\varphi)]\,\gamma_t}{(1-\varphi)}\frac{p_t}{p_{t+1}} + \frac{(1-\gamma_t)}{(1-\varphi)}R_t\right) f(\varphi)\,d\varphi
$$

subject to

$$
\begin{aligned}
0 &\le \gamma_t \le 1 \\
0 &\le \alpha_t(\varphi) \le 1.
\end{aligned}
$$

The function α_t, which is the fraction of bank reserves paid out to movers, is chosen after the realization of φ, while the function γ_t, the fraction of reserves in the bank's asset portfolio, is chosen before the realization of φ. Hence we can first determine the optimal value of α_t for fixed values of γ_t and φ. That is, we can choose α_t to solve

$$
\max_{0 \le \alpha_t \le 1} \varphi \ln\left(\frac{\alpha_t\gamma_t}{\varphi}\frac{p_t}{p_{t+1}}\right) + (1-\varphi)\ln\left(\frac{(1-\alpha_t)\,\gamma_t}{(1-\varphi)}\frac{p_t}{p_{t+1}} + \frac{(1-\gamma_t)}{(1-\varphi)}R_t\right).
$$

The solution to this problem sets

$$
\alpha_t(\varphi) = \begin{cases} \varphi\left(1 + \frac{1-\gamma_t}{\gamma_t}R_t\frac{p_{t+1}}{p_t}\right) & \text{for } \varphi \in (0;\varphi^*) \\ 1 & \text{for } \varphi \in [\varphi^*;1) \end{cases};
$$

where we have

$$
\varphi^* = \frac{\gamma_t\frac{p_t}{p_{t+1}}}{\gamma_t\frac{p_t}{p_{t+1}} + (1-\gamma_t)R_t}. \tag{6}
$$

Hence for realizations of the relocation shock below the critical value φ^*, the bank pays out only a fraction of its reserves to movers. However, when a relocation shock $\varphi > \varphi^*$ materializes, all reserves are paid out to movers, and repayments to non-movers are drawn from loan repayments only. In other words, the bank holds precautionary reserves. When the realization of the relocation shock is below the critical value φ^*, these cash reserves are sufficient to equalize the returns received by movers and non-movers. However, when the realization of the relocation shock is greater than

the critical value $\varphi^{\ast}$, the bank faces a "liquidity crisis". It pays out all its cash reserves to movers, while repayments to non-movers are drawn from loan repayments only. In a crisis, the bank cannot equalize the returns of movers and non-movers; movers must receive a lower return.

This result follows from the trade-off between two forces. First, the return on cash balances is lower than the return on loans. Therefore, everything else being equal, the bank would like to minimize on cash reserves. On the other hand, in a quest to maximize young lenders' utility, the bank strives to provide insurance by equalizing the returns between movers and non-movers for all realizations of φ. To do so, it must hold sufficient cash balances. At the margin, the welfare gains from equalizing the returns to movers and non-movers exactly offset the cost implied by the return dominance of loans over cash reserves.

It remains to determine the optimal value of θ_t: To do so, we substitute the optimal value of α_t into the bank's objective function so that the only remaining choice variable is θ_t: Doing so yields the problem

$$\max_{0 \le \theta_t \le 1} \int_{\varphi^{\ast}}^{1} \ln\left(\theta_t \frac{p_t}{p_{t+1}} + (1-\theta_t) R_t\right) f(\varphi)\, d\varphi + \int_{0}^{\varphi^{\ast}} \left[\varphi \ln\left(\frac{\theta_t}{\varphi} \frac{p_t}{p_{t+1}}\right) + (1-\varphi) \ln\left(\frac{(1-\theta_t)}{(1-\varphi)} R_t\right)\right] f(\varphi)\, d\varphi.$$

This formulation of the problem makes it clear that the return earned by both movers and non-movers will be the same when φ is less than $\varphi^{\ast}$, but will in general be different when φ is greater than $\varphi^{\ast}$. The first-order condition for this problem is

$$\frac{R_t - \frac{p_t}{p_{t+1}}}{\theta_t \frac{p_t}{p_{t+1}} + (1-\theta_t) R_t} F(\varphi^{\ast}) = \frac{1}{\theta_t} \int_{\varphi^{\ast}}^{1} \varphi f(\varphi)\, d\varphi - \frac{1}{1-\theta_t} \int_{\varphi^{\ast}}^{1} (1-\varphi) f(\varphi)\, d\varphi. \tag{7}$$

This can be reduced to[6]

$$\theta_t = 1 - \int_{\varphi^{\ast}}^{1} F(\varphi)\, d\varphi. \tag{8}$$

This implicitly defines the solution to the bank's problem when no lender of last resort services are provided. We now turn to an analysis of general equilibrium for this case.

[6] To facilitate the refereeing process, we provide the derivation in Referee's Appendix A.

3.2 Equilibrium

An equilibrium of this economy is characterized by the market clearing conditions for real balances and loans. Since the supply of real balances z_t is equal to $\frac{M}{p_t}$, while the demand for real balances is given by $\phi_t d$, market clearing for real balances and (2) require that we have

$$\frac{M}{p_t} = \phi_t \frac{\bar{\ }}{1+\bar{\ }} x.$$

Similarly, from (1), the demand for loans ℓ_t is equal to $\frac{y}{(1+\bar{\ })R_t}$, while the supply of loans is given by $(1-\phi_t)d$. This yields the following market clearing condition for loans

$$\frac{y}{(1+\bar{\ })R_t} = (1-\phi_t)\frac{\bar{\ }}{1+\bar{\ }} x.$$

These equations imply that we have

$$\phi_t \frac{p_t}{p_{t+1}} = \phi_{t+1} \tag{9}$$

and

$$R_t(1-\phi_t) = \frac{y}{\bar{x}}. \tag{10}$$

Substituting (9) and (10) into the expression for φ in (6) yields

$$\varphi = \frac{\phi_{t+1}}{\phi_{t+1} + \frac{y}{\bar{x}}},$$

which we can substitute into (8) to obtain the difference equation

$$\phi_t = 1 - \int_{\frac{\phi_{t+1}}{\phi_{t+1}+\frac{y}{\bar{x}}}}^{1} F(\varphi)\, d\varphi. \tag{11}$$

This implicitly defines the law of motion for ϕ_t. We can now formulate the following proposition.

Proposition 1 When there is no lender of last resort, the economy has a unique equilibrium. This equilibrium is stationary with $\phi_t = \phi_a$ for all t, and $\max\left\{E(\varphi); 1-\frac{y}{\bar{x}}\right\} < \phi_a < 1$.

The proof of Proposition 1 is presented in Appendix A.

Proposition 1 is illustrated in Fig. 1. The law of motion implicitly defined in (11) crosses the forty-five degree line exactly once, and this steady state is the only equilibrium of the economy. The fact that there are no inflationary equilibria here follows from the strong demand for currency

generated by relocations. In this model, there is a positive lower bound on the demand for real money balances. As the rate of return to holding currency goes to zero, real currency demand approaches $E[\frac{1}{4}]d$ (this follows from (8) using (6)). To see why this rules out inflation as an equilibrium outcome, suppose that the initial price level is higher than $p_a = \frac{M}{\circ_a d}$. In other words, suppose that in Fig. 1 the economy starts on the part of the law of motion that lies to the left of $\circ_a$: Then the initial supply of real balances is lower than $\frac{M}{p_a}$: For this to be an equilibrium, the demand for real balances must to be lower than $\circ_a d$. Hence the return on holding cash has to be lower than $\frac{p_a}{p_a} = 1$, and the price level has to rise. Since the nominal supply of money is constant, this results in a further decrease in the real supply of money, which requires a commensurate reduction in the demand for real balances in the next period. This, in turn, requires an even lower rate of return to holding money. Therefore, along a trajectory that starts to the left of $\circ_a$, the supply of real balances decreases continuously as the price level rises. For this to be an equilibrium, the rate of return to holding money would need to adjust so that real currency demand matches the ever-shrinking supply in each period. Since the supply goes to zero, this cannot occur. Roughly speaking, decreasing the demand for real balances below $E(\frac{1}{4})d$ would require a negative return to holding money, which is not consistent with equilibrium. Therefore, unlike the standard Samuelson-case economy discussed in Gale [9] , this model cannot have inflationary equilibria when the money supply is constant. The steady state is the unique equilibrium.

Notice, however, that this equilibrium is not Pareto efficient. There are states of the world in which the returns for relocated and non-relocated lenders are not equalized, even though there is no uncertainty about the aggregate resources of the economy. The problem is that banks must choose their currency holdings before currency demand is realized. If the bank could adjust these holdings once demand is known by, say, borrowing from a lender of last resort when currency demand is high, it seems possible that a more efficient outcome could be achieved. We study various such lending regimes in the remaining sections.

4. An Unrestricted Lender of Last Resort

We first analyze the regime in which the lender is willing to make one-period loans of currency at a zero nominal interest rate in any quantity that banks desire. Note that such a policy is always

feasible, in that it requires no real resources from the lender. After the realization of φ, a bank determines the real amount $b \geq 0$ that it would like to borrow at time t – which will depend on the realization of φ – and the lender simply prints bp_t dollars and gives them to the bank. In the following period, the bank must return these dollars to the lender and they are destroyed.

4.1 The Bank's Problem

With such borrowing, the bank's constraints become

$$\varphi d r_t^m(\varphi) = \alpha_t(\varphi) z_t \frac{p_t}{p_{t+1}} + b_t(\varphi) \frac{p_t}{p_{t+1}}$$

and

$$(1-\varphi) d r_t(\varphi) = [1-\alpha_t(\varphi)] z_t \frac{p_t}{p_{t+1}} + (d - z_t) R_t - b_t(\varphi) \frac{p_t}{p_{t+1}}.$$

Using our earlier notion for the reserve deposit ratio, $\gamma_t \equiv \frac{z_t}{d}$, and defining $\lambda_t \equiv \frac{b_t}{d}$ to be the liquidity loan to deposit ratio, these constraints can be expressed as

$$\varphi r_t^m(\varphi) = \alpha_t(\varphi) \gamma_t \frac{p_t}{p_{t+1}} + \lambda_t(\varphi) \frac{p_t}{p_{t+1}} \tag{12}$$

and

$$(1-\varphi) r_t(\varphi) = [1-\alpha_t(\varphi)] \gamma_t \frac{p_t}{p_{t+1}} + (1-\gamma_t) R_t - \lambda_t(\varphi) \frac{p_t}{p_{t+1}}. \tag{13}$$

After substituting (12) and (13) into the bank's objective function (5) and dropping the constant terms, we obtain the problem

$$\max_{\alpha_t(\varphi),\, \lambda_t(\varphi),\, \gamma_t} \int_0^1 \varphi \ln\left[\frac{\alpha_t(\varphi)\gamma_t}{\varphi} + \frac{\lambda_t(\varphi)}{\varphi} \frac{p_t}{p_{t+1}}\right] f(\varphi)\, d\varphi + \int_0^1 (1-\varphi) \ln\left[\frac{[1-\alpha_t(\varphi)]\gamma_t}{(1-\varphi)} \frac{p_t}{p_{t+1}} + \frac{(1-\gamma_t)}{(1-\varphi)} R_t - \frac{\lambda_t(\varphi)}{(1-\varphi)} \frac{p_t}{p_{t+1}}\right] f(\varphi)\, d\varphi.$$

subject to

$$0 \leq \gamma_t \leq 1$$

$$0 \leq \alpha_t(\varphi) \leq 1$$

$$\lambda_t(\varphi) \geq 0.$$

Both the fraction of bank reserves paid out to movers α_t and the real value of the liquidity loan λ_t

are chosen after the realization of θ, while γ_t, the fraction of reserves in the bank's asset portfolio, is chosen before the realization of θ: Hence we can first solve for the optimal values of α_t and β_t as functions of γ and θ: That is, we can choose α_t and β_t to solve

$$\max_{\alpha_t;\beta_t} \quad \theta \ln\left[\frac{\alpha_t \gamma_t}{\theta} + \frac{\beta_t}{\theta}\frac{p_t}{p_{t+1}}\right] + (1-\theta)\ln\left[\frac{(1-\alpha_t)\gamma_t}{(1-\theta)}\frac{p_t}{p_{t+1}} + \frac{(1-\gamma_t)}{(1-\theta)}R_t - \frac{\beta_t}{(1-\theta)}\frac{p_t}{p_{t+1}}\right]$$

subject to

$$0 \le \alpha_t \le 1$$

$$\beta_t \ge 0.$$

The solution to this problem sets

$$\alpha_t(\theta) = \left(\theta\left[1 + \frac{1-\gamma_t}{\gamma_t}R_t\frac{p_{t+1}}{p_t}\right]\right) \quad \text{and} \quad \beta_t(\theta) = \left(\begin{cases} 0 \\ \gamma_t\theta\left[1 + \frac{1-\gamma_t}{\gamma_t}R_t\frac{p_{t+1}}{p_t}\right] - 1 \end{cases}\right)$$

$$\text{for } \theta \in \begin{cases} [0;\theta^*) \\ [\theta^*;1] \end{cases};$$

where θ^* continues to be given by (6).

For realizations of the relocation shock below the critical value θ^*, the bank pays out only a fraction of its reserves to movers. When the relocation shock is larger than θ^*, the bank takes a liquidity loan from the lender of last resort, and pays out all its reserves, plus the liquidity it obtains from the loan, to movers. At the beginning of next period, non-movers are paid what is left from loan repayments after the bank has repaid the lender of last resort.

We now proceed to solve for the optimal value of γ_t: To do so, we substitute the optimal values of α_t and β_t into the bank's objective function so that the only remaining variable to be determined is γ_t: We thus obtain the problem

$$\max_{0 \le \gamma_t \le 1} \int_0^{\theta^*} \ln\left[\gamma_t\frac{p_t}{p_{t+1}} + (1-\gamma_t)R_t\right] f(\theta)\,d\theta + \int_{\theta^*}^{1} \ln\left[\gamma_t\frac{p_t}{p_{t+1}} + (1-\gamma_t)R_t\right] f(\theta)\,d\theta$$

$$= \int_0^1 \ln\left[\gamma_t\frac{p_t}{p_{t+1}} + (1-\gamma_t)R_t\right] f(\theta)\,d\theta.$$

Here we see clearly that the introduction of an unrestricted lender of last resort allows the bank to

offer complete insurance to lenders: Both movers and non-movers receive the average return. In order to maximize this average return, the optimal choice of reserve-deposit ratio θ_t must be given by

$$\theta_t = \begin{cases} 0 & \\ \in (0;1) & \\ 1 & \end{cases} \text{ if } R_t \begin{cases} > \\ = \\ < \end{cases} \frac{p_t}{p_{t+1}}.$$

4.2 Equilibrium

The market-clearing equations are the same as in the previous section, and hence (9) and (10) continue to hold. Moreover, in equilibrium we cannot have $\theta_t = 0$, since then movers would have zero old-age consumption. We cannot have $\theta_t = 1$ either, since then borrowers would have zero young-period consumption. Therefore, in equilibrium the pricing relationship

$$R_t = \frac{p_t}{p_{t+1}} \tag{14}$$

must obtain. After substituting (14) into (9) and (10), the market clearing conditions simplify into the law of motion for θ,

$$\theta_{t+1} = \frac{y}{x} \frac{\theta_t}{1 - \theta_t}. \tag{15}$$

We can now state the following proposition.

Proposition 2 When there is an unlimited lender of last resort, the economy displays a unique steady state equilibrium, $\theta_b = 1 - \frac{y}{x} \in (0; \theta_a)$: There is a stationary equilibrium path for $\theta_0 = \theta_b$, and a continuum of inflationary equilibrium paths for $\theta_0 \in (0; \theta_b)$.

The proof of Proposition 2 is presented in Appendix B.

The results of Proposition 2 are illustrated in Fig. 2. The economy with unlimited lender of last resort services has a unique steady state. Since banks can now borrow money when the demand for it is high, they no longer hold precautionary reserves and therefore the steady state reserve-deposit ratio is smaller than in the case without a lender of last resort.

The striking feature of the new law of motion is that it permits inflationary equilibria. Suppose again that the initial price level is higher than $p_b = \frac{M}{\theta_b d}$. That is, suppose that in Fig. 2 the economy starts on the part of the law of motion that lies to the left of θ_b: This implies a trajectory along which the real supply of base money, $\theta_t d = \frac{M}{p_t}$, decreases continuously as the price level rises, and falls

to zero as the price level approaches infinity. However, now the bank takes out a liquidity loan $\pm_t$ every time its reserves $\circ_t$ fall short of its liquidity needs. Using (9) and (15) shows that in the limit as $\circ_t$ goes to zero, the return to holding money approaches $\frac{y}{x} < 1$: Thus there is sustained inflation in such an equilibrium. The amount of real borrowing is the limit is given by

$$\lim_{\circ_t \to 0} \pm_t = \text{¼}_t:$$

Hence the long-run expected value of the total real money supply is $E[\text{¼}] > 0$: Since the inflation rate is positive, this means that the amount of nominal borrowing from the lender of last resort is increasing over time (in expected value). This is why inflation is consistent with equilibrium in the model with a lender of last resort: *a lender of last resort responds to an inflation by printing money*. In the long run, the (expected) total money supply is increasing at exactly the rate of inflation. That is, in the long run, a form of the quantity theory of money holds here. The provision of liquidity by the lender of last resort confirms inflationary expectations and, consequently, inflationary equilibria arise as self-fulfilling prophecies.

In the steady state, the provision of unlimited and costless lender of last resort services allows the economy to completely overcome the stochastic relocation friction. Note that in the steady state, p_t is constant even though the money supply is being changed every period as banks borrow from the lender. The money supply is being moved to exactly match the stochastic movements in demand, leaving the value of money unchanged. In fact, the law of motion (15) is identical to the one that would obtain if there were no relocations in this economy. It is well known that the steady state is Pareto optimal in this case, but that the inflationary equilibria are not.[7]

In summary, the introduction of an unlimited lender of last resort generates a Pareto optimal equilibrium. However, it also generates a continuum of inflationary equilibria that are not Pareto efficient. Is it better to have a lender of last resort or not? There are no clear criteria for answering such a question, since it involves comparing the *set* of equilibria generated by two different policies. Rather than address it directly, we take an approach suggestive of that in Shell [16] , Grandmont [10] , Woodford [22] , and Smith [17] (among others). We ask if it is possible to design a policy that captures some of the benefit of providing lender of last resort services without introducing

[7] This follows from Proposition 5.6 in Balasko and Shell [2] . See also p.838 in Champ, Smith and Williamson [4] .

inflationary equilibria. We study two policies, the first of which restricts the amount of lending that can be undertaken and the second of which involves charging interest on liquidity loans.

5. A Lender of Last Resort with an Upper Bound on Loanable Funds

The analysis above shows that inflation is an equilibrium outcome if and only if the (expected) money supply increases at the rate of inflation in the long run. If the lender of last resort could credibly commit to print no more than C dollars in any period, this would eliminate all inflationary equilibria for any finite value of C: However, there are clear credibility issues with such a promise. If the bound were ever reached, it would be (short-run) optimal to reset the bound to some higher number so that it did not bind in the current period. If this is done every time the bound is reached, then effectively there is no bound and we are back in the case of Section 4.

In this section we focus instead on a lender that faces a fixed upper bound on the *real value* of the lending it can undertake. As an extreme example, imagine an economy where all currency injections would have to be completely backed by holdings of foreign reserves. We use $c \in (0;1)$ to denote the real value of liquidity per unit deposited that can be lent to a bank in distress.

5.1 The Bank's Problem

For this case, the bank's constraints continue to be given by (12) and (13). Substituting these constraints into the bank's objective function (5), dropping the constant terms, and taking into account the upper bound on loanable funds yields the problem

$$\max_{\alpha_t(\pi);\, \beta_t(\pi);\, \gamma_t} \quad \int_1^1 \pi \ln\left[\frac{\alpha(\pi)\gamma_t}{\pi} + \frac{\beta_t(\pi)}{\pi}\frac{p_t}{p_{t+1}}\right] f(\pi)\, d\pi + \int_0^1 (1-\pi)\ln\left[\frac{[1-\alpha(\pi)]\gamma_t}{(1-\pi)}\frac{p_t}{p_{t+1}} + \frac{(1-\gamma_t)}{(1-\pi)}R_t - \frac{\beta_t(\pi)}{(1-\pi)}\frac{p_t}{p_{t+1}}\right] f(\pi)\, d\pi:$$

$$0 \le \gamma_t \le 1$$

$$0 \le \alpha_t(\pi) \le 1$$

$$0 \le \beta_t \le c:$$

Clearly, the upper bound on loanable funds can only be binding for some states if its value is smaller

than the value of the loan a bank would take for $\varphi = 1$ in the presence of an unlimited lender of last resort. Hence, the above program differs from the one in Section 4 only if $c < (1 - \gamma_t)\frac{R_t}{\frac{p_t}{p_{t+1}}} - e$ holds. We will solve the problem assuming that this condition holds. When it does not hold, the solution in the previous section applies.

As before, we can first solve for the optimal values of α_t and β_t for given values of γ_t and φ. This entails choosing α_t and β_t to solve

$$\max_{\alpha_t;\beta_t} \quad \varphi \ln\left[\frac{\alpha_t \gamma_t}{\varphi} + \frac{\beta_t}{\varphi}\frac{p_t}{p_{t+1}}\right] + (1-\varphi)\ln\left[\frac{(1-\alpha_t)\gamma_t}{(1-\varphi)}\frac{p_t}{p_{t+1}} + \frac{(1-\gamma_t)}{(1-\varphi)}R_t - \frac{\beta_t}{(1-\varphi)}\frac{p_t}{p_{t+1}}\right]$$

subject to

$$0 \le \alpha_t \le 1$$

$$0 \le \beta_t \le c.$$

The solution to this problem sets

$$\alpha_t(\varphi) = \begin{cases} \varphi\left(1 + \frac{1-\gamma_t}{\gamma_t}R_t\frac{p_{t+1}}{p_t}\right) \\ 1 \\ 1 \end{cases} \text{ and } \beta_t(\varphi) = \begin{cases} \gamma_t\left[\varphi\left(1 + \frac{1-\gamma_t}{\gamma_t}R_t\frac{p_{t+1}}{p_t}\right) - 1\right] \\ 0 \\ c \end{cases}$$

$$\text{for } \varphi \in \begin{cases} [0;\varphi^{*}) \\ [\varphi^{*};\varphi^{**}] \\ (\varphi^{**};1] \end{cases};$$

where φ^{*} continues to be given by (6) and φ^{**} is given by

$$\varphi^{**} = \frac{(\gamma_t + c)\frac{p_t}{p_{t+1}}}{\gamma_t\frac{p_t}{p_{t+1}} + (1-\gamma_t)R_t}. \tag{16}$$

For realizations of the relocation shock below the first critical value φ^{*}, the bank pays out only a fraction of its reserves to movers. When a relocation shock $\varphi \in [\varphi^{*};\varphi^{**}]$ materializes, all reserves are paid out to movers, and the bank obtains a liquidity loan $b(\varphi) < cd$. Finally, when the relocation shock is larger than φ^{**}, the bank pays out all its reserves to movers and, in addition, takes the maximum loan cd it can get from the lender of last resort. Notice that we have $\varphi^{**} = 1$ when $c = e$ holds, which confirms that this solution only applies for $c < e$.

We can now determine the bank's optimal portfolio in the presence of a constrained lender of last resort: To do so, we substitute the optimal values of α_t and b_t into the bank's objective function: This yields the problem

$$\max_{0 \le \theta_t \le 1} \int_{\pi}^{\pi^{**}} \left[\ln \left(\theta_t \frac{p_t}{p_{t+1}} + (1 - \theta_t) R_t \right) \right] f(\pi)\, d\pi + \int_{\pi^{**}}^{1} \left[\ln \left(\theta_t \frac{p_t}{p_{t+1}} + (1 - \theta_t) R_t \right) \right] f(\pi)\, d\pi + \int_{\pi^{**}}^{1} \left\{ \pi \ln \left(\frac{\theta_t}{\pi} + \frac{c}{\pi} \frac{p_t}{p_{t+1}} \right) + (1 - \pi) \ln \left[\frac{(1 - \theta_t)}{1 - \pi} R_t - \frac{c}{(1 - \pi)} \frac{p_t}{p_{t+1}} \right] \right\} f(\pi)\, d\pi;$$

which can be rewritten as

$$\max_{0 \le \theta_t \le 1} \int_{\pi^{**}}^{1} \left[\ln \left(\theta_t \frac{p_t}{p_{t+1}} + (1 - \theta_t) R_t \right) \right] f(\pi)\, d\pi + \int_{\pi^{**}}^{1} \left\{ \pi \ln \left(\frac{\theta_t}{\pi} + \frac{c}{\pi} \frac{p_t}{p_{t+1}} \right) + (1 - \pi) \ln \left[\frac{(1 - \theta_t)}{1 - \pi} R_t - \frac{c}{(1 - \pi)} \frac{p_t}{p_{t+1}} \right] \right\} f(\pi)\, d\pi.$$

Hence the return earned by both movers and non-movers will be the same when π is less than π^{**}, but will in general be different when π is greater than π^{**}. The first-order condition for this problem is

$$\frac{\frac{p_t}{p_{t+1}} - R_t}{\theta_t \frac{p_t}{p_{t+1}} + (1 - \theta_t) R_t} \int_0^{\pi^{**}} f(\pi) d\pi + \frac{1}{\theta_t + c} \int_{\pi^{**}}^{1} \pi f(\pi) d\pi = \frac{R_t}{(1 - \theta_t) R_t - c \frac{p_t}{p_{t+1}}} \int_{\pi^{**}}^{1} (1 - \pi) f(\pi) d\pi. \tag{17}$$

which can be reduced to[8]

$$\theta_t = 1 - \frac{1 - \pi^{**} + c}{1 - \pi^{**}} \int_{\pi^{**}}^{1} F(\pi) d\pi. \tag{18}$$

We are now ready to state the general equilibrium conditions for this case.

5.2 Equilibrium

The market-clearing conditions (9) and (10) continue to hold. Substituting these equations into the expression for π^{**} in (16) yields

$$\pi^{**} = \frac{\theta_{t+1} + \frac{\theta_{t+1}}{\theta_t} c}{\theta_{t+1} + \frac{y}{x}}.$$

[8] To facilitate the refereeing process, we provide the many intermediate steps in Referee's Appendix B.

Substituting this into (18), we obtain

$$\gamma_t = 1 - \frac{\left[\frac{y}{\bar{x}} - c\left(\frac{\gamma_{t+1}}{\gamma_t} - \gamma_{t+1} - \frac{y}{\bar{x}}\right)\right]}{\frac{y}{\bar{x}} - c\frac{\gamma_{t+1}}{\gamma_t}} \int_{\frac{\gamma_{t+1} + \frac{\gamma_{t+1}}{\gamma_t}c}{\gamma_{t+1} + \frac{y}{\bar{x}}}}^{1} F(\pi)d\pi. \tag{19}$$

The law of motion implicitly defined here applies when $c < \epsilon$ holds, or when

$$\gamma_{t+1} < \frac{y}{c\bar{x}}\gamma_t$$

holds. As shown in Fig. 3, the phase plane is effectively divided into two regions. Below the lower dashed line, the law of motion is given by (19). Above this line, it is given by (15). Both curves intersect the dashed line at $\gamma_t = 1 - c > 0$, and therefore the piecewise-defined law of motion is continuous. Whether or not the bound affects the steady state equilibrium simply depends on whether or not $c < \epsilon$ holds when $\gamma_{t+1} = \gamma_t$: That is, c is binding in some states in the stationary equilibrium if and only if

$$c < \frac{y}{\bar{x}}$$

holds.

Whether or not there exist inflationary equilibria is determined by (19), since this governs the law of motion near the origin. The demand for base money[9] is given by (18). Taking the limit as the return to holding money goes to zero, we have

$$\lim_{\frac{p_t}{p_{t+1}} \to 0} \gamma_t = E[\pi] - c(1 - E[\pi]);$$

which can be either positive or negative. If the upper bound is low enough for this to be positive, that is, if

$$c < \frac{E[\pi]}{1 - E[\pi]}$$

holds, then the demand for base money has a positive lower bound. This case is qualitatively similar to having no lender of last resort $(c = 0)$: Demand for base money never goes to zero, and therefore sustained inflations are not possible in equilibrium. This is the case depicted in Fig. 3, where the law of motion intersects the horizontal axis to the right of the origin.

[9] Recall that base money is circulating currency net of borrowing from the lender of last resort.

If instead we have

$$c > \frac{E[\varpi]}{1 - E[\varpi]};$$

then the demand for base money goes to zero when the return to holding money $\frac{p_t}{p_{t+1}}$ approaches some finite number. In this case the part of the law of motion given by [**?**] also begins at the origin, and hence the set of equilibria is qualitatively similar that when there is an unrestricted lender of last resort $(c = 1)$: There is a continuum of inflationary equilibria in which the inflation rate and the growth rate of the total (expected) money supply approach the same constant. None of these equilibria are Pareto optimal.

Finally, if we happen to have

$$c = \frac{E[\varpi]}{1 - E[\varpi]};$$

the demand for money goes to zero only as the rate of return to holding money goes to zero. In this case there are true hyperinflationary equilibria where the inflation rate (and the expected money growth rate) grows without bound. The following proposition formalizes this result.

Proposition 3 Suppose that there is a lender of last resort with an upper bound on loanable funds $c \in \left[\frac{y}{\bar{x}}; 1\right]$. Then there is a stationary equilibrium with $\theta_t = \theta_b$ for all t. If $c < \frac{E[\varpi]}{1 - E[\varpi]}$ holds, this is the unique equilibrium. If instead $c \geq \frac{E[\varpi]}{1 - E[\varpi]}$ holds, there is also a continuum of inflationary paths for $\theta_0 \in (0; \theta_b)$:

The proof of this proposition closely follows the reasoning given above and is therefore omitted. It is interesting to note that the condition for c to affect the steady state equilibrium and the condition for it to eliminate inflationary equilibria are unrelated. If the distribution of liquidity shocks satisfies

$$E[\varpi] > \frac{y}{y + \bar{x}}; \tag{20}$$

then an upper bound of this sort is an ideal policy. The cap can be chosen high enough to never bind in the steady state (making this equilibrium efficient), while still being low enough to eliminate inflationary equilibria. Note that (20) necessarily holds if the expected value of ϖ is at least $\frac{1}{2}$; as it is for the uniform distribution. If, however, high liquidity demand is a rare event (and hence $E[\varpi]$ is low), the bound required to eliminate the inflationary equilibria would be low and very little lending would take place.

6. A Lender of Last Resort that Charges Interest

We now return to a situation where the lender makes one-period loans of currency in any quantity that banks desire. However, banks are now required to pay interest on liquidity loans. This is in line with the recommendation of Bagehot [1] that "in a crisis, the lender of last resort should lend freely, at a penalty rate, ∶∶∶" After the realization of ϖ, a bank determines the real amount $b \geq 0$ that it would like to borrow at time t (which will depend on the realization of ϖ) and the lender prints bp_t dollars for the bank. Next period, the bank must return the same number of dollars and these dollars are destroyed so that the stock of base money remains unaffected. In addition, the bank must repay the lender of last resort br^d in goods $\left(r^d \geq 0\right)$, that is, r^d represents the real interest rate on liquidity loans.[10] Agents derive no utility from the revenue that the lender earns from these interest payments.[11]

6.1 The Bank's Problem

Under this arrangement, the bank's constraints become

$$\varpi r_t^m(\varpi) = \alpha_t(\varpi)\,\gamma_t \frac{p_t}{p_{t+1}} + \beta_t(\varpi)\frac{p_t}{p_{t+1}} \tag{21}$$

and

$$(1-\varpi)\,r_t(\varpi) = [1-\alpha_t(\varpi)]\,\gamma_t \frac{p_t}{p_{t+1}} + (1-\gamma_t)R_t - \beta_t(\varpi)\left(\frac{p_t}{p_{t+1}} + r^d\right). \tag{22}$$

Substituting (21) and (22) into the bank's objective function (5) and dropping the constant terms yields the problem

$$\max_{\alpha_t(\varpi);\,\beta_t(\varpi);\,\gamma_t} \quad \int_1^0 \varpi \ln\left(\frac{\alpha(\varpi)\,\gamma_t}{\varpi} + \frac{\beta_t(\varpi)}{\varpi}\frac{p_t}{p_{t+1}}\right) f(\varpi)\,d\varpi + \int_0^1 (1-\varpi)\ln\left(\frac{[1-\alpha(\varpi)]\,\gamma_t}{(1-\varpi)}\frac{p_t}{p_{t+1}} + \frac{(1-\gamma_t)}{(1-\varpi)}R_t - \frac{\beta_t(\varpi)}{(1-\varpi)}\left(\frac{p_t}{p_{t+1}} + r^d\right)\right) f(\varpi)\,d\varpi$$

subject to

[10] If the interest rate were instead fixed in nominal terms, the steady state would be the same as the one derived below, but it would be impossible to eliminate inflationary equilibria. With high inflation rates, the real interest payments would become small, and hence the economy would behave similarly to one where the real interest rate is zero.

[11] Effectively we are assuming that the lender throws this revenue away. If instead the revenue were rebated to banks as a state-contingent, lump-sum payment, the qualitative properties of the results would not change. Since such rebates complicate the algebra substantially, we present the simpler case here.

$$0 \leq \theta_t \leq 1$$

$$0 \leq \alpha_t(\pi) \leq 1$$

$$\beta_t(\pi) \geq 0.$$

Again, given the timing of the bank's decisions, we can first solve for the optimal values of α_t and β_t given θ_t and π. That is, we can choose α_t and β_t to solve

$$\max_{\alpha_t;\beta_t} \quad \pi \ln\left[\frac{\alpha_t \theta_t}{\pi} + \frac{\beta_t}{\pi}\frac{p_t}{p_{t+1}}\right] + (1-\pi)\ln\left[\frac{(1-\alpha_t)\theta_t}{(1-\pi)}\frac{p_t}{p_{t+1}} + \frac{(1-\theta_t)}{(1-\pi)}R_t - \frac{\beta_t}{(1-\pi)}\frac{p_t}{p_{t+1}} + r^d\right]$$

subject to

$$0 \leq \alpha_t \leq 1$$

$$\beta_t \geq 0.$$

The solution to this problem sets

$$\alpha_t(\pi) = \begin{cases} \pi\left[1 + \frac{1-\theta_t}{\theta_t}R_t\frac{p_{t+1}}{p_t}\right] \\ 1 \\ 1 \end{cases}, \quad \text{and } \beta_t(\pi) = \begin{cases} 0 \\ 0 \\ \theta_t\left[\pi\left(1 + \frac{1-\theta_t}{\theta_t}\frac{R_t}{\frac{p_t}{p_{t+1}} + r^d}\right) - 1\right] \end{cases},$$

$$\text{for } \pi \in \begin{cases} [0;\pi^{\square}) \\ [\pi^{\square};\pi^{\square\square}] \\ (\pi^{\square\square};1] \end{cases};$$

where $\pi^{\square}$ continues to be given by (6) and we have

$$\pi^{\square\square} = \frac{\theta_t\left[\frac{p_t}{p_{t+1}} + r^d\right]}{\theta_t\left[\frac{p_t}{p_{t+1}} + r^d\right] + (1-\theta_t)R_t}. \tag{23}$$

For realizations of the relocation shock below the first critical value $\pi^{\square}$, the bank pays out only a fraction of its reserves to movers. When a relocation shock $\pi \in [\pi^{\square};\pi^{\square\square}]$ materializes, all reserves are paid out to movers, but the bank does not resort to a liquidity loan. Finally, when the relocation shock is larger than $\pi^{\square\square}$, the bank pays out all its reserves to movers and, in addition, obtains a liquidity loan from the lender of last resort. The range of inaction $[\pi^{\square};\pi^{\square\square}]$ is generated by a kink in the bank's opportunity set. Once the level of reserves θ is set, the bank has a certain amount

of currency on hand and the return to holding that currency is $\frac{p_t}{p_{t+1}}$: If the bank wants to acquire additional currency (in order to increase total payments to movers), the cost per unit is higher, at $\left(\frac{p_t}{p_{t+1}} + r^d\right)$: Hence the increase in lenders' expected utility must be sufficiently large before the bank will undertake any borrowing at this rate.

We now proceed to solve for the optimal value of γ_t: To do so, we substitute the optimal values of α_t and β_t into the bank's objective function so that the only remaining variable to be determined is γ_t: We thus obtain the problem

$$\max_{0 \le \gamma_t \le 1} \begin{cases} \int_{\theta^*}^{\theta^{**}} \left[\ln\left(\gamma_t \frac{p_t}{p_{t+1}} + (1-\gamma_t) R_t\right)\right] f(\theta)\, d\theta + \\ \int_0^{\theta^{**}} \left[\theta \ln\left(\frac{\gamma_t}{\theta} \frac{p_t}{p_{t+1}}\right) + (1-\theta) \ln\left(\frac{(1-\gamma_t)}{(1-\theta)} R_t\right)\right] f(\theta)\, d\theta + \\ \int_{\theta^{**}}^{1} \left[\theta \ln\left(\gamma_t \frac{p_t}{p_{t+1}} + (1-\gamma_t) \frac{R_t}{\frac{p_t}{p_{t+1}} + r^d} \frac{p_t}{p_{t+1}}\right) + (1-\theta) \ln\left(\gamma_t \left(\frac{p_t}{p_{t+1}} + r^d\right) + (1-\gamma_t) R_t\right)\right] f(\theta)\, d\theta. \end{cases}$$

This formulation of the problem makes it clear that the return earned by both movers and non-movers will be the same when θ is less than θ^{**}, but will in general be different when θ is greater than θ^{**}. The first-order condition for this problem is

$$\frac{R_t - \frac{p_t}{p_{t+1}}}{\gamma_t \frac{p_t}{p_{t+1}} + (1-\gamma_t) R_t} F(\theta^*) + \frac{R_t - \left(\frac{p_t}{p_{t+1}} + r^d\right)}{\gamma_t \left(\frac{p_t}{p_{t+1}} + r^d\right) + (1-\gamma_t) R_t} [1 - F(\theta^{**})]$$
$$= \frac{1}{\gamma_t} \int_{\theta^*}^{\theta^{**}} \theta f(\theta)\, d\theta - \frac{1}{1-\gamma_t} \int_{\theta^*}^{\theta^{**}} (1-\theta) f(\theta)\, d\theta; \tag{24}$$

which can be reduced to[12]

$$\gamma_t = \theta^{**} - \int_{\theta^*}^{\theta^{**}} F(\theta)\, d\theta. \tag{25}$$

This implicitly defines the solution to the bank's problem in the presence of a lender of last resort who charges interest. We now turn to an analysis of general equilibrium for this case.

6.2 Equilibrium

Once again, the market-clearing equations are the same as in the benchmark case, and hence (9) and

[12] To facilitate the refereeing process, we provide the derivaton in Referee's Appendix C.

(10) continue to hold. Substituting (9) and (10) into the expression for $\pi^{\square}$ in (6) and the expression for $\pi^{\square\square}$ in (23) yields

$$\pi^{\square} = \frac{\phi_{t+1}}{\phi_{t+1} + \frac{y}{x}} \tag{26}$$

and

$$\pi^{\square\square} = \frac{\phi_{t+1} + r^d \phi_t}{\phi_{t+1} + r^d \phi_t + \frac{y}{x}}: \tag{27}$$

Substituting these into (25), we obtain the graph of the law of motion for ϕ_t,

$$\phi_t = \frac{\phi_{t+1} + r^d \phi_t}{\phi_{t+1} + r^d \phi_t + \frac{y}{x}} \int_{\frac{\phi_{t+1}}{\phi_{t+1} + \frac{y}{x}}}^{\frac{\phi_{t+1} + r^d \phi_t}{\phi_{t+1} + r^d \phi_t + \frac{y}{x}}} F(\pi)\, d\pi: \tag{28}$$

We can now state the following proposition.

Proposition 4 When there is a lender of last resort who charges an interest rate $r^d \geq 0$ on liquidity loans, the economy displays a unique steady state equilibrium $\phi_c \in [\phi_b; \phi_a]$: When $r^d \leq \frac{y}{x}$, there is a stationary equilibrium path for $\phi_0 = \phi_c$, and a continuum of inflationary equilibrium paths for $\phi_0 \in (0; \phi_c)$. When $r^d > \frac{y}{x}$, the economy displays a unique equilibrium, which is stationary with $\phi_t = \phi_c$ for all t:

The proof of Proposition 4 is presented in Appendix C. The case of $r^d > \frac{y}{x}$ is illustrated in Fig. 4.

The case of no lender of last resort studied in Section 3 corresponds to $r^d = 1$, while the case of an unrestricted lender (Section 4) corresponds to $r^d = 0$: The proposition here shows that when r^d is positive and finite, the steady state falls in between those of the first two cases. Having the ability to borrow leads banks to reduce reserve holdings below ϕ_a: However, since borrowing is costly, banks still hold some precautionary reserves and ϕ is above ϕ_b:

The proposition also shows that eliminating inflationary equilibria requires the interest rate to be sufficiently high. In particular, r^d must be above $\frac{y}{x}$; the asymptotic rate of return to real lending along any inflationary path. The reason for this is that such a high interest rate generates a lower bound on the demand for base money. To see why, suppose that the economy follows an inflationary trajectory; that is, suppose that the economy starts on the part of the law of motion that lies to the left of ϕ_c in Fig. 4. In each period, the value of the stock of base money decreases and therefore the rate of return to holding money must also decrease. As this happens, banks engage in more real lending and the rate of return to real lending falls as well. Imagine a situation where ϕ has become very close to zero, that is, where there has been sustained inflation for many periods. This implies

that in practically every period, the bank will be borrowing currency at a cost of $\frac{p_t}{p_{t+1}} + r^d$: At the same time, the return the bank is receiving from its real lending is close to $\frac{y}{x}$: If $r^d > \frac{y}{x}$ holds, then regardless of $\frac{p_t}{p_{t+1}}$ the bank would be better off holding more reserves and engaging in less lending, simply because borrowing is so expensive. This means there is a lower bound on the demand for base money, even as the rate of return to holding money goes to zero. For this reason there cannot be a sustained inflation. If, on the other hand, $r^d < \frac{y}{x}$ holds, then as $\frac{p_t}{p_{t+1}}$ becomes very low the real value of reserve holdings chosen by banks will also go to zero and inflation is an equilibrium outcome. Roughly speaking, when borrowing is not very expensive, banks prefer to put all of their deposits into real loans and hence there is no demand for base money.

7. Conclusions

We have studied a pure-exchange economy in which spatial separation, limited communication and random relocation combine to create a role for money, even when it is dominated in rate of return. Banks arise endogenously in this world to insure agents against the liquidity shocks implied by random relocation. When the money supply is constant, the economy displays a unique equilibrium that is not Pareto efficient. This equilibrium is marked by periodic crises in which high aggregate liquidity demand leads to low consumption levels for agents in need of liquidity.

When we introduce a lender of last resort providing unlimited, zero nominal interest rate liquidity loans to banks in distress, the stationary equilibrium is Pareto optimal. However, there is a continuum of inflationary equilibria that are not Pareto efficient. Thus, while allowing the economy to overcome the frictions associated with stochastic relocation, the introduction of an unrestricted lender of last resort also makes the economy vulnerable to currency instability. We then show that these inflationary equilibria disappear when the lender of last resort either (i) credibly commits to a cap on liquidity loans that is sufficiently low or (ii) charges an interest rate on liquidity loans that is sufficiently high.

There are several directions in which the present analysis could be extended to address additional issues that figure prominently in discussions of the desirability and optimal design of lender of last resort services. First, our model is set up so that the provision of liquidity loans does not affect the government's intertemporal budget constraint. Yet the fiscal cost of bank bailouts is a primary

concern in the design of lender of last resort arrangements. Changes in the structure of the model could be made to address this issue. Second, it is often argued that the explicit or implicit access to liquidity loans provides banks with an incentive to take on "excessive" risk in its asset portfolio. This could be addressed by adding technologies to the model that give banks a choice regarding the riskiness of their investments. Third, in many emerging economies countries, and certainly in those that are contemplating dollarization, a large fraction of banks' liabilities and assets is denominated in foreign currency. Hence, the provision of lender of last resort services, because of its effect on the money supply and thus on the exchange rate, may affect the real value of that part of the portfolio that is denominated in foreign exchange. Addressing this issue would require either a two-country or an open-economy version of the model. We plan to address all these important questions in future research.

APPENDIX A: PROOF OF PROPOSITION 1

We cannot solve (11) explicitly for $\circ_{t+1}$ as a function of $\circ_t$, but we can derive some properties of the implicit function. First note that when $\circ_{t+1} = 0$, which is its minimum value, $\circ_t$ is given by $\circ_t = 1 - \int_0^1 F(\alpha)\,d\alpha = \int_0^1 \alpha f(\alpha)\,d\alpha = E(\alpha)$. Thus the implicit function is not defined for values of $\circ_t$ below the expected value of α. Next, when $\circ_{t+1} = 1$, which is its maximum value, $\circ_t$ is strictly below one. Since the implicit function is continuous, there exists at least one steady state for $\circ$. Moreover, at a steady state (11) implies that we have

$$\circ = 1 - \int_{\frac{\circ}{\circ + \frac{y}{\bar{x}}}}^{1} F(\alpha)\,d\alpha.$$

We know $F < 1$ always holds, so any steady state $\circ$ must satisfy

$$\circ > 1 - \int_{\frac{\circ}{\circ + \frac{y}{\bar{x}}}}^{1} d\alpha = \frac{\circ}{\circ + \frac{y}{\bar{x}}}.$$

Hence for any steady state

$$\circ > 1 - \frac{y}{\bar{x}} \tag{29}$$

must hold.

For the slope of (11), we can use Leibnitz's integral rule to obtain

$$\frac{d\circ_t}{d\circ_{t+1}} = F\left(\frac{\circ_{t+1}}{\circ_{t+1} + \frac{y}{\bar{x}}}\right)\left[\frac{\frac{y}{\bar{x}}}{\left(\circ_{t+1} + \frac{y}{\bar{x}}\right)^2}\right] > 0.$$

Thus $\circ_t$ as a function of $\circ_{t+1}$ is always increasing and hence is invertible. The inverse function is the law of motion for $\circ_t$, and it is also strictly increasing. By the inverse function rule we obtain the slope of the law of motion,

$$\frac{d\circ_{t+1}}{d\circ_t} = \frac{\left(\circ_{t+1} + \frac{y}{\bar{x}}\right)^2}{\frac{y}{\bar{x}}}\left[\frac{1}{F\left(\frac{\circ_{t+1}}{\circ_{t+1} + \frac{y}{\bar{x}}}\right)}\right] > 0. \tag{30}$$

We can now evaluate the derivative of the law of motion, (30), at any steady state. Given (29) and taking into account that the economy is a Samuelson case economy, which implies that $y < \bar{x}$, the first term of (30) is greater than one for any steady state. The second term is always greater than or

equal to one, so for any steady state the derivative itself must be greater than one. The law of motion for $\circ_t$ must therefore cross the 45± line from below at every steady state. This implies that there is exactly one steady state, which we will denote by $\circ_a$ and that this steady state is in the open interval $(E(\tfrac{1}{4}); 1)$: The steady state is globally unstable, and all nonstationary trajectories eventually leave the feasible region. Hence the steady state is the only equilibrium of this economy. ■

APPENDIX B: PROOF OF PROPOSITION 2

The law of motion (15) starts at the origin, and is increasing and convex. The slope of (15) at its origin is $\frac{y}{\bar{x}} < 1$, and hence there is a unique steady state, the value of which is given by $\circ_b = 1 - \frac{y}{\bar{x}} > 0$. From Proposition 1, we know that $1 - \frac{y}{\bar{x}} < \circ_a$, therefore $\circ_b < \circ_a$. The slope of (15) evaluated at the steady state is $\frac{\bar{x}}{y} > 1$: Hence the unique steady state is unstable. Each $\circ_0 > \circ_b$ generates a trajectory that leaves the feasible region and hence cannot be an equilibrium. Each $\circ_0 \in (0; \circ_b]$ is associated with a trajectory that remains within the feasible region, and hence there is a continuum of equilibrium paths. For $\circ_0 = \circ_b$, the equilibrium path is stationary. For $\circ_0 \in (0; \circ_b)$, we have $\lim_{t \to \infty} \circ_t = \lim_{t \to \infty} z_t = 0$, and therefore we have $\lim_{t \to \infty} p_t = \infty$ and the equilibrium trajectories display sustained positive inflation rates. ■

APPENDIX C: PROOF OF PROPOSITION 4

We cannot solve (28) explicitly for $\circ_{t+1}$ as a function of $\circ_t$, but we can nevertheless derive some properties of the law of motion for $\circ_t$. First note that when $\circ_{t+1} = 1$, $\circ_t$ is strictly below one. Moreover, when $\circ_{t+1} = 0$, (28) implies that we have

$$\int_0^{\frac{r^d \circ_t}{r^d \circ_t + \frac{y}{\bar{x}}}} F(\tfrac{1}{4})\, d\tfrac{1}{4} = \circ_t \left(\frac{r^d}{r^d \circ_t + \frac{y}{\bar{x}}} - 1 \right) : \tag{31}$$

Clearly, $\circ_t = 0$ satisfies (31), so the law of motion starts at the origin. Since (28) is continuous, we can conclude that there exists at least one steady state.

However, $\circ_t = 0$ may not be the only value for which $\circ_{t+1} = 0$. Indeed, when $r^d > \frac{y}{\bar{x}}$ then there will exists at least one other $\circ_t \in (0; 1)$ for which $\circ_{t+1} = 0$:

Further, at any steady state, θ, (28) simplifies to

$$\theta - \left(\frac{1+r^d}{(1+r^d)\theta + \frac{y}{x}}\right)\theta = \int_{\frac{\theta}{\theta+\frac{y}{x}}}^{\frac{(1+r^d)\theta}{(1+r^d)\theta+\frac{y}{x}}} F(\varphi)\,d\varphi.$$

Notice that this implies that for $r^d = 0, \theta = 1 - \frac{y}{x} = \theta_b$. Moreover, we know $F < 1$ always holds. Thus, for $r^d > 0$, the steady state level of θ must satisfy

$$\theta - \left(\frac{1+r^d}{(1+r^d)\theta + \frac{y}{x}}\right)\theta > \int_{\frac{\theta}{\theta+\frac{y}{x}}}^{\frac{(1+r^d)\theta}{(1+r^d)\theta+\frac{y}{x}}} d\varphi = \frac{\theta}{\theta + \frac{y}{x}} - \left(\frac{1+r^d}{(1+r^d)\theta + \frac{y}{x}}\right)\theta.$$

Hence, at any steady state, (29) is satisfied.

Next, it can be shown that

$$\frac{d\theta_{t+1}}{d\theta_t} = \frac{1 - \frac{r^d \frac{y}{x}}{\left(\theta_{t+1} + r^d\theta_t + \frac{y}{x}\right)^2}\left[1 - F\left(\frac{\theta_{t+1} + r^d\theta_t}{\theta_{t+1} + r^d\theta_t + \frac{y}{x}}\right)\right]}{\frac{\frac{y}{x}}{\left(\theta_{t+1} + r^d\theta_t + \frac{y}{x}\right)^2}\left[1 - F\left(\frac{\theta_{t+1} + r^d\theta_t}{\theta_{t+1} + r^d\theta_t + \frac{y}{x}}\right)\right] + \frac{\frac{y}{x}}{\left(\theta_{t+1} + \frac{y}{x}\right)^2} F\left(\frac{\theta_{t+1}}{\theta_{t+1} + \frac{y}{x}}\right)} \tag{32}$$

holds. Clearly the law of motion is increasing everywhere if $r^d \leq \frac{y}{x}$. However, when $r^d > \frac{y}{x}$, that is no longer the case.

Indeed, for $\theta_{t+1} = \theta_t = 0$, we obtain

$$\left.\frac{d\theta_{t+1}}{d\theta_t}\right|_{\theta_{t+1}=\theta_t=0} = \frac{y}{x} - r^d.$$

Hence we have

$$\left.\frac{d\theta_{t+1}}{d\theta_t}\right|_{\theta_{t+1}=\theta_t=0} \begin{cases} \geq 0, & \frac{y}{x} < 1 \\ < 0 & \text{if } r^d > \frac{y}{x}. \end{cases}$$

Moreover, using (26), (27), and (32), it is clear that at any steady state $\theta > 0$, we have

$$\left.\frac{d\theta_{t+1}}{d\theta_t}\right|_{\theta_{t+1}=\theta_t=\theta} > 1$$

if and only if

$$\left[\frac{\left(1 + r^d\right)\frac{y}{x}}{(1 + r^d)\theta + \frac{y}{x}}\right]^2 [1 - F(\varphi^{**})] + \left[\frac{\frac{y}{x}}{\theta + \frac{y}{x}}\right]^2 F(\varphi^{*}) < 1 \tag{33}$$

holds. Given that (29) holds at any steady state, (33) is equivalent to

$$\left(\frac{1+r^d}{(1+r^d)\theta+\frac{y}{x}}\right)[1-F(\varphi^{**})]+\left(\frac{\theta}{\theta+\frac{y}{x}}\right)F(\varphi^{*})<\frac{\theta}{\frac{y}{x}};$$

which, using (26) and (27), can be expressed as

$$\varphi^{**}[1-F(\varphi^{**})]+\varphi^{*}F(\varphi^{*})<\frac{\theta}{\frac{y}{x}}:$$

But from (41) in Referee's Appendix B, we have

$$\varphi^{**}[1-F(\varphi^{**})]+\varphi^{*}F(\varphi^{*})=\theta-\int_{\varphi^{*}}^{\varphi^{**}}\varphi f(\varphi)\,d\varphi<\theta<\frac{\theta}{\frac{y}{x}}:$$

Hence $\frac{d\theta_{t+1}}{d\theta_t}\Big|_{\theta_{t+1}=\theta_t=\theta}>1$ always holds, and the law of motion must cross the 45° line from below at every steady state. This implies that there is exactly one steady state, which we will denote by θ_c. From (29) and Proposition 2, it is clear that $\theta_c>\theta_b$ for $r^d>0$, while $\theta_c=\theta_b$ for $r^d=0$.

Moreover, $\theta_c<\theta_a$ also holds. Indeed, it can be shown that

$$\frac{d\theta_c}{dr^d}=\frac{\theta_c\frac{y}{x}\left(1-F\left(\frac{(1+r^d)\theta}{(1+r^d)\theta+\frac{y}{x}}\right)\right)}{\left(1-F\left(\frac{\theta}{\theta+\frac{y}{x}}\right)\frac{\frac{y}{x}}{(\theta+\frac{y}{x})^2}\right)\left((1+r^d)\theta+\frac{y}{x}\right)^2+(1+r^d)\frac{y}{x}\left(1-F\left(\frac{(1+r^d)\theta}{(1+r^d)\theta+\frac{y}{x}}\right)\right)}:$$

Clearly, given (29), we have $\frac{d\theta_c}{dr^d}>0$: Moreover, as r^dgoes to infinity, the case without a lender of last resort, the expression $\frac{\theta_{t+1}+r^d\theta_t}{\theta_{t+1}+r^d\theta_t+\frac{y}{x}}$ approaches unity and hence (28) collapses to (11). Hence for $r^d\to\infty$, $\theta_c\to\theta_a$ holds, while $\theta_c<\theta_a$ for $r^d<\infty$.

Given that the law of motion crosses the 45° line from below at any steady state, θ_c is globally unstable. When $r^d\le\frac{y}{x}$, each $\theta_0\in(0;\theta_c]$ is associated with a trajectory that remains within the feasible region. Hence there is a continuum of equilibrium paths. For $\theta_0=\theta_c$, the equilibrium path is stationary. For $\theta_0\in(0;\theta_c)$, we have $\lim_{t\to\infty}\theta_t=\lim_{t\to\infty}m_t=0$, and hence $\lim_{t\to\infty}p_t=\infty$ and the equilibrium trajectories display sustained inflation. On the other hand, when $r^d>\frac{y}{x}$, all nonstationary trajectories eventually leave the feasible region. Hence the steady state is the only equilibrium. ■

REFEREE'S APPENDIX A: DERIVATION OF (8)

Using (6), the first-order condition (7) can be written as

$$\varphi^{*}F(\varphi^{*}) + \int_{\varphi^{*}}^{1} \varphi f(\varphi)\,d\varphi = \frac{p_{t+1}}{p_t} R_t \varphi^{*} F(\varphi^{*}) + \frac{\delta_t}{1-\delta_t} \int_{\varphi^{*}}^{1} (1-\varphi) f(\varphi)\,d\varphi;$$

or

$$\varphi^{*}F(\varphi^{*}) + \int_{\varphi^{*}}^{1} \varphi f(\varphi)\,d\varphi = \frac{p_{t+1}}{p_t} R_t \varphi^{*} F(\varphi^{*}) + \frac{\delta_t}{1-\delta_t} \left\{ [1-F(\varphi^{*})] - \int_{\varphi^{*}}^{1} \varphi f(\varphi)\,d\varphi \right\}.$$

If we add $\frac{\delta_t}{1-\delta_t}\varphi^{*}F(\varphi^{*})$ to both sides, we have

$$\varphi^{*}F(\varphi^{*}) + \frac{\delta_t}{1-\delta_t}\varphi^{*}F(\varphi^{*}) + \int_{\varphi^{*}}^{1} \varphi f(\varphi)\,d\varphi =$$
$$\left(\frac{p_{t+1}}{p_t} R_t + \frac{\delta_t}{1-\delta_t}\right) \varphi^{*}F(\varphi^{*}) + \frac{\delta_t}{1-\delta_t} \left\{ [1-F(\varphi^{*})] - \int_{\varphi^{*}}^{1} \varphi f(\varphi)\,d\varphi \right\};$$

which reduces to

$$\varphi^{*}F(\varphi^{*}) + \int_{\varphi^{*}}^{1} \varphi f(\varphi)\,d\varphi = \left[(1-\delta_t)\frac{p_{t+1}}{p_t} R_t + \delta_t\right] \varphi^{*}F(\varphi^{*}) + \delta_t - \delta_t F(\varphi^{*}).$$

Making use of (6) again, we obtain

$$\varphi^{*}F(\varphi^{*}) + \int_{\varphi^{*}}^{1} \varphi f(\varphi)\,d\varphi = \delta_t F(\varphi^{*}) + \delta_t - \delta_t F(\varphi^{*}) = \delta_t \tag{34}$$

This can be written in another form by noting that

$$\frac{d}{dx}[xF(x)] = F(x) + xf(x) \tag{35}$$

holds, which allows us to write

$$\begin{aligned}
\int_{\varphi^{*}}^{1} \varphi f(\varphi)\,d\varphi &= \int_{\varphi^{*}}^{1} \left\{ \frac{d}{d\varphi}[\varphi F(\varphi)] - F(\varphi) \right\} d\varphi \\
&= \varphi F(\varphi)\big|_{\varphi^{*}}^{1} - \int_{\varphi^{*}}^{1} F(\varphi)\,d\varphi \\
&= 1 - \varphi^{*}F(\varphi^{*}) - \int_{\varphi^{*}}^{1} F(\varphi)\,d\varphi.
\end{aligned}$$

This demonstrates that

$$1 - \int_{\varphi^*}^{1} F(\varphi)\, d\varphi = \varphi^* F(\varphi^*) + \int_{\varphi^*}^{1} \varphi f(\varphi)\, d\varphi.$$

Substituting this into (34) yields the solution presented in (8).

REFEREE'S APPENDIX B: DERIVATION OF (18)

Using the definition of φ^* given by (6), the first order condition (17) can be rewritten as:

$$\left(1 - R_t \frac{p_{t+1}}{p_t}\right) \frac{\varphi^*}{\theta_t} F(\varphi^{**}) + \frac{1}{\theta_t + c} \int_{\varphi^{**}}^{1} \varphi f(\varphi) d\varphi = \frac{R_t}{(1-\theta_t) R_t - c \frac{p_t}{p_{t+1}}} \int_{\varphi^{**}}^{1} (1-\varphi) f(\varphi) d\varphi. \quad (36)$$

Now, note that (6) and (16) imply that we have

$$\frac{1}{\theta_t + c} = \frac{\varphi^*}{\varphi^{**}} \frac{1}{\theta_t}$$

and

$$\frac{R_t}{(1-\theta_t) R_t - c \frac{p_t}{p_{t+1}}} = \frac{\varphi^*}{1-\varphi^{**}} R_t \frac{p_{t+1}}{p_t} \frac{1}{\theta_t}.$$

This allows us to write (36) as

$$\left(1 - R_t \frac{p_{t+1}}{p_t}\right) \varphi^{**} F(\varphi^{**}) + \int_{\varphi^{**}}^{1} \varphi f(\varphi) d\varphi = \frac{\varphi^{**}}{1-\varphi^{**}} R_t \frac{p_{t+1}}{p_t} \int_{\varphi^{**}}^{1} (1-\varphi) f(\varphi) d\varphi. \quad (37)$$

Using (35), note that we have

$$\begin{aligned} \int_{\varphi^{**}}^{1} \varphi f(\varphi)\, d\varphi &= \int_{\varphi^{**}}^{1} \left[\frac{d}{d\varphi} [\varphi F(\varphi)] - F(\varphi) \right] d\varphi \\ &= \varphi F(\varphi) \Big|_{\varphi^{**}}^{1} - \int_{\varphi^{**}}^{1} F(\varphi)\, d\varphi \\ &= 1 - \varphi^{**} F(\varphi^{**}) - \int_{\varphi^{**}}^{1} F(\varphi) d\varphi. \end{aligned}$$

Hence, (37) reduces to

$$1 - \int_{\varphi^{**}}^{1} F(\varphi) d\varphi = \frac{\varphi^{**}}{1-\varphi^{**}} R_t \frac{p_{t+1}}{p_t} \int_{\varphi^{**}}^{1} F(\varphi) d\varphi. \quad (38)$$

Multiplying (38) by $(1 - \varphi^{**})$ and rearranging gives

$$1 - \varphi^{**} = \left(1 - \varphi^{**} + \varphi^{**} R_t \frac{p_{t+1}}{p_t} \int_{\varphi^{**}}^{\infty} F(\varphi)\, d\varphi\right). \tag{39}$$

But (16) implies that we have

$$\varphi^{**} R_t \frac{p_{t+1}}{p_t} = \frac{\theta_t (1 - \varphi^{**}) + c}{(1 - \theta_t)}.$$

Therefore, (39) becomes

$$1 - \varphi^{**} = \left(1 - \varphi^{**} + \frac{\theta_t (1 - \varphi^{**}) + c}{(1 - \theta_t)} \int_{\varphi^{**}}^{\infty} F(\varphi)\, d\varphi\right). \tag{40}$$

Multiplying (40) $(1 - \theta_t)$ and rearranging terms yields

$$1 - \theta_t = \left(\frac{(1 - \theta_t)(1 - \varphi^{**}) + \theta_t (1 - \varphi^{**}) + c}{(1 - \varphi^{**})} \int_{\varphi^{**}}^{\infty} F(\varphi)\, d\varphi\right).$$

This clearly reduces to (18).

REFEREE'S APPENDIX C: DERIVATION OF (25)

Using (6) and (24) in the first-order condition **??**, we obtain

$$\varphi^{*} F(\varphi^{*}) + \varphi^{**}[1 - F(\varphi^{**})] + \int_{\varphi^{*}}^{\varphi^{**}} \varphi f(\varphi)\, d\varphi =$$
$$\frac{p_{t+1}}{p_t} R_t \varphi^{*} F(\varphi^{*}) + \frac{R_t}{\frac{p_t}{p_{t+1}} + r^d} \varphi^{**}[1 - F(\varphi^{**})] + \frac{\theta_t}{1 - \theta_t} \int_{\varphi^{*}}^{\varphi^{**}} (1 - \varphi) f(\varphi)\, d\varphi;$$

or

$$\varphi^{*} F(\varphi^{*}) + \varphi^{**}[1 - F(\varphi^{**})] + \int_{\varphi^{*}}^{\varphi^{**}} \varphi f(\varphi)\, d\varphi =$$
$$\frac{p_{t+1}}{p_t} R_t \varphi^{*} F(\varphi^{*}) + \frac{R_t}{\frac{p_t}{p_{t+1}} + r^d} \varphi^{**}[1 - F(\varphi^{**})] + \frac{\theta_t}{1 - \theta_t}\left[F(\varphi^{**}) - F(\varphi^{*})\right] - \int_{\varphi^{*}}^{\varphi^{**}} \varphi f(\varphi)\, d\varphi\Bigg].$$

If we add $\frac{\delta_t}{1-\delta_t}\pi^{*}F(\pi^{*}) + \frac{\delta_t}{1-\delta_t}\pi^{**}[1-F(\pi^{**})]$ to both sides, we have

$$\pi^{*}F(\pi^{*}) + \pi^{**}[1-F(\pi^{**})] + \int_{\pi^{*}}^{\pi^{**}} \pi f(\pi)\,d\pi + \frac{\delta_t}{1-\delta_t}\pi^{*}F(\pi^{*}) + \frac{\delta_t}{1-\delta_t}\pi^{**}[1-F(\pi^{**})] =$$

$$\left(\frac{p_{t+1}}{p_t}R_t + \frac{\delta_t}{1-\delta_t}\right)\pi^{*}F(\pi^{*}) + \left(\frac{R_t}{\frac{p_t}{p_{t+1}} + r^d} + \frac{\delta_t}{1-\delta_t}\right)\pi^{**}[1-F(\pi^{**})] +$$

$$\frac{\delta_t}{1-\delta_t}[F(\pi^{**}) - F(\pi^{*})] - \int_{\pi^{*}}^{\pi^{**}} \pi f(\pi)\,d\pi :$$

This reduces to

$$\pi^{*}F(\pi^{*}) + \pi^{**}[1-F(\pi^{**})] + \int_{\pi^{*}}^{\pi^{**}} \pi f(\pi)\,d\pi =$$

$$(1-\delta_t)\frac{p_{t+1}}{p_t}R_t + \delta_t\,\pi^{*}F(\pi^{*}) + (1-\delta_t)\frac{R_t}{\frac{p_t}{p_{t+1}} + r^d} + \delta_t\,\pi^{**}[1-F(\pi^{**})] + \delta_t[F(\pi^{**}) - F(\pi^{*})]$$

Making use of (6) and (23) again, we obtain

$$\pi^{*}F(\pi^{*}) + \pi^{**}[1-F(\pi^{**})] + \int_{\pi^{*}}^{\pi^{**}} \pi f(\pi)\,d\pi$$
$$= \delta_t F(\pi^{*}) + \delta_t[1-F(\pi^{**})] + \delta_t F(\pi^{**}) - \delta_t F(\pi^{*})$$

Therefore, the solution to the problem is

$$\delta_t = \pi^{*}F(\pi^{*}) + \pi^{**}[1-F(\pi^{**})] + \int_{\pi^{*}}^{\pi^{**}} \pi f(\pi)\,d\pi. \tag{41}$$

Using (35), note that we have

$$\int_{\pi^{*}}^{\pi^{**}} \pi f(\pi)\,d\pi = \int_{\pi^{*}}^{\pi^{**}} \left[\frac{d}{d\pi}[\pi F(\pi)] - F(\pi)\right] d\pi$$
$$= \pi F(\pi)\Big|_{\pi^{*}}^{\pi^{**}} - \int_{\pi^{*}}^{\pi^{**}} F(\pi)\,d\pi$$
$$= \pi^{**}F(\pi^{**}) - \pi^{*}F(\pi^{*}) - \int_{\pi^{*}}^{\pi^{**}} F(\pi)\,d\pi.$$

Therefore, it is the case that

$$-\int_{\pi^{*}}^{\pi^{**}} F(\pi)\,d\pi = \pi^{*}F(\pi^{*}) - \pi^{**}F(\pi^{**}) + \int_{\pi^{*}}^{\pi^{**}} \pi f(\pi)\,d\pi$$

holds. Substituting this into (41) yields the solution presented in (25).

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Figure 1: No Lender of Last Resort

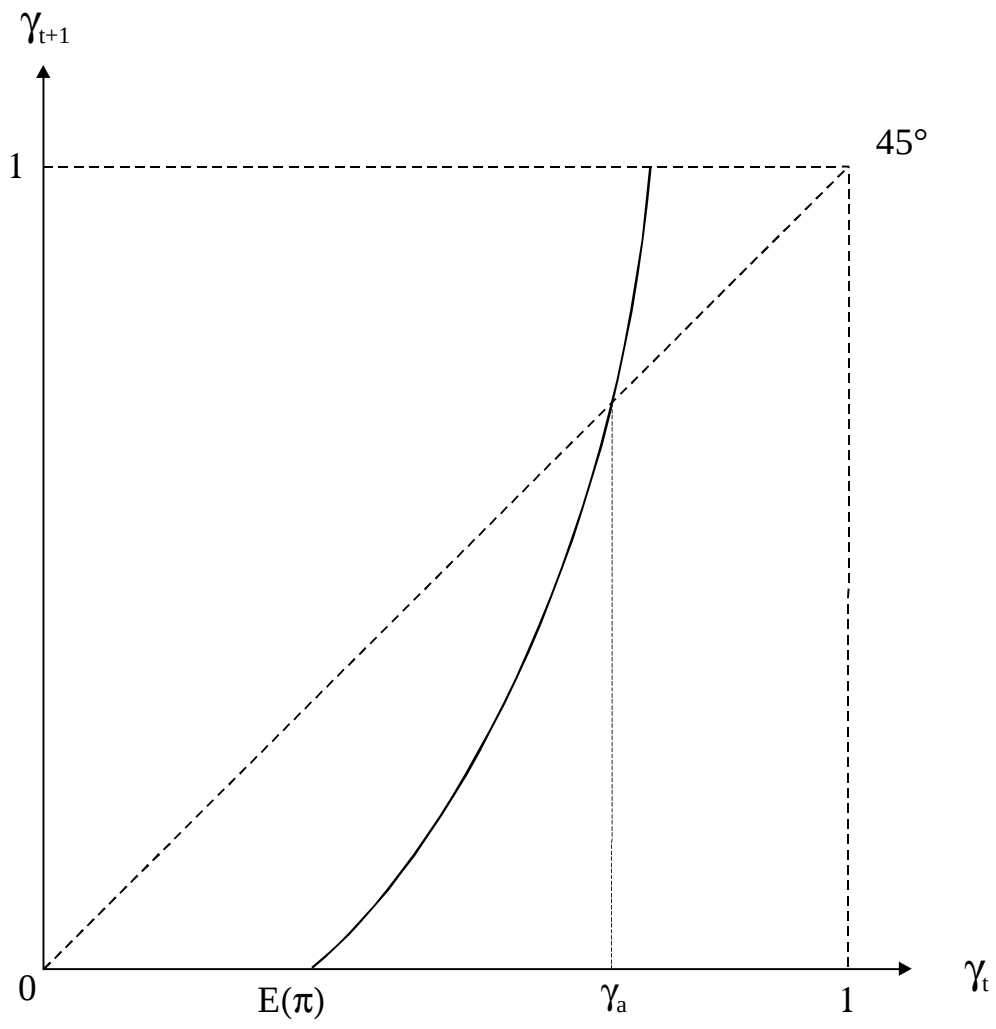

Figure 2: An Unrestricted Lender of Last Resort

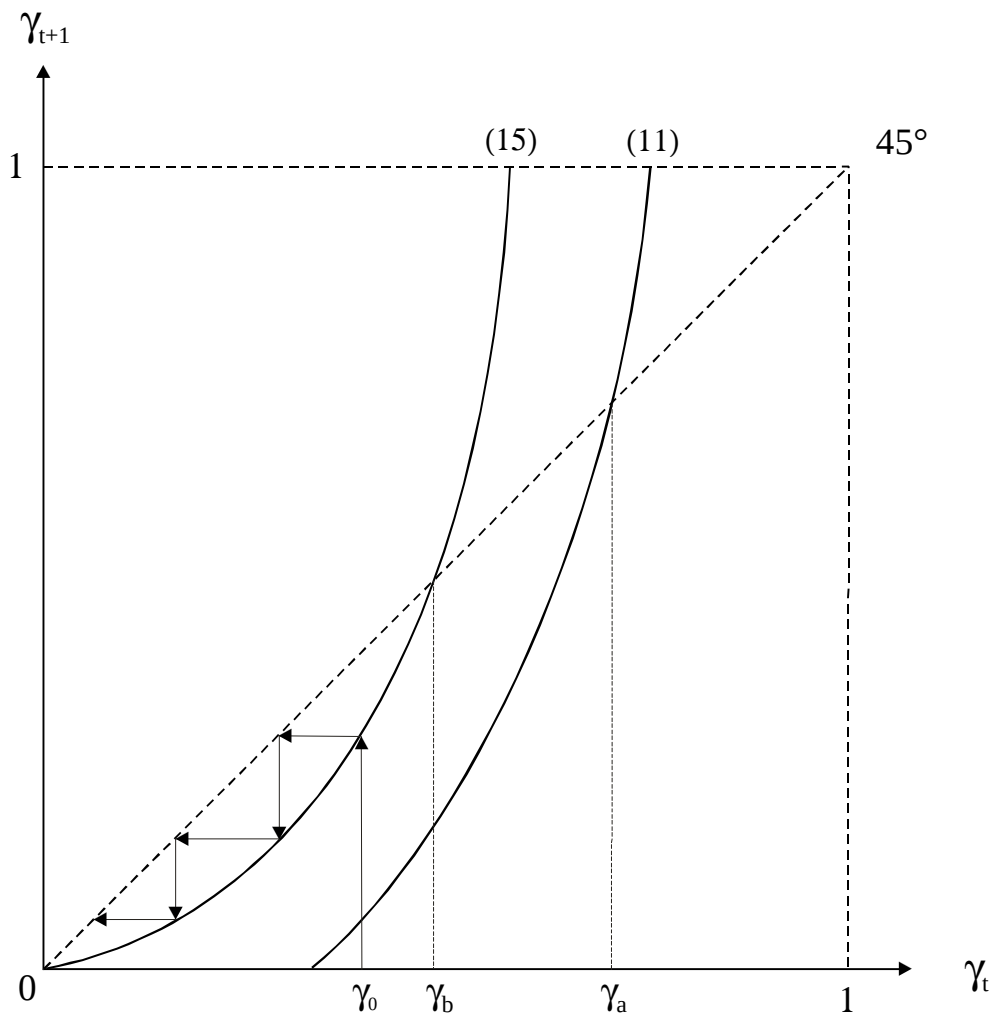

Figure 3: An Upper Bound on Loanable Funds

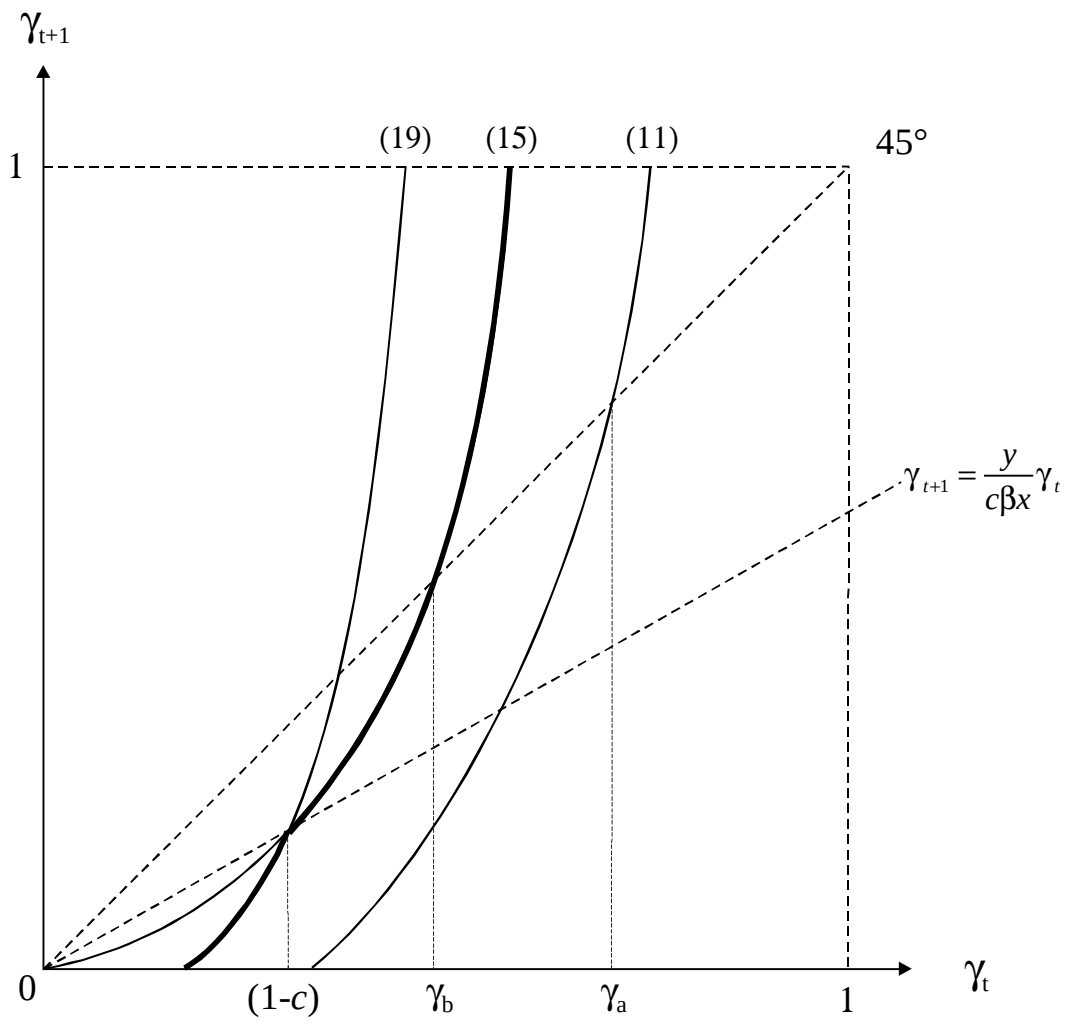
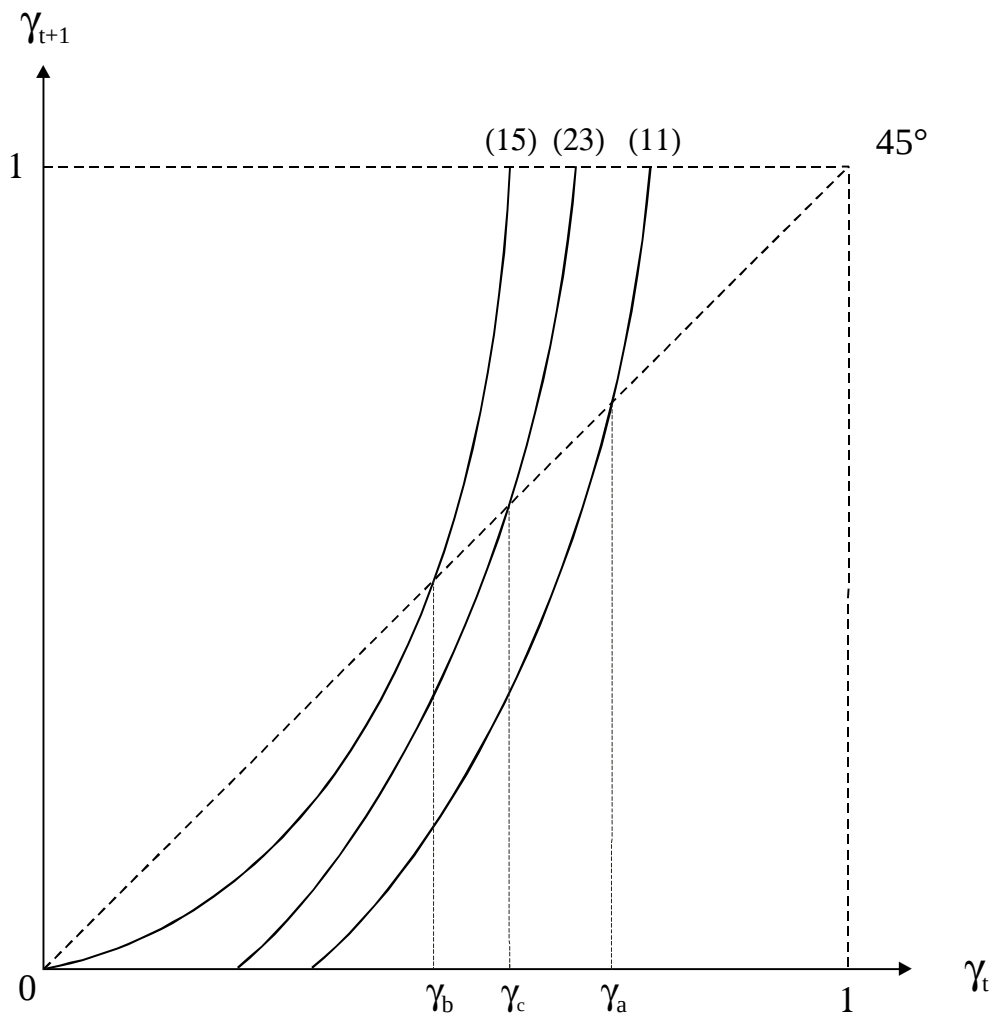
Figure 4: A Lender of Last Resort that Charges Interest