Size of Middleman in a Barter Economy
Preliminary Paper

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Abstract

This paper develops a model of intermediation in search environment with complete information and explains the optimal choice of the size of a middleman in an economy with an arbitrary finite number of goods. The model with linear storage cost predicts the existence of stationary equilibrium with the same number of shelves in every store and uniform distribution of agents over states. The efficiency consideration shows that the market might not create enough intermediaries in the economy allowing for Pareto improving government intervention. The paper elaborates very useful framework for further extensions of the problem.

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1 Introduction

In our century a vast majority of people could not imagine their life without money: we are paid by cash or check on our jobs, we use paper money or credit cards shopping for goods and services taking as given well developed financial sector. But at the same time there are countries where due to the transition from one organization of the economy (command) to another (market) governments while trying to suppress inflation create a situation when common for well elaborated market economies system of payments between different agents does not work, and these agents are forced to use barter, i.e. exchange goods for other goods. For instance, some economists in Russia (A.Yavlinskii) claim that 80% of all transactions use barter exchange. That is why the study of barter economies is not only of theoretical interest but also helps to explain real facts that some modern countries are experiencing in everyday life. One can observe that in these economies while money is washing away from all the transactions it is accompanied by the significant reduction in the variety of goods produced by individual plants making it optimal in the new barter environment. To explain this optimal behavior we are going to concentrate our attention on the trade-off that agents face in the exchange process: maximization of the variety of goods that increases the probability of trade versus minimization of production costs. In our model we will translate this situation into the optimal decision of a middleman who makes a choice as to the number of shelves in the store that she owns.

As it was pointed out by A.Rubinstein and Wolynsky (1987): "despite the important role played by intermediation in most markets, it is largely ignored by the standard theoretical literature... because the study of intermediation requires a basic model that describes explicitly the trade frictions that give rise to the function of intermediation". That is why we are going to use search theoretical methods to emphasize the role of middlemen in the exchange process. In search literature there are several different approaches in which intermediaries emerge endogenously. Usually authors assume that middlemen have some type of an advantage with respect to other agents in the economy, for example, an advantage in terms of more efficient search process (A.Rubinstein and Wolynsky, 1987) or an advantage in terms of private information (Yiting Li, 1996). In our model we do not give any privileges of described types to middlemen: all information is available to all participants and search process is completely symmetric ex-anti. The main difference of this model is the possibility for middlemen to store more than one good and

this feature drives all the results of the exercise.

The paper proceeds as follows. After describing the model we will solve it for the equilibrium in the simplest case of 2 goods in the economy. Having in mind the logic for this simple case it will be easier to proceed with the general case of K goods. After that we will analyze the welfare features of the model. At the end we will discuss some possible extensions of this exercise and summarize the results.

2 The Model

The model is similar to the economies considered in Kiyotaki and Wright (1993) and Burdett, Coles, Kiyotaki and Wright (1995) except that money is not introduced at this stage. There is a continuum of infinitely lived agents with total population normalized to one who produce and consume goods at discrete periods of time. There are K different consumption goods produced in the economy. Production side is modeled exactly in the same way as in Burdett, Coles, Kiyotaki and Wright (1995): there are K types of producers in equal proportions where type i produces commodity i. Cost of production is normalized to zero. Every period produces's desire for consumption good is a random draw from the uniform distribution. A producer can not store more than one good and produces her specific commodity immediately after consumption. If a random desire coincides with production specialization the producer consumes her own good, otherwise she enters a trading sector. Trade is characterized by bilateral random matching process when a producer can meet either a middleman or another producer. In the last case if both parties desire each others' goods one-for-one swap occurs followed by immediate consumption and production. In the case of a meeting with a middleman there is no problem of double coincidence of wants due to the specific assumption on preferences of middlemen: we assume that a middleman satisfies her random desire (to keep balanced nutrition) automatically having random flow of buyers into the store[1]. Every store has k shelves and it takes exactly one shelve to store every good. Storage cost in terms of disutility is an increasing function in the number of shelves: $c(k)$. We assume that only

[1]We could have assumed the same preferences for a middleman as for a producer but it would make our computations much more difficult without giving any additional insights to the model. To justify these preferences we assume that a middleman has a technology that allows her to transform any good into a desired consumption good.

buyers move searching for their consumption goods while sellers stay[2].

A buyer walks into a random store and if she finds the desired good bargaining process over the price occurs where the final price depends on the outside options for both parties (we assume complete information in the sense that the buyer observes all inventories in the store). Obviously, trade requires mutual consent. After the price is set both agents consume their shares of the good and as a result the seller has a new configuration of goods in the store (total utility derived from consuming one good will be normalized to one). A seller chooses the optimal size of the store at the beginning of time. After the choice has been made the seller is not allowed to change the number of shelves: she is committed to the original choice. It might be possible to show that ex-anti optimal decision is optimal choice ex-post, but at this stage of the research it has not been done so we have to assume the existence of the commitment technology. At least in the case of very patient agents that choice is endogenous both ex-anti and ex-post.

Now, when we already know how the model works, it is easy to understand the main trade-off for a middleman in the economy: the bigger the size of the store the higher the probability that a random buyer will find her desired good and consumption will take place. But this increase in the number of shelves requires additional cost which can offset possible gains.

3 Equilibrium for $K = 2$ Case

At the beginning of this section we will study the optimization problem for a middleman.

Let $V(k_1, k_2)$ denotes the value function for middlemen where k_1 is the number of the first good in the store and k_2 is the number of the second good. If the number of shelves in the store is k then equality $k_1 + k_2 = k$ should hold[3]. The arrival rate for middlemen is proportional to the number of buyers, i.e. it is equal to βn_b, where n_b is the fraction of buyers in the economy. Respectively, let n_s be the total fraction of middlemen, s.t. $n_b +$

[2]The justification for this assumption could be found in Burdett, Coles, Kiyotaki and Wright (1995), taking into account an additional assumption that moving cost for middlemen is much higher than for buyers.

[3]Note, that the number of shelves may be bigger than the number of goods in the economy: there is a positive probability that several buyers in a raw would ask for the same commodity which may make it profitable to keep more than K number of shelves.

$n_s = 1$. These numbers will be determined endogenously in the equilibrium. Let r denote the rate of time preference.

Then using standard Bellman's equations from dynamic programming we are ready to write down the expected value in terms of flow return for any configuration of goods in a store:

$$\begin{aligned}
rV_0 &= \beta n_b \frac{1}{2} \max\left[u_0^1 + V_1 - V_0, 0\right] - c \\
rV_1 &= \beta n_b \left\{ \frac{1}{2} \max\left[u_1^0 + V_0 - V_1, 0\right] + \frac{1}{2} \max\left[u_1^2 + V_2 - V_1, 0\right] \right\} - c \\
&\dots \\
rV_i &= \beta n_b \left\{ \frac{1}{2} \max\left[u_i^{i-1} + V_{i-1} - V_i, 0\right] + \frac{1}{2} \max\left[u_i^{i+1} + V_{i+1} - V_i, 0\right] \right\} - c \\
&\dots \\
rV_k &= \beta n_b \frac{1}{2} \max\left[u_k^{k-1} + V_{k-1} - V_k, 0\right] - c
\end{aligned} \tag{1}$$

where $V_i \equiv V(k-i, i)$, $c \equiv c(2)$ and u_i^j is the consumption share of a middleman after the bargaining process when configuration of goods in the store changes from i to j. The maximization operator reflects the fact that the seller accepts the terms of trade only if her gains from exchange are nonnegative[4]. We have got the system of $k+1$ equations and $k+1$ unknown value functions V_i. Every equation shows that the flow return to a middleman with given configuration of goods is proportional to the arrival rate βn_b, multiplied by the probability that a random buyer finds her favorite good ($\frac{1}{2}$ for $(k, 0)$ configuration and 1 for (k_1, k_2), $k_1, k_2 \neq 0$) and multiplied by the gains from trade. We have ruled out the possibility of meetings between the middlemen which is justified by relatively high moving costs for sellers.

Let's proceed by considering the price determination process. As it was mentioned before we will assume that the prices are set in the generalized Nash bargaining process where the current states are taken as the threat points. It means that after mutual investigation of each others' inventories both agents can see how the configuration of goods in the store changes from V_i to V_j. Therefore, the price is determined from the following simple maximization problem:

$$u_i^j = \arg\max_u (u + V_j - V_i)^\theta (1 - u + V_b - V_b)^{1-\theta}$$

[4]We will assume that if a seller is indifferent between two options she would prefer to trade.

$$s.t.\ 0 \leq u_i^j \leq 1 \tag{2}$$

since the total utility derived from consumption of one good was normalized to one. As a result we get:

$$\begin{aligned} u_i^j &= \theta - (1-\theta)(V_j - V_i) \\ s.t.\ 0 &\leq u_i^j \leq 1 \end{aligned} \tag{3}$$

The logic of decision making process for an agent in the economy goes as follows. A person evaluates the expected value functions of being a producer or a middleman with k number of shelves. It would be easy to do so if she could know the probability distribution over configurations of goods in the stores and if she were able to compute all value functions V_i^j. Moreover, at this moment it is not obvious that there exists the only one optimal size of a store. Let's see if we can answer all these questions.

First and interesting result that we are going to show describes the behavior of a middleman in the trade process.

Proposition 1 *A trade occurs whenever a random buyer finds her desired good in the store*[5].

Proof.

This proposition claims that every time when a buyer wants to trade the middleman would agree to trade which means that we can get rid of max operators in the system (1):

$$\max\left[u_i^j + V_j - V_i, 0\right] = \max\left[\theta(1 + V_j - V_i), 0\right] = \theta(1 + V_j - V_i)$$

To prove the proposition we first assume that the statement is true and then will show that $\theta(1 + V_j - V_i) > 0$ or $|V_j - V_i| < 1$ meaning that our original assumption was right. If the statement is true then system (1) would look like:

$$\begin{aligned} V_0 &= \beta n_b \theta \frac{1}{2r}\left[1 + V_1 - V_0\right] - \frac{c}{r} \\ V_1 &= \beta n_b \theta \frac{1}{2r}\left\{2 + V_0 + V_2 - 2V_1\right\} - \frac{c}{r} \end{aligned}$$

[5] In other words, the distribution of middlemen over different configurations of goods is ergodic.

$$
\begin{aligned}
& \cdots \\
V_i &= \beta n_b \theta \frac{1}{2r} \left\{ 2 + V_{i-1} + V_{i+1} - 2V_i \right\} - \frac{c}{r} \\
& \cdots \\
V_k &= \beta n_b \theta \frac{1}{2r} \left[1 + V_{k-1} - V_k \right] - \frac{c}{r}
\end{aligned} \tag{4}
$$

Introducing notations $\widetilde{\beta} = \beta n_b \theta \frac{1}{2r}, \widetilde{c} = \frac{c}{r}$, taking into account the symmetry of the problem and assuming that $k = 2N$[6] the system (3) can be rewritten:

$$
\begin{aligned}
V_0 &= \widetilde{\beta} \left[1 + V_1 - V_0 \right] - \widetilde{c} \\
& \cdots \\
V_i &= \widetilde{\beta} \left\{ 2 + V_{i-1} + V_{i+1} - 2V_i \right\} - \widetilde{c} \\
& \cdots \\
V_N &= 2\widetilde{\beta} \left[1 + V_{N-1} - V_N \right] - \widetilde{c}
\end{aligned} \tag{5}
$$

or

$$
\begin{pmatrix} V_{n+1} \\ V_n \\ 1 \end{pmatrix} = \Upsilon \begin{pmatrix} V_n \\ V_{n-1} \\ 1 \end{pmatrix} \tag{6}
$$

where transformation Υ is described by the following matrix:

$$
\Upsilon = \begin{pmatrix} 2 + \frac{1}{\widetilde{\beta}} & -1 & \frac{\widetilde{c}}{\widetilde{\beta}} - 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}
$$

Eigenvalues and eigenvectors corresponding to the transformation Υ are calculated in a standard way:

$$
\begin{aligned}
\lambda_1 &= 1, \lambda_{2,3} = \frac{1}{2\widetilde{\beta}} \left\{ 1 + 2\widetilde{\beta} \pm \sqrt{1 + 4\widetilde{\beta}} \right\} \\
e_1 &= \begin{pmatrix} 2\widetilde{\beta} - \widetilde{c} \\ 2\widetilde{\beta} - \widetilde{c} \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} \lambda_2 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} \lambda_3 \\ 1 \\ 0 \end{pmatrix}
\end{aligned} \tag{8}
$$

[6] using the same type of arguments it is easy to prove the statement for the case $k = 2N + 1$

Note that $0 < \lambda_2 < 1$ and $\lambda_3 > 1$. Then the general solution for (6) looks like:

$$\begin{pmatrix} V_{n+1} \\ V_n \\ 1 \end{pmatrix} = \gamma_1 e_1 + \gamma_2 \lambda_2^n e_2 + \gamma_3 \lambda_3^n e_3$$

or

$$V_n = 2\widetilde{\beta} - \widetilde{c} + \gamma_2 \lambda_2^n + \gamma_3 \lambda_3^n \tag{9}$$

since $\gamma_1 = 1$. Using the formulas for V_0 and V_N we can solve (9) for the coefficients γ_1 and γ_2. Finally we have got the analytical solution for our original system with respect to V_i:

$$V_n = 2\widetilde{\beta} - \widetilde{c} - \frac{2\widetilde{\beta}\left(\lambda_2^{N-n} + \lambda_3^{N-n}\right)}{\left(\sqrt{1+4\widetilde{\beta}}+1\right)\lambda_3^N - \left(\sqrt{1+4\widetilde{\beta}}-1\right)\lambda_2^N} \tag{10}$$

With this close form solution it is just a matter of simple calculations to show that V_n is increasing and concave function in n and that $V_n - V_{n-1} < 1$. ■

It is interesting to note that in addition to the statement of the proposition we have shown that the best configuration of the goods in the store corresponds to the biggest variety of goods which is rather intuitive.

As usually, we are going to study only stationary equilibria, in which the fractions of middlemen in each configuration do not change over time. To find this stationary distribution we have to equate inflows and outflows for every possible states taking into account that for the case $K = 2$ there are at most two ways to go from any state:

$$\begin{aligned} n_{s0} &\rightleftarrows n_{s1} \rightleftarrows \cdots \rightleftarrows n_{si} \rightleftarrows \cdots \rightleftarrows n_{sk} \\ \sum_{i=0}^{k} n_{si} &= n_s \end{aligned} \tag{11}$$

where n_{si} is a fraction of sellers in $(k-i, i)$ state.

Proposition 2 *For the case of 2 goods in the economy there always exists a stationary equilibrium with uniform distribution of sellers over configurations of goods in the stores; moreover, the equilibrium is stable.*

Proof.

For an equilibrium to be stationary n_{si} should satisfy the following system of $k+1$ equations with $k+1$ unknowns:

$$\begin{aligned}
\beta n_b \frac{1}{2} n_{s0} &= \beta n_b \frac{1}{2} n_{s1} \\
\beta n_b n_{s1} &= \beta n_b \frac{1}{2} n_{s0} + \beta n_b \frac{1}{2} n_{s2} \\
&\cdots \\
\beta n_b n_{si} &= \beta n_b \frac{1}{2} n_{s(i-1)} + \beta n_b \frac{1}{2} n_{s(i+1)} \\
&\cdots \\
\beta n_b \frac{1}{2} n_{sk} &= \beta n_b \frac{1}{2} n_{s(k-1)}
\end{aligned} \tag{12}$$

where on the left hand side of every equation we have the fraction of sellers leaving n_{si} state and on the right hand side there is the fraction of sellers coming to the state.

The existence of an equilibrium with uniform distribution over states follows immediately after observing that the above system (12) has a solution

$$n_{s0} = n_{s1} = \cdots = n_{si} = \cdots = n_{sk}$$

We are left to show that this is the only stationary equilibrium which is stable in the sense that the distribution over states converges to this uniform distribution from any initial point in the space of states. To show this we are going to reinterpret our problem in terms of Markov transition matrix. There are $k+1$ states and we know all the probabilities of moving from each state to all others. Therefore, Markov matrix describing transformation of the system over time looks as follows:

$$B = \begin{bmatrix} 1-\frac{b}{2} & \frac{b}{2} & 0 & 0 & \cdots & 0 \\ \frac{b}{2} & 1-b & \frac{b}{2} & 0 & \cdots & 0 \\ 0 & \frac{b}{2} & 1-b & \frac{b}{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \frac{b}{2} & 1-b & \frac{b}{2} \\ 0 & \cdots & 0 & 0 & \frac{b}{2} & 1-\frac{b}{2} \end{bmatrix} \tag{13}$$

where $b \equiv \beta n_b$ and elements of matrix B are b_{ij} = the probability that a system in state i at time t will be in state j at time $t+1$, $\sum_{i=0}^{k} b_{ij} = \sum_{j=0}^{k} b_{ij} = 1$[7].

[7]Note, that matrix B looks almost like the transition matrix for a symmetric random

Stationary equilibria of our model correspond to the solutions of the system:

$$Bn = n \tag{14}$$

where n is a $(k+1) * 1$ vector of states. Thus if we show that matrix B has only one eigenvalue $\lambda = 1$ while for all others $|\lambda_i| < 1$ then it proves that there exists the only stationary equilibrium which is stable[8].

To compute the eigenvalues of B matrix let us rewrite it (to simplify algebra) in the following form:

$$D = \begin{bmatrix} c+d & d & 0 & 0 & \cdots & 0 \\ d & c & d & 0 & \cdots & 0 \\ 0 & d & c & d & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & d & c & d \\ 0 & \cdots & 0 & 0 & d & c+d \end{bmatrix}$$

Then $B = D$ if $c = 1 - b$, $d = \frac{b}{2}$.

Eigenvalues of D can be determined from the characteristic equation:

$$X_m \equiv \det(D - \lambda I) = 0 \tag{15}$$

where I is the identity matrix: $I = (\delta_{ij})$, δ_{ij} is the Kronecker delta symbol and m is the dimension of matrix D. Expanding the determinant X_m in the elements of the first and the last rows we obtain:

$$X_m = (c + d - \lambda)^2 Y_{m-2} + 2d^2 (c + d - \lambda) Y_{m-3} + d^4 Y_{m-4} = 0 \tag{16}$$

where

$$Y_m = \det \begin{pmatrix} c-\lambda & d & 0 & \cdots & \cdots & 0 \\ d & c-\lambda & d & 0 & \cdots & 0 \\ 0 & d & c-\lambda & d & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & d & c-\lambda \end{pmatrix}$$

walk with reflecting barriers except that diagonal elements of it $b_{ii} \neq 0$.

[8]Any path $n(t)$ can be represented as a linear combination of eigenvectors generated by matrix B and associated with eigenvalues λ_i. Over time all the terms with $|\lambda_i| < 1$ will converge to 0 leaving nonzero the only term with $\lambda = 1$.

Doing similar expansion of the determinant Y_m in the elements of the rows we get the following recursive equation:

$$Y_m - (c-\lambda)\, Y_{m-1} + d^2 Y_{m-2} = 0 \tag{17}$$

The solution of this second-order linear difference equation with constant coefficients is:

$$Y_m = A_1 \left[\frac{(c-\lambda) + \sqrt{(c-\lambda)^2 - 4d^2}}{2} \right]^m + A_2 \left[\frac{(c-\lambda) - \sqrt{(c-\lambda)^2 - 4d^2}}{2} \right]^m$$

with A_1 and A_2 determined from the conditions: $Y_0 = 1$, $Y_1 = c - \lambda$. To be able to proceed let us use a simple but nice trick substituting $\cos\phi \equiv \frac{c-\lambda}{2d}$. Then expression for Y_m takes the form

$$Y_m = A_1 d^m \left[\exp\left(i\phi m\right)\right] + A_2 d^m \left[\exp\left(-i\phi m\right)\right]$$

where $\exp\left(\pm i\phi m\right) \equiv \cos\left(\phi m\right) \pm \sin\left(\phi m\right)$, $i \equiv \sqrt{-1}$. Finally we obtain for Y_m:

$$Y_m = \frac{d^m}{\exp\left(i\phi\right) - \exp\left(-i\phi\right)} \left\{\exp\left[i\phi\left(m+1\right)\right] - \exp\left[-i\phi\left(m+1\right)\right]\right\} \tag{18}$$

Plugging this expression into (16) we get:

$$X_m = \left(1 + \exp\left(i\phi\right) + \exp\left(-i\phi\right)\right)^2 \left\{\exp\left[i\phi\left(m-1\right)\right] - \exp\left[-i\phi\left(m-1\right)\right]\right\} -$$

$$2\left(1 + \exp\left(i\phi\right) + \exp\left(-i\phi\right)\right) \left\{\exp\left[i\phi\left(m-2\right)\right] - \exp\left[-i\phi\left(m-2\right)\right]\right\} +$$

$$\left\{\exp\left[i\phi\left(m-3\right)\right] - \exp\left[-i\phi m - 3\right]\right\} = 0$$

which has a solution $\phi = \frac{\pi l}{m}$, $l = 1, 2, \ldots, m$. Thus for the eigenvalues λ we finally obtain

$$\lambda_l = c - 2d\cos\frac{\pi l}{m}$$

or for our specific matrix B:

$$\lambda_l = 1 - \beta n_b \left(1 + \cos\frac{\pi l}{k+1}\right), \; l = \overline{1, k+1} \tag{19}$$

These $k+1$ eigenvalues completely characterize the dynamics of our system. Just staring at (19) we can easily conclude that there are exactly $k+1$ distinct eigenvalues all of them satisfying the condition $|\lambda_i| < 1$ except one $\lambda_{k+1} = 1$. This result completes our proof. ■[9].

Having the result of 2 we are able to compute the expected value V^{ex} function of a middleman with k shelves in the store in the steady state and to determine the optimal size of the store.

Proposition 3 *For the case of 2 goods in the economy and linear cost $c(k) \equiv ck$ there exists only one optimal value of k.*

Proof.

To obtain the expected value function V^{ex} we have to sum up all the value functions V_i multiplied by the probabilities of being in corresponding state $(k-i, i)$:

$$V^{ex} = \sum_{i=1}^{k+1} \frac{n_{si}}{n_s} V_i \tag{20}$$

But we just proved that $n_{s0} = \ldots = n_{si} = \ldots n_{sk}$ which means:

$$V^{ex} = \sum_{i=1}^{k+1} \frac{n_{s0}}{\sum\limits_{i=1}^{k+1} n_{si}} V_i = \sum_{i=1}^{k+1} \frac{n_{s0}}{(k+1)\, n_{s0}} V_i = \frac{1}{k+1} \sum_{i=1}^{k+1} V_i$$

To compute $\sum\limits_{i=1}^{k+1} V_i$ let us divide each equation in (1) by r and after that sum up all of them:

$$\sum_{i=1}^{k+1} V_i = \frac{\beta n_b}{2r} \left[\left(u_1^0 + u_0^1\right) + \left(u_2^1 + u_1^2\right) + \ldots + \left(u_j^i + u_i^j\right) + \ldots + \left(u_k^{k-1} + u_{k-1}^k\right) \right] -$$

$$- (k+1) \frac{ck}{r} \tag{21}$$

It is not difficult to notice that in (21) each sum $\left(u_j^i + u_i^j\right)$ gives us exactly 2θ. Therefore finally we obtain:

$$V^{ex} = \frac{1}{k+1} \sum_{i=1}^{k+1} V_i = \frac{\theta \beta n_b}{r} \frac{k}{k+1} - \frac{ck}{r} \tag{22}$$

[9] Later, using the eigenvalues λ_i we can talk about the dynamic properties of our system describing the process of convergence to the stationary equilibrium and impulse responses.

This value function is maximized at some optimal value of $k = k_{eq}$ that will be determined later.

Since we are looking for integer k then

$$k_{opt} = \begin{cases} \arg\max \left\{ V^{ex}\left[int\left(k_{eq}\right)\right], V^{ex}\left[int\left(k_{eq}\right)+1\right] \right\}, \text{ if } k_{eq} > 1 \\ 1, \text{ otherwise} \end{cases} \tag{23}$$

where $int\left(k_{eq}\right)$ is the integer part of k_{eq} [10]. ■

Note, that the optimal number of shelves in a store may be greater than the number of goods in the economy (2 in this case) which reflects the fact that the probability of getting several customers successively demanding the same good may make it profitable to keep additional inventories of every good.

To close the model we have to consider the maximization problem for a buyer. We mentioned at the beginning that buyer in contrast with a middleman can move while searching for the consumption opportunity. As a result she can meet both a middleman or another buyer (and producer at the same time). In the first type of meetings the buyer consumes whenever she finds her favorite good on the shelves of a store that she comes in randomly. In the second type of meetings there is a problem of double coincidence of wants: trading occurs if and only if both parties like each others' goods. Then the expected value function of being a buyer can be written in the form:

$$rV_b^{ex} = \beta \left\{ n_{s0}\frac{1}{2}\left(1-u_0^1\right) + \ldots + n_{si}\left[\frac{1}{2}\left(1-u_i^{i-1}\right) + \frac{1}{2}\left(1-u_i^{i+1}\right)\right] + \right.$$

$$\left. + n_{sk}\frac{1}{2}\left(1-u_k^{k-1}\right)\right\} + \beta n_b \frac{1}{2} \tag{24}$$

Using 2 and (21) we can simplify (24):

$$V_b^{ex} = \frac{\beta}{r}\left((1-\theta)n_s\frac{k}{k+1} + \frac{n_b}{2}\right) \tag{25}$$

At this point we are ready to determine the fractions of people in the economy who engage in different types of activities. To do this we will use

[10] The probability that $V^{ex}\left[int\left(k_{eq}\right)\right] = V^{ex}\left[int\left(k_{eq}\right)+1\right]$ is negligibly small.

an additional free entry condition: in the equilibrium the value functions for a middleman and a producer should equal to each other:

$$\theta\frac{\beta n_b}{r}\frac{k}{k+1}-\frac{ck}{r}=\frac{\beta}{r}\left((1-\theta)(1-n_b)\frac{k}{k+1}+\frac{n_b}{2}\right)\Rightarrow$$

$$n_b=\frac{2k}{k-1}\left(\frac{c}{\beta}(k+1)+(1-\theta)\right),\ n_s=1-n_b \tag{26}$$

We are ready to write down the equation for the optimal number of shelves k_{eq}. Maximizing V^{ex} with respect to k and solving the system of two equations with two unknowns k and n_b we get the following equation for the k_{eq}:

$$k^3+(1-2\theta)k^2-(1+2\theta+2\theta\frac{\beta}{c}(1-\theta))k-1=0$$

We can notice from this equation that for any arbitrary small value of θ there exists a value for the ratio $\frac{\beta}{c}$ such that the optimal number of shelves is greater than 1. If the middlemen were able to cooperate on their behavior meaning that they would maximize V^{ex} taking into account the fact that their choice affects the fraction of buyers in the economy then we would get different equation for k^c_{eq} :

$$(2\theta-1)k^4-2(3\theta-1)k^2-4\theta\left(1+\frac{\beta(1-\theta)}{c}\right)k-1=0$$

It is easy to see that in this case for all values of parameters β and c there is no space for middlemen in the economy with two goods when $\theta<\theta^*$ ($\theta^*>1/2$). One could check that in this case the welfare of a middleman is always less than the welfare of a producer. This negative result is pretty intuitive: with two goods a producer experiences relatively big probability to meet another producer with her favorite good while a middleman who has even higher probability to satisfy random demand consumes only a part of the good. That is why a seller (in cooperative equilibrium) should get pretty big share of the good in each trade (θ should be big enough) to chose the role of a middleman in the economy with 2 goods.

Let us see what is going on in the economy with number of goods greater than two.

4 Equilibrium for General Case of K Goods in the Economy

In this section we are going to explore and generalize the same logic that was used in the previous section.

As before we get started by solving the maximization problem for a middleman. Let us $V(k_1, k_{2,} \ldots, k_K)$ be the value function of a middleman where k_i is the number of i's good in the store. Total number of goods in the store should be equal to the number of shelves:

$$\sum_{i=1}^{K} k_i = k \tag{27}$$

Bellman's equations are of the form:

$$rV(k,0,..,0) = \beta n_b \frac{1}{K} \left\{ \frac{1}{K-1} \max\left[u_1 + V(k-1,1,..,0) - V(k,0,..,0), 0\right] + \right.$$

$$\left. +... + \frac{1}{K-1} \max\left[u_{K-1} + V(k-1,0,..,1) - V(k,0,..,0), 0\right] \right\} - c(k)$$

.....

$$rV(k_1,k_2,0,..,0) = \beta n_b \frac{2}{K} \left\{ \frac{1}{2(K-1)} \max[u_{K-2} + V(k_1-1,k_2+1,..,0) - \right.$$

$$-V(k_1,k_2,0,..,0), 0] + ... + \frac{1}{2(K-1)} \max[u_{3K-4} + V(k_1,k_2-1,0,..,0,1) -$$

$$\left. -V(k_1,k_2,0,..,0), 0] \right\} - c(k)$$

.....

$$rV(k_1,k_2,..k_i,0,..,0) = \beta n_b \frac{i}{K} \left\{ \frac{1}{i(K-1)} \max[u_l + V(k_1-1,k_2+1,k_{3,}..,0) - \right.$$

$$-V(k_1,k_2,..k_i,0,..,0), 0] + ... + \frac{1}{i(K-1)} \max[u_m + V(k_1,k_2,..k_i-1,0,..,0,1) -$$

$$\left. -V(k_1,k_2,..k_i,0,..,0), 0] \right\} - c(k)$$

.....

where u_i is the share of the good that goes to the seller. This share depends on the configuration of goods in the store before and after the trading and it is determined in a generalized Nash bargaining process (2). Every equation in our system shows that the flow return to a middleman with given configuration of goods is proportional to the arrival rate βn_b, multiplied by the probability that a random buyer finds her favorite good $\frac{i}{K}$ and multiplied by the gains from trade. There are $i\,(K-1)$ possible ways to go from the configuration $(k_1, k_2, ..k_i, 0, .., 0)$: if the buyer likes one of the goods in the store then she produces one of the $K-1$ goods other than the one she desires which is true for every i's good. As in the previous section we will be looking for stationary equilibria and as before the statement of the 2 is true:

Proposition 4 *For the case of K goods in the economy there always exists a stationary equilibrium with uniform distribution of sellers over the configurations of goods in the stores; moreover, the equilibrium is stable.*

Proof.

The system of outflow-inflow equations which similar to (12) in the general case takes the form:

$$\begin{aligned}
\frac{\beta n_b}{K} n_0 &= \frac{\beta n_b}{K\,(K-1)} n_1 + ... + \frac{\beta n_b}{K\,(K-1)} n_{K-1} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{\beta n_b i}{K} n_i &= \underbrace{\frac{\beta n_b i}{K\,(K-1)} n_j + ... + \frac{\beta n_b i}{K\,(K-1)} n_l}_{i(K-1)\text{ terms}} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

One can observe that the uniform distribution over states is a solution of this system of equations.

To prove the proposition it suffices to show that the Markov matrix that corresponds to the transition of middlemen distribution over states has the only eigenvalue $\lambda = 1$ and all others satisfy the condition $|\lambda_i| < 1$. We will not bother trying to write this matrix down. It is enough to emphasize some properties of the matrix (B). B is a non-negative symmetric matrix, all the elements of B in the principal diagonal are different from zero and equal to

$1 - \frac{\beta n_b i}{K}$, $i = \overline{1, K}$, while all off-diagonal elements are either 0 or $\frac{\beta n_b}{K(K-1)}$. Moreover, the sum of elements in each row is 1. One can show that the greatest eigenvalue λ of a non-negative matrix has the following property: it lies between the smallest and the largest row sums. Thus we immediately get the result for our matrix B:

$$\lambda_{\text{largest}} = 1, \ |\lambda_i| \leq 1$$

This means that a barter economy with K goods has at least one stationary equilibrium which is stable. We want to show that there exists only one stationary equilibrium (which would coincide with the uniform solution of the above system). To prove this we are going to use a remarkable property of positive (Perron) and non-negative matrices (Frobenius) which is formulated in the following theorem:

Let D be an irreducible and aperiodic[11] non-negative matrix. Then D has an eigenvalue λ which is real, positive and simple. For any other eigenvalue λ_i of D (which are distinct eigenvalues) we have $|\lambda_i| < \lambda$. To this maximal eigenvalue λ there corresponds a strictly positive eigenvector.

Our matrix B is irreducible (there are no absorbing states, i.e. every state can be reached from every other state) and aperiodic (any state can be reached in any arbitrary number of steps starting from it). Therefore we can apply the above theorem finally proving that the uniform stationary distribution is the only stationary equilibrium in our economy. ■

Fortunately, due to the 4 we do not have to solve the system of Bellman's equations for all the value functions. That is why we are able to prove the following proposition that is similar to the 3:

Proposition 5 *For the case of K goods in the economy and linear cost $c(k) \equiv ck$ there exists only one optimal value of k.*

Proof.

Let us compute the expected value function of a middleman with k number of shelves in the store using formula (20):

$$V^{ex} = \sum_{i=1}^{N} \frac{n_{si}}{n_s} V_i = \frac{1}{N} \sum_{i=1}^{N} V_i$$

[11] Criterion for irreducibility: a chain is irreducible if, and only if, every state can be reached from every other state.

An irreducible non-negative B matrix is called aperiodic if $B^k > 0$ for some k, or equivalently, state i of some Markov chain has a period k if starting from this state we can return back only in nk number of steps. Otherwise, the chain is aperiodic.

where N is the total number of possible configurations of the goods in a store which will be computed later. Dividing every Bellman'sssss equation by r and adding them up we obtain:

$$\sum_{i=1}^{N} V_i = N_1 \frac{\theta\beta n_b}{rK} + 2N_2 \frac{\theta\beta n_b}{rK} + ... + iN_i \frac{\theta\beta n_b}{rK} + ... - N\frac{ck}{r} \tag{28}$$

where N_i is the number of different configurations of the type $(k_1, k_{2,}..k_{i,}0, ..0)$, and $\sum\limits_i N_i = N$. To compute N_i we have to multiply the number of ways in which we can choose i elements from K by the number of ways in which we can choose $i-1$ elements from $k-1$. Using common notation $C_n^m = \frac{n!}{m!(n-m)!}$ $(n! = 1 \cdot 2 \cdot .. \cdot n)$ we can obtain for N_i:

$$N_i = C_K^i C_{k-1}^{i-1}$$

Then finally we can get an expression for V^{ex}:

$$V^{ex} = \frac{\theta\beta n_b}{rK} \frac{\sum\limits_{i=1}^{\min[k,K]} C_K^i C_{k-1}^{i-1} i}{\sum\limits_{i=1}^{\min[k,K]} C_K^i C_{k-1}^{i-1}} - \frac{ck}{r} \tag{29}$$

Fortunately, these sums are computable and we can simplify the above expression:

$$\begin{aligned} V^{ex} &= \frac{\theta\beta n_b}{rK} \frac{\sum\limits_{i=1}^{\min[k,K]} \frac{K!}{i!(K-i)!} \frac{(k-1)!}{(i-1)!(k-i)!} i}{\sum\limits_{i=1}^{\min[k,K]} \frac{K!}{i!(K-i)!} \frac{(k-1)!}{(i-1)!(k-i)!}} - \frac{ck}{r} = \frac{\theta\beta n_b}{r} \frac{\frac{\Gamma(K+k-1)}{\Gamma(K)\Gamma(k)}}{\frac{\Gamma(K+k)}{\Gamma(K)\Gamma(k+1)}} - \frac{ck}{r} = \\ &= \frac{\theta\beta n_b}{r} \frac{k}{k+K-1} - \frac{ck}{r} \end{aligned} \tag{30}$$

where $\Gamma(n)$ is a Gamma function[12].

We have got very nice function with the first term $\frac{\theta\beta n_b}{r} \frac{k}{k+K-1}$ which is increasing and concave in k. The optimal value of k is:

$$k_{opt} = \begin{cases} \arg\max \{V^{ex}[int(k_{eq})], V^{ex}[int(k_{eq})+1]\}, \text{ if } \sqrt{\frac{\beta n_b (K-1)}{2c}} > K \\ 1, \text{ otherwise} \end{cases} \tag{31}$$

[12]Notice that this formula coincides with (22) if $K = 2$.

where k_{eq} will be determined a bit later. The proof is completed. ■

Let us try to figure out if it might be profitable to be a middleman in the economy with $K > 2$. We have to calculate the expected value function for a buyer in the stationary equilibrium. By the same type of argument as in the previous section V_b^{ex} looks like:

$$V_b^{ex} = \frac{\beta}{r}\left((1-\theta)\frac{k}{k+K-1}n_s + \frac{1}{K(K-1)}n_b\right) \tag{32}$$

The free entry condition ($V^{ex} = V_b^{ex}$) gives us:

$$\frac{\theta\beta n_b}{r}\frac{k}{k+K-1} - \frac{ck}{r} = \frac{\beta}{r}\left((1-\theta)\frac{k}{k+K-1}(1-n_b) + \frac{1}{K(K-1)}n_b\right) \Rightarrow$$

$$n_b = (K-1)K\frac{\frac{c}{\beta}(k+K-1)+(1-\theta)}{k(K^2-K-1)-(K+k-1)}k \tag{33}$$

It is just a matter of simple algebraic calculations to show that there are some values of parameters c and β such that there exists an equilibrium with a positive fraction of middlemen in the economy for $K = 3$.

To make the calculations and the interpretations of the results more attractive let us assume that our economy is very large, i.e. $K \gg 1$. Then the simplification of (33) gives

$$n_b = (1-\theta) + \frac{c}{\beta}(k+K) \tag{34}$$

Maximizing the expression for V^{ex} with respect to k and solving the system of equations we can obtain the following formulas for optimal k and n_b:

$$\begin{aligned} k_{eq} &= K\left[\sqrt{\frac{\theta^2}{4} + \theta(1-\theta)\frac{\beta}{cK}} - \left(1-\frac{\theta}{2}\right)\right] \\ n_{beq} &= (1-\theta) + \frac{cK\theta}{2\beta} + \sqrt{\left(1-\theta+\frac{cK\theta}{2\beta}\right)^2 - (1-\theta)^2} \end{aligned} \tag{35}$$

A solution for the problem would be different if we assumed cooperative behavior of agents meaning that choosing the number of shelves in the store

every middleman takes into account an impact of her decision on the optimal fraction of each type of agents in the economy. To consider this case we should proceed as follows.

Let us plug the expression (34) into V^{ex} :

$$V^{ex} = \frac{\beta\theta(1-\theta)}{r}\frac{k}{k+K} - (1-\theta)\frac{c}{r}k$$

Maximizing V^{ex} over k we get the optimal value $\widetilde{k}$:

$$k^c_{eq} = \sqrt{\frac{\theta\beta K}{c}} - K \tag{36}$$

Then the fraction of buyers in the economy equals

$$n^c_{beq} = (1-\theta) + \sqrt{\frac{\theta c K}{\beta}} \tag{37}$$

One can observe that this solution is always positive, and it is less than 1, if and only if, $\frac{cK}{\theta\beta} < 1$. But if we solve the condition for $k^c_{eq} > 0$ ($K \gg 1$) we will get

$$\frac{cK}{\theta\beta} < 1 \tag{38}$$

which means that whenever it is optimal to have the number of shelves in a store greater than zero then we have the only equilibrium, which can be described by (36) and (37).

5 Welfare

It is interesting to compare the equilibrium solution for our problem with the optimal solution. For this purpose let us consider the planner's problem who is maximizing the total welfare of the society $W = n_s V^{ex} + n_b V^{ex}_b$ (for the case $K \gg 1$) without being constrained by the free entry condition ($V^{ex} = V^{ex}_b$)[13]:

$$\max_{\{k,n_b\}} [n_s V^{ex} + n_b V^{ex}_b] = \max_{\{k,n_b\}} \left\{ (1-n_b)\left(n_b \frac{\beta}{r}\frac{k}{k+K} - \frac{ck}{r} \right) + \frac{\beta}{rK^2} n_b \right\} \tag{39}$$

[13]Social planner can redistribute benefits from production and trading processes making everybody better off.

where the social planner is choosing the size and the fraction of the middlemen in the economy. Notice, that the function W does not depend on the bargaining power θ. This result is very intuitive: for the social planner it does not matter who has the higher bargaining power - the planner cares only about the total gain received in the trading process. It is easy to check the first and the second order conditions for the following maximum for $W(k, n_b)$ ($W_{kk}, W_{nn} < 0$; $W_{kk} \cdot W_{nn} - W_{kn}^2 > 0$):

$$k_{optsp} = \begin{cases} \arg\max \left\{ V^{ex}\left[int\left(k_{sp}\right)\right], V^{ex}\left[int\left(k_{sp}\right)+1\right] \right\}, \text{ if } \sqrt{\frac{\beta n_b K}{c}} > K \\ 1, \text{ otherwise} \end{cases}$$

where

$$k_{sp} = \sqrt{\frac{\beta n_{bsp} K}{c}} - K \tag{40}$$

and

$$n_{bsp} = \frac{1}{2}\left(1 + \frac{c}{\beta}(k_{sp} + K)\right) \tag{41}$$

or finally we can get the following expressions characterizing the solution of the social planner problem:

$$\begin{aligned} k_{sp} &= K\left(\sqrt{\frac{1}{16} + \frac{\beta}{2cK}} - \frac{3}{4}\right) \\ n_{bsp} &= \frac{1}{2}\left\{1 + \frac{Kc}{4\beta} + \sqrt{\left(1 + \frac{Kc}{4\beta}\right)^2 - 1}\right\} \end{aligned} \tag{42}$$

sp stands for social planner.

Let us see how this solution can be compared to the equilibrium solution described by (35). The results are presented in the following proposition:

Proposition 6 *If the economy under consideration is large ($K \gg 1$) and per unit cost c is small enough to make it profitable to keep more than one shelve in a store, then the economy always has a unique stationary equilibrium characterized by (35). The equilibrium solution for k_{eq} is always smaller than k_{sp} and $\theta_{opt1} > 1/2$. If $c \to 0$ then $\theta_{opt1} \to 1/2$. The market underprovides*

the fraction of middlemen in the economy for $\theta < \theta_{opt2}$ and overprovides this fraction for $\theta > \theta_{opt2}$. $\theta_{opt2} < 1/2$ and $\theta_{opt2} \to 1/2$ when $c \to 0$.[14]

Proof of the proposition is trivial: it suffices to compare (42) and (35).

Now let us compare these results with the cooperative equilibrium solution:

$$k_{eq}^{c} = K\left(\sqrt{\frac{\theta\beta}{cK}} - 1\right) \tag{43}$$

$$n_{beq}^{c} = (1-\theta) + \sqrt{\frac{\theta cK}{\beta}}$$

First of all, we can observe that the relationship between the social planner solution and the equilibrium solution depends on the value of θ. Necessary and sufficient condition for the existence of a middleman in the equilibrium is $\theta > \frac{cK}{\beta}$[15]. The same condition for the social planner's problem is $\frac{cK}{\beta} < 1$. Both solutions coincide if θ satisfies the following condition:

$$\theta_{opt}^{c} = \frac{1}{2}\left\{1 + \frac{Kc}{4\beta} + \sqrt{\left(1 + \frac{Kc}{4\beta}\right)^{2} - 1}\right\} = n_{bsp} \tag{44}$$

This interesting result claims that the equilibrium solution is optimal if and only if θ equals n_{bsp}.

6 Conclusions and Some Extensions

This paper has analyzed the role of middlemen in a barter economy when the agents can either produce different goods or open a store with arbitrary number of shelves. In an equilibrium an agent is indifferent between these two options. After choosing to be a middleman she has to determine the optimal size of the store trying to extract all benefits of having a larger variety of goods net of storage cost, which was assumed to be linear in our model. This

[14] θ_{opt1} is the value of θ that minimizes $k_{sp} - k_{eq}$ and θ_{opt2} is the value of θ at which $n_{bsp} = n_{beq}$.

[15] The same condition holds for "noncooperative" equilibrium.

is the first time in the search literature to our knowledge when a middleman is allowed to store an arbitrary number of goods.

It has been shown that this economy in the limit would converge to the stationary equilibrium in which there would be the same measure of sellers with every possible configuration of goods on the shelves. Every middleman chooses the same optimal level of shelves in the store maximizing her expected utility. The described equilibrium exists if the storage cost is small enough. Using general equilibrium framework we have closed the model introducing buyers who are able to produce and to store one good after every consumption possibility. We have shown that there will be distribution of prices determined endogenously in the generalized Nash bargaining process and depending on the configurations of goods in the store before and after the trade.

Intermediation in our model has welfare improving role since it helps to soften the severity of the "double coincidence of wants" problem increasing the probability of consumption in every period. Moreover, the market in this model underprovides the size of intermediation both in the number of shelves in the stores and in the fraction of middlemen if the sellers do not have enough bargaining power, even though it takes space from the production sector. This fact goes in the opposite direction in comparison with other models mentioned at the beginning of this paper: usually there are too many middlemen. It provides very simple intuition for the increasing fraction of intermediaries in modern market economies when storage and other costs are decreasing significantly, not to mention the well observed fact in transition economies when the proportion of middlemen has been rising dramatically during the first years of market liberalization.

To make our model more realistic it would be very interesting to extend it by introducing money to facilitate exchange. As you remember, it was the original idea to show how money allows to increase the variety of goods produced and stored in an economy.

We could consider more deeply the dynamics of the distribution over states predicting oscillatory responses of our system after an exogenous shocks (by the way, matrices, considered in the third section are called oscillatory matrices).

Another possible extension is relaxing the assumption of linearity of the storage cost (or assuming that there exists the distribution of storage costs) and taking into account financial constraints of agents to show the possibility of the store size distribution.

This model can be rephrased in terms of some other environment. For example, one can be interested in the optimal number of costly acquiring skills which increase the probability of, say, finding a job.

Therefore, this model, being somewhat abstract and artificial, provides a clear theoretical framework to address different type of questions regarding not only intermediation but also a lot of other decision-theoretical problems.

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