MATCHING AND MONEY*

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Abstract

We analyze matching models of the exchange process, with particular emphasis on the role of money, under the assumption that agents get to choose endogenously the individuals (or at least the types of individuals) they meet, rather than having them matched exogenously and at random, as in most of the previous literature. We show that as long as agents are restricted to one bilateral trade per period and specialization entails a double coincidence problem, there is still a role for money. We use the framework to revisit some issues that had been addressed in the literature with random matching. In particular, we characterize conditions under which fiat money may be used as a medium of exchange and may be essential for supporting efficient allocations. We also show how the implications of a standard commodity money model are affected by making matching endogenous. Some basic insights from random matching theory go through, although the details change; other times the results are quite different when we assume endogenous rather than exogenous random matching.

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1 Introduction

The standard search or matching model of monetary exchange assumes a large number of agents who meet bilaterally and at random (see, e.g., Kiyotaki and Wright [1989, 1991]). Given specialization in production or consumption, bilateral meetings can make exchange difficult due to the double coincidence problem. This seems like a reasonable friction upon which to base a theory of the role of money as a medium of exchange. However, bilateral exchange is not the only friction in the typical version of the model in the literature. Other frictions include an absence of commitment, so that agents cannot simply agree to any outcome that may seem desirable ex ante, and an absence of publicly observable trading histories (memory), so that agents cannot use punishment strategies to implement allocations to which they would like to commit. It seems clear that we need some sort of double coincidence problem combined with a lack of commitment and memory in order to generate an essential role for money. What is not so clear is the extent to which we need *random* matching.

In this paper we explore models with endogenous meetings, rather than exogenous random meetings. This seems desirable not only because it is easy to criticize the random matching assumption as being "unrealistic," but also because one would like to know the extent to which a role for money does or does not depend on whether we allow agents to have some control over who they meet. Our way of modeling the meeting process is related, at least in spirit, to the game-theoretic literature on matching dating back to Gale and Shapley (1962). In that framework, agents are matched into pairs subject to a stability condition – roughly, no agents prefer to be matched with each other, or by themselves, rather than with their current partners. However, unlike this literature, we analyze an intertemporal environment because we are interested in money. We study the implications of endogenizing the meeting process for several monetary exchange models in the literature, and discuss some new models as well.

The first thing we do is to try to formalize what one might mean by endogenous matching in a dynamic framework, a discussion of which is contained in Section 2. In Section 3, we apply these ideas to a very simple model, exploring in particular the role played by memory or the lack thereof. This is similar in spirit to work by Kocherlakota (1998) and Kocherlakota and Wallace (1998) in random matching monetary models, although we pay particular attention to the distinction between private and public memory (which is relevant because we assume a finite number of agents). As in the random matching models, as long as agents are relatively patient, it is simple to show that with complete public memory we can sustain an efficient pattern of exchange without money, and that with no memory money becomes essential. However, one thing we emphasize is that money works better in an endogenous matching model than in a random matching model: in our framework, money can substitute perfectly for memory, while in a random matching model it cannot.

In Section 4 we pursue these ideas by extending the framework proposed in Kiyotaki and Wright (1993) for analyzing fiat money to allow endogenous matching. We show that many of the basic insights of this model continue to hold, although some results can change in interesting ways. For example, with random matching, money can be valued only if specialization is not too extreme, while in our endogenous matching framework the degree of specialization does not matter for the existence of monetary equilibrium. In Section 5 we consider the of extension of this model by Shi (1995) and Trejos and Wright (1995) to include bilateral bargaining, and see how it is affected by moving to an endogenous matching formulation. Again, some of the basic insights go through, although other can change in interesting ways.

Finally, in Section 6, we consider a version of the commodity money model in Kiyotaki and Wright (1989), where again we make matching endogenous instead of random. In this framework, we are interested in determining which goods will be used as media of exchange. The results here can differ a lot between the random and endogenous matching models. For example, for one specification, we find that the so-called fundamental equilibrium, in which the commodity with the best intrinsic properties emerges as money, always exists, and indeed is the unique (symmetric, steady state, pure strategy) equilibrium. In this version of the model with random matching, a fundamental equilibrium exists only for some parameters, while for other parameters there exists a so-called speculative equilibrium where another good circulates as money.

The general conclusions of our analysis are as follows. First, in terms of details, it is possible to ask the same kinds of questions here that one asks in the random matching model, and while some of the basic results go through, others change in interesting ways. Second, in terms of the bigger picture, we conclude that idea of building a theory of monetary exchange based on specialization and a lack of commitment and memory can be pursued using matching models, but one does not especially require randomness in matching.

2 Matching

We begin with a somewhat abstract description of an environment, and the applications to follow will constitute special cases. Time is discrete: $t = 0, 1, \ldots$

T, where T may or may not be finite. Let A be a set of agents; in some applications, A may be finite while in others it may be a continuum. Agents are, in general, characterized by two indices: their type i (e.g., their tastes or technology); and their individual state j, which typically will describe their inventory (e.g., which good they are holding). We assume i and j are elements of finite sets, $i = 1, 2, \ldots I$ and $j = 1, 2, \ldots J$. At every date, the aggregate state is given by the distribution $p_t = [\ldots p_t(i,j) \ldots]$, where $p_t(i,j)$ is the measure of agents of type i in state j at t.

The approach in the random matching literature is to assume that any individual meets and trades with other agents bilaterally and at random; typically, it is assumed that the probability of any agent meeting type i in state j is proportional to $p_t(i,j)$. The approach adopted here is quite different. Although agents in our framework are still restricted to meet and trade bilaterally, they are allowed to choose who they meet. In some cases they may get to choose to meet a particular individual, in other cases we will allow them to choose a type i in state j but not a particular individual (e.g., they draw an individual at random from the set of agents who are of type i in state j).

Given a sequence $\{p_t\}$, a pattern of matches can be described by a sequence of assignment rules $\{\psi_t\}$, where $\psi_t : A \to A$ is a bijection that assigns to every individual a partner of a certain type in a certain state. Note that an agent could be assigned to himself, meaning he is in autarchy that period. We need to impose some consistency conditions. First, we need to assume $\psi(\psi(i)) = i$ (you are your partner's partner). Second, in the case where A is a continuum, we need to assume that ψ is *measure-preserving*: if I_{ij}^{hk} is the set of agents of type i in state j assigned by ψ to a type h agent in state k, we require

$\mu\left(I_{ij}^{hk}\right) = \mu\left(I_{hk}^{ij}\right)$ where μ is Lebesgue measure.[1] The sequence $\{\psi_t\}$ simply describes who meets whom. In principle, one may also consider randomizing over different assignment rules, for example, if agents use mixed strategies for choosing who to meet.

At every date t, ψ_t induces a *partition* θ_t of A into subsets of size 1 or 2. Let Θ be the set of feasible partitions. In general, after agents match (i.e., given θ_t), they must decide whether to "trade," which may generate an instantaneous payoff and can change their state (e.g., they could swap inventories). For now, however, we simplify the presentation by assuming that matching and trading are the same thing. Thus, given θ_t, we can generate instantaneous payoffs for all agents as functions of their state (e.g., if different inventories have different storage costs) and their partners that period (e.g., if they trade for a good they consume), and we can also generate a new state, $p_{t+1} = f(p_t, \theta_t)$. A history at t is a list $h_t = (p_0, \theta_0, p_1, \theta_1, ..., p_{t-1}, \theta_{t-1}, p_t)$ which describes current and past states and past partitions. Let H_t be the set of all possible histories at t.

A *matching process* is a sequence $\Phi = \{\Phi_t\}_{t=0}^{\infty}$ where $\Phi_t : H_t \to \Theta$ gives the current partition (and hence the current trades and payoffs, since here we assume that matching and trading are the same thing) as a function of history. A *memoryless matching process* is one for which the current partition depends only on the current state and date, $\theta_t = \Phi_t(p_t)$. By randomizing over such matching processes, one can generate a random matching model in which each agent meets a partner of type i with inventory j with a probability proportional to $p_t(i, j)$. A matching process Φ generates a sequence of instantaneous payoffs,

[1] The model with a continuum of agents is closely related to the ideas in Kaneko and Wooders (1986), except that they allow coalitions of any finite size k, while we restrict $k \leq 2$ (you can match with at most one other agent per period). See also Cole, Mailath and Postlewaite (1998). A critical difference, however, is that our environment is genuinely dynamic.

from which we can compute agents' lifetime utility at any date from any history (if $T = \infty$ we will assume that agents discount). We will be more precise about agents' preferences over alternative matches, but for now it will suffice to simply say that Φ generates payoffs.

Definition 1 *We say Φ^* is an* equilibrium matching process *if there is no date t and history h_t such that at h_t there exists a partition $\theta'_t \neq \Phi_t(h_t)$ with the property that some subset in θ'_t is better off than they were under the partition implied by Φ^*.*

Remark 1 *In some applications, one might want to require that* every *agent in θ'_t in the above definition is strictly better off, while in other applications one may only want to require that* some *agent in θ'_t is strictly better off. We will use each notion in different applications below.*

We can illustrate this with a very simple example. There are two agents and two dates: $A = \{1, 2\}$ and $t = 0, 1$. The set of possible partitions Θ at each date includes only two elements: the partition where the agents are in autarchy, or *single*, $[\{1\}, \{2\}]$; and the partition where they are matched, or *married*, $\{1, 2\}$. Agents carry no inventories, so we can get away with ignoring the state variable p for now; however, we assume preferences are history dependent in the following way. At $t = 0$ a married agent receives period utility $U_m > 0$ and one who is single receives $U_s > U_m$. At $t = 1$ a married agent also receives U_m, but a single agent receives U_s if he was single at $t = 0$ and $U_s - D$ if he was married at $t = 0$. Here D is the cost of divorce – which can only occur at $t = 1$ in the history where agents were married at $t = 0$, i.e. where $h_1 = (\{1, 2\})$. Assume $D > U_s - U_m$. Agents do not discount in this example.

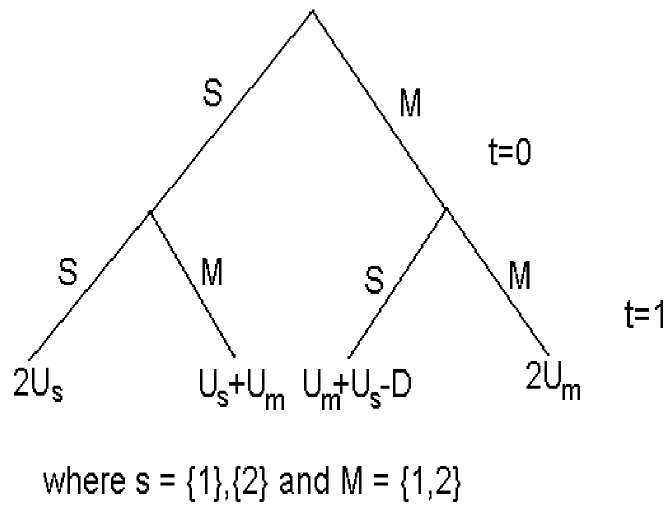

Figure 1: Possible histories in the marriage model

Figure 1 shows all possible histories. A matching process Φ gives a partition in Θ at each node in the figure (i.e. for each history). Using backward induction, given the parameter assumptions, married agents at $t = 1$ agree to stay married and single agents stay single. Given this, at $t = 0$, agents choose to stay single. Hence, the unique equilibrium Φ^* is as follows: $\Phi_0^* = [\{1\}, \{2\}]$ (stay single at $t = 0$); $\Phi_1^*([\{1\}, \{2\}]) = [\{1\}, \{2\}]$ (given you are single at $t = 1$, stay single); and $\Phi_1^*(\{1, 2\}) = \{1, 2\}$ (given you are married at $t = 1$, stay married). The equilibrium path is $\theta_0^* = (\{1\}, \{2\})$ and $\theta_1^* = (\{1\}, \{2\})$.[2]

This example is different from most of what we do in the paper for several

[2] One could also describe an equilibrium by the assignment rule ψ_t at each node. With two agents, it suffices to specify the assignment of agent 1. In the example, the equilibrium is given by: at $t = 0$, $\psi_0^*(1) = 1$ (1 is matched with himself); at $t = 1$ and $h_1 = ([\{1\}, \{2\}])$, $\psi_1^*(1) = 1$; and at $t = 1$ and $h_1 = (\{1, 2\})$, $\psi_1^*(1) = 2$.

reasons. First, $T < \infty$, which allows us to use backward induction. Second, there are only 2 agents, which drastically limits one's options (you can leave your spouse for bachelorhood but not for anther person). Finally, at each date, the two agents have a so-called a double coincidence of wants: agent 1 always wants to be with agent 2, and agent 2 always wants to be with agent 1. In more interesting environments considered below, there is not always this double coincidence at any given point in time, and this means that intertemporal considerations may have to play a crucial role. These intertemporal considerations are the focus of most of the remainder of the paper.

3 Matching, Memory, and Money

In this section we assume there is a double coincidence of wants problem, so that intertemporal considerations are crucial, and explore the implications of various scenarios regarding memory and money. Something similar in spirit has been done in random matching models by Kocherlakota (1998) and Kocherlakota and Wallace (1998), in models with a continuum of agents, which combined with random matching (and no public memory) implies that agents are anonymous: they have no knowledge of the histories of people they meet. We proceed instead, for now, with a finite set of agents, $A = \{1, 2, ..., N\}$, where $N \geq 3$. Even with (especially with) a finite set A, it is possible to discuss the implications of different assumptions regarding memory, including: complete public memory; no memory; and private but not public memory, by which we mean that each agent can perfectly recall anything he has directly observed and can make inferences from this information, but cannot directly observe what happens in meetings that do not involve him directly. And some new implications arise with a finite A.

We need to be more precise about goods and preferences here. Let us assume, to keep things close to the simplest random matching models, there are N (the same as the number of agents) indivisible and non-storable goods.[3] Each agent i produces only good i and consumes only good $i+1$, mod N, which implies that in any bilateral meeting there is not a double coincidence of wants. When agent i produces he receives instantaneous disutility $-c$ and when he consumes he receives utility $u > c$. Hence, if i is matched with $i+1$ his instantaneous payoff is u, if i is matched with $i-1$ his instantaneous payoff is $-c$, and if i is matched with anyone else (including himself) his instantaneous payoff is 0. Note that we continue to assume matching and trading are the same thing, for now, where in this context trading means $i+1$ produces for i but not vice-versa, since $i+1$ does not want the good i produces. Agents discount the future at rate $\beta = 1/(1+r)$, $r > 0$. A maintained assumption will be $c < \beta u$, or, equivalently, $r < (u-c)/c$.

Consider for now the case $N = 3$ (the interesting results easily generalize to any N; see below). The set Θ of feasible partitions of A is simply the following: autarchy, $[\{1\},\{2\},\{3\}]$, which we denote a; 1 and 2 matched, $[\{1,2\},\{3\}]$; 2 and 3 matched, $[\{2,3\},\{1\}]$; and 3 and 1 matched, $[\{3,1\},\{2\}]$.

We begin by considering the case of perfect public memory, which means that everyone knows the history of matches h_t at t. One equilibrium is given by global autarchy each period for any possible history: $\Phi^a = \{\Phi^a_t\}$, where $\Phi^a_t = a$ for every $h_t = (\theta_0, \theta_1, ...\theta_{t-1})$ (note that we do not include p_t in the history here because there is no state variable as of yet). This implies a lifetime payoff of 0.

[3] Note that there is no money in the model as of yet.

To see that it is an equilibrium, consider a history where there is a deviation from Φ^a by a pair of agents who decide to match. Without loss of generality, suppose it is $\{1, 2\}$ at $t = 0$; i.e., suppose, agent 2 and agent 1 get together, which entails 2 producing for 1. As this provides an instantaneous payoff to agent 2 of $-c$, and a continuation payoff the same as before (since 2 takes as given that agents will play Φ^a in the future), clearly he will not deviate from Φ^a. Hence, autarchy is an equilibrium.

Given we have public memory, however, there are other equilibria.. One class of equilibria contains those in which along the equilibrium path all agents iterate between production, consumption, and autarchy every three periods, and off the equilibrium path trigger to permanent global autarchy. Consider the case where agent 1 consumes first, then 2, then 3, denoted Φ^1. That is, along the equilibrium path, $\theta_0^1 = [\{1, 2\}, \{3\}]$ (first 1 and 2 match while 3 sits out); then $\theta_1^1 = [\{2, 3\}, \{1\}]$ (2 and 3 match while 1 sits out); then $\theta_2^1 = [\{3, 1\}, \{2\}]$ (3 and 1 match while 2 sits out); then we return to $\theta_3^1 = \theta_0^1$; and so on. And off the equilibrium path – i.e., at any history other than those induced by Φ^1 – agents go to permanent global autarchy. This case is illustrated in Figure 2. The lifetime payoffs along the equilibrium path are: $(u - \beta^2 c)/(1 - \beta^3)$ when one is a consumer; $(-c + \beta u)/(1 - \beta^3)$ when one is a producer; and $\beta(-c + \beta u)/(1 - \beta^3)$ when one is in autarchy that period. These are all positive by the maintained assumptions on preferences.

We argued above that Φ^a is an equilibrium, and so it is indeed an equilibrium for agents to go to autarchy at any node off the equilibrium path. To check the equilibrium path, we need to check that there is no coalition consisting of one or two members that can improve the payoffs of its members by deviating from

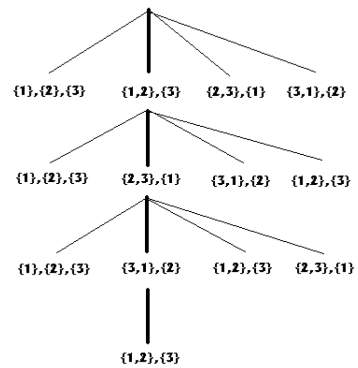

Figure 2: Possible Histories Under $\Phi^1$with Public Memory

the candidate equilibrium matching process. Without loss in generality, consider what happens when agent 2 is supposed to match with (i.e., produce for) agent 1. There are several potential deviations. First, 2 could stay in autarchy that period, giving a lifetime payoff of 0, which is less than what he gets along the equilibrium path since $c < \beta u$. Next, 2 could try to match with 3, but it is clear that 3 would not agree, since he would get instantaneous payoff $-c$ and then we move to permanent autarchy. Similarly, no other potential deviating coalition can improve its payoff.

Hence, Φ^1 is an equilibrium.[4] Notice that Φ^1 is efficient, in the sense that there is no way to improve the lot of all the agents over the outcome where they consume once every three periods, given preferences, technology and the restriction to bilateral trade. One interpretation of this equilibrium is that

[4]Similarly, so is Φ^2 or Φ^3, which are the same except that agent 2 or 3 gets to consume first. For sufficiently high β, there are other equilibria where, say, agent i matches with $i+1$ several periods in a row; agents stay in autarchy for a while and then begin matching according to one of these rules; agents match randomly each period; and so on.

agent i is willing to extend *credit* to $i-1$, which is repaid two periods hence, by the third party $i+1$. Moreover, the generalization to N types is simple. For example, with N an even number, we can support the outcome where every agent consumes and produces every second period (say, i matches with $i+1$ if t is even and with $i-1$ if t is odd) in exactly the same manner. With N odd the only complication is that i has to sit out once every N periods. But in any case, the binding constraint is always to get i to produce, which we can always do by allowing him to consume one period hence as long as $c < \beta u$.

To compare this to an exogenous random matching model, suppose now that each period every agent meets someone else with probability $\alpha \leq 1$, and when he does he meets every agent i with equal probability, $1/(N-1)$. Now agents choose to produce or not in each meeting. The efficient outcome is for i to produce every time he meets $i-1$. If agents have complete memory, we can support this as an equilibrium by triggering to permanent global autarchy (which is always an equilibrium) iff $\beta \geq 1/[1+\frac{\alpha}{N}\frac{u-c}{c}]$. This condition is necessarily more difficult to satisfy than the condition we needed in the endogenous matching model, $\beta \geq c/u$, as long as $\frac{\alpha}{N} < 1$, which it obviously must be. Hence, it is more difficult to support an efficient credit arrangement in this model, simply because the long term payoffs to extending credit are lower, while the cost is still c, when you are forced to match randomly rather than endogenously selecting the appropriate partner. Moreover, observe that credit becomes harder to sustain the more agents there are in the random matching world, while credit can be supported in our model whenever $\beta \geq c/u$, which does not depend on N.

Let us now suppose that there is no memory: for whatever reason, agents simply do not know h_t.[5] In this case, Φ^a is the unique equilibrium, simply because agent i has absolutely no incentive to match with (produce for) $i-1$ when Φ_t cannot depend on what happened at $s<t$. Clearly, without memory there can be no credit and the only possible equilibrium outcome is autarky. However, let us now introduce *money*. Thus, in addition to the above-mentioned goods, there now exists $M \leq N$ indivisible but perfectly storable units of an object that no one produces or consumes, say coins. We initially give coins to some of the agents. We now need to re-introduce the state variable for an agent, which will be his money inventory, $j \in \{0,1\}$. Agents know each other's state, although they cannot observe history. As the main point can be made with $N=3$ and $M=1$ (three agents, one coin), this is what we assume for now.

First, it is obvious that permanent global autarchy is still an equilibrium after we introduce money, since agents can always ignore coins. However, now there are other equilibria. To proceed, first note that agent i would ever match with (produce for) agent $i-1$ unless the latter gave him money, since the matching process cannot depend on whether he produced, *unless the state changes* (that is, a memoryless matching process must have the partition at t depend only on p_t). As always, the state can be described by p_t, where $p_t(i,1)$ is the probability that agent i has money, but in this case it suffices to simply keep track of which agent has the money. Letting $p_t(i)=1$ if i has money and $p_t(i)=0$ otherwise, a non-autarchic equilibrium is described as follows. In each period where $p_t(i)=1$, i matches with $i+1$ and swaps money for goods while $i-1$ sits out. Given

[5] As mentioned above, anonymity is what is often assumed in a random matching environment with a large number of agents, where it is realtively easy to motivate. One way to motivate no memory in a model with a small number of agents is to assume that every period agents reproduce offspring who are indentical to themselves except that they have no recollection of the past. In any case, the concern here is mainly to highlight the interaction between memory and money, which is easiest to see by simply ruling out memory, regardless of the motivation. We move to a large number of agents in the next section.

no agents will match unless money changes hands, the only feasible alternative when i has money is for either i or $i+1$ to deviate by remaining in autarchy that period. This clearly makes i no better off, and makes $i+1$ no better off given the maintained assumption $\beta u > c$. Hence, this is an equilibrium.

For example, if we initially give the money to agent 1, this equilibrium replicates the efficient allocation achieved by the credit arrangement Φ^1, even though agents have no knowledge of history, except what is conveyed by money holdings. This illustrates the role of money in overcoming the lack of memory in a world with a double coincidence problem. What is different, as compared to previous analyses in random matching models, is that money works much better when you get to pick who you meet. First, as should become clear in the next section (if not from then above analysis of credit), it is harder to support monetary exchange equilibria with random matching than with endogenous matching (i.e., β needs to be greater in the former case). Second even if a monetary equilibrium exists in the random matching environment, it does not work that well, in the sense that it is an imperfect substitute for the credit arrangement.

To see why money does not work very well in the random matching model consider the following argument. One problem is that you may meet an agent who produces the good you like but you have run out of money. Another reason is that you may meet an agent who produces the good you like, but, even though you do have money, he will not produce for you because he does not want your money (either because there is an exogenous bound on how much money he can hold, or because he already has so much money he endogenously decides not to trade).[6] Our endogenous matching model precludes such meetings – you would

[6]This second reason may be less convincing than the first, as it depends on some agents sometimes not wanting money. No matter, since the first reason is inescapable: sometimes

never choose to meet an agent who produces your good if you cannot trade, for any reason, whether because you have run out of money or he will not take your money. With endogenous matching, money is a perfect substitute for memory: it is as effective as credit in supporting the efficient outcome.

Before pursuing monetary economics further, as we do in the rest of the paper in models with a large number of anonymous agents, where the assumption on no public memory is perhaps easier to motivate, we want to present two results about the case of private but not public memory in the finite agent economy. Assume now that h_t is not observable to an agent, but he does only know the history of *his own* previous matches. First, note that Φ^a remains an equilibrium under this assumption. Now consider as a candidate equilibrium the process that is identical to Φ^1, except that agents go to permanent global autarchy after they have *directly* observed someone not match with (produce for) them. This case is illustrated in Figure 3. Consider agent 2 at $t = 0$, when he is supposed to match with 1, and suppose he deviates to autarchy that period. Agent 1 observes this but agent 3 does not; as far as 3 knows, at $t = 1$ we are still on the equilibrium path. We represent agent 3's lack of information by the set (rather than singleton) of nodes in his information partition in Figure 3. Thus, at $t = 1$ agent 3 matches with 2. Only at $t = 2$ will agent 3 experiences a deviation, when agent 1 refuses to match with him, at which point we are in global autarchy. Again, we must also check that agent 2 does not choose to match with agent 3 at $t = 0$. But in this case, all agents know about 2's deviation from Φ^1 and their payoff is simply 0 from punishment to autarchy.

Agent 2 chooses not to deviate to autarchy at $t = 0$ only if $\beta^4 u \geq c$. This is a

you run out of money.

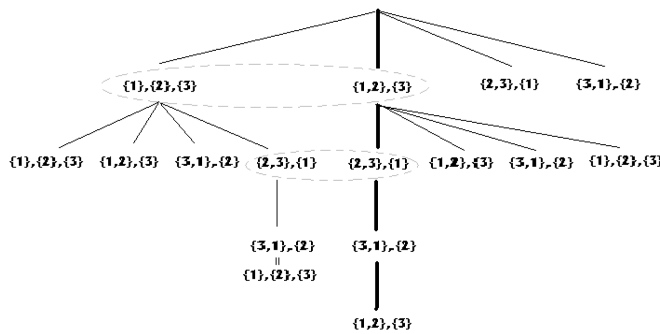

Figure 3: Possible Histories Under $\Phi^1$with Private Memory

much stronger condition than what we needed with publicly observable histories, $\beta u \geq c$. However, notice that the monetary equilibrium still exists with private but no public memory under the original parameter restriction, $\beta u \geq c$. Thus, we have demonstrated that for some parameters the credit arrangement breaks down, while money still works to support an efficient outcome (with endogenous matching in both cases). The result is, in fact general, the simple intuition is as follows. Since the actions of agents are not publicly observable, a deviation will trigger the autarchy only by the person that experienced it. This will create a chain of punishments by other agents that takes time to reach the deviator. So if agents are sufficiently impatient, they will want to deviate and credit will collapse. Summarizing, we have the following.

Proposition 1 *There exist regions of the parameter space where credit with private memory cannot support the first best outcome while money can.*

A direct implication of this is the next proposition, which asserts that equilibria involving credit become harder to support as the number of agents grows, but no so for monetary equilibria. The intuition is simple. The larger the population, the longer it takes a deviator to be discovered and punished in the credit arrangement. But we can always give money to some agents in such a way as to support the efficient process of exchange. For example, suppose N is an even number, and we give every second agent a unit of money to start. There is a monetary equilibrium in which they each consume every second period, again under the maintained assumption that $\beta u > c$.

Proposition 2 *Credit becomes harder to support as the number of agents in the economy, N, increases. Monetary exchange does not.*

To close this section, we mention that punishment to permanent global autarchy may seem extreme, but our goal was to give credit the best chance by making deviations as painful as possible. Even given this, the monetary equilibrium that implements the efficient allocation exists under more general conditions than the credit equilibrium. In our setup, an agent will meet with the same producer every two periods regardless of the number of agents in the economy. In a more general setup, with many potential producers producing the good that agent i likes, it could be the case that agent i might not wish to trigger the autarchy punishment if he experiences a deviation and, rather, try to hide it, since he might hope that he will not interact with the same deviator-producer in the near future. Clearly, the credit equilibrium studied here will be even harder to support in that case. In fact, in the next section we will move to a finite number of agent types but large number of agents, and assume that while you get to choose the type you meet you cannot choose to meet a particular agent, which implies that punishment strategies are useless and credit cannot work. Hence, some form of money is obviously essential, and the goal will be to see how this works.

4 Generalizing the Monetary Model

The following version of the environment in Kiyotaki and Wright (1993) is the simplest model of monetary exchange with a continuum of agents, $A = [0,1]$, equally divided into N types. By analogy to the previous section, now there are N indivisible and non-storable goods, each agent of type i can produce good i and can consume good $i+1$ (mod N). As before, production costs c, consumption yields u, and agents discount at rate $\beta = 1/(1+r)$. Now the amount of money is given by $M \in (0,1)$. Further, we now let $\gamma \geq 0$ be the per period storage cost of holding money; if $\gamma = 0$ this reduces to fiat money, as in the previous section. The state variable for agent i is again his money inventory, $j \in \{0,1\}$. In the symmetric outcomes considered below, the fraction of agents with money is always the same for all types: $p_{i1}(t) = M$, $\forall i, t$.

In Kiyotaki and Wright (1993) agents meet randomly, and the histories of people you meet are not observable to you (traders are anonymous). As in the previous section, this precludes any exchange that is not quid pro quo, and the only possible trades involve an agent of type i giving money to an agent of type $i+1$ in exchange for good $i+1$. If and when such trades actually do occur is what needs to be determined. Let π denote the probability that a random agent accepts money in exchange, and let V_j be the value function of an agent with j units of money in inventory, $j \in \{0,1\}$. Since we only consider symmetric outcomes, π and V_j are not indexed by type. If α is the probability of meeting someone at any date and $x = 1/N$, the probability a type i agent with money meeting a type $i+1$ without money is $a_1^r = \alpha x(1-M)$ and the probability a type $i+1$ agent without money meeting a type i with money is $a_0^r = \alpha xM$ (the superscript indicates we are in a random matching environment).

Hence, the value functions in steady state satisfy the flow Bellman equations:

$$\begin{aligned} rV_1 &= a_1^r \pi(u + V_0 - V_1) - \gamma & (1) \\ rV_0 &= a_0^r \pi(-c + V_1 - V_0). & (2) \end{aligned}$$

An equilibrium is defined as a list (V_1, V_0, π) satisfying (1) and (2), as well as the best response condition: $\pi = 1$ if $-c+V_1-V_0 > 0$; $\pi = 0$ if $-c+V_1-V_0 < 0$; and $\pi \in (0,1)$ implies $-c + V_1 - V_0 = 0$. The set of (symmetric, steady state) equilibria is described by the following result, the proof of which is omitted, as it simply involves a routine check of the above conditions.

Proposition 3 *In the random matching model, there is a critical value* $\overline{\gamma} = a_1^r(u-c) - rc$ *such that:* $\gamma < \overline{\gamma}$ *implies there are three equilibria,* $\pi = 0$, $\pi = 1$, *and* $\pi = \frac{rc+\gamma}{a_1^r(u-c)} \in (0,1)$*; and* $\gamma > \overline{\gamma}$ *implies the only equilibrium is* $\pi = 0$.

We want to reformulate the above model with endogenous, as opposed to random, matching. We want to maintain the assumption that traders are anonymous. If we assume that an agent can literally meet any individual he chooses, we must rule out memory by brute force, as in the previous section. Alternatively, we can assume that an agent gets to choose the type i and state j of the person he meets, but not a particular individual: he draws a partner at random from the set of agents who are type i with money inventory j. We will frame the discussion here as though an agent can choose the type and state of the person he meets but not the individual.

It is clear along the equilibrium path, the only relevant matches specified by Φ are between type i agents with money and type $i+1$ without money, since these are the only meetings in which there are possible trades. So we focus on equilibria where each period Φ_t simply specifies the following: if $M < 1/2$, then every type i agent with money meets a type $i+1$ agent without money drawn at random from the set of such agents, while exactly $\frac{M}{1-M}$ of the type $i+1$ agents without money drawn at random meet a type i agent with money; and if $M > 1/2$, then every type $i+1$ agent without money meets a type i agent with money drawn at random from the set of such agents, while exactly $\frac{1-M}{M}$ of the type i agents with money drawn at random meet a type $i+1$ agent with money. This implies the probabilities of agents with and without money meeting someone each period are $a_1^e = \min\left\{\frac{1-M}{M}, 1\right\}$ and $a_0^e = \min\left\{\frac{M}{1-M}, 1\right\}$. By way of contrast with the random search model, in this model everyone on the short side of the market meets someone each period, and meetings always involve the right types, and hence $a_j^r < a_j^e$, $j = 0, 1$.

Here we want to distinguish between matching and trading. Thus, after agents are matched according to Φ, the following happens: the type i agent with money proposes (this will be his equilibrium strategy) to exchange his money for good $i+1$, and the type $i+1$ accepts with probability π.[7] Now an equilibrium is defined by a pattern of meetings, Φ with the property that no coalition wants to deviate in the sense of Definition 1 (we cannot find an agent who strictly prefers to be unmatched in any period rather than to be

[7] If the $i+1$ agent rejects the trade offer, the agent with money cannot meet with any one else until next period; it would be equivalent here to say that the type $i+1$ agent accepts the proposed meeting with probability π, rather than agreeing to meet and then accepting the trade offer with probability π. These assumptions are made to keep the model as close as possible to the random matching model along dimensions other than the meetings. In any case, this is only relevant to the extent than one finds mixed strategy equilibria $\pi \in (0, 1)$ especially interesting.

matched according to Φ, and we cannot find two agents who strictly prefer to be matched with each other rather than to be matched according to Φ), as well as a list (V_1, V_0, π) satisfying analogs of the Bellman equations and best response conditions in the random matching model given below.

Without going into too much detail about what happens after a deviation, it should be clear that no individual or pair strictly prefers to deviate from the matching pattern Φ described above (i.e., the pattern where if $M < 1/2$ every type i agent with money matches with an agent of type $i+1$ without money, etc.). Although some agents on the long side of the market meet no one and would strictly prefer to meet someone, they cannot find anyone who strictly prefers to meet them over the meetings specified by the above pattern. Given these observations, the value functions satisfy

$$rV_1 = a_1^e \pi (u + V_0 - V_1) - \gamma \tag{3}$$

$$rV_0 = a_0^e \pi (-c + V_1 - V_0). \tag{4}$$

Comparing (1) and (2) with (3) and (4), we see that, at least in this very simple model, making the matching pattern endogenous alters the Bellman equations simply by changing the arrival rates. Moreover, the best response condition for trading is exactly the same as above: $\pi = 1$ if $-c + V_1 - V_0 > 0$; $\pi = 0$ if $-c + V_1 - V_0 < 0$; and $\pi \in (0,1)$ implies $-c + V_1 - V_0 = 0$.

Hence, this simple model actually ends up looking qualitatively quite similar whether we have random or endogenous matching. The analogue of the previous proposition is the following.[8]

Proposition 4 *In the endogenous matching model, there is a critical value* $\widehat{\gamma} =$

[8] The discussion that follows depends on the maintained assumption $c < u/(1+r)$.

$a_1^e(u-c) - rc$ *such that:* $\gamma < \widehat{\gamma}$ *implies there are three equilibria,* $\pi = 0$*,* $\pi = 1$*, and* $\pi = \frac{rc+\gamma}{a_1^e(u-c)} \in (0,1)$*; and* $\gamma > \widehat{\gamma}$ *implies the only equilibrium is* $\pi = 0$*.*

Recall that in the random matching model monetary equilibria exist iff γ is below $\overline{\gamma}$, while here monetary equilibria exist iff γ is below $\widehat{\gamma}$. The first thing to note is that $\overline{\gamma} < \widehat{\gamma}$, and so monetary equilibria are easier to support in the endogenous matching model. Intuitively, this is because it is easier to spend money, and so money tends to be more valuable, when you get to choose the type and inventory of the agents that you meet.[9] Moreover, since $\overline{\gamma}$ is increasing in $x = 1/N$, it becomes more difficult in the random matching model to support monetary equilibria when the number of goods/types grows. However, since $\widehat{\gamma}$ is independent of N, the existence of monetary equilibria in no way depends on the number of goods in the endogenous matching model, again because you get to choose the type and inventory of the agents you meet.

The results can also be stated in terms of the maximum value of M for which monetary equilibria exist, say $\overline{M}$ and $\hat{M}$ in the random and endogenous matching models, respectively. Then we have $\overline{M} < \hat{M}$. In the case where $\gamma = 0$, it is worth mentioning that in the limit as $r \to 0$ the two models converge, in the sense that both $\overline{M}$ and $\hat{M}$ converge to 1 and, in both models, the mixed strategy acceptance probability π converges to 0. The intuition for this is simple. As mentioned earlier, what distinguishes the two models in this section is essentially the arrival rates, and in particular the fact that $a_j^r < a_j^e$. This distinction becomes less relevant as agents become more patient. However,

[9] A related point is that, given the parameters, the probability that money is accepted in the mixed strategy equilibrium π is higher in the random matching model. Intuitively, if agents are to randomize they must be indifferent between accepting and rejecting money, and since in the endogenous matching version money tends to be more valuable, it has to have a lower acceptance rate to keep agents indifferent.

this conclusion depends on $\gamma = 0$; if $\gamma > 0$, the two models are different even when $r \to 0$, because by reducing the average amount of time an agent needs to hold money before he can spend it, endogenous matching allows them to reduce their storage costs.

So far we have focused on stationary equilibria, where strategies and payoffs do not change over time. At least in the random matching version, however, there are other equilibria, and it seems interesting to ask how moving to an endogenous matching technology affects the potential for such equilibria. Following the procedure in Wright (1997) for a random matching model, we can look for *sunspot equilibria*, for the endogenous matching model, as follows. Suppose there is a Markov process for an aggregate state variable $s_t \in \{m, n\}$ which switches according to $\Pr\{s' = n \mid s = m\} = \lambda_{mn}$ and $\Pr\{s' = m \mid s = n\} = \lambda_{nm}$. We now construct equilibria where $\{\phi(t)\}$ is as above (if $M < 1/2$ then every type i agent with money matches with an agent of type $i+1$ without money, etc.), but now, even though nothing fundamental in the economy depends on s, agents accept money in one state but not the other: $\pi_m = 1$ and $\pi_n = 0$.

If V_j^s is the value function of an agent with j units of money in state s, then we have:[10]

$$\begin{aligned}
rV_1^m &= (1-\lambda_{mn})a_1^e(u+V_0^m-V_1^m)+\lambda_{mn}(V_1^n-V_1^m) \\
rV_1^n &= \lambda_{nm}a_1^e(u+V_0^m-V_1^n)+\lambda_{nm}(1-a_1^e)(V_1^m-V_1^n) \\
rV_0^m &= (1-\lambda_{mn})a_0^e(-c+V_1^m-V_0^m)+\lambda_{mn}(V_0^n-V_0^m) \\
rV_0^n &= \lambda_{nm}a_0^e(-c+V_1^m-V_0^n)+\lambda_{nm}(1-a_0^e)(V_0^m-V_0^n).
\end{aligned}$$

For $\pi_m = 1$ and $\pi_n = 0$ to be an equilibrium, we require $V_1^m - V_0^m - c \geq 0$

[10] For simplicity, in the rest of this section we set $\gamma = 0$.

and $\pi_n = 0$ if $V_1^n - V_0^n - c \leq 0$. Straightforward algebra implies that these conditions are satisfied in the region in $(\lambda_{nm}, \lambda_{mn})$ space bounded by two lines:

$$\lambda_{mn} = \frac{a_1^e(u-c) - rc}{r[a_1^e(u-c)+c]}(r + \lambda_{nm}) \tag{5}$$

$$\lambda_{mn} = \frac{[a_1^e(1+r)(u-c) - rc(1-a_1^e-a_0^e)]\lambda_{nm} - rc(a_1^e + a_0^e + r)}{rc(1-a_1^e-a_0^e)}. \tag{6}$$

Figure **??** shows this region. It also shows the region in which a sunspot equilibrium with $\pi_m = 1$ and $\pi_n = 0$ exists in the random matching model, which is defined by replacing a_0^e by a_0^r and a_1^e by a_1^r (5) and (6).

As can be seen in the diagram, we are more likely to have a sunspot equilibrium in the random matching model than in the endogenous matching model when $\lambda_{nm}/\lambda_{mn}$ is large, and more likely to have a sunspot equilibrium in the endogenous matching model when $\lambda_{nm}/\lambda_{mn}$ is small. These results seem interesting for the following reason. First, it is understood that monetary economies, especially those with fiat money, are susceptible to endogenous fluctuations like those that occur in these sunspot equilibria. One might have thought that the region in which sunspot equilibria exist would be smaller in an endogenous matching model – that is, eliminating the randomness in meetings would reduce the parameter values for which sunspots matter. This is not the case. Intuitively, since money is easier to spend and hence tends to be more valuable in the endogenous matching model, it is easier to satisfy the condition for $\pi_m = 1$ and harder to satisfy the condition for $\pi_n = 0$ in that model; hence, we need to have λ_{nm} smaller or λ_{mn} larger with endogenous matching for sunspot equilibria to exist.

We conclude this section by saying, at least in the very simplest environment, it is possible to construct a theory of money based on a double coincidence prob-

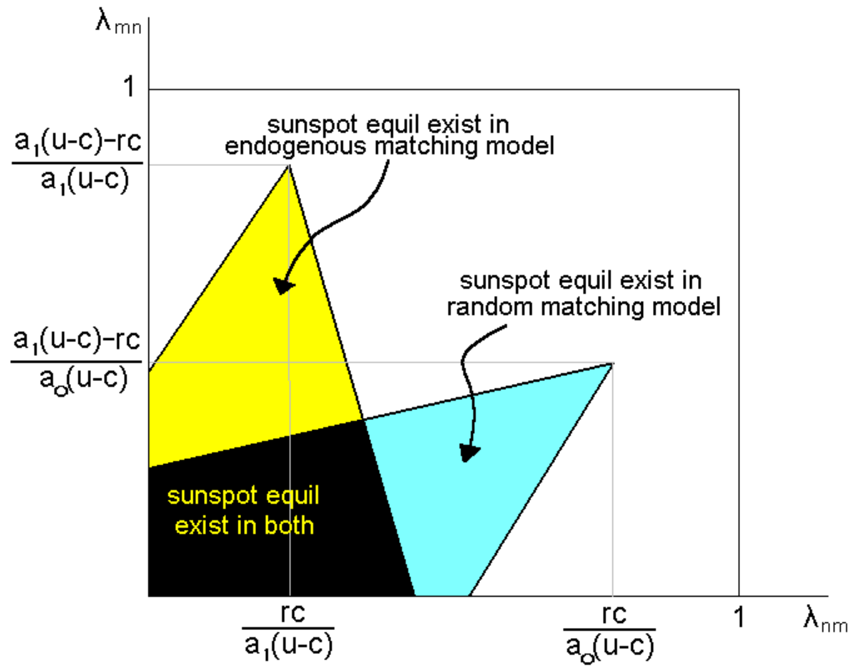
λmn
1
a1(u-c)-rc / a1(u-c)
a1(u-c)-rc / ao(u-c)
sunspot equil exist in
endogenous matching model
sunspot equil exist in
random matching model
sunspot equil
exist in both
rc / a1(u-c)
rc / ao(u-c)
1
λnm

Figure 4:

lem arising out of a restriction to bilateral trade (in combination with a lack of commitment and memory) without assuming random matching. Some of the basic insights of the random matching model go through, although some of the results change in interesting ways, such as the result that the existence of monetary equilibrium does not depend on the number of goods in the endogenous matching model. The next step will be to see what happens in more general or alternative matching models, including models where goods are divisible and hence quantities (and hence prices) are endogenous, models with commodity money, and so on.

5 Endogenous Prices

In this section, following Shi (1995) and Trejos and Wright (1995), we now assume that goods are divisible (but retain the assumption that money is indivisible). Any $q \geq 0$ units of good i good can be produced by type i at cost $c(q)$ and, consumed by an type $i+1$ for utility $u(q)$. Assume $c'(q) > 0$, $c''(q) \geq 0$, $\lim_{q \to 0} c'(q) = 0$, and $\lim_{q \to \infty} c'(q) = \infty$, as well as $u'(q) > 0$, $u''(q) \leq 0$, $\lim_{q \to 0} u'(q) = \infty$, and $\lim_{q \to \infty} u'(q) = 0$. Also, there exists $\hat{q} > 0$ such that $u(\hat{q}) = c(\hat{q})$. Let q^* satisfy $u'(q^*) = c'(q^*)$. Further, let $\gamma \geq 0$ be a storage cost of holding money. All other assumptions regarding the physical environment are identical to the preceding section.

If we assume random matching, for any given q, the value functions satisfy

$$rV_1 = a_1^r \left[u(q) + V_0 - V_1\right] - \gamma \tag{7}$$

$$rV_0 = a_0^r \left[-c(q) + V_1 - V_0\right]. \tag{8}$$

where as in the previous section $a_1^r = \alpha x(1-M)$ and $a_0^r = \alpha x M$. To determine

q, and the price level $p = 1/q$, the literature assumes that when agents meet q is determined as the equilibrium of a bargaining game as in Rubinstein (1982), or – what amounts to the same thing – q is given by the generalized Nash (1950) solution. We adopt the generalized Nash solution here:

$$\max_q \left[u(q) + V_0 - V_1\right]^{\omega} \left[-c(q) + V_1 - V_0\right]^{1-\omega} \tag{9}$$

$$s.t.\ u(q) + V_0 \geq V_1 \text{ and } - c(q) + V_1 \geq V_0, \tag{10}$$

where ω is the bargaining power of an agent with money and the threat points of an agent with j units of money is equal to V_j.[11] An equilibrium is defined as a list (q, V_1, V_0) satisfying the Bellman equations (7), (8), and the bargaining solution.

There always exists a nonmonetary equilibrium with $q = 0$; from now on, we focus on equilibria with $q > 0$. Taking the first order conditions from the bargaining equation (9), inserting the value functions, and simplifying we see that an equilibrium q satisfies the following condition

$$T(q) = \omega \left[a_1^r(u(q) - c(q)) - \gamma - rc(q)\right] u'(q) - (1-\omega) \left[a_0^r(u(q) - c(q)) + ru(q) + \gamma\right] c'(q) = 0.$$

One can show (as in Trejos and Wright (1995)) that for $\gamma = 0$, $T(0) = 0$ and there exists a unique $q \in (0, \hat{q})$ such that $T(q) = 0$. See Figure 2. For $\gamma > 0$, $T(q)$ shifts down. Hence there exists a critical $\bar{\gamma}$ such that if $\gamma \in (0, \bar{\gamma})$, there exists an even number of monetary equilibria (i.e. an even number of positive solutions to $T = 0$), and if $\gamma > \bar{\gamma}$, there exists no monetary equilibrium. Moreover, $T(q)$ is increasing in a_1^r and decreasing in a_0^r, which implies $\partial q/\partial M < 0$ in the case of a unique monetary equilibrium, $\gamma = 0$. Similarly, $\partial q/\partial r < 0$ in this case. Also,

[11] See Coles and Wright (1998) for the details of the relationship between strategic and axiomatic bargaining in monetary theory.

when $\gamma = 0$, as $r \to 0$,

$$\frac{u'(q)}{c'(q)} = \frac{(1-\omega)a_0^r}{\omega a_1^r} = \frac{(1-\omega)M}{\omega(1-M)}.$$

Since $u'(q) = c'(q)$ describes the efficient outcome, even for $r \to 0$, deviations from efficiency can occur if ω or M is not right.

We want to consider an endogenous matching version of this model. One issue that arises immediately is the extent to which agents can commit or equivalently when does bargaining occur. One possibility is that agents can choose any type of agent that they want, but then leave the matching process and bargain over the terms of trade bilaterally with their partner. Alternatively, agents can negotiate the terms of trade while deciding with whom they should match. But if there is no commitment, this reduces to the same thing (since agents may renegotiate q once they leave the matching process but before actually executing the trade). We will consider each case in turn, beginning with the case without commitment.

In the case where there is lack of commitment, the model with endogenous meeting looks very similar to the random matching model except that we change a_1^r to $a_1^e = \min\left\{\frac{1-M}{M}, 1\right\}$ and a_0^r to $a_0^e = \min\left\{\frac{M}{1-M}, 1\right\}$, once one recognizes that an equilibrium matching pattern Φ^* implies the only consequential matches are between type i agents with money and type $i+1$ agents without money. In particular, the bargaining solution is qualitatively the same as in the random matching model: due to the lack of commitment, q is still given by (9) even though meetings are endogenous. Hence, equilibria are given by solutions to $T(q) = 0$ with a_j^e substituting for a_j^r. This implies that the set of equilibria in the endogenous matching model without commitment is qualitatively the same as that in the exogenous matching model: if $\gamma = 0$, there exists a unique monetary equilibrium; there exists a critical $\hat{\gamma}$ such that if $\gamma \in (0, \hat{\gamma})$ then there exists an even number of monetary equilibria; and if $\gamma > \hat{\gamma}$ then there exists no monetary equilibrium.

Further, as in the random matching model, in the case of a unique monetary equilibrium, we have $\partial q/\partial M < 0$ and $\partial q/\partial r < 0$. Also, when $\gamma = 0$, as $r \to 0$ the unique equilibrium q solves

$$\frac{u'(q)}{c'(q)} = \frac{(1-\omega)\min\left(1, \frac{M}{1-M}\right)}{\omega \min\left(1, \frac{1-M}{M}\right)} = \frac{(1-\omega)M}{\omega(1-M)}.$$

As in the indivisible goods model in the previous section, if γ and r are negligible, endogenous matching (with no commitment) yields exactly the same outcome and random matching. Intuitively, sense since if agents do not discount the future and if there are no storage costs, then the advantages of being able to choose the type and inventory of your partner does not matter.

Compare the two models, let $T^r(q)$ and $T^e(q)$ denote the T functions in the random and endogenous matching models, respectively. At any q such that $T^r(q) = 0$, we can substitute

$$c'(q) = \frac{\omega\left[a_1^r\left(u(q) - c(q)\right) - \gamma - rc(q)\right]u'(q)}{(1-\omega)\left[a_0^r\left(u(q) - c(q)\right) + \gamma + ru(q)\right]},$$

into $T^r(q) - T^e(q)$ to show that it is equal in sign to:

$$\begin{aligned} D &= \left(a_1^r - a_1^e\right)\left[a_0^r\left(u(q) - c(q)\right) + ru(q) + \gamma\right] - \left(a_0^r - a_0^e\right)\left[a_1^r\left(u(q) - c(q)\right) - rc(q) - \gamma\right] \\ &= ru(q)\left(a_1^r - a_1^e\right) + rc(q)\left(a_0^r - a_0^e\right) + \gamma\left[a_0^r - a_0^e + a_1^r - a_1^e\right] < 0 \end{aligned}$$

(the last equality uses $a_0^e a_1^r = a_1^e a_0^r$). Hence, at any q such that $T^r(q) = 0$, we have $T^e(q) > 0$. This means that whenever a monetary equilibrium exists in the random matching model, one also exist in the endogenous matching model; thus, the critical storage cost below which monetary equilibria exist is greater

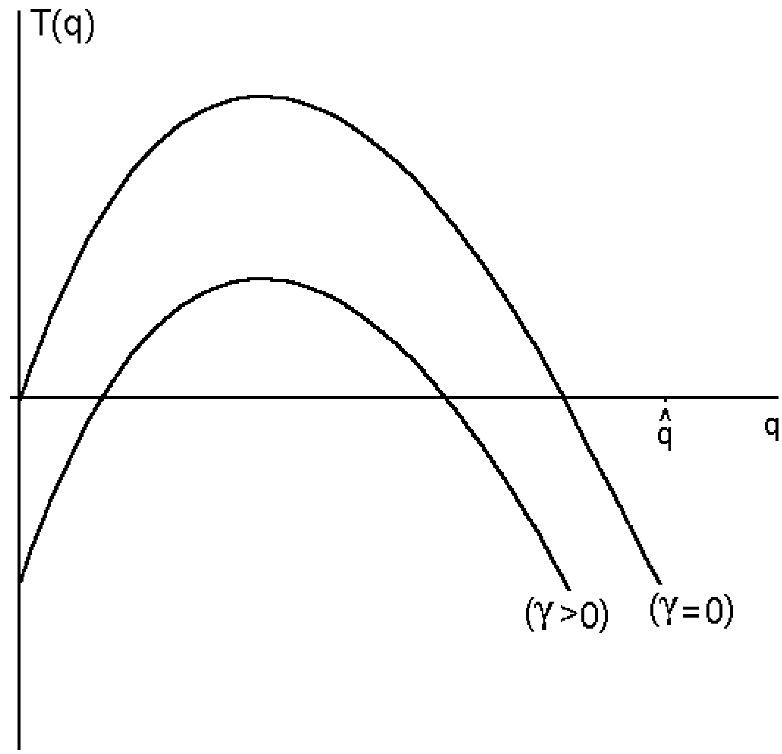

Figure 5:

in the endogenous matching model: $\bar{\gamma} < \hat{\gamma}$. Moreover, if $\gamma = 0$ so that there is a unique equilibrium in both models, say q^r and q^e, we know that $q^e > q^r$. Intuitively, these results hold because money is more valuable in the endogenous matching case.

All of the above results are for the case of no commitment. In the case of full commitment, the model with endogenous meetings is fundamentally different than the random matching model. When $M \leq 1/2$, $a_1^e = 1$ and $a_0^e = M/(1-M)$ so that agents with money constitute the short side of the market. In that case, agents with money have all the bargaining power, and effectively make take-it-or-leave-it offers to agents without money.This yields $V_1 - V_0 - c(q) = 0$, which implies $V_0 = 0$ by (8). Hence, q satisfies

$$S(q) = u(q) - \gamma - (1+r)\,c(q) = 0. \tag{11}$$

It is easy to see that for $\gamma = 0$, $S(0) = 0$ and there exists a unique $q \in (0, \hat{q})$ such that $S(q) = 0$. For $\gamma > 0$, $S(q)$ shifts down. Hence there exists a critical $\grave{\gamma}$ such that if $\gamma \in (0, \grave{\gamma})$, there exists an even number of monetary equilibria (i.e. an even number of positive solutions to $S = 0$), and if $\gamma > \grave{\gamma}$, there exists no monetary equilibrium. Moreover, $S(q)$ is independent of arrival rates (and hence M). When $M > 1/2$, $a_0^e = 1$ and $a_1^e = (1-M)/M$ so that agents without money constitute the short side of the market and effectively make take-it-or-leave-it offers to agents with money. In this case, $u(q) + V_0 - V_1 = 0$, which implies the only equilibrium is $q = 0$.

Therefore, unlike the random matching model, there is a possibility of monetary equilibria only when $M \leq 1/2$ in the endogenous matching model with commitment. Furthermore, since all of the surplus in this case goes to the buyer,

$q^e > q^r$ when $M \leq 1/2$. This can be shown formally since $S(q) - T(q) > 0$ at $M = 0$ and $q^e$is independent of M when $M \leq 1/2$. See Figure **??** for the case where $\gamma = 0$.

6 Commodity Money

The model is very similar to Kiyotaki and Wright (1989). There is a continuum of agents on $[0, 1]$. There are equal proportions of 3 types and there are 3 goods, where type i agents consume good i and produce good $i + 1$ mod 3. Each agent derives utility $u > 0$ from his consumption good. Agents discount the future at rate $\beta \in (0, 1)$. Good are indivisible and storable, and the cost of storing good j is given by c_j. We consider two versions of the model: in model A, $c_3 > c_2 > c_1 > 0$; in model B, $c_1 > c_2 > c_3 > 0$.[12] As for the matching technology, we assume that an individual can choose the type of agent with whom he wishes to trade, he draws an individual of that type at random. Hence, the probability of meeting a particular agent with whom one has traded before is zero, since there is a continuum of each type (the same thing would be true with a countable infinity of agents).

As in Section 2, the aggregate inventory distribution at time t is denoted by $[...p_t(i, j)...]$, where $p_t(i, j)$ is the proportion of type i agents holding good j at t. We assume that any agent of type i who acquires good i immediately consumes it and produces good $i + 1$, which is not restrictive provided that $\frac{u}{1-\beta} - c_i > -c_j$. Hence, we know that $p_t(i, i) = 0$, and so the distribution of commodity holdings at any point in time is completely characterized by three

[12] These are the only two relevant versions of the model – everything else is a relabeling. Also, note that reordering storage costs here is equivalent to changing the preference-technology specification between the two models, as was done in the original Kiyotaki and Wright (1989) analysis.

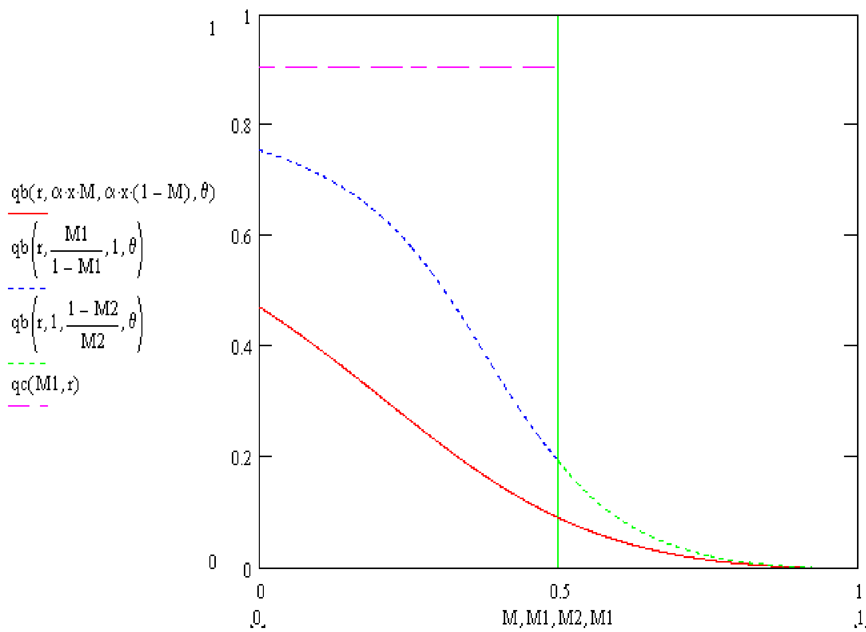
qb(r, α·x·M, α·x·(1 − M), θ)
qb(r, M1/(1 − M1), 1, θ)
qb(r, 1, (1 − M2)/M2, θ)
qc(M1, r)
1
0.8
0.6
0.4
0.2
0
0
0.5
1
M, M1, M2, M1

Figure 6:

numbers $p_t = [p_t(1,2), p_t(2,3), p_t(3,1)]$. Given a sequence $\{p_t\}$, we define an assignment rule $\{\psi_t\}$ which induces a partition $\{\theta_t\}$ of A into subsets of size 1 or 2 (i.e. the set of agents who are in autarchy or bilateral pairs) as in Section 2. In this section, we look for memoryless equilibrium matching processes.

To fix ideas, consider the aggregate state where all type i agents are inventorying their production good, $p = (111)$. We can easily enumerate all symmetric allocation rules from this state. In particular, types 1 and 2 can be matched and trade leading to state (011), types 1 and 3 can be matched and trade leading to state (110), or types 2 and 3 can be matched and trade leading to state (101). From each of these states, we can again enumerate all symmetric allocation rules that change the aggregate state.[13] From state (011), types 1 and 3 can be matched and trade leading back to state (111) or types 2 and 3 can be matched and trade leading to state (001). We call the path of matches (and trades) $(111) \overset{1,2}{\to} (011) \overset{1,3}{\to} (111)$ a two-cycle. From (001), all types 1 and 2 can trade leading back to state (111) or all types 1 and 3 can trade leading to state (101). We call the path of trades $(111) \overset{1,2}{\to} (011) \overset{2,3}{\to} (001) \overset{1,2}{\to} (111)$ a three-cycle. Table 1 in Appendix 1 enumerates all the possible cyclic patterns (we can have up to 8 cycles) associated with symmetric allocation rules.There are also states associated with asymmetric allocation rules. For instance, one candidate asymmetric allocation rule has $\frac{1}{2}$ of type 2 agents holding commodity 1 trade with $\frac{1}{2}$ of type 1 agents holding commodity 2 while $\frac{1}{2}$ of type 2 agents holding commodity 3 trade with $\frac{1}{2}$ of type 3 agents holding commodity 1. Such an allocation rule is associated with state $(1, \frac{1}{2}, 1)$ in each period. This allocation rule is a steady state version of a $111 \overset{2,3}{\to} 101 \overset{1,2}{\to} 111$ two-cycle. In much of what follows,

[13]To economize on notation, we do not consider bilateral matches among agents inventorying the same good.

we will use the following Lemma.

Lemma 1 *: It is a dominant strategy to trade for your consumption good when holding your production good.*

The first two Propositions assert the existence and uniqueness (among symmetric equilibria) of the fundamental equilibrium for our model A. In particular, there is no equilibrium in which the high storage good serves as a medium of exchange.

Proposition 5 *The fundamental equilibrium* $111 \overset{2,3}{\to} 101 \overset{1,2}{\to} 111$ *is a symmetric pure strategy equilibrium cycle for model A.*

Proposition 6 *Model A has no other symmetric pure strategy equilibria.*

Finally, the next Proposition asserts the existence for at least some parameter values of an asymmetric equilibrium. This is an 1-period analog of the symmetric fundamental equilibrium described earlier.

Proposition 7 *In Model A, the asymmetric equilibrium* $(1, \frac{1}{2}, 1)$ *exists for all* t *provided* $\frac{\beta u}{2} < (c_3 - c_2)$ *and* $\frac{\beta u}{2} < (c_2 - c_1)$.

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7 Appendix

7.1 Cycles from Initial State 111 in the Commodity Money Model

Table 1

Cycle							
				(111)			
		$1,2 \swarrow$		$\downarrow 1,3$		$\searrow 2,3$	
		(011)		(110)		(101)	
	$1,3 \swarrow$	$\downarrow 2,3$	$2,3 \swarrow$		$\searrow 1,2$	$\downarrow 1,3$	$\searrow 1,2$
2	(111)	(001)	(111)		(010)	(100)	(111)
	$1,2 \swarrow$	$\downarrow 1,3$		$1,3 \swarrow$	$\downarrow 2,3$	$\downarrow 1,2$	$\searrow 2,3$
3	(111)	(101)		(111)	(011)	(110)	(111)
	$1,2 \swarrow$	$\downarrow 1,3$		$1,3 \swarrow$	$\downarrow 2,3$	$\downarrow 1,2$	$\searrow 2,3$
4	(111)	(100)		(111)	(001)	(010)	(111)
	$2,3 \swarrow$	$\downarrow 1,2$		$1,2 \swarrow$	$\downarrow 1,3$	$\downarrow 2,3$	$\searrow 1,3$
5	(111)	(110)		(111)	(101)	(011)	(111)
	$2,3 \swarrow$	$\downarrow 1,2$		$1,2 \swarrow$	$\downarrow 1,3$	$\downarrow 2,3$	$\searrow 1,3$
6	(111)	(010)		(111)	(100)	(001)	(111)
	$1,3 \swarrow$	$\downarrow 2,3$		$2,3 \swarrow$	$\downarrow 1,2$	$\downarrow 1,3$	$\searrow 1,2$
7	(111)	(011)		(111)	(110)	(101)	(111)

7.2 Proofs for Money and Memory

Proposition 1. There exist regions of the parameter space where credit with private memory cannot support the first best outcome while money can.

Proof. Since the text proved the case for $N = 3$ and $M = 1$, here we consider the case with $N = 4$ and $M = 2$. With no public observability of histories, a producer who considers deviating from s^G by not producing when he is supposed to and instead choosing to consume his autarchic amount will be punished only after this deviation is discovered. Let $i \to j$ stand for "agent i produces for agent j." Without loss of generality, suppose agent 2 does not produce for agent 1 (i.e., $2 \nrightarrow 1$ in t). This deviation by 2 does not effect the play of 3 or 4 pair at t. At $t+1$ agent 1's strategy calls for him not to produce for 4, but 3's strategy calls for him to produce for 2 and 2 has no reason to reveal that he is a deviator. At $t+2$, agent 4's strategy calls for him not to produce for 3, while 1 and 2 are in autarchy. It is only at $t+3$ that 2 suffers the effects of his deviation, whence he is forever into permanent autarchy and receives $\frac{0}{1-\beta}$. This is summarized in the following table:

$\underline{period}$	$\underline{actions}$		$\underline{payoff\ to\ deviator}$
t	$2 \nrightarrow 1$	$4 \to 3$	0
$t+1$	$1 \nrightarrow 4$	$3 \to 2$	u
$t+2$	$2 \nrightarrow 1$	$4 \nrightarrow 3$	0
$t+3$	$1 \nrightarrow 4$	$3 \nrightarrow 2$	$\frac{0}{1-\beta}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Agent 2's payoff if he deviates is

$$V^D = 0 + \beta \left[u + \beta \left(0 + \frac{\beta 0}{1-\beta} \right) \right]. \tag{12}$$

Hence, a producer will not deviate if $V^G \geq V^D$, or

$$c \leq \beta^3 u, \tag{13}$$

which differs from the condition in the monetary equilibrium. In particular, it is harder to satisfy this condition. To complete the proof, notice that the monetary equilibrium with no memory remains an equilibrium with private memory; i.e., even though other equilibria where agents condition their actions on their private histories may exist, there is always one equilibrium where agents do not condition on their histories. ■

Proposition 2. "Credit" equilibria become harder to support as the number of agents in the economy, N, increases.

Proof. Given preferences, the payoff along the credit equilibrium path does not depend on N; i.e., an agent along that path always consumes and produces every other period. It is therefore sufficient to study how the payoff of a deviator changes with N. Consider two economies, one with N agents and one with $N+2$ agents, for any even number N. We have

$$V_N^D = 0 + \beta u + \beta^2 0 + ... + \beta^{N-3} u + \beta^{N-2} 0 + \beta^{N-1} \frac{0}{1-\beta}, \tag{14}$$

$$V_{N+2}^D = 0 + \beta u + \beta^2 0 + ... + \beta^{N-1} u + \beta^{N} 0 + \beta^{N+1} \frac{0}{1-\beta}. \tag{15}$$

Now, $V_{N+2}^D > V_N^D \Leftrightarrow u > \frac{0}{1-\beta}$, which is always true since $\beta \in (0,1)$. The result follows by a straightforward induction argument. ■

7.3 Proofs for the Commodity Money Model

Lemma 1. It is a dominant strategy to trade for your consumption good when holding your production good.

Proof. Doing so allows you to consume and produce your production good immediately, putting you back on the equilibrium path with a boost in utility. ■

Lemma A.1. In model A's FE, no agent of type 3 accepts good 2.

Proof. Suppose 3 accepts 2 in state 111. First assume he holds it for exactly 1 period. Since next period the state is 101, he must trade for 1, which takes him back to the equilibrium path, where he reverts to the candidate strategy (using unimprovability) with a lower payoff since he incurred higher storage cost $c_2 > c_1$. Now assume he holds it for exactly 2 periods, when the state is again 111. If he trades for 3, he consumes and produces 1, putting him back on the equilibrium path, where he reverts to the candidate strategy with a lower payoff since he incurred higher storage cost. If he trades for 1, he is back to the equilibrium path without consuming, and he reverts to the candidate strategy with an even lower payoff. Now assume he holds it more than 2 periods; by stationarity, he should hold it forever, which obviously is a bad idea. Hence, accepting 2 in state 111 and holding it any number of periods is a bad idea. Now suppose 3 accepts 2 in state 101. If he holds it 1 period and trades for 3, he consumes and produces 1, putting him back on the equilibrium path with a lower payoff since $c_2 > c_1$. If he holds it exactly 1 period and trades for 1, he is back to the equilibrium path without consuming, with an even lower payoff. If he holds it exactly 2 periods, the state will again be 101, and he must trade it for 1, putting him back to the equilibrium path with lower payoff. If he holds it more than 2 periods then, by stationarity, he should hold it forever, which obviously is a bad idea. Hence, accepting 2 in state 101 is also a bad idea. We conclude that accepting good 2 always makes 3 strictly worse off. ■

Lemma A.2. In model A's FE, no agent of type 1 accepts good 3.

Proof. Suppose 1 accepts 3, which must happen in 111. First assume he holds it exactly 1 period and trades for 1, whereupon he consumes and is back on the equilibrium path with a lower payoff since he would have consumed anyway and $c_3 > c_2$. If he holds it for exactly 1 period and trades for 2 he is back he is on the equilibrium path without consuming for an even lower payoff. Hence it is a bad idea to accept 3 and hold it for 1 period. Now assume he holds it for exactly 2 periods, where the state is again 111. If he trades for 2 he is back on the equilibrium path with a lower payoff. If he trades for 1 he consumes and is back on the equilibrium path, but he has consumed one period later than he would have on the equilibrium path and incurred higher storage cost. Hence it is a bad idea to accept 3 and store it for 2 periods. If he holds it more than 2 periods then, by stationarity, he should hold it forever, which obviously is a bad idea. We conclude that accepting 3 always makes I strictly worse off. ■

Proposition 5. The fundamental equilibrium $111 \stackrel{2,3}{\to} 101 \stackrel{1,2}{\to} 111$ is a symmetric pure strategy equilibrium cycle for model A.

Proof. Possible deviations in 111 are:

1. $(2,3) \to (1,2)$. Suboptimal by lemma A.2 for agent type 1.

2. $(2,3) \to (1,3)$. Suboptimal by lemma 1.

3. $(2,3) \to 2$. Suppose 2 stores commodity 3 in state 111. After he holds it for one period, the agent has two options in 101. Let $V_2^D(101)$ be the utility for a deviating agent type 2 associated with holding one's production good when all other type 2s are holding commodity 1. First, assume he holds it another period. This puts him back on the equilibrium path and he has incurred the highest storage costs for two periods rather than hold the lowest storage cost and then consume. Obviously this is suboptimal. The only other possibility in state 101 is to trade with a type 3 agent (since by lemma A.2 type 1 won't trade for commodity 3). In this case, in 111 he is holding commodity 1 (not his production good). He then has two options. First, he can store commodity 1 another period. This puts him back onto the equilibrium path with payoff:

$$-c_3 - \beta c_1 - \beta^2 c_1 + \beta^3 V_2(101)$$

which is obviously dominated by

$$-c_1 + \beta(u - c_3) - \beta^2 c_1 + \beta^3 V_2(101).$$

Second, he can trade commodity 1 to agent 1 (agent 1 will accept by lemma 1) and consume which means he enters state 101 holding his production good. But this puts him back to the two options he had in 101 and the payoffs associated with this deviation strategy are uniformly lower

$$-c_3 - \beta c_1 + \beta^2(u - c_3) + \beta^3 V_2^D(101).$$

By stationarity, other deviations are suboptimal as well.

4. $(2,3) \to 3$. Suboptimal by lemma 1.

Possible deviations in 101 are:

1. $(1,2) \to (1,3)$. While agent type 1 would be indifferent, type 3 does not trade by lemma A.1.

2. $(1,2) \to (2,3)$. Since both agents are inventorying the same good, this amounts to storage of commodity 1 by type 2. See below.

3. $(1,2) \to 1$. Suboptimal by lemma 1.

4. $(1,2) \to 2$. Suppose 2 stores commodity 1 in state 101. After he holds it for one period, the agent has two options in 111. First, assume he holds it another period (since type 3 are also holding 1 this option is subsumed here). This puts him back on the equilibrium path. The benefit is that he has held the lowest storage cost good for two periods, the cost is he has not consumed. This is suboptimal provided $-c_1 < u - c_3$. Second, assume he trades the commodity to a type 1 agent in 111 (which is possible applying lemma 1 for type 1). In this case, he consumes and enters 101 holding his production good rather than commodity 1 as all other type 2s are doing. Then, in 101, the deviating agent again has two options. If he holds commodity 3 for one more period, he is back on the equilibrium path. The payoff to this deviation is:

$$-c_1 + \beta(u - c_3) - \beta^2 c_3 + \beta^3 V_2(111)$$

which is dominated by the equilibrium strategy since $-(1-\beta)c_1 < (1-\beta)(u-c_3) + \beta^2 u$. If instead, he trades commodity 3 to agent type 3 in 101 (which is possible applying lemma 1 for type 3) he lowers his storage cost and enters state 111 holding a commodity other than his production good (i.e. off the equilibrium path). But he has exactly the same two options for 111 as already discussed. By stationarity, provided he does no better along the path, the strategy is suboptimal. The payoff is:

$$-c_1 + \frac{\beta\left[u - c_3 - \beta c_1\right]}{1-\beta^2}.$$

But this is dominated by the equilibrium strategy if $-c_1 < u - c_3$.[14] By stationarity, other deviations are suboptimal as well. Thus, we have exhausted all possibilities. ■

Proposition 6. Model A has no other symmetric pure strategy equilibrium cycles.

Proof. We exhaust the possibilities below:

2 Cycles

$\boxed{111 \to 011 \to 111}$

$V_2(011) = -c_3 + \beta(u - c_3) + \beta^2 V_2(011)$

$V_2^D(011) = -c_1 + \beta(u - c_3) + \beta^2 V_2(011)$

$V_1(111) = -c_3 + \beta(u - c_2) + \beta^2 V_1(111)$

$V_1^D(111) = u - c_2 - \beta c_2 + \beta^2 V_1(111)$

$\boxed{111 \to 110 \to 111}$

$V_2(111) = -c_3 + \beta(u - c_3) + \beta^2 V_2(111)$

$V_2^D(111) = -c_1 + \beta(u - c_3) + \beta^2 V_2(111)$

$V_3(111) = -c_2 + \beta(u - c_1) + \beta^2 V_3(111)$

[14] This deviation simply put off the proposed equilibrium strategy by one period.

$V_3^D(111) = u - c_1 - \beta c_1 + \beta^2 V_3(111)$

$V_1(110) = -c_2 + \beta V_1(111)$

$V_1^D(110) = u - c_2 + \beta V_1(111)$

3 Cycles

$\boxed{111 \to 101 \to 100 \to 111}$

$V_3(101) = -c_2 - \beta c_1 + \beta^2 V_3(111)$

$V_3^D(101) = -c_1 - \beta c_1 + \beta^2 V_3(111)$

$\boxed{111 \to 110 \to 010 \to 111}$

$V_3(111) = -c_2 - \beta c_2 + \beta^2(u - c_1) + \beta^3 V_3(111)$

$V_3^D(111) = -c_1 - \beta c_1 + \beta^2(u - c_1) + \beta^3 V_3(111)$

$V_1(010) = -c_2 + \beta V_1(111)$

$V_1^D(010) = u - c_2 + \beta V_1(111)$

$\boxed{111 \to 011 \to 001 \to 111}$

$V_1(111) = -c_3 - \beta c_3 + \beta^2(u - c_2) + \beta^3 V_1(111)$

$V_1^D(111) = -c_2 - \beta c_2 + \beta^2(u - c_2) + \beta^3 V_1(111)$

$V_2(001) = -c_3 + \beta V_2(111)$

$V_2^D(001) = u - c_3 + \beta V_2(111)$

4 Cycles

$\boxed{111 \to 011 \to 001 \to 101 \to 111}$

$V_1(011) = -c_3 + \beta(u - c_2) + \beta^2 V_1(101)$

$V_1^D(011) = u - c_2 - \beta c_2 + \beta^2 V_1(101)$

$\boxed{111 \to 110 \to 010 \to 011 \to 111}$

$V_1(110) = -c_3 - \beta c_3 + \beta^2(u - c_2) + \beta^3 V_1(111)$

$V_1^D(110) = -c_2 - \beta c_3 + \beta^2(u - c_2) + \beta^3 V_1(111)$

$\boxed{111 \to 101 \to 100 \to 110 \to 111}$

$V_3(101) = -c_2 - \beta c_2 + \beta^2(u - c_1) + \beta^2 V_3(111)$

$V_3^D(101) = -c_1 - \beta c_2 + \beta^2(u - c_1) + \beta^2 V_3(111)$

5 Cycles

$\boxed{111 \to 110 \to 010 \to 011 \to 001 \to 111}$

$V_1(110) = -c_3 - \beta c_3 - \beta^2 c_3 + \beta^3 V_1(001)$

$V_1^D(110) = -c_2 - \beta c_2 - \beta^2 c_3 + \beta^3 V_1(001)$

$\boxed{111 \to 011 \to 001 \to 101 \to 100 \to 111}$

$V_3(101) = -c_2 - \beta c_1 + \beta^2 V_3(111)$

$V_3^D(101) = -c_1 - \beta c_1 + \beta^2 V_3(111)$

$\boxed{111 \to 101 \to 100 \to 110 \to 010 \to 111}$

$V_3(101) = -c_2 - \beta c_2 - \beta^2 c_2 + \beta^3(u - c_1) + \beta^4 V_3(111)$

$V_3^D(101) = -c_1 - \beta c_1 - \beta^2 c_1 + \beta^3(u - c_1) + \beta^4 V_3(111)$

6 Cycles

$\boxed{111 \to 011 \to 001 \to 101 \to 100 \to 110 \to 111}$

$V_3(101) = -c_2 - \beta c_2 + \beta^2(u - c_1) + \beta^3 V_3(111)$

$V_3^D(101) = -c_1 - \beta c_2 + \beta^2(u - c_1) + \beta^3 V_3(111)$

$V_1(100) = u - c_2 + \beta V_1(110)$

$V_1^D(100) = u - c_2 + \beta V_1(110)$

$\boxed{111 \to 101 \to 100 \to 110 \to 010 \to 011 \to 111}$

$V_3(101) = -c_2 - \beta c_2 - \beta^2 c_2 + \beta^3(u - c_1) + \beta^4 V_3(011)$

$V_3^D(101) = -c_1 - \beta c_2 - \beta^2 c_2 + \beta^3(u - c_1) + \beta^4 V_3(011)$

$\boxed{111 \to 110 \to 010 \to 011 \to 001 \to 101 \to 111}$

$V_1(110) = -c_3 - \beta c_3 - \beta^2 c_3 + \beta^3(u - c_2) + \beta^4 V_1(101)$

$V_1^D(110) = -c_2 - \beta c_3 - \beta^2 c_3 + \beta^3(u - c_2) + \beta^4 V_1(101)$

$\boxed{111 \to 011 \to 001 \to 101 \to 100 \to 110 \to 010 \to 011}$

$V_3(101) = -c_2 - \beta c_2 - \beta^2 c_2 + \beta^3(u - c_1) + \beta^4 V_3(111)$

$V_3^D(101) = -c_1 - \beta c_1 - \beta^2 c_1 + \beta^3(u - c_1) + \beta^4 V_3(111)$

$\boxed{111 \to 101 \to 100 \to 110 \to 010 \to 011 \to 001 \to 101}$

$V_1(110) = -c_3 - \beta c_3 - \beta^2 c_3 + \beta^3(u - c_2) + \beta^4 V_1(111)$

$V_1^D(110) = -c_2 - \beta c_2 - \beta^2 c_2 + \beta^3(u - c_2) + \beta^4 V_1(111)$

$\boxed{111 \to 110 \to 010 \to 011 \to 001 \to 101 \to 111 \to 110}$

$V_1(110) = -c_3 - \beta c_3 - \beta^2 c_3 + \beta^3 V_1(001)$

$V_1^D(110) = -c_2 - \beta c_2 - \beta^2 c_3 + \beta^3 V_1(001)$

7 Cycles

$\boxed{111 \to 110 \to 010 \to 011 \to 001 \to 101 \to 100 \to 111}$

$V_3(101) = -c_2 - \beta c_1 + \beta^2 V_3(111)$

$V_3^D(101) = -c_1 - \beta c_1 + \beta^2 V_3(111)$

$\boxed{111 \to 011 \to 001 \to 101 \to 100 \to 110 \to 010 \to 111}$

$V_3(101) = -c_2 - \beta c_2 - \beta^2 c_2 + \beta^3(u - c_1) + \beta^4 V_3(111)$

$V_3^D(101) = -c_1 - \beta c_1 - \beta^2 c_1 + \beta^3(u - c_1) + \beta^4 V_3(111)$

$\boxed{111 \to 101 \to 100 \to 110 \to 010 \to 011 \to 001 \to 111}$

$V_1(110) = -c_3 - \beta c_3 - \beta^2 c_3 + \beta^3(u - c_2) + \beta^4 V_1(111)$

$V_1^D(110) = -c_2 - \beta c_2 - \beta^2 c_2 + \beta^3(u - c_2) + \beta^4 V_1(111)$ ■

Proposition 7. Model B has no symmetric pure strategy equilibrium cycles.

Proof. We exhaust the possibilities below:

2 Cycles

$\boxed{111 \to 011 \to 111}$

$V_1(111) = -c_3 + \beta(u - c_2) + \beta^2 V_1(111)$

$V_1^D(111) \geq u - c_2 + \beta(u - c_2) + \beta\frac{[-c_3+\beta(u-c_2)]}{1-\beta^2}$ spell out

$V_3(111) = -c_1 + \beta(u - c_1) + \beta^2 V_3(111)$

$V_3^D(111) = -c_2 + \beta(u - c_1) + \beta^2 V_3(111)$

$\boxed{111 \rightarrow 101 \rightarrow 111}$

$V_2(111) = -c_3 + \beta(u - c_3) + \beta^2 V_2(111)$

$V_2^D(111) = u - c_3 - \beta c_3 + \beta^2 V_2(111)$

$V_1(111) = -c_2 + \beta(u - c_2) + \beta^2 V_1(111)$

$V_1^D(111) = -c_3 + \beta(u - c_2) + \beta^2 V_1(111)$

$\boxed{111 \rightarrow 110 \rightarrow 111}$ spell out

$V_1(110) = -c_2 + \beta(u - c_2) + \beta^2 V_1(110)$

$V_1^D(110) = -c_3 + \beta(u - c_2) + \beta^2 V_1(110)$

$V_2(110) = u - c_3 + \beta V_2(111)$

$V_2^D(110) = u - c_3 + \beta V_2(111)$

$V_3(111) = u - c_1 - \beta c_2 - \beta^2 c_2 + \beta^3 V_3(110)$

$V_3^D(111) = -c_2 + \beta(u - c_1) - \beta^2 c_2 + \beta^3 V_3(110)$

$V_3^D(111) > V_3(111) \Leftrightarrow u - c_1 - \beta c_2 > \beta(u - c_1) - c_2$

3 Cycles

$\boxed{111 \rightarrow 101 \rightarrow 100 \rightarrow 111}$

$V_2(111) = -c_1 - \beta c_1 + \beta^2(u - c_3) + \beta^3 V_2(111)$

$V_2^D(111) = -c_3 - \beta c_3 + \beta^2(u - c_3) + \beta^3 V_2(111)$

$\boxed{111 \rightarrow 110 \rightarrow 010 \rightarrow 111}$

$V_3(110) = -c_2 - \beta(u - c_1) + \beta^2 V_3(111)$

$V_3^D(110) = u - c_1 + \beta(u - c_1) + \beta^2 V_3(111)$

$\boxed{111 \rightarrow 011 \rightarrow 001 \rightarrow 111}$

$V_3(001) = -c_1 + \beta V_3(111)$

$V_3^D(001) = u - c_1 + \beta V_3(111)$

4 Cycles

$\boxed{111 \to 011 \to 001 \to 101 \to 111}$

$V_3(101) = -c_1 - \beta c_1 + \beta^2 V_3(011)$

$V_3^D(101) = -c_2 + \beta(u - c_1) + \beta^2 V_3(011)$

$\boxed{111 \to 110 \to 010 \to 011 \to 111}$ spell out

$V_3(110) = -c_2 + \beta(u - c_1) + \beta^2 V_3(011)$

$V_3^D(110) = u - c_1 - \beta c_1 + \beta^2 V_3(011)$

$\boxed{111 \to 101 \to 100 \to 110 \to 111}$

$V_1(110) = -c_2 - \beta c_2 + \beta^2 V_1(101)$

$V_1^D(110) = -c_3 + \beta(u - c_1) + \beta^2 V_1(101)$

5 Cycles

$\boxed{111 \to 110 \to 010 \to 011 \to 001 \to 111}$

$V_3(001) = -c_1 + \beta V_3(111)$

$V_3^D(001) = u - c_1 + \beta V_3(111)$

$\boxed{111 \to 011 \to 001 \to 101 \to 100 \to 111}$

$V_2(011) = -c_1 - \beta c_1 + \beta^2 V_2(101)$

$V_2^D(011) = -c_3 - \beta c_1 + \beta^2 V_2(101)$

$\boxed{111 \to 101 \to 100 \to 110 \to 010 \to 111}$

$V_3(110) = -c_2 + \beta(u - c_1) + \beta^2 V_3(111)$

$V_3^D(110) = u - c_1 + \beta(u - c_1) + \beta^2 V_3(111)$

$V_1(010) = -c_2 + \beta V_1(111)$

$V_1^D(010) = u - c_2 + \beta V_1(111)$

6 Cycles

$\boxed{111 \to 011 \to 001 \to 101 \to 100 \to 110 \to 111}$

$V_2(011) = -c_1 - \beta c_1 + \beta^2 V_2(101)$

$V_2^D(011) = -c_3 - \beta c_1 + \beta^2 V_2(101)$

$\boxed{111 \to 101 \to 100 \to 110 \to 010 \to 011 \to 111}$ spell out

$V_2(111) = -c_1 - \beta c_1 + \beta^2(u - c_3) + \beta^3 V_2(110)$

$V_2^D(111) = -c_3 - \beta c_3 + \beta^2(u - c_3) + \beta^3 V_2(110)$

$V_3(100) = -c_2 - \beta c_2 + \beta^2 V_3(110)$

$V_3^D(100) = u - c_1 - \beta c_2 + \beta^2 V_3(110)$

$V_1(110) = c_3 - \beta c_3 - \beta^2 V_1(011)$

$V_1^D(110) = u - c_2 - \beta c_3 + \beta^2 V_1(011)$

$\boxed{111 \to 110 \to 010 \to 011 \to 001 \to 101 \to 111}$

$V_2(011) = -c_1 - \beta c_1 + \beta^2 V_2(101)$

$V_2^D(011) = -c_3 - \beta c_1 + \beta^2 V_2(101)$

$\boxed{111 \to 011 \to 001 \to 101 \to 100 \to 110 \to 010 \to 011}$

$V_2(011) = -c_1 - \beta c_1 + \beta^2 V_2(101)$

$V_2^D(011) = -c_3 - \beta c_1 + \beta^2 V_2(101)$

$\boxed{111 \to 101 \to 100 \to 110 \to 010 \to 011 \to 001 \to 101}$ spell out

$V_2(111) = -c_1 - \beta c_1 + \beta^2(u - c_3) + \beta^3 V_2(110)$

$V_2^D(111) = -c_3 - \beta c_3 + \beta^2(u - c_3) + \beta^3 V_2(110)$

$V_3(100) = -c_2 - \beta c_2 + \beta^2 V_3(110)$

$V_3^D(100) = u - c_1 - \beta c_2 + \beta^2 V_3(110)$

$V_1(110) = c_3 - \beta c_3 - \beta^2 V_1(011)$

$V_1^D(110) = u - c_2 - \beta c_3 + \beta^2 V_1(011)$

$\boxed{111 \to 110 \to 010 \to 011 \to 001 \to 101 \to 111 \to 110}$

$V_2(011) = -c_1 - \beta c_1 + \beta^2 V_2(101)$

$V_2^D(011) = -c_3 - \beta c_1 + \beta^2 V_2(101)$

7 Cycles

$\boxed{111 \to 110 \to 010 \to 011 \to 001 \to 101 \to 100 \to 111}$

$V_3(101) = -c_2 - \beta c_1 + \beta^2 V_3(111)$

$V_3^D(101) = -c_1 - \beta c_1 + \beta^2 V_3(111)$

$\boxed{111 \to 011 \to 001 \to 101 \to 100 \to 110 \to 010 \to 111}$

$V_3(101) = -c_2 - \beta c_2 - \beta^2 c_2 + \beta^3(u - c_1) + \beta^4 V_3(111)$

$V_3^D(101) = -c_1 - \beta c_1 - \beta^2 c_1 + \beta^3(u - c_1) + \beta^4 V_3(111)$

$\boxed{111 \to 101 \to 100 \to 110 \to 010 \to 011 \to 001 \to 111}$

$V_1(110) = -c_3 - \beta c_3 - \beta^2 c_3 + \beta^3(u - c_2) + \beta^4 V_1(111)$

$V_1^D(110) = -c_2 - \beta c_2 - \beta^2 c_2 + \beta^3(u - c_2) + \beta^4 V_1(111)$ ■

Lemma A.3. Suppose $(1, \frac{1}{2}, 1)$ exists for all t in Model A. Then agent type 1 does not accept commodity 3 if $\frac{\beta u}{2} < (c_3 - c_2)$ and agent type 3 does not accept commodity 2 if $\frac{\beta u}{2} < (c_2 - c_1)$.

Proof. Suppose agent type 1 accepts commodity 3 in the current period then plays according to the allocation rule that generates $(1, \frac{1}{2}, 1)$ exists for all t. Thus, 1 assures himself consumption next period (since he can always obtain his consumption good from a type 3 trader inventorying commodity 1 by the UL). In that case he receives:

$$-c_3 + \beta(u - c_2) + \frac{\beta^2\left(\frac{1}{2}u - c_2\right)}{(1 - \beta)}$$

If instead, agent type 1 holds commodity 2 he receives:

$$-c_2 + \frac{\beta\left(\frac{1}{2}u - c_2\right)}{(1 - \beta)}$$

The inequality follows. The proof for agent type 3 is analogous. ■

Proposition 7. In Model A, the asymmetric equilibrium $(1, \frac{1}{2}, 1)$ exists for all t provided $\frac{\beta u}{2} < (c_3 - c_2)$ and $\frac{\beta u}{2} < (c_2 - c_1)$.

Proof. Let 2_i denote agent 2's holding of good $i = 1, 3$. Possible deviations are listed below:

1. $(1, 2_1) \rightarrow (1, 3)$. Suboptimal by lemma A.3 for agent type 3.
2. $(1, 2_1) \rightarrow (1, 2_3)$. Suboptimal by lemma 1.
3. $(1, 2_1) \rightarrow (2_1, 3)$. Indifference (same good).
4. $(2_3, 3) \rightarrow (1, 3)$. Suboptimal by lemma 1.
5. $(2_3, 3) \rightarrow (2_3, 1)$. Suboptimal by lemma A.3 for agent type 1.
6. $(2_3, 3) \rightarrow (2_1, 3)$. Indifference (same good).
7. $(1, 2_1) \rightarrow 1$. Suboptimal by lemma 1.
8. $(1, 2_1) \rightarrow 2_1$. Suboptimal by lemma 1.
9. $(2_3, 3) \rightarrow 2_3$. Agent type 2 delays obtaining the low cost commodity which assures him his consumption good in order to hold the high cost good and thus cannot be optimal.
10. $(2_3, 3) \rightarrow 3$. Suboptimal by lemma 1.
11. Unmatched 1, unmatched $3 \rightarrow (1, 3)$. Suboptimal by lemma A.3 for agent type 3. ■